# GAZETA MATEMATICĂ

## SERIA A

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## ARTICOLE

### Exotic series with fractional part function

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**Abstract**. The paper is about the calculation of special classes of series involving the floor function as well as the fractional part function.

**Keywords:** Fractional part function, floor function, summation of series, Abel's summation formula, generating functions, Taylor's formula.

**MSC:** Primary 32A05; Secondary 41A58, 65B10.

### 1. Introduction and the main results

In this paper we calculate some classes of exotic series involving the fractional part function as well as the floor function. Throughout this paper $\{x\}$ denotes the fractional part of $x$ and $\lfloor x \rfloor = x - \{x\}$ is the integer part of $x$, also known as the floor of $x$.

Before we state the main results of the paper we collect a formula that we need in our analysis. Recall that *Abel's summation by parts formula* ([1, p. 55], [2, Lemma A.1, p. 258]) states that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two sequences of real numbers and $A_n = \sum\limits_{k=1}^{n} a_k$, then

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1}).$$

We will be using the *infinite version* of this formula

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} (A_n b_{n+1}) + \sum_{k=1}^{\infty} A_k (b_k - b_{k+1}). \tag{1}$$

Now we are ready to state the main results of this paper.

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**Theorem 1.** (a) **A series that equals** 1. *The following equality holds*

$$\sum_{n=1}^{\infty} \frac{\{n!\mathrm{e}\}}{n!} = \sum_{n=1}^{\infty} \left( \mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!} \right) = 1.$$

(b) **The generating function of** $\left(\frac{\{n!\mathrm{e}\}}{n!}\right)_{n\geq 1}$**.**
*Let* $x \in \mathbb{R}$, $x \neq 1$. *Then*

$$\sum_{n=1}^{\infty} \frac{\{n!\mathrm{e}\}}{n!} x^n = \frac{\mathrm{e}^x - \mathrm{e}x}{x-1} + 1.$$

*Proof.* (a) We have, based on Taylor's formula, that

$$\mathrm{e} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{\mathrm{e}^{\theta}}{(n+1)!}, \quad \theta \in (0,1),$$

and this implies that

$$n!\mathrm{e} = n! \left( 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \right) + \frac{\mathrm{e}^{\theta}}{n+1}, \quad \theta \in (0,1).$$

It follows that

$$\{n!\mathrm{e}\} = n!\mathrm{e} - \lfloor n!\mathrm{e} \rfloor = n! \left( \mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!} \right). \tag{2}$$

Using formula (1) with $a_n = 1$ and $b_n = \mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!}$, we get that

$$\sum_{n=1}^{\infty} \frac{\{n!\mathrm{e}\}}{n!} = \sum_{n=1}^{\infty} \left( \mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!} \right) = \sum_{n=1}^{\infty} \frac{n}{(n+1)!}$$
$$= \sum_{n=1}^{\infty} \left( \frac{1}{n!} - \frac{1}{(n+1)!} \right) = 1.$$

(b) We have, based on (2), that

$$\sum_{n=1}^{\infty} \frac{\{n!\mathrm{e}\}}{n!} x^n = \sum_{n=1}^{\infty} \left( \mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!} \right) x^n.$$

This power series converges for all $x \in \mathbb{R}$ and for calculating it one can either apply Abel's summation formula with $a_n = x^n$ and $b_n = \mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!}$, or a method based on an integration technique. For the sake of diversity we use the later technique for computing the power series.

We have, based on (2), that

$$\begin{aligned}\{n!\mathrm{e}\} &= n!\left(\mathrm{e}-1-\frac{1}{1!}-\frac{1}{2!}-\cdots-\frac{1}{n!}\right)\\ &= n!\left(\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\cdots\right)\\ &= \sum_{k=1}^{\infty}\frac{1}{(n+1)(n+2)\cdots(n+k)}.\end{aligned} \tag{3}$$

This implies, since

$$\frac{1}{n(n+1)\cdots(n+k)} = \frac{1}{k!}\int_0^1 (1-x)^k x^{n-1}\mathrm{d}x, \tag{4}$$

that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\{n!\mathrm{e}\}}{n!}x^n &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{x^n}{n!(n+1)\cdots(n+k)}\\ &= \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!}\sum_{k=1}^{\infty}\frac{1}{k!}\int_0^1(1-t)^k t^{n-1}\mathrm{d}t\\ &= \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!}\int_0^1\left(\mathrm{e}^{1-t}-1\right)t^{n-1}\mathrm{d}t\\ &= x\int_0^1\left(\mathrm{e}^{1-t}-1\right)\mathrm{e}^{tx}\mathrm{d}t\\ &= \frac{\mathrm{e}^x-\mathrm{e}x}{x-1}+1,\end{aligned}$$

and the theorem is proved. □

**Remark 2.** One can prove more generally that, if $f$ is a function which has a Taylor series representation at 0 with radius of convergence $R > 1$, then

$$\sum_{n=1}^{\infty}\frac{\{n!\mathrm{e}\}}{n!}f^{(n)}(0) = \sum_{n=1}^{\infty}\left(\mathrm{e}-1-\frac{1}{1!}-\frac{1}{2!}-\cdots-\frac{1}{n!}\right)f^{(n)}(0) = \mathrm{e}\int_0^1 \mathrm{e}^{-t}f(t)\mathrm{d}t.$$

**Corollary 3.** *The following equalities hold*:

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\{n!\mathrm{e}\}}{n!} = \sum_{n=1}^{\infty}(-1)^n\left(\mathrm{e}-1-\frac{1}{1!}-\frac{1}{2!}-\cdots-\frac{1}{n!}\right) = 1-\cosh 1$;

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{\lfloor n!\mathrm{e}\rfloor}{n!}\,x^n = \frac{\mathrm{e}^x}{1-x}-1$, for $x\in(-1,1)$.

*Proof.* (a) This follows from part (b) of Theorem 1 when $x=-1$.

(b) We have, based on part (b) of Theorem 1, that

$$\sum_{n=1}^{\infty} \frac{\lfloor n!\mathrm{e} \rfloor}{n!}\, x^n = \sum_{n=1}^{\infty} \left( \mathrm{e}x^n - \frac{\{n!\mathrm{e}\}}{n!}\, x^n \right) = \frac{\mathrm{e}x}{1-x} - \left( \frac{\mathrm{e}^x - \mathrm{e}x}{x-1} + 1 \right)$$
$$= \frac{\mathrm{e}^x}{1-x} - 1,$$

and the corollary is proved. □

**Theorem 4.** (a) **A binomial series.**

*Let $k \geq 0$ be an integer. The following equality holds*

$$\sum_{n=k}^{\infty} \binom{n}{k} \left( \mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!} \right) = \frac{\mathrm{e}}{(k+1)!}.$$

(b) **A special series with the fractional part function.**
*Let $k \geq 0$ be an integer. The following equality holds*

$$\sum_{n=1}^{\infty} \frac{\{(n+k)!\mathrm{e}\}}{n!} = \frac{\mathrm{e}}{k+1} - k! \left( \mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{k!} \right).$$

*Proof.* (a) One can check, using integration by parts, that the following formula holds

$$\mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!} = \frac{1}{n!} \int_0^1 (1-t)^n \mathrm{e}^t \mathrm{d}t, \quad n \geq 0.$$

It follows that

$$\sum_{n=k}^{\infty} \binom{n}{k} \left( \mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!} \right) = \sum_{n=k}^{\infty} \binom{n}{k} \frac{1}{n!} \int_0^1 (1-t)^n \mathrm{e}^t \mathrm{d}t$$
$$= \frac{1}{k!} \int_0^1 (1-t)^k \mathrm{e}^t \left( \sum_{n=k}^{\infty} \frac{(1-t)^{n-k}}{(n-k)!} \right) \mathrm{d}t$$
$$= \frac{\mathrm{e}}{k!} \int_0^1 (1-t)^k \mathrm{d}t$$
$$= \frac{\mathrm{e}}{(k+1)!}.$$

(b) We have, based on (2), that

$$\{(n+k)!\mathrm{e}\} = (n+k)! \left( \mathrm{e} - 1 - \frac{1}{1!} - \cdots - \frac{1}{(n+k)!} \right),$$

and it follows

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\{(n+k)!\mathrm{e}\}}{n!} &= \sum_{n=1}^{\infty}\frac{(n+k)!}{n!}\left(\mathrm{e}-1-\frac{1}{1!}-\cdots-\frac{1}{(n+k)!}\right)\\ &= k!\sum_{n=1}^{\infty}\binom{n+k}{k}\left(\mathrm{e}-1-\frac{1}{1!}-\cdots-\frac{1}{(n+k)!}\right)\\ &= k!\sum_{m=k+1}^{\infty}\binom{m}{k}\left(\mathrm{e}-1-\frac{1}{1!}-\cdots-\frac{1}{m!}\right)\\ &\overset{(a)}{=} k!\left[\frac{\mathrm{e}}{(k+1)!}-\left(\mathrm{e}-1-\frac{1}{1!}-\cdots-\frac{1}{k!}\right)\right]\\ &= \frac{\mathrm{e}}{k+1}-k!\left(\mathrm{e}-1-\frac{1}{1!}-\frac{1}{2!}-\cdots-\frac{1}{k!}\right).\end{aligned}$$

The theorem is proved. □

**Remark 5.** One can prove more generally (see [3]) that, if $x \in \mathbb{R}$, the following power series formula holds

$$\sum_{n=k}^{\infty}\binom{n}{k}\left(\mathrm{e}-1-\frac{1}{1!}-\frac{1}{2!}-\cdots-\frac{1}{n!}\right)x^n$$
$$=\begin{cases}\dfrac{\mathrm{e}}{(k+1)!} & \text{if, } x=1,\\ \dfrac{k!}{(1-x)^{k+1}}\left(\mathrm{e}^{1-x}-1-\dfrac{1-x}{1!}-\dfrac{(1-x)^2}{2!}-\cdots-\dfrac{(1-x)^k}{k!}\right), & \text{if } \ x\neq 1.\end{cases}$$

**Theorem 6. Variations on the same theme.**

*The following equalities hold:*

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{\{(2n)!\mathrm{e}\}}{(2n)!}=\sum_{n=1}^{\infty}\left(\mathrm{e}-1-\frac{1}{1!}-\frac{1}{2!}-\cdots-\frac{1}{(2n)!}\right)=1-\frac{\cosh 1}{2};$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{\{(2n-1)!\mathrm{e}\}}{(2n-1)!}=\sum_{n=1}^{\infty}\left(\mathrm{e}-1-\frac{1}{1!}-\frac{1}{2!}-\cdots-\frac{1}{(2n-1)!}\right)=\frac{\cosh 1}{2}.$

*Proof.* (a) We have, based on the first equality in (3), that

$$\{(2n)!\mathrm{e}\}=(2n)!\left(\mathrm{e}-1-\frac{1}{1!}-\frac{1}{2!}-\cdots-\frac{1}{(2n)!}\right)$$

and it follows that

$$\sum_{n=1}^{\infty}\frac{\{(2n)!\mathrm{e}\}}{(2n)!}=\sum_{n=1}^{\infty}\left(\mathrm{e}-1-\frac{1}{1!}-\frac{1}{2!}-\cdots-\frac{1}{(2n)!}\right).$$

An application of formula (1) with $a_n = 1$ and $b_n = \mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{(2n)!}$ shows that

$$\sum_{n=1}^{\infty}\left(\mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{(2n)!}\right) = \sum_{n=1}^{\infty} n\left(\frac{1}{(2n+1)!} + \frac{1}{(2n+2)!}\right)$$
$$= \sum_{n=1}^{\infty}\left(\frac{1}{2(2n)!} - \frac{1}{(2n+2)!}\right)$$
$$= 1 - \frac{\cosh 1}{2}.$$

(b) Observe that

$$\sum_{n=1}^{\infty}\frac{\{(2n)!\mathrm{e}\}}{(2n)!} + \sum_{n=1}^{\infty}\frac{\{(2n-1)!\mathrm{e}\}}{(2n-1)!} = \sum_{n=1}^{\infty}\frac{\{n!\mathrm{e}\}}{n!} \overset{\text{Th.1(a)}}{=} 1,$$

and part (b) of the theorem follows from part (a). □

**Theorem 7.** *The following equalities hold*:

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\{n!\mathrm{e}\} = 1 - \mathrm{e} + \ln 2 + \int_0^1 \mathrm{e}^x \ln(2-x)\mathrm{d}x;$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{\{n!\mathrm{e}\}}{n} = \int_0^1 \frac{\mathrm{e}^x - 1}{x}\mathrm{d}x;$

(c) $\displaystyle\sum_{n=1}^{\infty}\left(\{n!\mathrm{e}\} - \frac{1}{n+1}\right) = 2 - \mathrm{e} + \int_0^1 \frac{\mathrm{e}^x - 1}{x}\mathrm{d}x.$

*Proof.* We prove only parts (b) and (c) of the theorem and leave the proof of part (a) to the interested reader. We mention that part (a) of the theorem can be proved by using the formula

$$\{n!\mathrm{e}\} = n!\left(\mathrm{e} - 1 - \frac{1}{1!} - \cdots - \frac{1}{n!}\right) = \int_0^1 (1-t)^n \mathrm{e}^t \mathrm{d}t.$$

(b) We have, based on (3), that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\{n!\mathrm{e}\}}{n} &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(n+1)(n+2)\cdots(n+k)}\\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\int_0^1(1-x)^k x^{n-1}\mathrm{d}x\\ &= \int_0^1\sum_{k=1}^{\infty}\frac{(1-x)^k}{k!}\sum_{n=1}^{\infty}x^{n-1}\mathrm{d}x\\ &= \int_0^1\frac{\mathrm{e}^{1-x}-1}{1-x}\mathrm{d}x\\ &= \int_0^1\frac{\mathrm{e}^{x}-1}{x}\mathrm{d}x.\end{aligned}$$

We used in the preceding calculations formula (4).

(c) Using formula (3) one has that

$$\{n!\mathrm{e}\}-\frac{1}{n+1}=\sum_{k=2}^{\infty}\frac{1}{(n+1)(n+2)\cdots(n+k)}.$$

It follows, based on (4), that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\{n!\mathrm{e}\}-\frac{1}{n+1}\right) &= \sum_{n=1}^{\infty}\sum_{k=2}^{\infty}\frac{1}{(n+1)\cdots(n+k)}\\ &= \sum_{n=1}^{\infty}\sum_{k=2}^{\infty}\frac{n}{k!}\int_0^1(1-x)^k x^{n-1}\mathrm{d}x\\ &= \int_0^1\sum_{k=2}^{\infty}\frac{(1-x)^k}{k!}\sum_{n=1}^{\infty}nx^{n-1}\mathrm{d}x\\ &= \int_0^1\frac{\mathrm{e}^{1-x}-1-(1-x)}{(1-x)^2}\mathrm{d}x\\ &= \int_0^1\frac{\mathrm{e}^{x}-1-x}{x^2}\mathrm{d}x\\ &= 2-\mathrm{e}+\int_0^1\frac{\mathrm{e}^{x}-1}{x}\mathrm{d}x,\end{aligned}$$

and the theorem is proved. □

**Theorem 8. The generating function of $\left(\left\{\frac{n!}{\mathrm{e}}\right\}\frac{1}{n!}\right)_{n\geq 1}$.**

*The following equalities hold*:

(a) $\displaystyle\sum_{n=1}^{\infty}\left\{\frac{n!}{\mathrm{e}}\right\}\frac{1}{n!}=\frac{\mathrm{e}}{2}-\frac{3}{2\mathrm{e}}$;

(b) $\displaystyle\sum_{n=1}^{\infty}\left\{\frac{n!}{\mathrm{e}}\right\}\frac{x^n}{n!}=\frac{x}{\mathrm{e}(1-x)}+\frac{\mathrm{e}^x}{2}+\frac{x+1}{x-1}\cdot\frac{\mathrm{e}^{-x}}{2}$, *for* $x\neq 1$;

(c) $\displaystyle\sum_{n=1}^{\infty}\left\lfloor\frac{n!}{\mathrm{e}}\right\rfloor\frac{x^n}{n!}=-\frac{1}{2}\left(\mathrm{e}^x+\frac{x+1}{x-1}\mathrm{e}^{-x}\right)$, *for* $x\in[-1,1)$.

*Proof.* (b) We start proving part (b) of the theorem. We have, based on Taylor's formula, that

$$\mathrm{e}^{-1}=1+\frac{-1}{1!}+\frac{(-1)^2}{2!}+\cdots+\frac{(-1)^n}{n!}+\frac{(-1)^{n+1}}{(n+1)!}+\frac{\mathrm{e}^{\theta}}{(n+2)!},\quad \theta\in(-1,0).$$

This implies that

$$\frac{n!}{\mathrm{e}}=n!\left(1+\frac{-1}{1!}+\frac{(-1)^2}{2!}+\cdots+\frac{(-1)^n}{n!}\right)+\frac{(-1)^{n+1}}{n+1}+\frac{\mathrm{e}^{\theta}}{(n+1)(n+2)}.$$

When $n$ is an even integer we have that $\left\lfloor\frac{(-1)^{n+1}}{n+1}+\frac{\mathrm{e}^{\theta}}{(n+1)(n+2)}\right\rfloor=-1$ and when $n$ is an odd integer one has that $\left\lfloor\frac{(-1)^{n+1}}{n+1}+\frac{\mathrm{e}^{\theta}}{(n+1)(n+2)}\right\rfloor=0$.

It follows that

$$\begin{aligned}\sum_{n=1}^{\infty}\left\{\frac{n!}{\mathrm{e}}\right\}\frac{x^n}{n!}&=\sum_{k=1}^{\infty}\left\{\frac{(2k)!}{\mathrm{e}}\right\}\frac{x^{2k}}{(2k)!}+\sum_{k=1}^{\infty}\left\{\frac{(2k-1)!}{\mathrm{e}}\right\}\frac{x^{2k-1}}{(2k-1)!}\\&=\sum_{k=1}^{\infty}\left[\mathrm{e}^{-1}-\left(1+\frac{(-1)}{1!}+\cdots+\frac{(-1)^{2k}}{(2k)!}\right)+\frac{1}{(2k)!}\right]x^{2k}\\&\quad+\sum_{k=1}^{\infty}\left[\mathrm{e}^{-1}-\left(1+\frac{(-1)}{1!}+\cdots+\frac{(-1)^{2k-1}}{(2k-1)!}\right)\right]x^{2k-1}\\&=\sum_{k=1}^{\infty}\left[\mathrm{e}^{-1}-\left(1+\frac{(-1)}{1!}+\cdots+\frac{(-1)^{2k}}{(2k)!}\right)\right]x^{2k}+\cosh x-1\\&\quad+\sum_{k=1}^{\infty}\left[\mathrm{e}^{-1}-\left(1+\frac{(-1)}{1!}+\cdots+\frac{(-1)^{2k-1}}{(2k-1)!}\right)\right]x^{2k-1}\\&=\sum_{k=1}^{\infty}\left[\mathrm{e}^{-1}-\left(1+\frac{(-1)}{1!}+\cdots+\frac{(-1)^{k}}{k!}\right)\right]x^{k}+\cosh x-1.\end{aligned}$$

We calculate this series by using formula (1) with $a_k = x^k$ and $b_k = \mathrm{e}^{-1} - \left(1 + \frac{(-1)}{1!} + \cdots + \frac{(-1)^k}{k!}\right)$, and we have that

$$\begin{aligned}
\sum_{n=1}^{\infty} \left\{\frac{n!}{\mathrm{e}}\right\} \frac{x^n}{n!} &= \lim_{n\to\infty} \frac{x(1-x^n)}{1-x} \left[\frac{1}{\mathrm{e}} - \left(1 + \frac{-1}{1!} + \frac{(-1)^2}{2!} + \cdots + \frac{(-1)^{n+1}}{(n+1)!}\right)\right] \\
&\quad + \sum_{k=1}^{\infty} (x + x^2 + \cdots + x^k) \frac{(-1)^{k+1}}{(k+1)!} + \cosh x - 1 \\
&= \sum_{k=1}^{\infty} (x + x^2 + \cdots + x^k) \frac{(-1)^{k+1}}{(k+1)!} + \cosh x - 1 \\
&= \frac{x}{1-x} \sum_{k=1}^{\infty} \left(\frac{(-1)^{k+1}}{(k+1)!} + \frac{(-x)^k}{(k+1)!}\right) + \cosh x - 1 \\
&= \frac{x}{1-x} \left(\mathrm{e}^{-1} - \frac{\mathrm{e}^{-x} - 1 + x}{x}\right) + \cosh x - 1 \\
&= \frac{x}{\mathrm{e}(1-x)} + \frac{\mathrm{e}^x}{2} + \frac{x+1}{x-1} \cdot \frac{\mathrm{e}^{-x}}{2}.
\end{aligned}$$

We used that

$$\lim_{n\to\infty} \frac{x(1-x^n)}{1-x} \left[\frac{1}{\mathrm{e}} - \left(1 + \frac{-1}{1!} + \frac{(-1)^2}{2!} + \cdots + \frac{(-1)^{n+1}}{(n+1)!}\right)\right] = 0,$$

which can be proved by using the formula (see [4, Problem 1.35, p. 9])

$$\lim_{n\to\infty} (-1)^{n-1}(n+1)! \left(\frac{1}{\mathrm{e}} - \sum_{k=0}^{n} (-1)^k \frac{1}{k!}\right) = 1.$$

(a) We follow the same steps as in part (b) and we get that

$$\sum_{n=1}^{\infty} \left\{\frac{n!}{\mathrm{e}}\right\} \frac{1}{n!} = \sum_{k=1}^{\infty} \left[\mathrm{e}^{-1} - \left(1 + \frac{(-1)}{1!} + \cdots + \frac{(-1)^k}{k!}\right)\right] + \cosh 1 - 1.$$

We calculate this series by using formula (1) with $a_k = 1$ and $b_k = \mathrm{e}^{-1} - \left(1 + \frac{(-1)}{1!} + \cdots + \frac{(-1)^k}{k!}\right)$, and we get that

$$\begin{aligned}
\sum_{n=1}^{\infty} \left\{\frac{n!}{\mathrm{e}}\right\} \frac{1}{n!} &= \sum_{k=1}^{\infty} k \frac{(-1)^{k+1}}{(k+1)!} + \cosh 1 - 1 \\
&= \sum_{k=1}^{\infty} \left(\frac{(-1)^{k+1}}{k!} - \frac{(-1)^{k+1}}{(k+1)!}\right) + \cosh 1 - 1 \\
&= \frac{\mathrm{e}}{2} - \frac{3}{2\mathrm{e}}.
\end{aligned}$$

(c) We have that

$$\sum_{n=1}^{\infty}\left\lfloor\frac{n!}{\mathrm{e}}\right\rfloor\frac{x^n}{n!}=\frac{1}{\mathrm{e}}\sum_{n=1}^{\infty}x^n-\sum_{n=1}^{\infty}\left\{\frac{n!}{\mathrm{e}}\right\}\frac{x^n}{n!}=\frac{x}{\mathrm{e}(1-x)}-\sum_{n=1}^{\infty}\left\{\frac{n!}{\mathrm{e}}\right\}\frac{x^n}{n!},$$

and the result follows from part (b) of the theorem. □

**Theorem 9. A bouquet of special series.**
*The following equalities hold*:

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{\{(2n-1)!\sinh 1\}}{(2n-1)!}=\frac{1}{2\mathrm{e}}$;

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{\{(2n)!\cosh 1\}}{(2n)!}=1-\cosh 1+\frac{\sinh 1}{2}$;

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{\{(2n)!\cos 1\}}{(2n)!}=\frac{1}{2}(\cosh 1-\cos 1-\sin 1)$;

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{\{(2n-1)!\sin 1\}}{(2n-1)!}=\frac{1}{2}(\cos 1+\sinh 1)$.

