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Some results concerning fixed points of φ-contractions

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Abstract. Existence and uniqueness of fixed points of φ-contractions, in the general sense, are given. Three proofs of a variant of the Boyd-Wong theorem are presented. Using the conclusion of the theorem, some other results concerning the attractor of a GIFS of φ-contractions are discussed.

Keywords: fixed point; Boyd-Wong theorem; φ-contraction.

MSC: Primary 54H25; Secondary 47H10.

1. Introduction and preliminaries

In the last few decades, fixed point theory has had a very flourishing development, mainly because of its large range of applications.

The beginning of the theory was in a certain way the classical Banach theorem, which states that if X is a complete metric space and f is a mapping of X into itself which satisfies

$$d(f(x), f(y)) \leq \rho d(x, y), \text{ for some } \rho \in [0, 1) \text{ and all } x, y \in X,$$

then f has a fixed point $\overline{x}$, and the successive approximations $f^n(x)$ converge to $\overline{x}$ for $x \in X$.

Nevertheless, the condition $d(f(x), f(y)) \leq d(x, y)$ does not insure that f has a fixed point.

In 1969 appeared the theorem of Boyd-Wong, which replaces the condition of Banach with

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \ \forall x, y \in X,$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a comparison function.

Definition 1. *A mapping* $\varphi : [0, \infty) \to [0, \infty)$ *is called a* comparison function *if*

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a) φ *is continuous;*
b) $\varphi(r) < r, \ \forall\, r > 0$.

Definition 2. *Let* $f : X \to X$ *and* $\varphi : [0, \infty) \to [0, \infty)$ *be a comparison function.* f *is called a* φ-function*, or a* φ-contraction*, if*

$$\forall x, y \in X, \text{ we have } d(f(x), f(y)) \leq \varphi(d(x, y)).$$

The class of φ-contractions enlarges the class of contractions in the Banach sense (every contraction being a φ-contraction).

In [4], the following result is called Matkowski's Theorem (1975).

Theorem 3. *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be a* φ*-contraction. Let* F_f *denote the set of fixed points for* f*. Then we have:*

i) $F_f = F_{f^n} = \{x^*\}$*, for each* $n \in \mathbb{N}^*$*;*
ii) for each $x \in X$ *the sequence of successive approximations* $f^n(x)$ *of* f *starting from* x *converges to* x^**;*
iii) if, additionally, φ *is a strict comparison function, then*

$$d(x, x^*) \leq \varphi\big(d(x, f(x))\big).$$

In [4] also appear other basic fixed point principles, like the Contraction Principle, Ćirić-Reich-Rus's Theorem (1971), Meir-Keeler's Theorem (1969), Krasnoselskii's Theorem (1972), Graphic Contraction Principle, Caristi-Browder's Theorem, Clarke's Theorem, Niemytzki-Edelstein's Theorem. It is also proved the next general result.

Theorem 4. *Let* X *be a nonempty set and* $f : X \to X$ *be an operator. Then the following statements are equivalent:*

(P1) *There exists a metric* d *on* X *such that* $f : (X, d) \to (X, d)$ *is a Picard operator;*
(P2) f *is a Bessaga operator;*
(P3) *There exist a comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *and a complete metric* d *on* X *such that* $f : (X, d) \to (X, d)$ *is a* φ*-contraction;*
(P4) *There exists a metric* d *on* X *such that the fixed point problem is well-posed for* f *with respect to* d*.*

The main result of [5] is the following theorem.