*Proof.* We prove only parts (a) and (c) and leave the proofs of parts (b) and (d) to the interested reader.

(a) We have, based on Taylor's formula, that when $n$ is an even integer one has that

$$\sinh 1=1+\frac{1}{3!}+\cdots+\frac{1}{(2n-1)!}+\frac{1}{(2n+1)!}\cosh\theta,\quad \theta\in(0,1),$$

and when $n$ is on odd integer one has that

$$\sinh 1=1+\frac{1}{3!}+\cdots+\frac{1}{(2n-1)!}+\frac{1}{(2n)!}\sinh\theta,\quad \theta\in(0,1).$$

It follows, in both cases, that

$$\lfloor(2n-1)!\sinh 1\rfloor=(2n-1)!\left(1+\frac{1}{3!}+\cdots+\frac{1}{(2n-1)!}\right)$$

and this implies that

$$\{(2n-1)!\sinh 1\}=(2n-1)!\left(\sinh 1-1-\frac{1}{3!}-\cdots-\frac{1}{(2n-1)!}\right).$$

Thus,

$$\sum_{n=1}^{\infty}\frac{\{(2n-1)!\sinh 1\}}{(2n-1)!}=\sum_{n=1}^{\infty}\left(\sinh 1-1-\frac{1}{3!}-\cdots-\frac{1}{(2n-1)!}\right).$$

An application of formula (1) with $a_n = 1$ and $b_n = \sinh 1 - 1 - \frac{1}{3!} - \cdots - \frac{1}{(2n-1)!}$ shows that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\sinh 1 - 1 - \frac{1}{3!} - \cdots - \frac{1}{(2n-1)!}\right) &= \sum_{n=1}^{\infty}\frac{n}{(2n+1)!}\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{(2n)!} - \frac{1}{(2n+1)!}\right)\\ &= \frac{1}{2\mathrm{e}}.\end{aligned}$$

(c) We have, based on Taylor's formula, that when $n$ is an even integer, then

$$\cos 1 = 1 - \frac{1}{2!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{(2n)!} + (-1)^{n+1}\frac{1}{(2n+1)!}\sin\theta, \quad \theta \in (0,1)$$

and when $n$ is an odd integer, then

$$\cos 1 = 1 - \frac{1}{2!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{(2n)!} + (-1)^{n+1}\frac{1}{(2n+2)!}\cos\theta, \quad \theta \in (0,1).$$

This implies that when $n$ is an even integer, then

$$\lfloor (2n)!\cos 1\rfloor = (2n)!\left(1 - \frac{1}{2!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{(2n)!}\right) - 1$$

and when $n$ is an odd integer one has that

$$\lfloor (2n)!\cos 1\rfloor = (2n)!\left(1 - \frac{1}{2!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{(2n)!}\right).$$

It follows that

$$\{(2n)!\cos 1\} = \begin{cases}(2n)!\left[\cos 1 - \left(1 - \frac{1}{2!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{(2n)!}\right)\right] + 1, & \text{if } n \text{ is even}\\ (2n)!\left[\cos 1 - \left(1 - \frac{1}{2!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{(2n)!}\right)\right], & \text{if } n \text{ is odd.}\end{cases}$$

Therefore

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\{(2n)! \cos 1\}}{(2n)!} &= \sum_{n \text{ even}} \frac{\{(2n)! \cos 1\}}{(2n)!} + \sum_{n \text{ odd}} \frac{\{(2n)! \cos 1\}}{(2n)!} \\
&= \sum_{n \text{ even}} \left[\cos 1 - \left(1 - \frac{1}{2!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{(2n)!}\right)\right] + \sum_{n \text{ even}} \frac{1}{(2n)!} \\
&\quad + \sum_{n \text{ odd}} \left[\cos 1 - \left(1 - \frac{1}{2!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{(2n)!}\right)\right] \\
&= \sum_{n=1}^{\infty} \left[\cos 1 - \left(1 - \frac{1}{2!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{(2n)!}\right)\right] + \sum_{n=1}^{\infty} \frac{1}{(4n)!} \\
&= \sum_{n=1}^{\infty} \left[\cos 1 - \left(1 - \frac{1}{2!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{(2n)!}\right)\right] \\
&\quad + \frac{\cosh 1 + \cos 1}{2} - 1.
\end{aligned}$$

Using formula (1) with $a_n = 1$ and $b_n = \cos 1 - \left(1 - \frac{1}{2!} + \frac{1}{4!} + \cdots + \frac{(-1)^n}{(2n)!}\right)$, we have that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\{(2n)! \cos 1\}}{(2n)!} &= \sum_{n=1}^{\infty} n \frac{(-1)^{n+1}}{(2n+2)!} + \frac{\cosh 1 + \cos 1}{2} - 1 \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{(2n+1)!} - 2\frac{(-1)^{n+1}}{(2n+2)!}\right) + \frac{\cosh 1 + \cos 1}{2} - 1 \\
&= \frac{1}{2}(1 - \sin 1) - \left(\cos 1 - 1 + \frac{1}{2}\right) + \frac{\cosh 1 + \cos 1}{2} - 1 \\
&= \frac{1}{2}(\cosh 1 - \cos 1 - \sin 1),
\end{aligned}$$

and part(c) of the theorem is proved. □

We mention that the power series versions of the series in Theorems 6 and 9 can be obtained by the same techniques given in this paper. We stop our line of investigation here and we invite the reader to explore further.

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# The Lebesgue dominated convergence theorem in action

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**Abstract**. We prove two results in which the Lebesgue dominated convergence theorem plays a key role. Various examples are given.

**Keywords:** Lebesgue dominated convergence theorem, Riemann integral, improper Riemann integral, limit of sequences of integrals.

**MSC:** Primary 26A42; Secondary 28A20.

## 1. Introduction, notations and background

The main purpose of this paper is to prove various results in which the Lebesgue dominated convergence theorem plays a key role. These results where suggested to us by the following limit given at Vojtěch Jarník International Mathematical Competition 2002,

$$\lim_{n\to\infty} n^2 \int_0^1 \left(\sqrt[n]{1+x^n}-1\right) \mathrm{d}x = \frac{\pi^2}{12}, \tag{$*$}$$

see [11]; for different results see [9]. More precisely, if $f:[0,1]\to\mathbb{R}$, $v:[0,1]\to\mathbb{R}$ are continuous functions and $\varphi:[0,1]\to\mathbb{R}$ is a Riemann integrable function continuous at 1, we find, under suitable assumptions, the limit of the sequence

$$I_n = n^2 \int_0^1 v\left(x^n\right)\left(\sqrt[n]{f\left(x^n\right)}-1\right)\varphi\left(x\right) \mathrm{d}x.$$

However, as we will see, from these results we can deduce not only the limit $(*)$, but also other well-known limits: $\lim\limits_{n\to\infty} n\int_0^1 x^n f\left(x\right)\mathrm{d}x = f\left(1\right)$, where $f:[0,1]\to\mathbb{R}$ is Riemann integrable and continuous at 1, see [6, Exercise 1, p. 54], [5, 8] and $\lim\limits_{n\to\infty} n\int_0^1 x^n f\left(x^n\right)\mathrm{d}x = \int_0^1 f\left(x\right)\mathrm{d}x$, where $f:[0,1]\to\mathbb{R}$ is continuous (see [8]).

The strategy of the proof is natural: We pass first from the Riemann integral to the improper Riemann integral and then to the Lebesgue integral. Here we apply the Lebesgue dominated convergence theorem and then return to the improper Riemann integral or Riemann integral. If $f:[0,1]\to\mathbb{R}$ is Riemann integrable on $[0,1]$ we denote by $\int_0^1 f\left(x\right)\mathrm{d}x$ the Riemann integral. We recall some definitions from the theory of improper Riemann integrable functions. We recommend the reader to consult the book [7] for a treatment of this concept and the book [4] for a great number of very instructive and suggestive examples concerning the improper Riemann integrable functions.

A function $f:(0,1]\to\mathbb{R}$ is *locally Riemann integrable on* $(0,1]$ if for each $0<u<1$ the function $f$ is Riemann integrable on $[u,1]$. A locally

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Riemann integrable function $f:(0,1]\to\mathbb{R}$ is *improper Riemann integrable on* $(0,1]$ if $\lim\limits_{\substack{u\to 0,\\ u>0}}\int_u^1 f(x)\,\mathrm{d}x\in\mathbb{R}$. In this case, by definition,

$$\int_{0+0}^{1} f(x)\,\mathrm{d}x=\lim_{\substack{u\to 0,\\ u>0}}\int_u^1 f(x)\,\mathrm{d}x.$$

A function $f:(0,1]\to\mathbb{R}$ is *absolutely improper Riemann integrable on* $(0,1]$ if $f$ is a locally Riemann integrable function and $\int_{0+0}^1|f(x)|\,\mathrm{d}x:=\lim\limits_{\substack{u\to 0,\\ u>0}}\int_u^1|f(x)|\,\mathrm{d}x<\infty$.

A function $f:(0,1)\to\mathbb{R}$ is *locally Riemann integrable on* $(0,1)$ if for each $0<u'<u''<1$ the function $f$ is Riemann integrable on $[u',u'']$. A locally Riemann integrable function $f:(0,1)\to\mathbb{R}$ is *improper Riemann integrable on* $(0,1)$ if $\lim\limits_{\substack{(u',u'')\to(0,1),\\ u'>0,\ u''<1}}\int_{u'}^{u''} f(x)\,\mathrm{d}x\in\mathbb{R}$. In this case, by definition,

$$\int_{0+0}^{1-0} f(x)\,\mathrm{d}x=\lim_{\substack{(u',u'')\to(0,1),\\ u'>0,\ u''<1}}\int_{u'}^{u''} f(x)\,\mathrm{d}x.$$

A function $f:(0,1)\to\mathbb{R}$ is *absolutely improper Riemann integrable on* $(0,1)$ if $f$ is a locally Riemann integrable function and

$$\int_{0+0}^{1-0}|f(x)|\,\mathrm{d}x:=\lim_{\substack{(u',u'')\to(0,1),\\ u'>0,\ u''<1}}\int_{u'}^{u''}|f(x)|\,\mathrm{d}x<\infty.$$

It is well-known and easy to prove, that if $f:[0,1]\to\mathbb{R}$ is Riemann integrable, then $f$ is improper Riemann integrable on $(0,1]$ and $(0,1)$ and, in addition, $\int_{0+0}^1 f(x)\,\mathrm{d}x=\int_{0+0}^{1-0} f(x)\,\mathrm{d}x=\int_0^1 f(x)\,\mathrm{d}x$.

If $f:[0,1]\to\mathbb{R}$ is integrable on $[0,1]$ with respect to the Lebesgue measure, for short *Lebesgue integrable*, by $\int_{[0,1]} f(x)\,\mathrm{d}x$ we denote the integral of $f$ on $[0,1]$ with respect to the Lebesgue measure (see [2, 10]). Let us recall the Lebesgue dominated convergence theorem.

**The Lebesgue dominated convergence theorem.** *Let $f_n:[0,1]\to\mathbb{R}$ be a sequence of Lebesgue integrable functions, $f:[0,1]\to\mathbb{R}$ such that $f_n\to f$ pointwise almost everywhere and there exists a Lebesgue integrable function $g:[0,1]\to[0,\infty)$ such that for every natural number $n$ the inequality $|f_n(x)|\le g(x)$ holds almost everywhere on $[0,1]$, then $f$ is Lebesgue integrable, $\lim\limits_{n\to\infty}\int_{[0,1]}|f_n(x)-f(x)|\,\mathrm{d}x=0$; in particular, $\lim\limits_{n\to\infty}\int_{[0,1]} f_n(x)=\int_{[0,1]} f(x)\,\mathrm{d}x$.*

A proof can be found in [2, 10].

## 2. PRELIMINARY RESULTS

We need some results which connect Riemann integrability and absolute improper Riemann integrability with the Lebesgue integrability.

**Theorem 1.** *If $f:[0,1]\to\mathbb{R}$ is Riemann integrable, then $f$ is integrable on $[0,1]$ with respect to the Lebesgue measure and $\int_0^1 f(x)\,\mathrm{d}x=\int_{[0,1]} f(x)\,\mathrm{d}x$.*

*Proof.* See [10, Teorema 8.3.1, pp. 160-162]. □

**Proposition 2.** *To each function $f:(0,1]\to\mathbb{R}$ we associate the function $\overline{f}:[0,1]\to[0,\infty)$ defined by $\overline{f}(x)=\begin{cases}0, & x=0,\\ f(x), & x\in(0,1].\end{cases}$*

(i) *Let $f:(0,1]\to[0,\infty)$ be a locally Riemann integrable function. Then $f$ is improper Riemann integrable if and only if $\overline{f}$ is Lebesgue integrable. Moreover, $\int_{0+0}^1 f(x)\,\mathrm{d}x=\int_{[0,1]}\overline{f}(x)\,\mathrm{d}x$.*

(ii) *Assume $f:(0,1]\to\mathbb{R}$ is locally Riemann integrable on $(0,1]$. Then $f$ is absolutely improper Riemann integrable if and only if $\overline{f}$ is Lebesgue integrable. Moreover, $\int_{0+0}^1 f(x)\,\mathrm{d}x=\int_{[0,1]}\overline{f}(x)\,\mathrm{d}x$.*

(iii) *Let $f:(0,1]\to\mathbb{R}$ be an absolutely Riemann integrable function on $(0,1]$ and $w:[0,1]\to\mathbb{R}$ a Riemann integrable function. Define another function $\overline{f}w:[0,1]\to\mathbb{R}$ by $\left(\overline{f}w\right)(x)=\overline{f}(x)\,w(x)$. Then $\overline{f}w$ is Lebesgue integrable and $\int_{0+0}^1 f(x)\,w(x)\,\mathrm{d}x=\int_{[0,1]}\overline{f}(x)\,w(x)\,\mathrm{d}x$.*

*Proof.* (i) Let us suppose that $f$ is improper Riemann integrable, that is, $\int_{0+0}^1 f(x)\,\mathrm{d}x<\infty$. For every natural number $n$ let $f_n:[0,1]\to[0,\infty)$ be defined by $f_n(x)=\begin{cases}0, & 0\le x<\frac{1}{n},\\ f(x), & \frac{1}{n}\le x\le 1,\end{cases}$ i.e., $f_n=f\chi_{[\frac{1}{n},1]}$, where $\chi_A$ is the characteristic function of the set $A$, $\chi_A(x)=\begin{cases}1, & x\in A,\\ 0, & x\notin A.\end{cases}$ Since $f$ is locally Riemann integrable, by a standard result, see [3, Observaţie, p. 61], $f_n$ is Riemann integrable and $\int_0^1 f_n(x)\,\mathrm{d}x=\int_{\frac{1}{n}}^1 f(x)\,\mathrm{d}x$. Then, by Theorem 1, all $f_n$ are Lebesgue integrable and $\int_0^1 f_n(x)\,\mathrm{d}x=\int_{[0,1]} f_n(x)\,\mathrm{d}x$. Since $f$ takes non-negative values, it easily follows that $f_n\nearrow\overline{f}$ pointwise. By the Beppo-Levi theorem, $\lim\limits_{n\to\infty}\int_{[0,1]} f_n(x)\,\mathrm{d}x=\int_{[0,1]}\overline{f}(x)\,\mathrm{d}x$, i.e., $\lim\limits_{n\to\infty}\int_{\frac{1}{n}}^1 f_n(x)\,\mathrm{d}x=\int_{[0,1]}\overline{f}(x)\,\mathrm{d}x$. But, $\int_{0+0}^1 f(x)\,\mathrm{d}x=\lim\limits_{n\to\infty}\int_{\frac{1}{n}}^1 f_n(x)\,\mathrm{d}x$ and thus $\int_{[0,1]}\overline{f}(x)\,\mathrm{d}x=\int_{0+0}^1 f(x)\,\mathrm{d}x<\infty$, that is $\overline{f}$ is Lebesgue integrable.

Conversely, let us suppose that $\overline{f}$ is Lebesgue integrable, which means $\int_{[0,1]}\overline{f}(x)\,\mathrm{d}x<\infty$. Let $(u_n)_{n\in\mathbb{N}}\subset(0,1]$ be a decreasing sequence with $\lim\limits_{n\to\infty}u_n=0$ and define $h_n:[0,1]\to[0,\infty)$ by $h_n(x)=\begin{cases}0, & 0\le x<u_n,\\ f(x), & u_n\le x\le 1,\end{cases}$

i.e., $h_n = f\chi_{[u_n,1]}$. Since $f$ is locally Riemann integrable, by [3, Observaţie, p. 61], $h_n$ is Riemann integrable and $\int_0^1 h_n(x)\,\mathrm{d}x = \int_{u_n}^1 f(x)\,\mathrm{d}x$. Then, by Theorem 1, all $h_n$ are Lebesgue integrable and $\int_0^1 h_n(x)\,\mathrm{d}x = \int_{[0,1]} h_n(x)\,\mathrm{d}x$. Since $f$ takes non-negative values, it easily follows that $h_n \nearrow \overline{f}$ pointwise. Again, by the Beppo-Levi theorem, $\lim\limits_{n\to\infty} \int_{[0,1]} h_n(x)\,\mathrm{d}x = \int_{[0,1]} \overline{f}(x)\,\mathrm{d}x$, i.e., $\lim\limits_{n\to\infty} \int_{u_n}^1 f_n(x)\,\mathrm{d}x = \int_{[0,1]} \overline{f}(x)\,\mathrm{d}x$. Thus $\lim\limits_{u\to 0,\ u>0} \int_u^1 f(x)\,\mathrm{d}x = \int_{[0,1]} \overline{f}(x)\,\mathrm{d}x$, so $f$ is improper Riemann integrable and $\int_{0+0}^1 f(x)\,\mathrm{d}x = \int_{[0,1]} \overline{f}(x)\,\mathrm{d}x$.

(ii) Let us suppose that $f$ is absolutely improper Riemann integrable. Then, since $f$ is locally Riemann integrable, it follows that $|f| : (0,1] \to [0,\infty)$ is locally Riemann integrable and $\int_{0+0}^1 |f(x)|\,\mathrm{d}x < \infty$. By (i), $\left|\overline{f}\right|$ is Lebesgue integrable i.e. $\overline{f}$ is Lebesgue integrable.