Theorem 5. *Let* (X, d) *be a complete metric space and* $f : X^m \to X$ *be such that there exists* $\varphi : \mathbb{R}^m \to \mathbb{R}$ *with the following properties:*

(a) $(r \leq s, r, s \in \mathbb{R}^m_+) \Rightarrow (\varphi(r) \leq \varphi(s))$*;*
(b) $(r \in \mathbb{R}_+, r > 0) \Rightarrow (\varphi(r, \ldots, r) < r)$*;*
(c) φ *is continuous;*

(d) $\sum_{k=0}^{\infty} \varphi^k(r) < +\infty$;

(e) for all $r \in \mathbb{R}_+$

$$\varphi(r, 0, \ldots, 0) + \varphi(0, r, 0, \ldots, 0) + \cdots + \varphi(0, 0, \ldots, 0, r) \leq \varphi(r, r, \ldots, r);$$

(f) for all $x_0, x_1, \ldots, x_m \in X$,

$$d\big(f(x_0, \ldots, x_{m-1}), f(x_1, \ldots, x_m)\big) \leq \varphi\big(d(x_0, x_1), \ldots, d(x_{m-1}, x_m)\big).$$

Then:

(i) $F_f = \{x^*\}$;

(ii) for any $\tilde{x}_0 \in X$, *the sequence* $(\tilde{x}_n)_{n\in\mathbb{N}}$, $\tilde{x}_n = f(\tilde{x}_{n-1}, \ldots, \tilde{x}_{n-1})$, *converges to* x^*;

(iii) for all $x_0, \ldots, x_{m-1} \in X$, *the sequence* $(x_{m+n})_{n\in\mathbb{N}}$ *defined by* $x_{m+n} = f(x_n, \ldots, x_{n+m-1})$ *converges to* x^* *and*

$$d(x_n, x^*) \leq m \sum_{k=0}^{\infty} \varphi(d_0)^{\left\lfloor \frac{n}{m} \right\rfloor + k},$$

$$\textit{where} \quad d_0 = \max\big(d(x_0, x_1), \ldots, d(x_{m-1}, x_m)\big).$$

Conditions to obtain iterates to the fixed point of the equation $x = f(x, \ldots, x)$ are presented in [5]. All the papers treat the fixed point theory, mainly because the theory is important in solving operatorial equations. The topic of [6] (which is the resumé of the author's Ph.D. thesis) is a chapter of this domain, namely the metrical fixed point theory. The approach is mainly the one of successive approximations (first initiated by E. Picard in the years 1890–1894, and later developed by St. Banach (1922) and R. Caccioppoli (1930)).

The metrical fixed point theory was treated by many authors, like R. Kannan, S. Reich, M. G. Maia, F. E. Browder, S. B. Nadler etc. After the year 1968, the development of the metrical fixed point theory has been explosive, its subjects being mainly the treatment of the following types of operators: contraction operators, contractive operators (Edelstein), and non-expansive operators.

The main aim in [6] is the study of contraction operators that satisfy a condition of generalized contraction, a condition that assures the convergence of the sequence of successive approximations to the unique fixed point of the operator (called Picard type operator).

There are presented comparison functions, and generalizations of them, like (c)-comparison functions, p-dimensional comparison functions, abstract φ-contractions. They all serve to obtain results that generalize the classical Contraction Principle (even concerning the order of convergence of the sequence of successive approximations).

The notion of φ-contraction plays an important part in [7], too. Various metrical fixed point theorems are established for (X,d) a metric space and $f : X \to X$ a mapping or an operator satisfying one of a number of conditions of contraction type. Some of them are:

(i) (Banach (1922)) There exists a number $a \in [0,1)$ such that

$$d\big(f(x), f(y)\big) \leq ad(x,y), \forall x, y \in X.$$

(ii) (Nemytzki (1936), Edelstein (1962)) For all $x, y \in X, x \neq y$,

$$d\big(f(x), f(y)\big) < d(x,y).$$

(iii) (Rakotch (1962), Boyd and Wong (1969), Browder (1968)) There exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) < t$ for $t > 0$, φ is increasing, continuous and

$$d\big(f(x), f(y)\big) \leq \varphi\big(d(x,y)\big), \forall x, y \in X.$$

(iv) (Rus (1972)) There exists a number $a \in [0,1)$ such that

$$d\big(f^2(x), f(x)\big) \leq ad\big(x, f(x)\big), \forall x \in X.$$

(v) (Reich (1971), Rus (1971)) There exist $a, b \in \mathbb{R}_+$, $a + 2b < 1$, such that

$$d\big(f(x), f(y)\big) \leq ad(x,y) + b\Big[d\big(x, f(x)\big) + d\big(y, f(y)\big)\Big], \forall x, y \in X.$$

See also Guseman (1970), Yen (1972), Kannan (1968), Ćirić (1974), Zamfirescu (1972), Jachymski and Stein (1999).