Conversely, let us suppose that $\overline{f}$ is Lebesgue integrable. Since $f$ is locally Riemann integrable, $|f|$ is locally Riemann integrable. By (i), $|f| : (0,1] \to [0,\infty)$ is improper Riemann integrable. Let $(u_n)_{n\in\mathbb{N}} \subset (0,1]$ be a decreasing sequence with $\lim\limits_{n\to\infty} u_n = 0$ and define $h_n : [0,1] \to [0,\infty)$ by $h_n(x) = \begin{cases} 0, & 0 \le x < u_n, \\ f(x), & u_n \le x \le 1, \end{cases}$ i.e., $h_n = f\chi_{[u_n,1]}$. Since $f$ is locally Riemann integrable, $h_n$ is Riemann integrable and $\int_0^1 h_n(x)\,\mathrm{d}x = \int_{u_n}^1 f(x)\,\mathrm{d}x$. Then, by Theorem 1, all $h_n$ are Lebesgue integrable and $\int_0^1 h_n(x)\,\mathrm{d}x = \int_{[0,1]} h_n(x)\,\mathrm{d}x$. Also $h_n \to \overline{f}$ pointwise and $|h_n(x)| \le \left|\overline{f}(x)\right|$, $\forall x \in [0,1]$, $\forall n \in \mathbb{N}$. By the Lebesgue dominated convergence theorem it follows that $\lim\limits_{n\to\infty} \int_{[0,1]} f_n(x)\,\mathrm{d}x = \int_{[0,1]} \overline{f}(x)\,\mathrm{d}x$, i.e., $\lim\limits_{n\to\infty} \int_{u_n}^1 f_n(x)\,\mathrm{d}x = \int_{[0,1]} \overline{f}(x)\,\mathrm{d}x$. Then, we deduce that $\lim\limits_{u\to 0,\ u>0} \int_u^1 f(x)\,\mathrm{d}x = \int_{[0,1]} \overline{f}(x)\,\mathrm{d}x$, which means that $\int_{0+0}^1 f(x)\,\mathrm{d}x = \int_{[0,1]} \overline{f}(x)\,\mathrm{d}x$.

(iii) Since $f : (0,1] \to \mathbb{R}$ is locally Riemann integrable and $w$ is Riemann integrable, it follows that $fw$ is locally Riemann integrable. Since $w$ is Riemann integrable, it is bounded and $|f(x)\,w(x)| \le M\,|f(x)|$, $\forall\, x \in (0,1]$, where $M = \sup\limits_{x\in[0,1]} |w(x)|$. From $\int_{0+0}^1 |f(x)|\,\mathrm{d}x < \infty$, by the comparison criterion for non-negative improper integrals, see [4, 7], it follows that $\int_{0+0}^1 |f(x)\,w(x)|\,\mathrm{d}x < \infty$. We apply now (ii) for $fw$ instead of $f$. □

**Proposition 3.** *To each function* $f : (0,1) \to \mathbb{R}$ *we associate the function* $\overline{f} : [0,1] \to \mathbb{R}$, $\overline{f}(x) = \begin{cases} 0, & x = 0, \\ f(x), & x \in (0,1), \\ 0, & x = 1. \end{cases}$

(i) *Let* $f : (0,1) \to [0,\infty)$ *be a locally Riemann integrable function. If* $\int_{0+0}^{1-0} f(x)\,\mathrm{d}x < \infty$*, then* $\overline{f}$ *is Lebesgue integrable and* $\int_{0+0}^{1-0} f(x)\,\mathrm{d}x = \int_{[0,1]} \overline{f}(x)\,\mathrm{d}x$.

(ii) *Let* $f : (0,1) \to \mathbb{R}$ *be a locally Riemann integrable function. If* $\int_{0+0}^{1-0} |f(x)|\,\mathrm{d}x < \infty$*, then* $\overline{f}$ *is Lebesgue integrable and*

$$\int_{0+0}^{1-0} f(x)\,\mathrm{d}x = \int_{[0,1]} \overline{f}(x)\,\mathrm{d}x.$$

(iii) *Assume* $f : (0,1) \to \mathbb{R}$ *is an absolutely Riemann integrable function on* $(0,1)$ *and* $w : [0,1] \to \mathbb{R}$ *is a Riemann integrable function. Then, the function* $\overline{f}w : [0,1] \to \mathbb{R}$, $\left(\overline{f}w\right)(x) = \overline{f}(x)\,w(x)$ *is Lebesgue integrable and* $\int_{0+0}^{1-0} f(x)\,w(x)\,\mathrm{d}x = \int_{[0,1]} \overline{f}(x)\,w(x)\,\mathrm{d}x$.

*Proof.* It is analogous to the proof of Proposition 2. For (i)–(ii) consider $f_n(x) = \begin{cases} 0, & 0 \le x < \frac{1}{n}, \\ f(x), & \frac{1}{n} \le x \le 1 - \frac{1}{n}, \\ 0, & 1 - \frac{1}{n} < x \le 1, \end{cases}$ $f_n = f\chi_{\left[\frac{1}{n}, 1-\frac{1}{n}\right]}$, and so on. □

We need the following result whose proof is included for completeness.

**Proposition 4.** *The following inequalities hold:*

$$0 < \frac{e^a - 1}{a} < 1, \forall a < 0; \quad 0 < \frac{e^a - 1}{a} \le e - 1, \forall a \le 1, a \ne 0.$$

*Proof.* Let $f : \mathbb{R} \to \mathbb{R}$, $f(a) = \begin{cases} \frac{e^a - 1}{a}, & a \ne 0, \\ 1, & a = 0. \end{cases}$ It is a standard exercise to prove that $f$ is an increasing function. The inequalities from the statement follow from this fact. □

## 3. THE RESULTS

We begin with a result which is an extension of the limit (*).

**Theorem 5.** *Let* $f : [0,1] \to [1,\infty)$, $v : [0,1] \to \mathbb{R}$ *be continuous functions. If* $\int_{0+0}^{1} \frac{|v(x)| \ln f(x)}{x} < \infty$*, then for each Riemann integrable function* $\varphi : [0,1] \to \mathbb{R}$*, continuous at* $1$*, we have*

$$\lim_{n\to\infty} n^2 \int_0^1 v(x^n)\left(\sqrt[n]{f(x^n)} - 1\right)\varphi(x)\,\mathrm{d}x = \varphi(1)\int_{0+0}^{1} \frac{v(x)\ln f(x)}{x}\mathrm{d}x.$$

*Proof.* For each $n \in \mathbb{N}$ let $I_n = n^2 \int_0^1 v(x^n)\left(\sqrt[n]{f(x^n)} - 1\right)\varphi(x)\,\mathrm{d}x$ and note that $I_n = n^2 \int_{0+0}^1 v(x^n)\left(\sqrt[n]{f(x^n)} - 1\right)\varphi(x)\,\mathrm{d}x$. With the change of variable $x^n = t$, $x = \sqrt[n]{t}$, $\mathrm{d}x = \frac{\sqrt[n]{t}}{nt}\mathrm{d}t$ (in the improper integral), we deduce

$$I_n = n\int_{0+0}^{1} \frac{v(t)\left(\sqrt[n]{f(t)} - 1\right)}{t} \cdot \sqrt[n]{t}\varphi\left(\sqrt[n]{t}\right)\mathrm{d}t.$$

Let $h_n : [0,1] \to \mathbb{R}$ be defined by $h_n(t) = \begin{cases} 0, & t = 0, \\ \frac{n\left(\sqrt[n]{f(t)}-1\right)}{t}, & t \in (0,1] \end{cases}$. Let $n_0 \in \mathbb{N}$ be such that $n_0 \geq M := \ln\left(\sup_{t\in[0,1]} f(t)\right)$. Let also $n \geq n_0$. For each $t \in [0,1]$, $0 \leq \frac{1}{n}\ln f(t) \leq \frac{M}{n} \leq 1$ (we used $f(t) \geq 1$), and by the second inequality in Proposition 4 we have $0 \leq n\left(\sqrt[n]{f(t)} - 1\right) \leq (e-1)\ln f(t)$ and thus

$$0 \leq h_n(t) \leq \frac{(e-1)\ln f(t)}{t}, \forall t \in (0,1].$$

Then

$$0 \leq |v(t)|\, h_n(t) \leq \frac{(e-1)\,|v(t)|\ln f(t)}{t}, \forall t \in (0,1]$$

and, by hypothesis and the comparison criterion for improper integrals, we deduce $\int_{0+0}^{1} |v(t)|\, h_n(t)\,\mathrm{d}t < \infty$, which, by Proposition 2(iii) gives us

$$I_n = \int_{[0,1]} h_n(t)\, v(t)\, \sqrt[n]{t}\varphi\left(\sqrt[n]{t}\right) \mathrm{d}t,\ \forall n \geq n_0. \tag{1}$$

Let us note that at this moment we have passed from the Riemann integral to the Lebesgue integral. Let $g : [0,1] \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 0, & x = 0, \\ \dfrac{v(x)\ln f(x)}{x}, & x \in (0,1], \end{cases}$$

and note that by hypothesis and Proposition 2(ii), $g$ is Lebesgue integrable. Now from $\lim_{n\to\infty} n\left(\sqrt[n]{a} - 1\right) = \ln a$, $a > 0$, the continuity of $\varphi$ at 1, we deduce

$$\lim_{n\to\infty} h_n(t)\, v(t)\, \varphi\left(\sqrt[n]{t}\right) \sqrt[n]{t} = \varphi(1)\, v(t)\, \frac{\ln f(t)}{t} = \varphi(1)\, g(t), \forall t \in (0,1].$$

We also have

$$\begin{aligned} \left|h_n(t)\, v(t)\, \varphi\left(\sqrt[n]{t}\right) \sqrt[n]{t}\right| &\leq (e-1)\, L \frac{|v(t)|\ln f(t)}{t} \\ &= (e-1)\, L\, |g(t)|, \forall t \in (0,1], \forall n \geq n_0, \end{aligned}$$

where $L = \sup_{t\in[0,1]} |\varphi(t)|$. Now from the Lebesgue dominated convergence theorem we deduce

$$\lim_{n\to\infty} \int_{[0,1]} h_n(t)\, v(t)\, \sqrt[n]{t}\varphi\left(\sqrt[n]{t}\right) \mathrm{d}t = \int_{[0,1]} \varphi(1)\, g(t)\, \mathrm{d}t. \tag{2}$$

By Proposition 2 and the hypothesis we have

$$\int_{[0,1]} g(t)\, \mathrm{d}t = \int_{0+0}^{1} \frac{v(t)\ln f(t)}{t} \mathrm{d}t. \tag{3}$$

Let us note that at this moment we have passed from the Lebesgue integral to the improper Riemann integral. From (1)–(3), the statement follows. □

The next result is a natural completion of Theorem 5.

**Theorem 6.** *Let* $f:[0,1]\to[0,1]$, $v:[0,1]\to\mathbb{R}$ *be continuous functions. If* $\{x\in[0,1]\mid f(x)=0\}\subseteq\{0,1\}$ *and* $\int_{0+0}^{1-0}\frac{|v(x)\ln f(x)|}{x}\mathrm{d}x<\infty$*, then for each Riemann integrable function* $\varphi:[0,1]\to\mathbb{R}$*, continuous at* 1*, we have*

$$\lim_{n\to\infty} n^2\int_0^1 v(x^n)\left(\sqrt[n]{f(x^n)}-1\right)\varphi(x)\,\mathrm{d}x=\varphi(1)\int_{0+0}^{1-0}\frac{v(x)\ln f(x)}{x}\mathrm{d}x.$$

*Proof.* Let $I_n=n^2\int_0^1 v(x^n)\left(\sqrt[n]{f(x^n)}-1\right)\varphi(x)\,\mathrm{d}x$ and, as in the proof of Theorem 5, we deduce $I_n=n\int_{0+0}^1\frac{v(t)\left(\sqrt[n]{f(t)}-1\right)}{t}\cdot\sqrt[n]{t}\varphi\left(\sqrt[n]{t}\right)\mathrm{d}t$. For each $n\in\mathbb{N}$, let $h_n:[0,1]\to\mathbb{R}$ be defined by $h_n(t)=\begin{cases}0, & t=0,\\ \frac{n\left(\sqrt[n]{f(t)}-1\right)}{t}, & t\in(0,1),\\ 0, t=1\end{cases}$

and observe that

$$I_n=\int_{0+0}^{1-0}h_n(t)\,v(t)\,\sqrt[n]{t}\varphi\left(\sqrt[n]{t}\right)\mathrm{d}t.$$

From the hypothesis, it follows that $(0,1)\subseteq\{x\in[0,1]\mid f(x)\neq 0\}$, i.e., $f(t)\neq 0$, $\forall t\in(0,1)$, hence since $f(t)\in[0,1]$, we get $f(t)\in(0,1)$, $\forall t\in(0,1)$. Let $n\in\mathbb{N}$. For every $t\in(0,1)$, since $\ln f(t)<0$, by the first inequality in Proposition 4, $n\left|\sqrt[n]{f(t)}-1\right|\leq|\ln f(t)|$, i.e., $|h_n(t)|\leq\frac{|\ln f(t)|}{t}$ and thus $|v(t)\,h_n(t)|\leq\frac{|v(t)\ln f(t)|}{t}$. From $\int_{0+0}^{1-0}\frac{|v(x)\ln f(x)|}{x}\mathrm{d}x<\infty$, by Proposition 3(iii), we have

$$I_n=\int_{[0,1]}h_n(t)\,v(t)\,\sqrt[n]{t}\varphi\left(\sqrt[n]{t}\right)\mathrm{d}t. \tag{4}$$

Note that at this moment we have passed from the Riemann integral to the Lebesgue integral. Let us define the function $g:[0,1]\to\mathbb{R}$

$$g(x)=\begin{cases}0, & x=0,\\ \frac{v(x)\ln f(x)}{x}, & x\in(0,1),\\ 0, & x=1.\end{cases}$$

Since $\int_{0+0}^{1-0}\frac{|v(x)\ln f(x)|}{x}\mathrm{d}x<\infty$, by Proposition 3(ii), $g$ is Lebesgue integrable. From $\lim\limits_{n\to\infty}n\left(\sqrt[n]{a}-1\right)=\ln a$, $a>0$ and the continuity of $\varphi$ at 1, we deduce

$$\lim_{n\to\infty}h_n(t)\,v(t)\,\varphi\left(\sqrt[n]{t}\right)\sqrt[n]{t}=\varphi(1)\,\frac{v(t)\ln f(t)}{t}=\varphi(1)\,g(t),\ \forall t\in(0,1).$$

For each $n \in \mathbb{N}$ and $t \in (0,1)$ we have $|h_n(t)| \leq \frac{|\ln f(t)|}{t}$ and thus

$$\left| h_n(t)\, v(t)\, \varphi\left(\sqrt[n]{t}\right) \sqrt[n]{t} \right| \leq L \frac{|v(t) \ln f(t)|}{t} = L\,|g(t)|\,,$$

where $L = \sup\limits_{t \in [0,1]} |\varphi(t)|$. Since $g$ is Lebesgue integrable, from the Lebesgue dominated convergence theorem we obtain

$$\lim_{n\to\infty} \int_{[0,1]} h_n(t)\, v(t) \sqrt[n]{t} \varphi\left(\sqrt[n]{t}\right) \mathrm{d}t = \int_{[0,1]} \varphi(1)\, g(t)\, v(t)\, \mathrm{d}t. \tag{5}$$

From $\int_{0+0}^{1-0} \left| \frac{\ln f(t)}{t} v(t) \right| < \infty$ it follows, again by Proposition 3(iii), that the function $t \mapsto g(t)\, v(t)$ is Lebesgue integrable and

$$\int_{[0,1]} g(t)\, v(t)\, \mathrm{d}t = \int_{0+0}^{1-0} \frac{v(t) \ln f(t)}{t} \mathrm{d}t. \tag{6}$$

Let us note that at this moment we have passed from the Lebesgue integral to the improper Riemann integral. From (4)–(6), the statement follows. □

**Remark 7.** Let $A = \{x \in [0,1] \mid f(x) = 0\}$. The condition $A \subseteq \{0,1\}$ which appears in the statement of Theorem 6 means that one has one of the four possibilities $A = \varnothing$, $A = \{0\}$, $A = \{1\}$, $A = \{0,1\}$.

## 4. Applications

We begin by giving some applications of Theorem 5.

**Corollary 8.** (i) *Let $v : [0,1] \to \mathbb{R}$ be a continuous function such that $\int_{0+0}^{1} \frac{|v(x)|}{x} \mathrm{d}x < \infty$. For each Riemann integrable function $\varphi : [0,1] \to \mathbb{R}$, continuous at* 1*, we have*

$$\lim_{n\to\infty} n \int_0^1 v(x^n)\, \varphi(x)\, \mathrm{d}x = \varphi(1) \int_{0+0}^{1} \frac{v(x)}{x} \mathrm{d}x.$$

(ii) *Let $\varphi : [0,1] \to \mathbb{R}$ be a Riemann integrable function continuous at* 1*. Then*

$$\lim_{n\to\infty} n \int_0^1 x^n \varphi(x)\, \mathrm{d}x = \varphi(1)\,.$$

(iii) *Let $f : [0,1] \to \mathbb{R}$ be a continuous function. For each Riemann integrable function $\varphi : [0,1] \to \mathbb{R}$, continuous at* 1*, we have*

$$\lim_{n\to\infty} n \int_0^1 x^n f(x^n)\, \varphi(x)\, \mathrm{d}x = \varphi(1) \int_0^1 f(x)\, \mathrm{d}x.$$

*Proof.* (i) Take in Theorem 5, $f : [0,1] \to [1,\infty)$, $f(x) = e$, and use that $\lim\limits_{n\to\infty} n\left(\sqrt[n]{e} - 1\right) = 1$.

(ii) Take in (i), $v : [0,1] \to \mathbb{R}$, $v(x) = x$.

(iii) Take in (i), $v : [0,1] \to \mathbb{R}$, $v(x) = xf(x)$. □

**Corollary 9.** (i) *For each continuous function* $v : [0,1] \to \mathbb{R}$ *such that* $\int_{0+0}^{1} \frac{|v(x)|}{x} \mathrm{d}x < \infty$ *we have*

$$\lim_{n\to\infty} n^2 \int_0^1 v(x^n) \left( \sqrt[n]{\ln(e+x^n)} - 1 \right) \mathrm{d}x = \int_{0+0}^{1} \frac{v(x)\ln(e+x)}{x} \mathrm{d}x.$$

(ii) *The following equality holds*

$$\lim_{n\to\infty} n^2 \int_0^1 x^n \left( \sqrt[n]{\ln(e+x^n)} - 1 \right) \mathrm{d}x = (e+1)\ln(e+1) - e - 1.$$

*Proof.* (i) Take in Theorem 5, $f : [0,1] \to [1,\infty)$, $f(x) = \ln(e+x)$.
(ii) Take in (i) $v : [0,1] \to \mathbb{R}$, $v(x) = x$, and then integrate by parts. □

**Corollary 10.** (i) *Let* $g : [0,1] \to \mathbb{R}$ *be a continuous function and* $f : [0,1] \to [1,\infty)$ *a continuous function such that one of the following conditions is satisfied:*
*(a)* $f(x) > 1$, $\forall x \in [0,1]$,
*(b)* $f(x) = 1$ *if and only if* $x = 0$ *and* $f$ *is derivable at* $0$.
*Then, for each Riemann integrable function* $\varphi : [0,1] \to \mathbb{R}$ *continuous at* $1$ *we have*

$$\lim_{n\to\infty} n^2 \int_{0+0}^{1} \frac{x^n g(x^n)}{\ln f(x^n)} \left( \sqrt[n]{f(x^n)} - 1 \right) \varphi(x)\, \mathrm{d}x = \varphi(1) \int_0^1 g(x)\, \mathrm{d}x.$$

(ii) *Let* $g : [0,1] \to \mathbb{R}$ *be a continuous function. For each Riemann integrable function* $\varphi : [0,1] \to \mathbb{R}$ *continuous at* $1$ *we have*

$$\lim_{n\to\infty} n^2 \int_{0+0}^{1} \frac{x^n g(x^n)}{\ln(1+x^n)} \left( \sqrt[n]{1+x^n} - 1 \right) \varphi(x)\, \mathrm{d}x = \varphi(1) \int_0^1 g(x)\, \mathrm{d}x.$$

(iii) *Let* $g : [0,1] \to \mathbb{R}$ *be a continuous function. For each Riemann integrable function* $\varphi : [0,1] \to \mathbb{R}$ *continuous at* $1$ *we have*

$$\lim_{n\to\infty} n^2 \int_{0+0}^{1} \frac{x^n g(x^n)}{\ln(\ln(e+x^n))} \left( \sqrt[n]{\ln(e+x^n)} - 1 \right) \varphi(x)\, \mathrm{d}x = \varphi(1) \int_0^1 g(x)\, \mathrm{d}x.$$

*Proof.* (i) Let $v : [0,1] \to \mathbb{R}$ be defined by $v(x) = \begin{cases} \frac{xg(x)}{\ln f(x)}, & x \in (0,1], \\ 0, & x = 0, \end{cases}$
and note that under our hypothesis $v$ is continuous. Also $\frac{v(x)\ln f(x)}{x} = g(x)$, $\forall x \in (0,1]$. Apply now Theorem 5.

(ii) Take in (i), $f : [0,1] \to [1,\infty)$, $f(x) = 1 + x$.

(iii) Take in (i), $f : [0,1] \to [1,\infty)$, $f(x) = \ln(e+x)$. □

In applications we need the next well-known result. For the sake of completeness we include its proof.