The study is centered around these generalized contractions in terms of Picard operators, weakly Picard operators, Bessaga operators and Janos operators.

2. Main Results

We begin the presentation with a variant of the Boyd-Wong theorem.

Theorem 6. *Let (X,d) be a complete metric space and $\varphi : [0,\infty) \to [0,\infty)$ a function with the properties:*

(a) *φ is continuous;*
(b) *$\varphi(r) < r$, $\forall r > 0$ ($\varphi(0) = 0$);*
(c) *φ is nondecreasing.*

If $f : X \to X$ has the property that

$$d(f(x), f(y)) \leq \varphi(d(x,y)), \forall x, y \in X,$$

then f has a unique fixed point $\overline{x} \in X$.

Moreover, for any $x_0 \in X$, the sequence $(f^n(x_0))_{n\geq 1}$ converges to $\overline{x}$.

Remark 7. (1) A comparison function verifies the inequality

$$\varphi(r) \leq r, \forall r \in [0, \infty).$$

(2) Any $f : X \to X$ with the property that

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \forall x, y \in X$$

is non-expansive, meaning

$$d(f(x), f(y)) \leq d(x, y), \forall x, y \in X.$$

In particular, f is continuous.

We shall present three proofs for the theorem above.

2.1. **First proof.** *Uniqueness of the fixed point.* Suppose that there are $x_1, x_2 \in X$ such that $f(x_1) = x_1$ and $f(x_2) = x_2$. Then

$$r := d(x_1, x_2) = d(f(x_1), f(x_2)) \leq \varphi(d(x_1, x_2)) = \varphi(r).$$

Supposing that $r > 0$, we obtain $r \leq \varphi(r) < r$, which represents a contradiction. Therefore $r = 0$, so that $x_1 = x_2$.
Existence of a fixed point. Let $x_0 \in X$. We shall prove that the sequence $(x_n)_n$ defined by $x_n = f^n(x_0)$, $n \geq 1$, is a Cauchy sequence in the complete metric space (X, d), so that $(x_n)_n$ is a convergent sequence to $\overline{x} \in X$.

From the relation $x_{n+1} = f(x_n), n \geq 1$, and f continuous we then deduce that $\overline{x} = f(\overline{x})$ and $\overline{x}$ is the unique fixed point of f.

Lemma 8. *If $r \in [0, \infty)$, then $\varphi^n(r) \xrightarrow[n\to\infty]{} 0$.*

Proof. If $r = 0$ then $\varphi^n(0) = 0$ for all positive integers n.

If $r > 0$ then $\varphi^2(r) < \varphi(r) < r$ etc., so that $(\varphi^n(r))_{n\geq 0}$ is a decreasing sequence, minorated by 0. Put $\lim\limits_{n\to\infty} \varphi^n(r) = l \geq 0$. The requirements imposed on φ entail $\varphi(l) = l$ and therefore $l = 0$. □

Let p be a positive integer and $\varepsilon > 0$. We have

$$d\big(f^n(x_0), f^{n+p}(x_0)\big) \leq \varphi^n\big(d(x_0, f^p(x_0))\big).$$

According to Lemma 8, the right-hand side converges to 0 as $n \to \infty$. Thus, $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ such that for every $n \geq n_\varepsilon, \forall p \geq 1, d(f^n(x_0), f^{n+p}(x_0)) < \varepsilon$. This means $(f^n(x_0))_{n\geq 1}$ is a Cauchy sequence.

2.2. **Second proof.** We keep the notation introduced in the statement of Theorem 6.

Proposition 9. *Let $x_0 \in X$ and $K = \overline{\{x_0, f(x_0), f^2(x_0), \dots\}}$.*

If $d\big(f(x), f(y)\big) \leq \varphi\big(d(x, y)\big), \forall x, y \in X$, then K is a compact set.

Proof. We shall prove that the metric space $(K, d) \subset (X, d)$ is countably compact. This implies that (K, d) is a compact metric space.

Let $\{x_n\}_n \subseteq K$. Each such element can be written $x_n = \lim\limits_{p\to\infty} e_p^n$ for suitable e_p^n in the orbit $O(x_0) = \{x_0, f(x_0), f^2(x_0), \dots\}$ of x_0.

For each $n \in \mathbb{N}$ we denote $A_n = \{x_n, x_{n+1}, \dots\}$ and we shall prove that $(\overline{A_n})_{n\geq 1}$ fulfills the following conditions:

- $\overline{A_n}$ is a closed set for any $n \geq 1$;
- the sequence $(\overline{A_n})_{n\geq 1}$ is decreasing with respect to inclusion;
- $\operatorname{diam}(\overline{A_n}) \xrightarrow[n\to\infty]{} 0$.

As (X, d) is a complete metric space, using the Cantor theorem, it follows that $\bigcap\limits_{n\in\mathbb{N}} \overline{A_n} \neq \emptyset$, which implies that is (K, d) countably compact, so (K, d) will be a compact metric space.

As the first two of the desired properties are obviously satisfied, it remains to prove that one has $\operatorname{diam}(\overline{A_n}) \xrightarrow[n\to\infty]{} 0$. We notice that

$$d(x_n, x_m) = d(\lim_{p\to\infty} e_p^n, \lim_{p\to\infty} e_p^m) = \lim_{p\to\infty} d(e_p^n, e_p^m).$$

We consider the sets $B_n = \{x_n, x_{n+1}, \dots\}$ and suppose that x is x_{p_1}, $f(x)$ is x_{p_2}, and so on. We eliminate first $\max\{p_1, \dots, p_{n-1}\}$ positions. The elements left, meaning $f^n(x), f^{n+1}(x), \dots$, are in the sets from B_s onward (when $n \to \infty \Rightarrow s \to \infty$)

$$d(f^n(x), f^{n+k}(x)) \leq \varphi^n(d(x, f^k(x))) \xrightarrow[n\to\infty]{} 0 \Rightarrow \operatorname{diam}(B_s) \xrightarrow[s\to\infty]{} 0$$

and Proposition 9 is proved. □

For the promised proof of the Boyd-Wong theorem, we need a well-known result, recalled next.

Proposition 10. *If a compact metric space (K, d) and $f : K \longrightarrow K$ satisfy*

$$d(f(x), f(y)) < d(x, y), \forall x \neq y \in K,$$

then f has a unique fixed point $\overline{x}$.

Moreover, for any $x \in K$, the sequence $(f^n(x))_{n\geq 1}$ converges to $\overline{x}$.

In the conditions of the Boyd-Wong theorem, we can apply Proposition 10 with K an arbitrary closed orbit $O(x) = \overline{\{x, f(x), f^2(x), \dots\}}$ and f the given mapping (because $f\big(O(x)\big) \subseteq O(x)$ as f is continuous, Proposition 9 applies). So, f admits a unique fixed point, say, $\overline{x}$, on every set $O(x)$. As the hypotheses from the Boyd-Wong theorem entail

$$d\big(f(x), f(y)\big) \leq \varphi\big(d(x, y)\big) < d(x, y), \forall x \neq y \in X,$$

all the points $\overline{x}$ are the same.