**Proposition 11.** *Let $f_k : [0,1] \to [0,\infty)$ be a sequence of continuous functions such that $f_k \searrow 0$ pointwise on $[0,1]$. Then*

$$\int_0^1 \left( \sum_{k=1}^{\infty} (-1)^{k-1} f_k(x) \right) \mathrm{d}x = \sum_{k=1}^{\infty} (-1)^{k-1} \int_0^1 f_k(x) \, \mathrm{d}x.$$

*Proof.* By the well-known Dini's theorem it follows that $f_k \searrow 0$ uniformly on $[0,1]$. Then, by the Abel-Dirichlet criterion, the series $\sum\limits_{k=1}^{\infty} (-1)^{k-1} f_k$ is uniformly convergent. Now, as is well known, under the uniform convergence, we can permute $\int$ and $\sum$, whence the statement. □

**Corollary 12.** (i) *For each continuous function $v : [0,1] \to \mathbb{R}$ and each $b \in \mathbb{Q}$, $b > 0$ we have*

$$\lim_{n\to\infty} n^2 \int_0^1 v(x^n) \left( \sqrt[n]{1+x^{bn}} - 1 \right) \mathrm{d}x = \int_{0+0}^1 \frac{v(x) \ln\left(1+x^b\right)}{x} \mathrm{d}x.$$

(ii) *For each $b \in \mathbb{Q}$, $b > 0$, we have*

$$\lim_{n\to\infty} n^2 \int_0^1 \left( \sqrt[n]{1+x^{bn}} - 1 \right) \mathrm{d}x = \frac{\pi^2}{12b}.$$

(iii) *The following equalities hold:*

$$\begin{aligned} \lim_{n\to\infty} n^2 \int_0^1 x^n \left( \sqrt[n]{1+x^{2n}} - 1 \right) \mathrm{d}x &= \ln 2 + \frac{\pi}{2} - 2; \\ \lim_{n\to\infty} n^2 \int_0^1 x^{2n} \left( \sqrt[n]{1+x^{2n}} - 1 \right) \mathrm{d}x &= \ln 2 - \frac{1}{2}. \end{aligned}$$

*Proof.* (i) Take $f : [0,1] \to [1,\infty)$, $f(x) = 1 + x^b$, and apply Theorem 5.

(ii) Putting $v(x) = 1$ in (i), we get

$$\lim_{n\to\infty} n^2 \int_0^1 \left( \sqrt[n]{1+x^{bn}} - 1 \right) \mathrm{d}x = \int_{0+0}^1 \frac{\ln\left(1+x^b\right)}{x} \mathrm{d}x.$$

Now we use the well known formula $\ln(1+x) = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}$, $\forall x \in [0,1]$. Then, by Proposition 11 we have

$$\begin{aligned} \int_{0+0}^1 \frac{\ln\left(1+x^b\right)}{x} \mathrm{d}x &= \int_0^1 \left( \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{bk-1}}{k} \right) \mathrm{d}x \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} \int_0^1 \frac{x^{bk-1}}{k} \mathrm{d}x = \frac{1}{b} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{12b}. \end{aligned}$$

(iii) From (i) we get

$$\lim_{n\to\infty} n^2 \int_0^1 x^n \left(\sqrt[n]{1+x^{2n}}-1\right) \mathrm{d}x = \int_0^1 \ln\left(1+x^2\right) \mathrm{d}x = \ln 2 + \frac{\pi}{2} - 2;$$
$$\lim_{n\to\infty} n^2 \int_0^1 x^{2n} \left(\sqrt[n]{1+x^{2n}}-1\right) \mathrm{d}x = \int_0^1 x\ln\left(1+x^2\right) \mathrm{d}x = \ln 2 - \frac{1}{2}.$$

Above we integrated by parts. □

In the sequel we give applications of Theorem 6.

**Corollary 13.** (i) *For each continuous function $v : [0,1] \to \mathbb{R}$ and each $0 < a \leq 1$ we have*

$$\lim_{n\to\infty} n^2 \int_0^1 v\left(x^n\right) \left(\sqrt[n]{1-ax^n}-1\right) \mathrm{d}x = \int_{0+0}^1 \frac{v(x)\ln(1-ax)}{x} \mathrm{d}x.$$

(ii) *The following equalities hold:*

$$\lim_{n\to\infty} n^2 \int_0^1 \left(\sqrt[n]{1-x^n}-1\right) \mathrm{d}x = -\frac{\pi^2}{6};$$
$$\lim_{n\to\infty} n^2 \int_0^1 \left(\sqrt[n]{1-ax^n}-1\right) \mathrm{d}x = -\sum_{n=1}^{\infty} \frac{a^n}{n^2},\ 0 < a < 1.$$

(iii) *The following equalities hold:*

$$\lim_{n\to\infty} n^2 \int_0^1 x^n \left(\sqrt[n]{1-x^n}-1\right) \mathrm{d}x = -1;$$
$$\lim_{n\to\infty} n^2 \int_0^1 x^n \left(\sqrt[n]{1-ax^n}-1\right) \mathrm{d}x = \left(1-\frac{1}{a}\right)\ln(1-a) - 1,\ 0 < a < 1.$$

(iv) *The following equalities hold:*

$$\lim_{n\to\infty} n^2 \int_0^1 x^{2n} \left(\sqrt[n]{1-x^n}-1\right) \mathrm{d}x = -\frac{3}{4};$$
$$\lim_{n\to\infty} n^2 \int_0^1 x^{2n} \left(\sqrt[n]{1-ax^n}-1\right) \mathrm{d}x = \frac{\left(1-a^2\right)\ln(1-a)}{2} - \frac{1}{2a} - \frac{1}{4},$$

*where $0 < a < 1$.*

*Proof.* (i) Let $f : [0,1] \to [0,1]$ be defined by $f(x) = 1 - ax$. Then $\{x \in [0,1] \mid f(x) = 0\} = \varnothing$ for $a < 1$ and $\{1\}$ for $a = 1$. Also, as we will prove later, $\int_{0+0}^{1-0} \frac{|\ln f(x)|}{x} \mathrm{d}x = -\int_{0+0}^{1-0} \frac{\ln(1-ax)}{x} \mathrm{d}x < \infty$. From Theorem 6 we get the statement.

(ii) Taking in (i) $v(x) = 1$ and using that, $\ln(1-x) = -\sum\limits_{n=1}^{\infty} \frac{x^n}{n}$, $0 \le x < 1$, we get

$$\begin{aligned}\lim_{n\to\infty} n^2 \int_0^1 \left(\sqrt[n]{1-ax^n} - 1\right) \mathrm{d}x &= \int_{0+0}^1 \frac{\ln(1-ax)}{x} \mathrm{d}x \\ &= -\int_0^1 \left(\sum_{n=1}^{\infty} \frac{a^n x^{n-1}}{n}\right) \mathrm{d}x = -\sum_{n=1}^{\infty} \frac{a^n}{n^2}.\end{aligned}$$

Now we note that for $a = 1$, $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(iii) Taking in (i) $v(x) = x$, we get

$$\lim_{n\to\infty} n^2 \int_0^1 x^n \left(\sqrt[n]{1-ax^n} - 1\right) \mathrm{d}x = \int_{0+0}^1 \ln(1-ax)\, \mathrm{d}x = \left(1 - \frac{1}{a}\right) \ln(1-a) - 1.$$

and

$$\lim_{n\to\infty} n^2 \int_0^1 x^n \left(\sqrt[n]{1-x^n} - 1\right) \mathrm{d}x = \int_{0+0}^{1-0} \ln(1-t)\, \mathrm{d}t = \int_{0+0}^{1-0} \ln x \mathrm{d}x = -1.$$

(iv) Taking in (i) $v(x) = x^2$, we get

$$\begin{aligned}\lim_{n\to\infty} n^2 \int_0^1 x^{2n} \left(\sqrt[n]{1-x^n} - 1\right) \mathrm{d}x &= \int_{0+0}^{1-0} x \ln(1-x)\, \mathrm{d}x \\ &= \int_{0+0}^{1-0} (1-t) \ln t \mathrm{d}t = -\frac{3}{4};\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} n^2 \int_0^1 x^{2n} \left(\sqrt[n]{1-ax^n} - 1\right) \mathrm{d}x &= \int_{0+0}^1 x \ln(1-ax)\, \mathrm{d}x \\ &= \frac{\left(1-a^2\right) \ln(1-a)}{2} - \frac{1}{2a} - \frac{1}{4}.\end{aligned}$$

□

**Corollary 14.** *Let $g : [0,1] \to \mathbb{R}$ be a continuous function and $0 < a \le 1$. For each Riemann integrable function $\varphi : [0,1] \to \mathbb{R}$, continuous at 1, we have*

$$\lim_{n\to\infty} n^2 \int_{0+0}^{1-0} \frac{x^n g(x^n)}{\ln(1-ax^n)} \left(\sqrt[n]{1-ax^n} - 1\right) \varphi(x)\, \mathrm{d}x = \varphi(1) \int_0^1 g(x)\, \mathrm{d}x.$$

*Proof.* Let $v : [0,1] \to \mathbb{R}$ be defined by $v(x) = \begin{cases} -ag(0), & x = 0, \\ \frac{xg(x)}{\ln(1-ax)}, & x \in (0,1). \\ 0, & x = 1. \end{cases}$ Now we note that $v$ is continuous and $\frac{v(x)\ln(1-ax)}{x} = g(x)$, $\forall x \in (0,1]$. Apply now Theorem 6 for $f : [0,1] \to [0,1]$, $f(x) = 1 - ax$. □

**Corollary 15.** (i) *Let $v : [0,1] \to \mathbb{R}$ be a continuous function such that $\int_{0+0}^{1} \frac{|v(x)|}{x^2} \mathrm{d}x < \infty$. The following equality holds*

$$\lim_{n\to\infty} n^2 \int_{0+0}^{1} v(x^n) \left(e^{-\frac{1}{nx^n}} - 1\right) \mathrm{d}x = -\int_{0+0}^{1} \frac{v(x)}{x^2} \mathrm{d}x.$$

(ii) *Let $g : [0,1] \to \mathbb{R}$ be a continuous function. The following equality holds*

$$\lim_{n\to\infty} n^2 \int_{0+0}^{1} x^{2n} g(x^n) \left(e^{-\frac{1}{nx^n}} - 1\right) \mathrm{d}x = -\int_{0}^{1} g(x) \mathrm{d}x.$$

(iii) *The following equalities hold:*

$$\begin{aligned}\lim_{n\to\infty} n^2 \int_{0+0}^{1} x^{kn} \left(e^{-\frac{1}{nx^n}} - 1\right) \mathrm{d}x &= -\frac{1}{k-1},\ k \in \mathbb{Q},\ k \geq 2;\\ \lim_{n\to\infty} n^2 \int_{0+0}^{1} \ln\left(1 + x^{2n}\right) \left(e^{-\frac{1}{nx^n}} - 1\right) \mathrm{d}x &= -\ln 2 + \frac{\pi}{2}.\end{aligned}$$

*Proof.* (i) Let $f : [0,1] \to [0,1]$, $f(x) = \begin{cases} e^{-\frac{1}{x}}, & x \neq 0, \\ 0, & x = 0, \end{cases}$ and note that $f$ is continuous. Also $\{x \in [0,1] \mid f(x) = 0\} = \varnothing$ and $\frac{v(x) \ln f(x)}{x} = -\frac{v(x)}{x^2}$, $\forall x \in (0,1]$. Apply now Theorem 6.

(ii) Take in (i) $v : [0,1] \to \mathbb{R}$, $v(x) = x^2 g(x)$.

(iii) First take in (ii) $g : [0,1] \to \mathbb{R}$, $g(x) = x^{k-2}$. Then take in (ii) $g : [0,1] \to \mathbb{R}$, $g(x) = \begin{cases} \frac{\ln\left(1+x^2\right)}{x^2}, & x \in (0,1] \\ 1, & x = 0 \end{cases}$ and integrate by parts to get $\int_{0+0}^{1} \frac{\ln\left(1+x^2\right)}{x^2} \mathrm{d}x = -\ln 2 + \frac{\pi}{2}$. □

**Corollary 16.** *Let $g : [0,1] \to \mathbb{R}$ be a continuous function. For each Riemann integrable function $\varphi : [0,1] \to \mathbb{R}$, continuous at 1, we have*

$$\lim_{n\to\infty} n^2 \int_{0+0}^{1} \frac{x^n g(x^n)}{\ln(1 + nx^n \ln x)} \left(\sqrt[n]{1 + nx^n \ln x} - 1\right) \varphi(x) \mathrm{d}x = \varphi(1) \int_{0}^{1} g(x) \mathrm{d}x.$$

*Proof.* Let $v : [0,1] \to \mathbb{R}$ defined by $v(x) = \begin{cases} \frac{xg(x)}{\ln(1+x\ln x)}, & x \in (0,1], \\ 0, & x = 0. \end{cases}$ Note that $v$ is continuous and $\frac{v(x)\ln(1+x\ln x)}{x} = g(x)$, $\forall x \in (0,1]$. Now take in Theorem 6, $f : [0,1] \to [0,1]$, $f(x) = \begin{cases} 1 + x\ln x, & x \neq 0, \\ 1, & x = 0, \end{cases}$ which is a continuous function, and note that $\{x \in [0,1] \mid f(x) = 0\} = \varnothing$ (use the Rolle sequence). □

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# A tale of involutions

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**Abstract**. This material intends to encourage undergraduates exposed to an introductory course in group theory to undertake research on their own. The rôle of involutions is emphasized by considering two problems related to commutativity.

**Keywords:** Finite group, involution, automorphism, orbit.
**MSC:** Primary 20-01; Secondary 20D60, 20D45.

*Dedicated to Adi Jurjiu, who inspired us all*

## 1. INTRODUCTION

Throughout this paper $G$ will be a group with identity element 1 and $G^\star$ will denote the set of all non-identity elements of $G$. A problem proposed at the 1996 Romanian Mathematical Olympiad asked to determine all finite groups $G$ in which exactly two elements of $G^\star$ commute. The groups in question are the group of order 3 and the symmetric group $S_3$. For a subset $S$ of $G$ we will use $|S|$ to denote the cardinal of $S$ and we will solve here a far reaching extension of the mentioned Olympiad problem:

**Problem 1.** *Determine all groups $G$ if there exists $X \subseteq G^\star$ such that:*

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*i)* $|X| \geq 2$,
*ii) if* $x, y \in X$*, then* $xy = yx$,
*iii) two distinct elements of* $G^\star$ *commute if and only if* $x, y \in X$.

Twenty years later, A. Bors considered in [1] finite groups $G$ having an automorphism with a cycle whose size is larger than $\frac{|G|}{2}$. He proved that these groups are abelian and succeeded to determine them all in a twenty pages long technical paper. If we denote by $\mathrm{Aut}(G)$ the automorphism group of $G$, Bors' results suggest a more general (and, much harder) problem:

**Problem 2.** *Determine the finite groups* $G$ *which have an abelian subgroup* $A$ *of* $\mathrm{Aut}(G)$ *such that* $A$ *has an orbit of size larger than* $\frac{|G|}{2}$.

We are not able to solve Problem 2, but we will show that the odd order groups in question must be abelian. In doing so we are using elementary techniques and we illustrate the important rôle played by involutions in both $G$ and $\mathrm{Aut}(G)$. The reader we have in mind is the curious undergraduate having a good grasp of the basics of group theory. We are using the order of an element and of a subgroup, coset decomposition and Lagrange's Theorem. Normal subgroups and the Cauchy's Theorem are the other tools we need.

## 2. Preliminaries

Recall that $\mathrm{Aut}(G)$ is a group with respect to map composition and whose identity element is the identity map $\mathrm{id}_G$. We are using here the popular exponential notation. If $x, y \in G$ and $\alpha, \beta \in \mathrm{Aut}(G)$, we write $x^\alpha := \alpha(x)$, so we have $(xy)^\alpha = x^\alpha y^\alpha$ and $x^{\alpha\beta} = (x^\alpha)^\beta$. In particular, $x^\alpha = x^\beta$ means that $x^{\alpha\beta^{-1}} = x^{\beta\alpha^{-1}} = x$.

We write $C_G(\alpha) := \{g \in G \mid g^\alpha = g\}$ and note that $C_G(\alpha)$ is a subgroup of $G$. Lagrange's Theorem implies that $|C_G(\alpha)|$ divides $|G|$ when $G$ is finite.

When $A$ is a subgroup of $\mathrm{Aut}(G)$ one can define an equivalence relation $R$ on $G$ via $xRy$ if and only if there exists some $\alpha \in A$ such that $y = x^\alpha$. The equivalence class of $g \in G$ is the set $O_A(g) = \{g^\alpha \mid \alpha \in A\}$ and is called the *orbit* of $g$ under the action of $A$. So, if $x, y \in G$ we have that $O_A(x) = O_A(y)$ or $O_A(x) \cap O_A(y) = \emptyset$.

We use $|g|$ to denote the order of $g$ and we call $g \in G^\star$ an *involution* of $G$ if $|g| = 2$, that is, if $g = g^{-1}$. Cauchy's Theorem implies that if $G$ is finite then involutions exist in $G$ if and only if $|G|$ is even. The involutions have a very useful property which will be used repeatedly: if $x, y \in G$ are distinct involutions, then $xy$ is an involution if and only if $xy = yx$. When $xy \neq yx$ there is no restriction on the order of $xy$ other than $|xy| \geq 3$. The reader may try to supply an example of involutions $x, y$ such that $xy$ has infinite order by looking at $2 \times 2$ non-singular matrices.

## 3. The solution for Problem 1

We let $H := X \cup \{1\}$ and we proceed through a sequence of simple steps.

**Step 1.** If $G$ is abelian and $|G| \geq 3$, then $G$ is a solution and $X = G^\star$. From now on $G$ will denote a non-abelian solution to Problem 1, so $H \neq G$.

**Step 2.** The elements of $G \setminus H$ are involutions.

*Proof.* Let $g \in G \setminus H$, so $g \in G^\star$. If $|g| \geq 3$, then $g \neq g^2$ and since $gg^2 = g^2g$, one obtains by iii) that $g \in X \subset H$, a contradiction. □

**Step 3.** $H$ is an abelian subgroup of $G$ containing no involutions.

*Proof.* If $h \in H^\star$ and if $h \neq h^{-1}$, we get by iii) that $h^{-1} \in H$. If $h = h^{-1}$ then $h^{-1} \in H$, so inverses of elements in $H$ remain in $H$. Let now $h, k \in H$. If either is the identity, then clearly $hk \in H$, so we assume that $h, k \in H^\star$. But then $h \neq hk$ and since $hk = kh$, by ii) we get $h(hk) = (hk)h$, which forces $hk \in X \subset H$ by iii). So we proved that $H$ is a subgroup of $G$ and the fact that $H$ is abelian is trivial by ii).

Suppose now that $h \in H$ is an involution and pick $a \in G \setminus H$. Then $a$ is an involution by Step 2 and also $ha \notin H$ since $H$ is a subgroup of $G$. But then $ha$ is an involution by Step 2 again and then the property of involutions in the Preliminaries forces $ha = ah$. This contradicts iii) and shows that $H$ has no involutions. □

**Step 4.** If $a \in G \setminus H$, then $G = H \cup Ha$.

*Proof.* If $x, y \in G \setminus H$ are distinct, then $xy \neq yx$ by iii) and also $xy \in H$, for otherwise, by Step 2, $|xy| = 2$ and we get $xy = yx$, a contradiction. Fix $a \in G \setminus H$ and recall that the map $r_a : G \to G$, where $r_a(g) = ga$, is a bijection. Moreover, since $|a| = 2$, we see that $r_a = {r_a}^{-1}$. Note that $r_a(H) = Ha \subseteq G \setminus H$ and that $r_A(G \setminus H) \subseteq H$. Thus $G \setminus H = r_a(r_a(G \setminus H)) \subseteq r_a(H) \subseteq G \setminus H$. We thus have equality through and through and we get $r_a(H) = G \setminus H$, so $Ha = G \setminus H$ and the proof is complete. □

The index of $H$ in $G$ is thus 2 and the reader might recall that in this case $H$ is a normal subgroup of $G$. In fact, $G = H\langle a \rangle$, where $\langle a \rangle = \{1, a\}$ is the cyclic group generated by $a$. We leave it to the reader to verify that $ah = h^{-1}a$ for every $h \in H$. When $G$ is finite, then $|H| = \frac{|G|}{2}$ is odd since $H$ has no involutions, so $|X| = |H| - 1$ is even. When $G$ is infinite, then $H$ is an infinite abelian group with no involutions.

We can now list all solutions for Problem 1:

**Step 5.** A group $G$ with $|G| \geq 3$ is a solution for Problem 1 if either:

a) $G$ is abelian and $X = G^\star$.

b) $G$ is non-abelian, $G = H\langle a \rangle$, where $H$ is a normal abelian subgroup of $G$ with no involutions, $|a| = 2$ and $ah = h^{-1}a$ for every $h \in H$. Here $H = X \cup 1$ and $|X|$ is even when $G$ is finite.

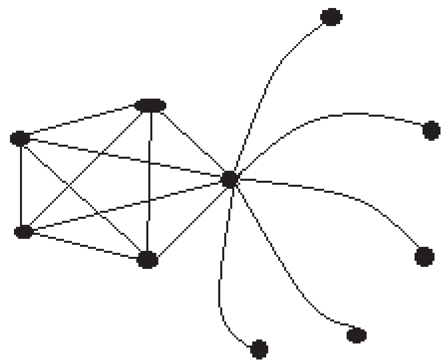

FIGURE 1.The commutativity graph of a nonabelian solution $G$ for Problem 1

As an amusing comment, we remark that the commutativity graph of a nonabelian solution $G$ for Problem 1 looks like an octopus with a number of "tentacles".

The commutativity graph of $G$ has the elements of $G$ as vertices and two vertices are joined if the corresponding elements commute. The subgraph corresponding to the vertices in $H$ is complete by Step 3 and the vertex 1 is joined with all vertices in $G \setminus H$ by "tentacles". The reader is encouraged to answer the question if exactly 8 such "tentacles" are possible.

## 4. A PARTICULAR CASE OF PROBLEM 2

This section is devoted to a proof of the following

**Proposition 1.** *Let $G$ be a group of odd order and let $A$ be an abelian subgroup of* $\mathrm{Aut}(G)$. *If $A$ has an orbit of length larger than $\frac{|G|}{2}$, then $G$ is abelian.*

*Proof.* There is nothing to prove if $|G| = 1$ so from now on we assume that there exists $g \in G^{\star}$ such that $|O_A(g)| > \frac{|G|}{2}$. Note that since $|G|$ is odd then $g \neq g^{-1}$. The inversion map $g \mapsto g^{-1}$ is a bijection and since $(g^{-1})^{\alpha} = (g^{\alpha})^{-1}$ for $\alpha \in A$, we see that $(O_A(g))^{-1} = O_A(g^{-1})$. Since $|O_A(g)| = |O_A(g^{-1})| > \frac{|G|}{2}$, we must have $O_A(g) = O_A(g^{-1})$, for otherwise $|O_A(g) \cup O_A(g^{-1})| > |G|$, a contradiction.

For a nonempty subset $X$ of $G$ we let $X^{-1}$ denote the set consisting of the inverses of the elements in $X$.

This was Bors' original idea in [1], for the case when $A$ is a cyclic group, but it can be adapted easily for the more general case when $A$ is an abelian subgroup of $\mathrm{Aut}(G)$ as follows. Since $O_A(g) = O_A(g^{-1})$, there exists $\tau \in A$ such that $g^{\tau} = g^{-1}$. If $y = g^{\alpha} \in O_A(g)$, then $y^{\tau} = (g^{\alpha})^{\tau} = g^{\alpha\tau} = g^{\tau\alpha} = (g^{-1})^{\alpha} = (g^{\alpha})^{-1} = y^{-1}$. This implies that $y^{\tau^2} = y$ for every $y \in O_A(g)$. Thus $O_A(g) \subseteq C_G(\tau^2)$ and since $|C_G(\tau^2)|$ divides $|G|$, this forces $G = C_G(\tau^2)$. Thus

$\tau^2 = \mathrm{id}_G$ and since $g^\tau = g^{-1} \neq g$, it follows that $\tau$ is in fact an involution of $\mathrm{Aut}(G)$.

There is a rich literature concerning involutions in $\mathrm{Aut}(G)$ and a subset of it deals with the case when $G$ is finite. One of the famous such results is more than 100 years old and is due to W. Burnside. It states that if $\tau$ is an involution of $\mathrm{Aut}(G)$ and if $C_G(\tau) = 1$, then $G$ is abelian of odd order.

Another useful result is Lemma 10.4.1 in D. Gorenstein's book [2]. It states that if $\tau$ is an involution of $\mathrm{Aut}(G)$ and if $G$ has odd order, then $|G| = |I||C_G(\tau)|$, where $I = \{g \in G \mid g^\tau = g^{-1}\}$. In fact, it is not hard to check that $I = \{g^{-1}g^\tau \mid g \in G\}$ and $I$ is a set of representatives for the cosets of $C_G(\tau)$ in $G$ — this information alone should be sufficient to help the reader fill in the gaps.

Returning to our proof, we note that $C_G(\tau) = 1$. For otherwise, since $|G|$ is odd, $|C_G(\tau)| \geq 3$. Also, we proved that $O_A(g) \subseteq I$ and so $I \geq |O_A(g)| > \frac{|G|}{2}$. Then $|G| = |I||C_G(\tau)| > 3\frac{|G|}{2}$, a contradiction. But then, Burnside's theorem forces $G$ to be abelian. □

Analyzing our proof will show that it is identical to that of Bors up to and including the existence of $\tau \in \mathrm{Aut}(G)$ which sends all elements of $O_A(g)$ into their inverses. Our proof is easier because $|G|$ being odd $\tau$ is an involution of $\mathrm{Aut}(G)$. If $O_A(g)$ is the orbit of an involution $g$ of $G$, then $\tau = \mathrm{id}_G$ and all control on the structure of $G$ is lost. In Bors' case, $A = \langle \tau \rangle$, so $\tau \neq \mathrm{id}_G$ and he goes on by using other, more complicated results in the literature.

These comments are added here to convince the reader that it is essential to understand and to analyze the proofs inside and out. Note that the conditions in Problem 2 imply the existence of a subset $X$ of $G$ containing more than $\frac{|G|}{2}$ elements of the same order. All sorts of questions now come to mind: what are the possible orders of such elements, do these elements have to be involutions, what are the finite groups where something like this is possible? For the beginners have to understand that the *statements* don't fall from the sky. They are suggested by reading other statements, by asking if their converse is true, by playing around with simple notions done in class etc.

## References


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# Traian Lalescu National Mathematics Contest for University Students, 2017 edition

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**Abstract**. This paper deals with the problems proposed at the 2017 edition of the Traian Lalescu National Mathematics Competition for university students, hosted by Constanţa Maritime University between May 4th and May 6th, 2017.

**Keywords:** Determinants, dominated convergence theorem, eigenvalues, Gamma function, Leibniz product rule

**MSC:** Primary 97U40; Secondary 11C20; 15A18; 33D05; 40A30.

## Introduction

Constanţa Maritime University organized between May 4th and May 6th the 2017 edition of the Traian Lalescu National Mathematics Contest for university students.

The contest saw a participation of 104 students representing 12 universities from Bucureşti, Cluj, Constanţa, Craiova, Iaşi and Timişoara, and was organised in five sections:

- Section A for students of faculties of Mathematics,
- Section B for first-year students in Electrical Engineering or Computer Science,
- Section C for first-year students of technical faculties, non-electrical specializations,
- Section D for first and second-year students in Electrical Engineering,
- Section E for second-year students of technical faculties, non-electrical specializations.

We present in the sequel the problems proposed in Sections A and B of the contest and their solutions.

## Section A

**Problem 1.** Decide whether the limit

$$\lim_{(x_1,x_2,\ldots,x_n)\to(0,0,\ldots 0)} \frac{x_1x_2\cdots x_n}{x_1^2+x_2^2+\cdots+x_n^2}$$

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exists or not. Compute it if it does.

**Ionel Rovenţa**

*The jury considered this problem to be easy. The contestants confirmed this opinion, about half of them managing to solve it.*

**Solution.** For $n = 1$ the limit does not exist, since

$$\lim_{x \nearrow 0} \frac{1}{x} = -\infty \neq +\infty = \lim_{x \searrow 0} \frac{1}{x}.$$

For $n = 2$ the limit does not exist, since

$$\lim_{n\to\infty} \frac{0 \cdot \frac{1}{n}}{0 + \left(\frac{1}{n}\right)^2} = 0 \neq \frac{1}{2} = \lim_{n\to\infty} \frac{\frac{1}{n} \cdot \frac{1}{n}}{\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)^2}.$$

For $n \geq 3$ we may write

$$\left|\frac{x_1 x_2 \cdots x_n}{x_1^2 + x_2^2 + \cdots + x_n^2}\right| \leq \left|\frac{x_1 x_2}{x_1^2 + x_2^2}\right| \cdot |x_3 x_4 \cdots x_n| \leq \frac{1}{2} |x_3 x_4 \cdots x_n|.$$

Since the limit of the right hand side of the relation above as $(x_1, x_2, \ldots, x_n)$ tends to $(0, 0, \ldots, 0)$ is zero, the limit we are concerned with exists and equals zero. $\square$

**Problem 2.** Show that two subgroups of $\mathbb{Q}/\mathbb{Z}$ are isomorphic if and only if they are equal.

**Cornel Băeţica**

*Although the jury considered this problem to be easy-medium, only the top four ranking contestants managed to find a suitable approach. These four contestants actually gave full solutions of the problem. Their solutions went along the lines of Solution 1 below.*

**Solution 1.** Let $G_1$ and $G_2$ be subgroups of $\mathbb{Q}/\mathbb{Z}$. If $G_1 = G_2$, they are isomorphic via identity. Suppose, conversely, that $G_1$ and $G_2$ are isomorphic, and let $\varphi : G_1 \to G_2$ be an isomorphism. Let $x = \widehat{\frac{a}{s}} \in G_1$ ($s \in \mathbb{N}^*$, $0 \leq a < s$, $\gcd(a, s) = 1$), and let $n \in \mathbb{Z}$. Then $n \cdot \widehat{\frac{a}{s}} = \widehat{0} \Leftrightarrow \frac{na}{s} \in \mathbb{Z} \Leftrightarrow s \mid na$. Since $\gcd(a, s) = 1$, this is equivalent to $s \mid n$. Therefore, $\operatorname{ord}(x) = s$.

Now let $\varphi(x) = \widehat{\frac{b}{t}}$ ($t \in \mathbb{N}^*$, $0 \leq b < t$, $\gcd(b, t) = 1$). As above, we get $\operatorname{ord}(\varphi(x)) = t$. Consequently, $s = t$. From $\gcd(b, s) = 1$ it follows that $b + s\mathbb{Z}$ is an invertible element of $\mathbb{Z}_s$. Denoting $(b + s\mathbb{Z})^{-1}$ by $c + s\mathbb{Z}$, we obtain $s \mid bc - 1$, whence $x = \widehat{\frac{a}{s}} = \widehat{\frac{a+a(bc-1)}{s}} = ac \cdot \widehat{\frac{b}{s}} = ac \cdot \varphi(x) \in G_2$. We conclude that $G_1 \subset G_2$. Similarly, $G_2 \subset G_1$, so $G_1 = G_2$. $\square$

**Solution 2.** For each positive integer $n$ let $H_n = \left\{\widehat{0}, \widehat{\frac{1}{n}}, \widehat{\frac{2}{n}}, \ldots, \widehat{\frac{n-1}{n}}\right\}$. Note that $H_n$ is finite and closed under addition, so it is a subgroup of $\mathbb{Q}/\mathbb{Z}$.

Now let $n$ be a positive integer and let $H$ be a subgroup of order $n$ of $\mathbb{Q}/\mathbb{Z}$. If $\widehat{\frac{a}{s}} \in H$, with $s \in \mathbb{N}^*$, $0 \leq a < s$ and $\gcd(a, s) = 1$, then $n \cdot \widehat{\frac{a}{s}} = \widehat{0}$, so $\frac{na}{s} \in \mathbb{Z}$ and therefore $s \mid n$. Denoting $\frac{n}{s}$ by $c$, we derive that $\widehat{\frac{a}{s}} = \widehat{\frac{ca}{n}} \in H_n$. Therefore, $H \subset H_n$. Since $H$ and $H_n$ are finite and have the same cardinality, we get $H = H_n$.

Now let $G_1$ and $G_2$ be two subgroups of $\mathbb{Q}/\mathbb{Z}$. If $G_1 = G_2$, they are isomorphic via identity. If, on the other hand, $G_1$ and $G_2$ are isomorphic via $\psi$, then let $x = \widehat{\frac{a}{s}} \in G_1$, with $s \in \mathbb{N}^*$, $0 \leq a < s$, and $\gcd(a, s) = 1$. As in Solution 1, $\operatorname{ord}(x) = s$. Then $\operatorname{ord}(\psi(x))$ is also $s$. Therefore, $\langle \psi(x) \rangle = H_s = \langle x \rangle$, so $x \in \langle \psi(x) \rangle \subset G_2$. We thus obtained $G_1 \subset G_2$. Replacing in the above $\psi$ by $\psi^{-1}$, we get $G_2 \subset G_1$ as well, so $G_1 = G_2$. □

**Problem 3.** Let $a, b, c, d \in \mathbb{R}$ be such that $a^2 + b^2 + c^2 \neq 0$. Find the distance between the plane $ax + by + cz + d = 0$ and the paraboloid $z = x^2 + y^2$.

**Gabriel Mincu**

*The jury considered the problem to be standard, of medium difficulty, and likely to offer contestants the possibility to apply a wide array of techniques for solving it. Despite these facts, only five contestants managed to find a consistent approach, and only three of them actually gave complete solutions. Their solutions were in the spirit of Solution 3 below.*

Throughout all the solutions below, we will denote the plane in the statement by $\alpha$, the paraboloid by $\mathcal{P}$ and the distance between them by $D$.

**Solution 1.** Let us notice that if $\alpha \cap \mathcal{P} = \emptyset$, then $D$ will be the distance between $\alpha$ and the point where one of the planes which are tangent to $\mathcal{P}$ and parallel to $\alpha$ touches $\mathcal{P}$. We will find these points of tangency by asking that the normal direction to the paraboloid at its generic point be the same as the normal direction to the plane $\alpha$.

If $c = 0$, then $\alpha \parallel Oz$ or $Oz \subset \alpha$, so, since the projection of $\mathcal{P}$ on the plane $(xOy)$ is the whole of $(xOy)$, we get $\alpha \cap \mathcal{P} \neq \emptyset$, and thus $D = 0$.

If $c \neq 0$, the condition on the normal directions yields the tangency point $(x_0, y_0, z_0) = \left(-\frac{a}{2c}, -\frac{b}{2c}, \frac{a^2+b^2}{4c^2}\right)$. We take this opportunity to notice that $\mathcal{P}$ has exactly one tangent plane parallel to $\alpha$. We will denote this plane by $\beta$; the above calculations show its equation to be $ax + by + cz + \frac{a^2+b^2}{4c} = 0$. If $\alpha = \beta$ or $\alpha$ and the origin (which is a point of $\mathcal{P}$) lie on the same side of $\beta$, i.e., if

$$\frac{a^2 + b^2}{4c} \left( \frac{a^2 + b^2}{4c} - d \right) \geq 0 \Leftrightarrow 4cd \leq a^2 + b^2,$$

then $\alpha \cap \mathcal{P} \neq \emptyset$, so $D = 0$. If, on the contrary, $\alpha$ and the origin lie on opposite sides of $\beta$, i.e., if $4cd > a^2 + b^2$, then $D$ equals the distance from the point

$(x_0, y_0, z_0)$ to $\alpha$, whence

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}} = \frac{4cd - a^2 - b^2}{4|c|\sqrt{a^2+b^2+c^2}}. \quad \square$$

**Solution 2.** $D$ is the smallest distance between the points of $\mathcal{P}$ and $\alpha$. Therefore, its value is

$$\min_{(x,y,z)\in\mathcal{P}} \frac{|ax+by+cz+d|}{\sqrt{a^2+b^2+c^2}}.$$

We notice that if there are points $(x, y, z) \in \mathcal{P}$ such that $ax+by+cz+d = 0$, these points belong to both $\mathcal{P}$ and $\alpha$, so $D$ is 0. This case occurs iff the system of equations

$$(\mathcal{S}): \quad \begin{cases} z = x^2 + y^2, \\ ax + by + cz + d = 0, \end{cases}$$

has solutions. If $c = 0$, the system obviously has solutions. If $c \neq 0$, the second equation is equivalent to $z = -\frac{ax+by+d}{c}$. Plugging this into the first equation yields

$$x^2 + \frac{a}{c}x + y^2 + \frac{b}{c}y + \frac{d}{c} = 0.$$

This equation may be rewritten as

$$\left(x + \frac{a}{2c}\right)^2 + \left(y + \frac{b}{2c}\right)^2 = \frac{a^2+b^2-4cd}{4c^2},$$

and it has solutions (and thus so does $(\mathcal{S})$) if and only if $4cd \leq a^2 + b^2$.

Consequently, in the case $4cd > a^2+b^2$, the function $f : \mathcal{P} \to \mathbb{R}$ defined by $f(x, y, z) = ax + by + cz + d$ does not assume the value 0 at any point of $\mathcal{P}$. Since $\mathcal{P}$ is connected and $f$ is continuous, $f(\mathcal{P})$ is also connected, so all the values in $f(\mathcal{P})$ have the same sign as $d$. Therefore, if $d > 0$ then $f(\mathcal{P})$ has an infimum (which, since $f$ is continuous, is actually a minimum), whilst if $d < 0$ then $f(\mathcal{P})$ has a supremum (which, since $f$ is continuous, is actually a maximum).

On the other hand, the system of Lagrange multipliers for the function $f$ subject to the constraint $z = x^2 + y^2$ is

$$(\mathcal{L}): \quad \begin{cases} a + 2\lambda x = & 0, \\ b + 2\lambda y = & 0, \\ c - \lambda = & 0, \\ z = & x^2 + y^2. \end{cases}$$

If $c = 0$, $(\mathcal{L})$ yields the contradiction $a = b = 0$ (together with the above remarks, this shows that for $c = 0$ we have $\mathcal{P} \cap \alpha \neq \emptyset$, so $D = 0$). If $c \neq 0$, the system $(\mathcal{L})$ has the unique solution $\lambda = c$, $x = -\frac{a}{2c}$, $y = -\frac{b}{2c}$, $z = \frac{a^2+b^2}{4c^2}$. This fact and the remark we made above on the existence of an extremum point for $f$ subject to the constraint $z = x^2 + y^2$ shows that this extremum point is $\left(-\frac{a}{2c}, -\frac{b}{2c}, -\frac{a^2+b^2}{4c^2}\right)$, whilst the value of $D$ is

$$\frac{\left|f\left(-\frac{a}{2c}, -\frac{b}{2c}, -\frac{a^2+b^2}{4c^2}\right)\right|}{\sqrt{a^2+b^2+c^2}} = \frac{\left|d - \frac{a^2+b^2}{4c}\right|}{\sqrt{a^2+b^2+c^2}} = \frac{|4cd - a^2 - b^2|}{4|c|\sqrt{a^2+b^2+c^2}}$$
$$= \frac{4cd - a^2 - b^2}{4|c|\sqrt{a^2+b^2+c^2}}.$$

The conclusion of the above discussion is that if $4cd \leq a^2 + b^2$ then $D = 0$, whilst if $4cd > a^2 + b^2$ then $D = \dfrac{4cd - a^2 - b^2}{4|c|\sqrt{a^2+b^2+c^2}}$. □

**Solution 3.** As in Solution 2, $D$ is the smallest distance between the points of $\mathcal{P}$ and $\alpha$, so its value is

$$\min_{(x,y,z)\in\mathcal{P}} \frac{|ax + by + cz + d|}{\sqrt{a^2+b^2+c^2}} = \min_{(x,y,z)\in\mathcal{P}} \sqrt{\frac{(ax+by+cz+d)^2}{a^2+b^2+c^2}}.$$

Since the square root function is strictly increasing, it is enough to find the minimum value of the expression

$$E(x, y) = (cx^2 + cy^2 + ax + by + d)^2$$

as $x$ and $y$ run through $\mathbb{R}$. Let $F(x, y) = cx^2 + cy^2 + ax + by + d$. If $c = 0$, then $F(x, y)$ assumes all possible real values, so the minimum value of $E(x, y)$ is 0. If $c \neq 0$, then

$$E(x, y) = c^2\left(x^2 + y^2 + \frac{a}{c}x + \frac{b}{c}y + \frac{d}{c}\right)^2$$
$$= c^2\left[\left(x + \frac{a}{2c}\right)^2 + \left(y + \frac{b}{2c}\right)^2 + \frac{4cd - a^2 - b^2}{4c^2}\right]^2.$$

Consequently:

- If $4cd \leq a^2 + b^2$, the minimum value of $E(x, y)$ is 0, and thus $D = 0$.
- If $4cd > a^2 + b^2$, the minimum value of $E(x, y)$ is $c^2\left(\dfrac{4cd - a^2 - b^2}{4c^2}\right)^2$,

and thus $D = \dfrac{4cd - a^2 - b^2}{4|c|\sqrt{a^2+b^2+c^2}}$. □

**Problem 4.** Suppose $K$ is a finite field of characteristic $\operatorname{char} K \neq 2$ and cardinality $|K| > 5$. Prove that for all $x \in K$ there exist $a, b, c \in K^{\times}$ such that $x = a^2 + b^2 + c^2$.

**Cornel Băeţica**

*The jury considered this problem to be difficult. The contestants' results confirmed this opinion: the winner of the contest managed to fully solve the problem, the second best scorer gave a partial solution, whilst none of the other contestants managed to find a suitable approach. Solution 2 below is a streamlined version of the solution given by the winner.*

**Solution 1.** Recall that in any finite group $(G, +)$, if $A$ is a subset of $G$ of size greater than $|G|/2$, then $A + A = G$.

Let $S$ be the set of non-zero squares in $K$ and $S_0 = S \cup \{0\}$. According to the above remark, $K = S_0 + S_0$. Let the notation $3S$ stand for $S + S + S$. Since $3S$ is closed under multiplication by non-zero squares, it is enough to prove that at least one square of $K^\times$ belongs to $3S$, at least one non-square of $K^\times$ belongs to $3S$, and $0 \in 3S$.

We notice that any $a \in K \setminus \{-1, 1\}$ can be written as $x^2 - y^2$ with $x, y \neq 0$ (just take $x = 2^{-1} \cdot (a+1)$ and $y = 2^{-1} \cdot (a-1)$). From $K = S_0 + S_0$, we know that every non-square may be written as $a^2 + b^2$. Moreover, in this case $a, b \neq 0$. Thus, every non-square lies in $S + S$.

Since $|K| \geq 7$, there are at least 3 non-zero squares in $K$, so there is $b \in K$ such that $b^2 \in K \setminus \{-1, 0, 1\}$. As noticed earlier, we may write $b^2$ as $x^2 - y^2$ with $x, y \neq 0$. Then $x^2 = b^2 + y^2$, and if $c^2$ is a non-zero square we have

$$c^2 = (cx^{-1})^2 x^2 = (cx^{-1}b)^2 + (cx^{-1}y)^2.$$

Therefore, all nonzero squares lie in $S + S$.