To finish the proof, it remains to remark that no matter how we chose the initial point $x_0 \in X$, the sequence $(f^n(x_0))_{n\geq 1}$ converges to $\overline{x_0}$, the unique fixed point of f.

2.3. **Third proof.** For any $x, y \in X, x \neq y$, we have

$$\overline{\lim_{n\to\infty}} d\big(f^n(x), f^n(y)\big) \leq \overline{\lim_{n\to\infty}} \varphi^n\big(d(x,y)\big) = 0,$$

so

$$\overline{\lim_{n\to\infty}} d\big(f^n(x), f^n(y)\big) = 0.$$

Let $a \in X$ be arbitrary, $(a_n)_n$ be a sequence of Picard iterates of f at the point a,

$$Y = \overline{\{a_n\}} = \overline{\{a, f(a), \dots, f^n(a), \dots\}},$$

and

$$F_n = \left\{ x \in Y \mid d\big(x, f^k(x)\big) \leq \frac{1}{n},\ \forall k = \overline{1,n} \right\}.$$

From $\lim\limits_{n\to\infty} d\big(f^n(x), f^n(y)\big) = 0$ it follows that F_n is nonempty. It is clear that one has $F_{n+1} \subseteq F_n, \forall n \in \mathbb{N}^*$. Since f is continuous, F_n is closed for any n.

Let $(x_n)_n$ and $(y_n)_n$ be arbitrary sequences with $x_n, y_n \in F_n,\ n \geq 1$, and $(n_j)_j$ a sequence of positive integers such that

$$\lim_{j\to\infty} d(x_{n_j}, y_{n_j}) = \overline{\lim_{n\to\infty}} d(x_n, y_n).$$

Now we have

$$\begin{aligned}\lim_{j\to\infty} d(x_{n_j}, y_{n_j}) &\leq \lim_{j\to\infty} \left[d(x_{n_j}, f^{n_j}(x_{n_j})) + d(f^{n_j}(x_{n_j}), f^{n_j}(y_{n_j})) + d(y_{n_j}, f^{n_j}(y_{n_j}))\right] \\ &\leq \lim_{j\to\infty} \varphi^{[n_j]}(d(x_{n_j}, y_{n_j})) = 0,\end{aligned}$$

so that $\lim\limits_{j\to\infty} d(x_{n_j}, y_{n_j}) = 0$ or equivalently $\overline{\lim\limits_{n\to\infty}} d(x_n, y_n) = 0$. This in turns implies $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$, that is, $\lim\limits_{n\to\infty} \operatorname{diam}(F_n) = 0$.

By completeness of Y, it follows that there exists $z \in X$ such that

$$\bigcap_{n=1}^{\infty} F_n = \{z\}.$$

Since $d(z, f(z)) \leq \frac{1}{n}$ for any n, we have $f(z) = z$. Then, from

$$\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0,$$

it follows that all sequences of Picard iterates defined by f converge to z.

3. Application for Boyd-Wong theorem

Definition 11. *Let (X,d) be a complete metric space and let $m \in \mathbb{N}$.*
A generalized iterated function system *of φ-contractions (for short, a GIFS) on X of order m, denoted by*

$$\mathcal{S} = \big(X, (f_i)_{i=\overline{1,n}}\big),$$

consists of a finite family of functions $(f_i)_{i=\overline{1,n}}$, $f_i : X^m \to X$, such that $d\big(f_i(x_1,\dots,x_m), f_i(y_1,\dots,y_m)\big) \le \varphi\big(\max_{k=\overline{1,m}} d(x_k,y_k)\big)$, $\forall i = \overline{1,n}$.

For a given a metric space (X,d), $\mathcal{K}(X)$ denotes the set of all nonempty compact subsets of X and $\mathcal{BC}(X)$ denotes the set of nonempty bounded closed subsets of X.