We thus obtained $K^\times \subset S+S$. It follows that $K^\times + S \subset 3S$. Therefore, $|K^\times| \leq |3S|$, so $3S$ misses at most one element of $K$. Using the fact that $3S$ is closed under multiplication by $S$, we derive that the element lacking from $3S$ has to be zero.

We now consider an arbitrary non-zero element $\alpha \in K$ and, according to the above, we write $-\alpha^2 = \beta^2 + \gamma^2$. We get $0 = \alpha^2 + \beta^2 + \gamma^2$, so $0 \in 3S$. We conclude that $3S$ does not miss any element of $K$ after all. Thus, $K = 3S$ and we are done. $\square$

**Solution 2.** The hypotheses imply that $\operatorname{char} K$ is an odd prime number and $|K| \geq 7$. The notations $S$ and $3S$ will have the meanings from Solution 1.

Since $\operatorname{char} K \neq 2$, for every $x^2 \in S$ there exist exactly two elements ($x$ and $-x$) whose squares equal $x^2$; this shows that $|S| = \frac{|K|-1}{2}$. The main idea is to prove that the following property holds under our hypotheses:

$$\textit{There exist non-zero } x, y \in K \textit{ such that } x^2 + y^2 = -1. \qquad (\mathcal{P})$$

Assume $(\mathcal{P})$ is proven. Then $0 = x^2 + y^2 + 1 \in 3S$. In order to prove that $k \in 3S$ for all $k \in K^\times$, one uses the fact that 2 is invertible in $K$ (because char$K \neq 2$) and the following identity, which is valid for any $a, b \in K$:

$$\begin{aligned} ab &= \left(2^{-1}(a+b)\right)^2 - \left(2^{-1}(a-b)\right)^2 \\ &= \left(2^{-1}(a+b)\right)^2 + \left(2^{-1}x(a-b)\right)^2 + \left(2^{-1}y(a-b)\right)^2. \end{aligned}$$

The identity shows that the problem is solved if for all $k \in K^*$ there exist $a, b \in K$ such that $ab = k$ and $a \neq \pm b$. But this statement is true, since $|S| = \frac{|K|-1}{2} \geq 3$, therefore we have $S \setminus \{k, -k\} \neq \emptyset$, so there certainly exist non-zero elements $a \in K$ with $a^2 \neq \pm k$.

Let us now prove $(\mathcal{P})$.

*Case I:* $-1 \notin S$. Consider $S_0 = \left\{x^2 \mid x \in K\right\}, T_0 = \left\{-1 - x^2 \mid x \in K\right\}$. Then $|S_0| = \frac{|K|+1}{2} = |T_0|$, so $|S_0| + |T_0| = |K| + 1 > |K|$. Thus, we must have $S_0 \cap T_0 \neq \emptyset$, so there exist $x, y \in K$ such that $x^2 = -1 - y^2$, which may be rewritten as $-1 = x^2 + y^2$. Notice that $x, y \neq 0$ (otherwise $-1 \in S$, contradiction).

*Case II:* $-1 \in S$. In this case, let $\alpha \in K$ be such that $\alpha^2 = -1$. Since char$K \neq 2$, we have $\alpha \neq \pm 1$.

*Subcase II.1.* char$K \neq 5$. We have the easily verified identity

$$\left(\alpha + 2^{-1}\right)^2 - \left(\alpha - 2^{-1}\right)^2 = (\alpha + 1)^2,$$

where $x = \alpha + 2^{-1}$, $y = \alpha - 2^{-1}$, $z = \alpha + 1$ are all non-zero. Indeed, if $\alpha = \pm 2^{-1}$, then $-1 = \alpha^2 = 4$, so $5 = 0$ in $K$, contradiction. Thus, $x^2 - y^2 = z^2$, which implies $\left(\alpha x z^{-1}\right)^2 + \left(y z^{-1}\right)^2 = -1$.

*Subcase II.2:* char$K = 5$. We further distinguish two cases:

*Subcase II.2.a):* $2 \in S$.

Let $\beta \in K$ such that $\beta^2 = 2$. The equality $\left(2\beta^{-1}\right)^2 + \left(2\beta^{-1}\right)^2 = -1$ settles this case.

*Subcase II.2.b):* $2 \notin S$.

The sets $S$ and $T = \left\{-1 - y^2 \mid x \in K^\times\right\}$ have $\frac{|K|-1}{2}$ elements each. Note that $(\mathcal{P})$ is equivalent to the statement $S \cap T \neq \emptyset$.

Assume, by contradiction, that $S \cap T = \emptyset$. This means that $|S \cup T| = |K| - 1$. Therefore, there exists exactly one element $t \in K$ such that $t \notin S \cup T$. Since $0 = -1 - \alpha^2 \in T$, we have $t \neq 0$. Also, $t \notin S$, so $t$ is not a square.

Note that, if we denote by $N$ the set $K^\times \setminus S$ of the non-squares in $K^\times$, one easily sees that $|N| = \frac{|K|-1}{2}$ and that for each $u \in N$ the set $uS$ equals $N$.

Now take $u \in N$, $u \neq \pm t$ (since $|N| \geq 3$, such elements $u$ do exist). Since $u \in S \cup T$, $u \notin S$, we must have $u \in T$. The same argument applies to $(-1)\,u \in N$ (recall that $-1$ is a square). Thus, there exist $a, b \in K^\times$ such that $u = -1 - a^2$, $-u = -1 - b^2$. This yields $2u = b^2 - a^2$.

Since $2 \notin S$, we have $2S = N$. This shows that the element $u$ above is of the form $2v^2$, for some $v \in K^*$. We may thus write $4v^2 = 2u = b^2 - a^2$.

This implies

$$-1 = \left((2v)^{-1}\alpha b\right)^2 + \left((2v)^{-1} a\right)^2,$$

so we reached a contradiction. This means that $S \cap T$ cannot be empty, and therefore $(\mathcal{P})$ also holds in this subcase. □

## SECTION B

**Problem 1.** Consider the matrix $A = I_n - 2XX^T$, where $X = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ has the properties $x_1 + \cdots + x_n \neq 0$ and $x_1^2 + \cdots + x_n^2 = 1$.

a) Prove that $A$ is an orthogonal matrix.

b) Find the eigenvalues of the matrix $A$, and a basis with respect to which $A$ has the Jordan canonical form.

c) Determine $X \in \mathbb{R}^n$, provided that $(1, \dots, 1)^T$ is an eigenvector of $A$.

**Marian Panţiruc**

*The jury considered this problem to be easy. The students tried to approach the problem either via calculations or along the lines of the first solution given in the sequel.*

**Solution 1.** a) Observe that $A^T = (I_n - 2XX^T)^T = I_n - 2(XX^T)^T = A$. Moreover, since $X^TX = \langle X, X\rangle = \|X\|^2 = 1$, one has

$$A^2 = (I_n - 2XX^T)(I_n - 2XX^T) = I_n - 4XX^T + 4XX^TXX^T = I_n,$$

so $A^2 = I_n$, whence $A^{-1} = A = A^T$, so $A$ is orthogonal.

b) Since $A$ is a symmetric matrix, it follows that it is diagonalizable and all its eigenvalues are real. Using also the fact that $A$ preserves Euclidean norm, it follows that if $\lambda$ is an eigenvalue of $A$ and $V$ an eigenvector corresponding to $\lambda$ then

$$\|V\| = \|AV\| = \|\lambda V\| = |\lambda|\,\|V\|,$$

so the only possible eigenvalues of $A$ are $\pm 1$.

Moreover, note that $\lambda = 1$ iff $\langle X, V\rangle = 0$, hence the eigenspace corresponding to the eigenvalue 1 consists precisely of the vectors which are orthogonal to $X$, and therefore its dimension equals $n-1$. Suppose $\max\limits_{i=1,..,n} |x_i| = |x_j|$ and note that, since $\|X\| = 1$, we have $|x_j| > 0$. For $k, i \in \{1, 2, \dots, n\}$, $k \neq j$, let

$$v_{ki} = \begin{cases} 1, & i = k, \\ -\frac{x_k}{x_j}, & i = j, \\ 0, & i \notin \{k, j\}, \end{cases}$$

and let $V_k = (v_{k1}, v_{k2}, \dots, v_{kn})^T$. Then $AV_k = V_k$ for all $k \in \{1, 2, \dots, n\} \setminus \{j\}$ and the vectors $V_k$ are linearly independent, so they form a basis of the eigenspace corresponding to the eigenvalue $\lambda = 1$ of $A$.

One easily checks that $V_n = X$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda = -1$. Therefore, a basis with respect to which $A$ has diagonal form is $\{V_1, \dots, V_n\}$.

c) Denote $(1, \dots, 1)^T$ by $U$. Taking item b) into account, we have either $AU = U$ or $AU = -U$.

1. $AU = U \Leftrightarrow U - 2XX^TU = U \Leftrightarrow 2\langle X, U\rangle X = 0 \Leftrightarrow \langle X, U\rangle = 0 \Leftrightarrow x_1 + \dots + x_n = 0$, which contradicts the assumptions made in the statement.

2. $AU = -U \Rightarrow X$ and $U$ are collinear. Since $\|X\| = 1$, it follows that

$$X = \pm\frac{1}{\sqrt{n}}(1, \dots, 1)^T;$$

one easily checks that both these matrices satisfy the conditions in the statement. $\square$

**Solution 2.** Another solution for item b) (and, in fact, another approach for the entire problem, since items a) and c) easily follow) is based upon the following useful characterization of the matrices $B \in \mathcal{M}_{n,m}(\mathbb{R})$ of rank 1 :

**Theorem.** *The matrix* $B = (b_{ij})_{\substack{i=\overline{1,n},\\ j=\overline{1,m}}}$ *has rank* 1 *iff there exist column vectors* $X = \mathcal{M}_{n,1}(\mathbb{R}) \setminus \{0\}$, *and* $Y = \mathcal{M}_{m,1}(\mathbb{R}) \setminus \{0\}$ *such that* $B = X \cdot Y^T$ *(more explicitly, if there exist real numbers* $x_1, \dots, x_n$ *and* $y_1, \dots, y_m$ *such that* $x_1^2 + \dots + x_n^2 \neq 0 \neq y_1^2 + \dots + y_m^2$ *and* $b_{ij} = x_iy_j$, $i = \overline{1,n}$, $j = \overline{1,m}$*).*

Using the previous theorem, it follows that in the case $m = n$ the quadratic matrices of rank 1 are of the form $B = (x_iy_j)_{i,j=\overline{1,n}}$, with $x_1^2 + \dots + x_n^2 \neq 0 \neq y_1^2 + \dots + y_n^2$. Since all the minors of order at least 2 of the matrix $B$ are 0, its characteristic polynomial is $f_B(\lambda) = \lambda^n - \operatorname{Tr} B \cdot \lambda^{n-1}$, hence the eigenvalues of $B$ are $\lambda_1 = \ldots = \lambda_{n-1} = 0$ and $\lambda_n = \operatorname{Tr} B = \sum_{i=1}^n x_iy_i$ (note that it is possible for $\lambda_n$ to be 0).

The eigenvectors corresponding to the eigenvalue $\lambda = 0$ are solutions to the system of equations (where, as in the theorem above, $X$ and $Y$ denote $(x_1, \dots, x_n)^T$ and $(y_1, \dots, y_n)^T$, respectively)

$$B \cdot U = 0 \Leftrightarrow X \cdot Y^T \cdot U = 0 \Leftrightarrow X \cdot \sum_{i=1}^n y_iu_i = 0 \Leftrightarrow \sum_{i=1}^n y_iu_i = 0.$$

Since $\operatorname{rank} B = 1$, the system reduces to a single equation, and it has $n-1$ solutions $V_1, \dots, V_{n-1}$ (for instance, those from Solution 1 in which we replace each $x_i$ by the corresponding $y_i$) which are linearly independent and span the subspace orthogonal to $Y$ in the Euclidean space $\mathbb{R}^n$ (endowed with the standard inner product $\langle U, W\rangle = \sum_{i=1}^n u_iw_i$).

If $\operatorname{Tr} B \neq 0$, the vector $X$ satisfies the relation

$$B \cdot X = (XY^T) \cdot X = X \cdot \left(Y^TX\right) = X \cdot \operatorname{Tr} B = \operatorname{Tr} B \cdot X = \lambda_n \cdot X,$$

hence the $n^{\text{th}}$ eigenvector is $X$.

Going back to the solution of the actual problem, we note that the matrix $B = XX^T$ is a special case of the above; we have $A = I_n - 2B$ and $\operatorname{Tr} B = \sum_{i=1}^{n} x_i^2 = 1$. The eigenvalues of $A$ will be $\mu_A = 1 - 2\lambda_B$, hence for $\lambda_1 = \ldots = \lambda_{n-1} = 0$ we get $\mu_1 = \ldots = \mu_{n-1} = 1$, and for $\lambda_n = \operatorname{Tr} B = 1$ we get $\mu_n = 1 - 2\cdot 1 = -1$. The eigenvectors of $A$ and $B$ being the same, we derive that the Jordan canonical form of the matrix $A$ is $J_A = \operatorname{diag}(1, \ldots, 1, -1)$ with respect to the basis $\{V_1, \ldots, V_{n-1}, X\}$. □

**Remark.** Observe that $A = I_n - 2XX^T$, where the Euclidean norm of $X$ equals 1, is known in numerical analysis under the name of Householder matrix, and it generates the corresponding well-known numerical method of solving linear systems.

**Problem 2.** Consider the real sequence $(x_n)_{n\geq 1}$ given by

$$x_n = \sum_{1\leq i<j\leq n} \frac{1}{i\cdot j} \quad \text{for all } n \geq 2.$$

Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{x_n^{\alpha}},$$

where $\alpha$ is a real parameter.

**Marian Panţiruc**

*Although the jury considered this problem to be easy-medium, the students had difficulties in solving it. We present in what follows two solutions of this problem. The complete solutions given by the contestants were in the spirit of Solution 1 below.*

**Solution 1.** Observe that

$$2x_n = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)^2 - \left(1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right) = y_n^2 - \sum_{k=1}^{n} \frac{1}{k^2}. \quad (1)$$

The sequence $y_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is unbounded, whilst the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, so the sequence $S_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is bounded. It follows that $(x_n)_n$ is unbounded.

Furthermore, denoting $\gamma_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n = y_n - \ln n$, one can write

$$2x_n = y_n^2 - \sum_{k=1}^{n} \frac{1}{k^2} = \gamma_n^2 + 2\gamma_n \ln n + \ln^2 n - \sum_{k=1}^{n} \frac{1}{k^2},$$

whence

$$\frac{2x_n}{\ln^2 n} = \frac{\gamma_n^2}{\ln^2 n} + 2\frac{\gamma_n}{\ln n} + 1 - \frac{1}{\ln^2 n} \cdot S_n.$$

But $(\gamma_n)$ converges to Euler's constant $\gamma \simeq 0.577\ldots$ and thus

$$\lim_{n\to\infty} \frac{x_n}{\ln^2 n} = \frac{1}{2},$$

hence the series $\sum_{n=1}^{\infty} \frac{1}{x_n^\alpha}$ and $\sum_{n=1}^{\infty} \frac{1}{\ln^{2\alpha} n}$ are either both convergent or both divergent.

Obviously, the second series diverges for any $\alpha \leq 0$. For $\alpha > 0$, one may apply the Cauchy condensation test, getting that $\sum_{n=1}^{\infty} \frac{1}{\ln^{2\alpha} n}$ converges iff the series $\sum_{n=1}^{\infty} 2^n \frac{1}{n^{2\alpha} \ln^{2\alpha} 2}$ converges. Since the last series diverges for any $\alpha > 0$ (apply, for instance, the ratio test), it follows that the first one diverges as well.

In conclusion, the series $\sum_{n=1}^{\infty} \frac{1}{x_n^\alpha}$ diverges for any $\alpha \in \mathbb{R}$. □

**Solution 2.** Observe that

$$x_n > \frac{1}{2} + \cdots + \frac{1}{n},$$

hence $x_n \to \infty$ and therefore the series $\sum_{n=1}^{\infty} \frac{1}{x_n^\alpha}$ diverges for any $\alpha \leq 0$.

Consider now $\alpha > 0$. We denote $y_n = \frac{1}{x_n^\alpha}$ and observe that

$$x_{n+1} = x_n + \frac{1}{n+1} \cdot \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

with $b_n = \frac{1}{n+1} \cdot \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \to 0$, hence $\frac{y_{n+1}}{y_n} \to 1$, as $n \to \infty$.

In order to apply the Raabe-Duhamel test, one must compute the limit

$$\ell = \lim_{n\to\infty} n \left(\frac{y_n}{y_{n+1}} - 1\right).$$

To this end, one writes

$$\frac{y_n}{y_{n+1}} = \left( \frac{x_n + \frac{1}{n+1} \cdot \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)}{x_n} \right)^{\alpha} = \left(1 + \frac{b_n}{x_n}\right)^{\alpha}.$$

Since $b_n \to 0$ and $x_n \to \infty$, one has $z_n = \dfrac{b_n}{x_n} \in (0,1)$ for $n$ sufficiently large and, by the binomial series formula, one has

$$(1+z_n)^{\alpha} = 1 + \alpha z_n + O\left(z_n^2\right),$$

hence

$$\ell = \lim_{n\to\infty} \left(n\alpha z_n + nO(z_n^2)\right).$$

Using relation (1), one has

$$\lim_{n\to\infty} nz_n = \lim_{n\to\infty} \frac{\frac{n}{n+1}\cdot\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}{\frac{1}{2}\cdot\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)^2 - \frac{1}{2}\cdot\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)} = 0.$$

It follows that $\lim\limits_{n\to\infty} nO(z_n^2) = 0$, hence $\ell = 0$ and the series diverges for any $\alpha \in \mathbb{R}$. □

**Problem 3.** Let $V$ be a real vector space of finite dimension, and $T, S : V \to V$ two linear mappings such that $T \circ S = T$ and $S \circ T = S$.

a) Prove that $\operatorname{rank} T = \operatorname{rank} S$.

b) Prove that if the mappings $T$ and $S$ have the same image, then $T = S$.

**Vasile Pop**

*This problem was considered by the jury as a medium-hard problem, and this was confirmed by the contestants. Only two contestants fully solved the problem, whilst a third gave an almost complete solution.*

**Solution.** a) From the rank theorem for linear mappings, one has

$$\dim(\operatorname{Im} T) = \dim V - \dim(\operatorname{Ker} T) \Leftrightarrow \operatorname{rank} T = n - \dim(\operatorname{Ker} T).$$

We will prove that, under our assumptions, $\operatorname{Ker} T = \operatorname{Ker} S$.

If $S(x) = 0$, then $T(S(x)) = 0 \Leftrightarrow T(x) = 0$, hence $\operatorname{Ker} S \subset \operatorname{Ker} T$. By a similar argument, the reverse inclusion also holds; it follows that $n - \operatorname{rank} T = n - \operatorname{rank} S \Leftrightarrow \operatorname{rank} T = \operatorname{rank} S$.

b) We prove that $S$ and $T$ are projections ($S^2 = S$ and $T^2 = T$). We have $(S \circ T) \circ S = S \circ S$ and $S \circ (T \circ S) = S \circ T = S$, hence $S \circ S = S$. Similarly, $T \circ T = T$.

Let $x \in V$ be arbitrary. Then $T(x) \in \operatorname{Im} T = \operatorname{Im} S$, so there exists $y \in V$ such that $T(x) = S(y)$, whence

$$S(T(x)) = S(S(y)) = S(y) = T(x).$$

On the other hand, the assumption $S \circ T = S$ means that $S(T(x)) = S(x)$ for any $x \in V$. It follows that $T(x) = S(x)$ for any $x \in V$, i.e., $T = S$.

□

**Remark.** 1) The conditions $T \circ S = T$ and $S \circ T = S$ characterize the pairs $(T, S)$ of projections of $V$ such that $\operatorname{Ker} T = \operatorname{Ker} S$.

Indeed, given $T \circ S = T$ and $S \circ T = S$, then $S$ and $T$ are projections as in the proof of item b); furthermore, for $x \in \operatorname{Ker} T$ we have

$$T(x) = 0 \Rightarrow T(S(x)) = 0 \Rightarrow S(T(S(x))) = 0 \Rightarrow S(S(x)) = 0 \Rightarrow S(x) = 0,$$

so $x \in \operatorname{Ker} S$. Thus, $\operatorname{Ker} T \subset \operatorname{Ker} S$; the reverse inclusion is proven in a similar manner.

Suppose now that $S$ and $T$ are projections of $V$ such that $\operatorname{Ker} T = \operatorname{Ker} S$. Let $x \in V$. Since $S(x - S(x)) = 0$, $x - S(x) \in \operatorname{Ker} S = \operatorname{Ker} T$, so $T(x - S(x)) = 0$. Therefore, $T(x) = T(S(x)) + T(x - S(x)) = T(S(x))$, so $T \circ S = T$. Similarly, $S \circ T = S$.

2) If $S$ and $T$ are as in 1) but $\operatorname{Im} T = \operatorname{Im} S$ also holds, then for every $x \in V$ we have an unique decomposition $x = y + z$ with $y \in \operatorname{Ker} T = \operatorname{Ker} S$ and $z \in \operatorname{Im} T = \operatorname{Im} S$. In this case, $T(x) = S(x) = z$.

3) Similarly, one can characterize the pairs of projections $T, S : V \to V$ which have the same image to be the pairs of endomorphisms of $V$ subject to the relations $T \circ S = S$ and $S \circ T = T$.

4) Item a) of the problem might have been formulated in terms of matrices: *If $A$ and $B$ are square matrices of the same size $n$ which satisfy $A \cdot B = A$ and $B \cdot A = B$, then* $\operatorname{rank} A = \operatorname{rank} B$. This is easily reduced to the actual problem by considering the endomorphisms of $\mathbb{R}^n$ that have $A$ and $B$, respectively, as matrices with respect to a suitable basis.

**Problem 4.** Let $c > 1$ and let $f : [0, c] \to \mathbb{R}$ be a continuous and increasing function, with $f(0) = 0$ and $f(1) = 1$. Compute the limit

$$\lim_{t \searrow 0} \frac{1}{t} \int_0^1 f(x) \left( f(x + t) - f(x) \right) \, \mathrm{d}x.$$

**Radu Gologan**

*This problem was considered to be hard by the jury, and proved to be too difficult for Section B, being perhaps more appropriate for Section A. The solution proposed by the author was not attempted by any student. The only (incomplete) solution given by a contestant uses the theorem of Weierstrass on uniform approximation on closed intervals of continuous functions by polynomials. The idea is quite natural, since for differentiable functions the problem becomes much easier. However, for a rigorous solution one should at some point prove the uniform convergence with respect to the variable $t$ of a sequence of functions given in integral form, and this can be done using a* *boundedness which can be proved along the lines of the first part of the official solution. In this way, this solution becomes quite technical, and cannot avoid the main idea behind the author's solution. For this reason, we only show the official solution in the sequel, leaving the one we sketched here as an exercise for the interested reader.*

**Solution.** We will prove firstly that

$$\lim_{t\searrow 0}\frac{1}{t}\int_0^1 (f(x+t)-f(x))\ \mathrm{d}x = f(1)-f(0). \tag{2}$$

Let $t\in[0,c-1]$. By the change of variable $y=x+t$, we get

$$\begin{aligned}\lim_{t\searrow 0}\frac{1}{t}\int_0^1 (f(x+t)-f(x))\ \mathrm{d}x &= \lim_{t\searrow 0}\frac{1}{t}\left(\int_t^{t+1} f(x)\ \mathrm{d}x - \int_0^1 f(x)\ \mathrm{d}x\right)\\ &= \lim_{t\searrow 0}(f(t+1)-f(t)) = f(1)-f(0),\end{aligned}$$

by l'Hospital's rule.

We will prove next that, for any continuous and increasing function $f:[0,c]\to\mathbb{R}$, one has

$$\lim_{t\searrow 0}\frac{1}{t}\int_0^1 (f(x+t)-f(x))^2\ \mathrm{d}x = 0.$$

Let $g:[0,c-1]\to\mathbb{R}$ be given by

$$g(t)=\begin{cases}\dfrac{1}{t}\displaystyle\int_0^1 (f(x+t)-f(x))\ \mathrm{d}x, & \text{if } t\in(0,c-1],\\ f(1)-f(0), & \text{if } t=0.\end{cases}$$

The function $g$ is continuous, hence bounded. Consequently, there exists $M>0$ such that, for any $t\in(0,c-1]$, $0\le g(t)\le M$. Being continuous on a compact interval, the function $f$ is uniformly continuous. It follows that for any $\varepsilon>0$ there exists $\delta_\varepsilon\in(0,c-1)$ such that, for any $x\in[0,1]$ and any $t\in(0,\delta_\varepsilon)$, one has

$$0\le f(x+t)-f(x)<\frac{\varepsilon}{2M}.$$

Then, for any $t\in(0,\delta_\varepsilon)$, we obtain

$$\begin{aligned}0 &\le \frac{1}{t}\int_0^1 (f(x+t)-f(x))^2\ \mathrm{d}x \le \frac{1}{t}\int_0^1 \frac{\varepsilon}{2M}(f(x+t)-f(x))\ \mathrm{d}x\\ &= \frac{\varepsilon}{2M}g(t)\le\frac{\varepsilon}{2}<\varepsilon,\end{aligned}$$

hence $\lim\limits_{t\searrow 0} \dfrac{1}{t}\displaystyle\int_0^1 \left(f(x+t)-f(x)\right)^2\ \mathrm{d}x = 0$.
Now, since

$$\begin{aligned} \frac{1}{t}\int_0^1 f(x)\left(f(x+t)-f(x)\right)\ \mathrm{d}x &= \frac{1}{2t}\int_0^1 \left(\left(f(x+t)\right)^2 - \left(f(x)\right)^2\right)\ \mathrm{d}x \\ &\quad -\frac{1}{2t}\int_0^1 \left(f(x+t)-f(x)\right)^2\ \mathrm{d}x, \end{aligned}$$

we apply relation (2) to the function $f(x)^2$ to obtain

$$\lim_{t\searrow 0} \frac{1}{t}\int_0^1 f(x)\left(f(x+t)-f(x)\right)\ \mathrm{d}x = \frac{1}{2}\left(\left(f(1)\right)^2 - \left(f(0)\right)^2\right) = \frac{1}{2}.$$

□

# PROBLEMS

Authors should submit proposed problems to `gmaproblems@rms.unibuc.ro`. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of November 2018**.

## PROPOSED PROBLEMS

**464.** Prove that the only twice differentiable functions $f : (0, \infty)^2 \to \mathbb{R}$ satisfying

$$x^2 \frac{\partial^2 f}{\partial x^2}(x, y) + 2xy \frac{\partial^2 f}{\partial x \partial y}(x, y) + y^2 \frac{\partial^2 f}{\partial y^2}(x, y) + \frac{1}{4} f(x, y) = 0$$

are

$$f(x, y) = C(x, y) + D(x, y) \log x,$$

where $C$ and $D$ are twice differentiable, homogeneous functions of degree $1/2$.

We say that a function $C$ is homogeneous of degree $r$ if

$$C(tx, ty) = t^r C(x, y) \text{ for all } t > 0.$$

`Proposed by George Stoica, Saint John, New Brunswick, Canada.`

**465.** Let $A \in M_{m,n}(\mathbb{C})$, $B \in M_{n,m}(\mathbb{C})$. Prove that

$$\operatorname{rank}(I_m - AB) - \operatorname{rank}(I_n - BA) = m - n.$$

`Proposed by Vasile Pop, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.`

**466.** Let $f : [1, \infty) \to \mathbb{R}$ be differentiable with $\lim_{x\to\infty} f(x) = 0$. Suppose that $f'$ is strictly monotone, does not vanish anywhere and has the property $\lim_{n\to\infty} \frac{f'(n+1)}{f'(n)} = 1$.

Prove that the limit

$$\gamma_f := \lim_{n\to\infty} \left( \sum_{k=1}^{n} f(k) - \int_1^n f(x)\mathrm{d}x \right),$$

which we call the generalized Euler constant, exists and is finite. Calculate

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} f(k) - \int_1^n f(x)\mathrm{d}x - \gamma_f}{f(n)}.$$

`Proposed by Mircea Ivan, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.`

**467.** Let $a, b, c, d, n \geq 0$ be some integers with $a + c, b + d \geq n$. Then

$$\sum_{k+l=n} k!l!\binom{a}{l}\binom{b}{l}\binom{d}{k}\binom{a+c-l}{k} = \sum_{k+l=n} k!l!\binom{a}{l}\binom{c}{k}\binom{d}{k}\binom{b+d-k}{l}.$$

(Here we make the convention that $\binom{n}{k} = 0$ if $k < 0$ or $k > n$.)

Proposed by Constantin-Nicolae Beli, IMAR, Bucureşti, Romania.

**468.** Find all functions $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying $f(xu + yv, -xv + yu) = f(x, y)f(u, v)$ for all $x, y, u, v \in \mathbb{R}$. Which ones are continuous?

Proposed by George Stoica, Saint John, New Brunswick, Canada.

**469.** Let $A, B \in M_3(\mathbb{R})$ such that $A^2 + BA + B^2 = AB$ and $S(A) \leq 0$, where $S(A)$ is the sum of minors corresponding to the entries on the diagonal of $A$. Prove that $(AB)^2 = A^2B^2$.

Proposed by Florin Stănescu, Şerban Cioculescu Highschool, Găeşti, Dâmboviţa, Romania.

**470.** Let $a \in \mathbb{R}$ and let $b, c \in \mathbb{R}$ with $bc > 0$. Calculate

$$\lim_{n\to\infty} \begin{pmatrix} 1 & \frac{b}{n} \\ \frac{c}{n} & 1 + \frac{a}{n^2} \end{pmatrix}^n.$$

Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

**471.** Let $R$ be a commutative ring of characteristic 2, i.e., where $2 = 0$. In the ring $R[[X]]$ we consider the formal power series $f(X) = \sum_{n\geq 0} X^{2^n-1}$. Prove that for any $\alpha, \beta \in R$ the formal power series $f(\alpha^2 X)f(\beta^2 X)f((\alpha + \beta)^2 X)$ is a square in $R[[X]]$.

Proposed by Constantin-Nicolae Beli, IMAR, Bucureşti, Romania.

## SOLUTIONS

**447.** Let $A \in \mathcal{M}_2(\mathbb{Z})$. Prove that $\mathrm{e}^A \in \mathcal{M}_2(\mathbb{Z})$ if and only if $A^2 = O_2$.

Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

**Disclaimer.** By an unfortunate error, this problem was published twice. It coincides with problem 446 from our 3–4/2015 issue. The solution appears in the issue 1–2/2017.

**448.** Let $f : [0, \infty) \to (0, \infty)$ be a continuous function and let $F(x) = \int_0^x f(t)\mathrm{d}t$. Let $k > 0$.

(i) Prove that if $\alpha < ke$ then there are $\varepsilon$, $A > 0$ with the property that

$$\int_0^{2A} e^{-\varepsilon(x-A)^2}(F(x+k) - \alpha f(x))\mathrm{d}x > 0.$$

Moreover $\varepsilon, A$ are independent of $f$. (Hint: take $\varepsilon$ small and $A$ large.)

(ii) Give an example of $f$ such that $F(x+k) < kef(x)$ $\forall x > 0$.

Conclude that if $k > 0$ and $\alpha \in \mathbb{R}$ then there is $f : [0,\infty) \to (0,\infty)$ such that $\int_0^{x+k} f(t)\mathrm{d}t < \alpha f(x)$ $\forall x > 0$ if and only if $\alpha \geq ke$.

`Proposed by Constantin-Nicolae Beli, IMAR, Bucureşti, Romania.`

*Solution by the author.* (ii) Since the exponential functions have the property that they are the same order of magnitude as their primitives, we are looking for examples of this type. Let $f(x) = e^{ax}$ with $a > 0$. Then $F(x) = \frac{e^{ax}-1}{a}$ and we must have $ke > \frac{F(x+k)}{f(x)} = \frac{1}{a}(e^{ak} - e^{-ax})$ for every $x > 0$. Since $e^{-ax} \to 0$ as $x \to \infty$ this is equivalent to $ke \geq \frac{1}{a}e^{ak}$. We actually have $\min_{a>0} \frac{1}{a}e^{ak} = ke$ and it is reached at $a = \frac{1}{k}$. Indeed $\frac{d}{da}\frac{1}{a}e^{ak} = (\frac{k}{a} - \frac{1}{a^2})e^{ak}$, which is negative if $a < \frac{1}{k}$ and positive if $a > \frac{1}{k}$. Hence $\frac{1}{a}e^{ak}$ reaches its minimum at $a = \frac{1}{k}$ and this minimum is $ke$.

(i) We have

$$\begin{aligned}\int_0^{2A} e^{-\varepsilon(x-A)^2}F(x+k)\mathrm{d}x &= \int_0^{2A} e^{-\varepsilon(x-A)^2}\int_0^{x+k} f(t)\mathrm{d}t\mathrm{d}x \\ &= \iint_D e^{-\varepsilon(x-A)^2} f(t)\mathrm{d}t\mathrm{d}x,\end{aligned}$$

where $D = \{(x,t) \in \mathbb{R}^2 \mid 0 \leq x \leq 2A,\ 0 \leq t \leq x+k\}$.

Note that $D$ is the trapezoid of vertices $(0,0), (2A,0), (2A, 2A{+}k), (0,k)$. We have $D = D_1 \cup D_2$, where

$$\begin{aligned}D_1 &= \{(x,t) \in \mathbb{R}^2 \mid 0 \leq t \leq k,\ 0 \leq x \leq 2A\} \quad \text{and} \\ D_2 &= \{(x,t) \in \mathbb{R}^2 \mid k \leq t \leq 2A+k,\ t-k \leq x \leq 2A\}.\end{aligned}$$

It follows that

$$\begin{aligned}\int_0^{2A} e^{-\varepsilon(x-A)^2}F(x+k)\mathrm{d}x &= \left(\int_0^k \int_0^{2A} + \int_k^{2A+k} \int_{t-k}^{2A}\right) e^{-\varepsilon(x-A)^2} f(t)\mathrm{d}x\mathrm{d}t \\ &= \int_0^{2A+k} g(t)f(t)\mathrm{d}t,\end{aligned}$$

where

$$g(t) = \begin{cases} \int_0^{2A} e^{-\varepsilon(x-A)^2}\mathrm{d}x & \text{if } 0 \leq t \leq k, \\ \int_{t-k}^{2A} e^{-\varepsilon(x-A)^2}\mathrm{d}x & \text{if } k \leq t \leq 2A+k. \end{cases}$$

Since $f(x) > 0\ \forall x$, we have $g(t) \geq 0\ \forall 0 \leq t \leq 2A + k$, with equality iff $t = 2A+k$. Also $g$ is decreasing (it is constant on $[0, k]$ and strictly decreasing on $[k, 2A + k]$).

We have $\int_0^{2A} e^{-\varepsilon(x-A)^2}\big(F(x + k) - \alpha f(x)\big)\mathrm{d}x = \int_0^{2A+k} g(t)f(t)\mathrm{d}t - \int_o^{2A} \alpha e^{-\varepsilon(t-A)^2} f(t)\mathrm{d}t > \int_0^{2A} g(t)f(t)\mathrm{d}t - \int_o^{2A} \alpha e^{-\varepsilon(t-A)^2} f(t)\mathrm{d}t$. It follows that $\int_0^{2A} e^{-\varepsilon(x-A)^2}\big(F(x + k) - \alpha f(x)\big)\mathrm{d}x > 0$ holds provided that for $0 \leq t \leq 2A$ we have $g(t) \geq h(t) := \alpha e^{-\varepsilon(t-A)^2}$. Therefore we must prove that there are $\varepsilon, A > 0$ such that $g(t) \geq h(t)$ for $0 \leq t \leq 2A$. This condition is independent of $f$.

Note that if $0 \leq u \leq A$ then $h(A+u) = h(A-u) = \alpha e^{-\varepsilon u^2}$ and, since $g$ is decreasing, $g(A + u) \leq g(A - u)$. Hence if $g(A + u) \geq h(A + u)$ holds then $g(A - u) \geq h(A - u)$ also holds. Therefore it suffices to prove our statement for $A \leq t \leq 2A$.

We will ask to have $A \geq k$, so that for $t \geq A$ the formula $g(t) = \int_{t-k}^{2A} e^{-\varepsilon(x-A)^2}\mathrm{d}x$ applies.

The relation to prove, $g(t) \geq h(t)\ \forall A \leq t \leq 2A$, also writes as $g(A+u) \geq h(A + u)\ \forall 0 \leq u \leq A$. This means $\int_{u+A-k}^{2A} e^{-\varepsilon(x-A)^2}\mathrm{d}x \geq \alpha e^{-\varepsilon u^2}$. After the change of variable $v = x - A - u$, so that $x - A = u + v$, this becomes $\int_{-k}^{A-u} e^{-\varepsilon(u+v)^2}\mathrm{d}v \geq \alpha e^{-\varepsilon u^2}$, i.e.,

$$\int_{-k}^{A-u} e^{-\varepsilon(2uv+v^2)}\mathrm{d}v \geq \alpha. \tag{3}$$

We prove the relation (3) in three cases, when $u$ is small, i.e., close to 0, when it is large, i.e., close to $A$, and when it is in between. In each case, in order that the proof works, we need conditions on $\varepsilon$ and $A$.

**Case 1.** If $u$ is small then $e^{-\varepsilon(2uv+v^2)}$ is large in the vicinity of 0. We ask that $A \geq u + \alpha e$ and $e^{-\varepsilon(2uv+v^2)} \geq \frac{1}{e}$ for $0 \leq v \leq \alpha e$. If this happens then

$$\int_{-k}^{A-u} e^{-\varepsilon(2uv+v^2)}\mathrm{d}v \geq \int_0^{\alpha e} e^{-\varepsilon(2uv+v^2)}\mathrm{d}v \geq \alpha e \cdot \frac{1}{e} = \alpha$$

and we are done.

The condition $e^{-\varepsilon(2uv+v^2)} \geq \frac{1}{e}$ is equivalent to $2uv + v^2 \leq \frac{1}{\varepsilon}$. We want this to happen for $0 \leq v \leq \alpha e$. Since $v \mapsto 2uv + v^2$ is increasing on $[0, \alpha e]$, this is equivalent to $2\alpha eu + \alpha^2e^2 \leq \frac{1}{\varepsilon}$. For this it is enough to have $\alpha^2e^2 \leq \frac{1}{2\varepsilon}$, i.e., $\varepsilon \leq c_1 := \frac{1}{2\alpha^2e^2}$, and $2\alpha eu \leq \frac{1}{2\varepsilon}$, i.e., $u \leq \frac{C_1}{\varepsilon}$, where $C_1 = \frac{1}{4\alpha e}$. In order that $A \geq u + \alpha e$ for every $u \leq \frac{C_1}{\varepsilon}$, we need that $A \geq \frac{C_1}{\varepsilon} + \alpha e$.

In conclusion, if $\varepsilon \leq c_1$ and $A \geq \frac{C_1}{\varepsilon} + \alpha e$ then (3) holds for $0 \leq u \leq \frac{C_1}{\varepsilon}$.

**Case 2.** When $u$ is large the map $v \mapsto e^{-\varepsilon(2uv+v^2)}$ decreases fast in the vicinity of 0. We ask that $e^{-\varepsilon(2uv+v^2)} \geq \frac{2\alpha}{k}$ when $\forall -k \leq v \leq -\frac{k}{2}$. If this happens then

$$\int_{-k}^{A-u} e^{-\varepsilon(2uv+v^2)}\mathrm{d}v > \int_{-k}^{-k/2} e^{-\varepsilon(2uv+v^2)}\mathrm{d}v \geq \frac{k}{2}\cdot\frac{2\alpha}{k} = \alpha$$

and we are done.

The condition $e^{-\varepsilon(2uv+v^2)} \geq \frac{2\alpha}{k}$ is equivalent to $2uv+v^2 \leq -\frac{1}{\varepsilon}\log\frac{2\alpha}{k}$. It must hold for $-k \leq v \leq -\frac{k}{2}$. But for these values of $v$ we have $2uv+v^2 \leq 2u(-\frac{k}{2})+(-k)^2 = -ku+k^2$, so it is enough to have $-ku+k^2 \leq -\frac{1}{\varepsilon}\log\frac{2\alpha}{k}$, that is, $u \geq k+\frac{C_2}{\varepsilon}$, where $C_2 = \frac{1}{k}\log\frac{2\alpha}{k}$.

In conclusion, (3) holds for every $t$ with $u \geq k+\frac{C_2}{\varepsilon}$.

**Case 3.** The idea of our proof for $\frac{C_1}{\varepsilon} \leq u \leq k+\frac{C_2}{\varepsilon}$ is the following. Note that $2C_1 \leq 2\varepsilon u \leq 2C_2+2k\varepsilon$, so $2\varepsilon u$ is bounded. Therefore, when $v$ is large, $e^{-\varepsilon(2uv+v^2)}$ is small, while if $v$ is not large, since $\varepsilon$ is small, we have $e^{-\varepsilon(2uv+v^2)} \cong e^{-2\varepsilon uv}$. Then we may expect to have

$$\int_{-k}^{A-u} e^{-\varepsilon(2uv+v^2)}\mathrm{d}v \cong \int_{-k}^{A-u} e^{-2\varepsilon uv}\mathrm{d}v,$$

which, if $A$ is large, is

$$\cong \int_{-k}^{\infty} e^{-2\varepsilon uv}\mathrm{d}v = \frac{1}{2\varepsilon u}e^{2\varepsilon uk} \geq ke > \alpha.$$

(Recall from the proof of (ii) that $\min_{a>0}\frac{1}{a}e^{ak} = ke$.) Hence we may expect that $\int_{-k}^{A-u} e^{-\varepsilon(2uv+v^2)}\mathrm{d}v > \alpha$ for $\varepsilon$ small enough and $A$ large enough.