For a metric space (X,d), one considers on $\mathcal{P}^*(X)$ the generalized Hausdorff-Pompeiu pseudometric

$$h : \mathcal{P}^*(X) \times \mathcal{P}^*(X) \to [0,\infty],$$

defined by

$$\begin{aligned} h(A,B) &= \max(d(A,B), d(B,A)) \\ &= \inf\{r \in [0,\infty) : A \subseteq B(B,r), B \subseteq B(A,r)\}, \end{aligned}$$

where

$$\begin{aligned} B(A,r) &= \{x \in X : d(x,A) < r\} \\ d(A,B) &= \sup_{x\in A} d(x,B) = \sup_{x\in A} (\inf_{y\in B} d(x,y)) \end{aligned}$$

Remark 12. The Hausdorff-Pompeiu pseudometric is a metric on $\mathcal{BC}(X)$ and, in particular, on $\mathcal{K}(X)$.

Remark 13. The metric spaces $(\mathcal{BC}(X), h)$ and $(\mathcal{K}(X), h)$ are complete, provided that (X,d) is a complete metric space. Moreover, $(\mathcal{K}(X), h)$ is compact, provided that (X,d) is a compact metric space.

Proposition 14. *Let (X,d) be a complete metric space, m a positive integer, $\delta : X \to X^m$ defined by $\delta(x) = (x,x,\dots,x)$, and $\varphi : [0,\infty) \to [0,\infty)$ be a function with the properties:*

(a) *φ is continuous,*
(b) *$\varphi(0) = 0$ and $\varphi(r) < r$ $\forall r > 0$,*
(c) *φ is nondecreasing.*

Then $(X^m, d_{\max})$ is a complete metric space, and if $f : X^m \to X$ has the property that for any $(x_1,\dots,x_m), (y_1,\dots,y_m) \in X^m$ it holds

$$d\big(f(x_1,\dots,x_m), f(y_1,\dots,y_m)\big) \le \varphi\big(\max_{k=\overline{1,m}} d(x_k,y_k)\big),$$

then there is a unique point $(\alpha, \alpha, \dots, \alpha) \in X^m$ *such that* $f(\alpha, \alpha, \dots, \alpha) = \alpha$. *Moreover, for any* $(x_1^0, \dots, x_m^0) \in X^m$, *the sequence* $\big([\delta \circ f]^n(x_1^0, \dots, x_m^0)\big)_{n\geq 1}$ *converges to* $(\alpha, \alpha, \dots, \alpha)$.

Proof. Let g denote the composed mapping $X^m \xrightarrow{f} X \xrightarrow{\delta} X^m$. Then

$$g(x_1, \dots, x_m) = \big(f(x_1, \dots, x_m), \dots, f(x_1, \dots, x_m)\big)$$

and for any $(x_1, \dots, x_m), (y_1, \dots, y_m) \in X^m$ it holds

$$\begin{aligned} &d\big(g(x_1, \dots, x_m), g(y_1, \dots, y_m)\big) \\ &= d\big((f(x_1, \dots, x_m), \dots, f(x_1, \dots, x_m)), (f(y_1, \dots, y_m), \dots, f(y_1, \dots, y_m))\big) \\ &\leq \max\{d(f(x_1, \dots, x_m), f(y_1, \dots, y_m))\} \leq \varphi(\max_{k=\overline{1,m}} d(x_k, y_k)). \end{aligned}$$

Using the Boyd-Wong theorem on $(X^m, d_{\max})$, we obtain that $g = \delta \circ f$ has a unique fixed point $(\overline{x_1}, \dots, \overline{x_m}) \in X^m$, which means that

$$\overline{x_1} = \overline{x_2} = \cdots = \overline{x_m} = f(\overline{x_1}, \dots, \overline{x_m}) = \alpha.$$

This α satisfies $f(\alpha, \alpha, \dots, \alpha) = \alpha$. Moreover, for any $(x_1^0, \dots, x_m^0) \in X^m$, the sequence $\big(g^n(x_1^0, \dots, x_m^0)\big)_{n\geq 1}$ converges to $(\alpha, \alpha, \dots, \alpha)$. □