To write the above reasoning rigorously, put

$$\beta = \frac{\alpha+ke}{2}, \quad l = \frac{1}{2C_1}\log\frac{1}{2C_1(ke-\beta)}.$$

Note that $\alpha < \beta < ke$. Since $u \geq \frac{C_1}{\varepsilon}$, we have $\frac{e^{-2\varepsilon ul}}{2\varepsilon u} \leq \frac{e^{-2C_1 l}}{2C_1} = ke-\beta$. Together with $\frac{1}{2\varepsilon u}e^{2\varepsilon uk} \geq ke$, which follows from $\min_{a>0}\frac{1}{a}e^{ak} = ke$, this implies

$$\int_{-k}^{l} e^{-2\varepsilon uv}\mathrm{d}v = \frac{1}{2\varepsilon u}e^{2\varepsilon uk} - \frac{1}{2\varepsilon u}e^{-2\varepsilon ul} \geq ke-(ke-\beta) = \beta.$$

We ask the condition that $\varepsilon \leq c_2 := \frac{1}{\max\{k^2,l^2\}}\log\frac{\beta}{\alpha}$. Then for every $-k \leq v \leq l$ we have $v^2 \leq \max\{k^2,l^2\}$, so that $\varepsilon v^2 \leq \log\frac{\beta}{\alpha}$ and therefore

$$e^{-\varepsilon(2uv+v^2)} = e^{-2\varepsilon uv}e^{-\varepsilon v^2} \geq e^{-2\varepsilon uv}\frac{\alpha}{\beta}.$$

It follows that

$$\int_{-k}^{l} e^{-\varepsilon(2uv+v^2)}\mathrm{d}v \geq \int_{-k}^{l} \frac{\alpha}{\beta}e^{-2\varepsilon uv}\mathrm{d}v = \frac{\alpha}{\beta}\cdot\beta = \alpha.$$

We impose the condition $A \geq k+l+\frac{C_2}{\varepsilon}$. Since $u \leq k+\frac{C_2}{\varepsilon}$, this implies $A \geq l+u$, i.e., $A-u \geq l$. It follows that $\int_{-k}^{A-u} e^{-\varepsilon(2uv+v^2)}\mathrm{d}v \geq \int_{-k}^{l} e^{-\varepsilon(2uv+v^2)}\mathrm{d}x > \alpha$, so we are done.

Combining the conditions from Cases 1–3 we get that if

$$\varepsilon \leq \min\{c_1, c_2\} \quad \text{and} \quad A \geq \max\left\{k, \frac{C_1}{\varepsilon}, k+l+\frac{C_2}{\varepsilon}\right\}$$

then $A \geq k$ and (3) holds $\forall 0 \leq u \leq A$, thus $\varepsilon$ and $A$ satisfy all required conditions.

If $\alpha < ke$ then by (i) for some $\varepsilon, A > 0$ we have

$$\int_0^{2A} e^{-\varepsilon(x-A)^2}\big(F(x+k) - \alpha f(x)\big)\mathrm{d}x > 0,$$

so there is $x \in [0, 2A]$ with $F(x+k) > \alpha f(x)$.

More generally, in each interval of lenght $2A$ there is some element $x$ with $F(x+k) > \alpha f(x)$. Indeed, if $B \geq 0$ and we consider the function $f_B : [0,\infty) \to (0,\infty)$, $f_B(x) = f(x+B)$, then

$$F_B(x) := \int_0^x f_B(t)\mathrm{d}t = \int_B^{x+B} f(t)\mathrm{d}t \leq F(x+B).$$

By the result above applied to $f_B$ there is some $x \in [0, 2A]$ with $F_B(x+k) > \alpha f_B(x)$. Hence $F(x+k+B) \geq F_B(x+k) > \alpha f_B(x) = \alpha f(x+B)$, so $y := x+B \in [B, B+2A]$ satisfies $F(y+k) > \alpha f(y)$.

On the other hand, if $\alpha \geq ke$ then from (i) one obtains that $f(x) = e^{\frac{1}{k}x}$ satisfies $F(x+k) < kef(x) \leq \alpha f(x)$ $\forall x > 0$. Hence the only values of $\alpha \in \mathbb{R}$ such that there is $f : [0,\infty) \to (0,\infty)$ continuous with $F(x+k) = \int_0^{x+k} f(t)\mathrm{d}t < \alpha f(x)$ $\forall x > 0$ are $\alpha \geq ke$. $\square$

**449.** Let $m, n$ be positive integers and let $f, g_1, \ldots, g_{n+1} \in \mathbb{Z}[X]$, where $f$ is monic with $\deg f = n$, such that all roots of $f$ are real and $|g_i(\alpha)| < m$ for $1 \leq i \leq n+1$ and for every root $\alpha$ of $f$. Show that there exist $n+1$ integers, not all zero, each of absolute value at most $(2m(n+1))^n$, such that every root of $f$ is also a root of $\sum_{i=1}^{n+1} a_i g_i$.

`Proposed by Marius Cavachi, Ovidius University, Constanţa, Romania.`

*Solution by the author.* Let $\alpha_1, \ldots, \alpha_n$ be the roots of $f$, counting multiplicities, and let $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the linear function whose $j$-th component is

$$F_j(x) = \sum_{i=1}^{n+1} x_i g_i(\alpha_j), \quad x = (x_1, \ldots, x_{n+1}).$$

Let further $M = (2m(n+1))^n$ and note that $F$ maps the set of latticial points

$$S = \{0, 1, \ldots, M\}^{n+1}$$

into a cube $[-r, r]^n$, where $r < m(n+1)M$. This is an $n$-dimensional cube with the edge of length $2r < 2m(n+1)M$. This cube can be divided into $(2m(n+1)M)^n = M^{n+1}$ small cubes with the edges of lenght $\frac{2r}{2m(n+1)} < 1$. Since $|S| = (M+1)^{n+1} > M^{n+1}$ there are $x, y \in S$, $x \neq y$, such that $F(x)$ and $F(y)$ belong to the same small cube. Let $x - y = a = (a_1, \ldots, a_{n+1})$. Since $F(x)$ and $F(y)$ belong to a cube with an edge of length $< 1$, all entries of $F(a) = F(x) - F(y)$ are $< 1$ in absolute value. This means that

$$\left|\sum_{i=1}^{n+1} a_i g_i(\alpha_j)\right| = |F_j(a)| < 1 \text{ for } j = 1, \ldots, n. \qquad (*)$$

Since $x, y \in S$, their entries belong to $\{0, 1, \ldots, M\}$, so the entries of $a$ are integers with $|a_i| \leq M = (2m(n+1))^n$ for $i = 1, \ldots, n+1$.

We now proceed to show that each root of $f$ is also a root of $g = \sum_{i=1}^{n+1} a_i g_i \in \mathbb{Z}[X]$. Begin by noticing that each root $\alpha$ of $f$ is also a root of some irreducible factor $h$ of $f$ in $\mathbb{Z}$. Since $f$ is monic, so is $h$ upon multiplying it by $\pm 1$. Also, by Gauss's lemma, $h$ is irreducible in $\mathbb{Q}[X]$, so it is the minimal polynomial of $\alpha$. By rearranging the roots of $f$ we may assume that the roots of $h$ are $\alpha_1, \ldots, \alpha_k$. Since the product $g(\alpha_1) \cdots g(\alpha_k)$ is a symmetric polynomial with coefficients in $\mathbb{Z}$ in $\alpha_1, \ldots, \alpha_k$, it can be written as a polynomial in the symmetric fundamental polynomials in $\alpha_1, \ldots, \alpha_k$, which, by Vieta's relations, are $\pm$ the coefficients of $h$, so they are integers. It follows that $g(\alpha_1) \cdots g(\alpha_k) \in \mathbb{Z}$. On the other hand, by (*) we have $|g(\alpha_1) \cdots g(\alpha_k)| < 1$, so this product must be zero. Therefore $g(\alpha_i) = 0$ for some $i$, $1 \leq i \leq k$. Since also $h(\alpha_i) = 0$, we have $\gcd(g, h) \neq 1$ in $\mathbb{Q}[X]$. Since $h$ is irreducible, this implies that $h$ divides $g$. Since $h(\alpha) = 0$, we get $g(\alpha) = 0$. □

**450.** Find all polynomials $P \in \mathbb{R}[X]$ with the property

$$P(\sin x) + P(\cos x) = 1, \ \forall\, x \in \mathbb{R}.$$

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

*Solution by the author.* We prove that

$$P(X) = X^2 + \left(X^2 - \frac{1}{2}\right) Q(X^4 - X^2), \quad Q \in \mathbb{R}[X].$$

It is clear that the polynomial $P_1(X) = X^2$ verifies the above relation. We substitute $P(X) = X^2 + R(X)$ to obtain, for $R$, the following relation

$$R(\sin x) + R(\cos x) = 0. \qquad (1)$$

By replacing $x$ by $-x$, we obtain

$$R(-\sin x)+R(\cos x)=0. \tag{2}$$

From (1) and (2) it follows:

$$R(\sin x)=R(-\sin x)\Longrightarrow R(t)=R(-t),\ \forall\, t\in[-1,1]$$
$$\Longrightarrow R(X)=R(-X)\Longrightarrow R(X)=S(X^2),\ S\in\mathbb{R}[X],$$

and for the polynomial $S$ we obtain from (1)

$$S(\sin^2 x)+S(\cos^2 x)=0\Longleftrightarrow S(\sin^2 x)+S(1-\sin^2 x)=0$$
$$\Longleftrightarrow S(t)=-S(1-t)\forall t\in[0,1]\Longleftrightarrow S(X)=-S(1-X)$$
$$\Longleftrightarrow S\left(\frac{1}{2}-X\right)=-S\left(\frac{1}{2}+X\right).$$

The polynomial $U(X)=S\left(\frac{1}{2}+X\right)$ is an odd function $(U(-X)=-U(X))$, hence $U(X)=XV(X^2)$, $V\in\mathbb{R}[X]$. It follows:

$$S\left(\frac{1}{2}+X\right)=XV(X^2)\Longrightarrow S(X)=\left(X-\frac{1}{2}\right)V\left(\left(X-\frac{1}{2}\right)^2\right)$$

$$\Longrightarrow R(X)=\left(X^2-\frac{1}{2}\right)V\left(\left(X^2-\frac{1}{2}\right)^2\right)$$
$$=\left(X^2-\frac{1}{2}\right)V\left(X^4-X^2+\frac{1}{4}\right)=\left(X^2-\frac{1}{2}\right)Q(X^4-X^2),$$

where $Q(X)=V(X+\frac{1}{4})$, and so

$$P(X)=X^2+\left(X^2-\frac{1}{2}\right)Q(X^4-X^2),$$

which verifies the given relation for any polynomial $Q\in\mathbb{R}[X]$ (since $\sin^4 x-\sin^2 x=\cos^4 x-\cos^2 x$ and $\sin^2 x-\frac{1}{2}+\cos^2 x-\frac{1}{2}=0$ for any real $x$.)

**451.** Let $V$ be the linear space of the polynomial functions with real coefficients, defined on $[0,1]$. Denote $e_n(x)=x^n$, $x\in[0,1]$, $n=0,1,2,\ldots$. Let $T:V\to V$ be a linear operator such that:

(1) $Te_0=e_0$, $Te_1=Te_2=e_1$;
(2) If $p\in V$, $p(x)\geq 0$, $\forall x\in[0,1]$, then $(Tp)(x)\geq 0$, $\forall x\in[0,1]$.

Prove that $T$ is a projection, and find the null space $\operatorname{Ker} T$ and the range $\operatorname{Im} T$.

`Proposed by Ioan Raşa, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.`

*Solution by the author.* Let $f,g:[0,1]\to\mathbb{R}$. We write $f\leq g$ if $f(x)\leq g(x)$ for all $x\in[0,1]$. Note that the second property of $T$ together with the linearity of $T$ imply that for every $f,g\in V$ with $f\geq g$ we have

$Tf \geq Tg$. Indeed, if $p := f - g$ then $p(x) = f(x) - g(x) \geq 0$ $\forall x \in [0,1]$, so $Tf(x) - Tg(x) = Tp(x) \geq 0$.

Let $n \geq 3$. Then $e_n \leq e_1$, hence $Te_n \leq Te_1 = e_1$. In order to establish the converse inequality, notice that a relation of the form $e_n \geq e_1 - a(e_1 - e_2)$, with a suitable $a \in \mathbb{R}$, would be useful, as it implies $T(e_n) \geq T(e_1 - a(e_1 - e_2)) = e_1$ and so $Te_n = e_1$. This can be achieved by the inequality of the arithmetic and geometric means. We have $\frac{x^n+(n-2)x}{n-1} \geq \sqrt[n-1]{x^n \cdot x^{n-2}} = x^2$, so $x^n + (n-2)x \geq (n-1)x^2$, i.e., $e_n \geq e_1 - (n-1)(e_1 - e_2)$. In conclusion, $Te_n = e_1$, for all $n \geq 1$.

Let now $p \in V$. Then $p = a_0e_0 + a_1e_1 + \cdots + a_ne_n$, so that $Tp = a_0e_0 + (a_1 + \cdots + a_n)e_1$. But $a_0 = p(0)$, $a_1 + \cdots + a_n = p(1) - p(0)$, hence

$$Tp = p(0)e_0 + (p(1) - p(0))e_1, \quad \text{for all } p \in V.$$

It follows that $T^2 = T$, so that $T$ is a projection. Moreover, it is plain that one has

$$\operatorname{Ker} T = \{p \in V : p(0) = p(1) = 0\},$$
$$\operatorname{Im} T = \{a_0e_0 + a_1e_1 : a_0, a_1 \in \mathbb{R}\}.$$

**452.** Let $(T_n)_{n\in\mathbb{N}}$ be the sequence of Chebyshev's polynomials, which on the interval $[-1,1]$ are defined by $T_n(x) = \cos(n \arccos x)$, $n \in \mathbb{N}$, and let $(F_n)_{n\geq 0}$ be the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$, for all $n \geq 1$. Prove that

$$T_n\left(-\frac{3}{2}\right) = 1 + (-1)^n \frac{5}{2} F_n^2, \quad \forall n \in \mathbb{N}.$$

`Proposed by Vasile Pop, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.`

*Solution by the author.* We solve the problem by mathematical induction.

For $n = 0$ the formula to prove is verified since $T_0(x) = 1$ and $F_0 = 0$. For $n = 1$ we have $T_1(x) = x$, $F_1 = 1$, and $-\frac{3}{2} = 1 - \frac{5}{2}$. We assume the identity to prove is true for all integers up to $n$ and we prove it holds for $n+1$. One can prove that the Chebyshev's polynomials verify the recurrence relation $T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x)$, for all $n \geq 1$ and all $x \in [-1,1]$, so it also holds for all $x \in \mathbb{R}$. (If $\theta = \arccos x$ this formula writes as $\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$.) Using this formula it sufices to prove that

$$1 + (-1)^{n+1}\frac{5}{2}F_{n+1}^2 = 2\left(-\frac{3}{2}\right)\left(1 + (-1)^n\frac{5}{2}F_n^2\right) - \left(1 + (-1)^{n-1}\frac{5}{2}F_{n-1}^2\right)$$

$\iff 2 = (-1)^n \left(F_{n+1}^2 - 3F_n^2 + F_{n-1}^2\right)$. We have

$$\begin{aligned}
F_{n+1}^2 - 3F_n^2 + F_{n-1}^2 &= F_{n+1}^2 - 2F_n^2 - (F_n^2 - F_{n-1}^2) \\
&= F_{n+1}^2 - 2F_n^2 - (F_n - F_{n-1})(F_n + F_{n-1}) \\
&= F_{n+1}^2 - F_{n+1}(F_n - F_{n-1}) - 2F_n^2 \\
&= F_{n+1}(F_{n+1} - F_n + F_{n-1}) - 2F_n^2 \\
&= 2(F_{n+1}F_{n-1} - F_n^2) \\
&= 2(-1)^n,
\end{aligned}$$

where the last equality can be proved by direct computation. For example one can use the Binet formula for the Fibonacci sequence $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. □

**Notes from the editor.**

1. We have $F_{n+2}F_n - F_{n+1}^2 = (F_{n+1}+F_n)F_n - F_{n+1}^2 = F_n^2 - F_{n+1}(F_{n+1} - F_n) = F_n^2 - F_{n+1}F_{n-1} = -(F_{n+1}F_{n-1} - F_n^2)$. This allows us to prove the formula $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ by induction, without using the Binet formula for $F_n$.

2. There is a direct solution that doesn't use induction. By definition $T_n(\cos\theta) = \cos n\theta \ \forall\theta \in [0,\pi]$. If $z = \exp(i\theta) = \cos\theta + i\sin\theta$, this writes as $T_n(\frac{z+z^{-1}}{2}) = \frac{z^n+z^{-n}}{2}$. Since this relation holds for an infinity of complex values of $z$, it must be an identity, i.e., we have $T_n(\frac{X+X^{-1}}{2}) = \frac{X^n+X^{-n}}{2}$ in $\mathbb{C}[X, X^{-1}]$.

Since $X = -\frac{3+\sqrt{5}}{2}$ is a root of $\frac{X+X^{-1}}{2} = -\frac{3}{2}$, we have

$$\begin{aligned}
T\left(-\frac{3}{2}\right) &= \frac{\left(-\frac{3+\sqrt{5}}{2}\right)^n + \left(-\frac{3+\sqrt{5}}{2}\right)^{-n}}{2} = (-1)^n \frac{\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3+\sqrt{5}}{2}\right)^{-n}}{2} \\
&= (-1)^n \frac{\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n}{2}.
\end{aligned}$$

But $\frac{3+\sqrt{5}}{2} = \alpha^2$ and $\frac{3-\sqrt{5}}{2} = \beta^2$. (Here we use the author's notation $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.) From $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ and $\alpha\beta = -1$ we get

$$\begin{aligned}
T_n\left(-\frac{3}{2}\right) &= (-1)^n \frac{\alpha^{2n} + \beta^{2n}}{2} = (-1)^n \left(\frac{(\alpha^n - \beta^n)^2}{2} + \alpha^n\beta^n\right) \\
&= (-1)^n \left(\frac{(\sqrt{5}F_n)^2}{2} + (-1)^n\right) = 1 + (-1)^n \frac{5}{2}F_n^2.
\end{aligned}$$

**453.** Let $f, g \in \mathbb{Q}[X]$ be monic and irreducible polynomials over $\mathbb{Q}$ with the property that there are $\alpha, \beta \in \mathbb{C}$ with $\alpha + \beta \in \mathbb{Q}$ and $f(\alpha) = g(\beta) = 0$. Prove that the polynomial $f^2 - g^2$ has a root in $\mathbb{Q}$.

**Remark.** A generalization of this problem is the following: Let $K \subseteq L$ be an extension of fields of characteristic different from 2 and $f, g \in K[X]$ two monic irreducible polynomials over $K$ for which there exist $\alpha, \beta \in L$ with $\alpha + \beta \in K$ and $f(\alpha) = g(\beta) = 0$.

Prove that the polynomial $f^2 - g^2$ has a root in $K$.

Proposed by Bogdan Moldovan, student, Babeş-Bolyai University, Cluj-Napoca, Romania.

*Solution by the author.* From the hypothesis it follows that $f$ and $g$ are the minimal polynomials of $\alpha$ and $\beta$, respectively, over $K$. Then the polynomial $h = g(\alpha + \beta - x) \in K[X]$ is irreducible over $K$ (Gauss: the function $\phi : K[X] \to K[X]; \phi(f(X)) = f(aX + b)$ with $a, b \in K$, $a$ invertible, is an automorphism of polynomial rings) and, in addition, we have $h(\alpha) = g(\beta) = 0$. Therefore the polynomial $\pm h$ is the minimal polynomial of $\alpha$ over $K$. (Here the $+$ sign is used when $\deg g$ is even, and $-$ when $\deg g$ is odd.) But from the uniqueness of the minimal polynomial it results that $f(x) = \pm g(\alpha + \beta - x)$ and so $f^2(x) = g^2(\alpha + \beta - x)$. If we replace in the last equality $x = 2^{-1}(\alpha + \beta)$ (it is possible because char$(K) \neq 2$, so $2 \in K$ is invertible), we get that $\frac{\alpha+\beta}{2}$ is a root of $f^2 - g^2$ which belongs to $K$. □

**454.** Let $A \in M_3(\mathbb{Z})$ with $A^{30} - 2A^{25} = A^5 - 2I_3$. Prove that $\mathrm{tr}(A) \not\equiv 2 \pmod 5$.

Dumitru Popa, Department of Mathematics, Ovidius University, Constanţa, Romania.

*Solution by the author.* Let $\overline{A} \in M_3(\mathbb{Z}_5)$ be the reduction of $A$ modulo 5. We must prove that $\mathrm{tr}(\overline{A}) \neq 2$ in the field $\mathbb{Z}_5$. We have $\mathrm{tr}(\overline{A}) = \alpha_1 + \alpha_2 + \alpha_3$, where $\alpha_1, \alpha_2, \alpha_3$ are the roots of the characteristic polynomial $P_{\overline{A}}$ in some extension of $\mathbb{Z}_5$. But by hypothesis we have $P(\overline{A}) = 0$, where $P \in \mathbb{Z}_5[X]$, $P(X) = X^{30} - 2X^{25} - X^5 + 2$. So $P$ is divisible by the minimal polynomial of $\overline{A}$ and therefore $\alpha_1, \alpha_2, \alpha_3$ are amongst the roots of $P$. But over $\mathbb{Z}_5$ we have $P = (X^5 - 2)(X^{25} - 1) = (X - 2)^5(X - 1)^{25}$. Hence $\alpha_1, \alpha_2, \alpha_3 \in \{1, 2\}$. Therefore $\mathrm{tr}(\overline{A}) = \alpha_1 + \alpha_2 + \alpha_3 \in \{1+1+1, 1+1+2, 1+2+2, 2+2+2\} = \{3, 4, 0, 1\}$, which shows that $\mathrm{tr}(\overline{A}) \neq 2$. □