Proposition 15. *Let* (X, d) *be a complete metric space,* $m \in \mathbb{N}$, *and* $\mathcal{S} = \big(X, (f_i)_{i=\overline{1,n}}\big)$ *a GIFS of* φ*-contractions on* X *of order* m. *Suppose that for any* $(x_1, \dots, x_m)$, $(y_1, \dots, y_m) \in X^m$ *it holds*

$$d\big(f_k(x_1, \dots, x_m), f_k(y_1, \dots, y_m)\big) \leq \varphi\big(\max_{j=\overline{1,m}} d(x_j, y_j)\big), \forall k = \overline{1,n}.$$

Then for $K_1, \dots, K_m, H_1, \dots, H_m \in \mathcal{K}(X)$, *we have*

$$h\big(f_k(K_1 \times \cdots \times K_m), f_k(H_1 \times \cdots \times H_m)\big) \leq \varphi\big(\max_{j=\overline{1,m}} h(H_j, K_j)\big), \forall k = \overline{1,n}.$$

Proof. Since $(\mathcal{K}(X), h)$ is compact (see Remark 13), for any $j = \overline{1,m}$ there exist $\overline{x_j} \in K_j$, $\overline{y_j} \in H_j$ such that

$$d(\overline{x_j}, \overline{y_j}) = d(\overline{x_j}, H_j) \leq d(K_j, H_j) \leq h(K_j, H_j).$$

Then

$$h(f_k(K_1 \times \cdots \times K_m), f_k(H_1 \times \cdots \times H_m)) = h(f_k(\overline{x_1}, \dots, \overline{x_m}), f_k(H_1 \times \cdots \times H_m))$$

$$\begin{aligned} &= \inf\{d(f_k(\overline{x_1}, \dots, \overline{x_m}), f_k(y_1, \dots, y_m)) \mid y_1 \in H_1, \dots y_m \in H_m\} \\ &\leq d(f_k(\overline{x_1}, \dots, \overline{x_m}), f_k(\overline{y_1}, \dots, \overline{y_m})) \leq \varphi(\max_{j=\overline{1,m}} d(\overline{x_j}, \overline{y_j})) \leq \varphi(\max_{j=\overline{1,m}} d(H_j, K_j)), \end{aligned}$$

as φ is nondecreasing. □

Our result is presented in the following proposition.

Proposition 16. *Let (X,d) be a complete metric space and $\mathcal{S}=\big(X,(f_i)_{i=\overline{1,n}}\big)$ a GIFS of φ-contractions on X. We define $\mathcal{F}_s:\mathcal{K}(X)^m\to\mathcal{K}(X)$ by*

$$\mathcal{F}_s(K_1,\dots,K_m)=\bigcup_{k=1}^{n}\mathcal{F}_{f_k}(K_1,\dots,K_m)=\bigcup_{k=1}^{n}f_k(K_1\times\cdots\times K_m).$$

Then $\mathcal{F}_s:\mathcal{K}^m(X)\to\mathcal{K}(X)$ is a φ-contraction in the sense that for any $H_j,K_j\in\mathcal{K}(X)$ $(j=\overline{1,n})$ it holds

$$h\big(\mathcal{F}_s(K_1,\dots,K_m),\mathcal{F}_s(H_1,\dots,H_m)\big)\le\varphi\big(\max_{j=\overline{1,m}}h(K_j,H_j)\big).$$

Moreover, there exists a unique $A(\mathcal{S})\in\mathcal{K}(X)$ such that

$$\mathcal{F}_s\big(A(\mathcal{S}),\dots,A(\mathcal{S})\big)=A(\mathcal{S}).$$

Proof. From Proposition 15 and the hypotheses we get

$$h\big(f_k(K_1,\dots,K_m),f_k(H_1,\dots,H_m)\big)\le\varphi\big(\max_{j=\overline{1,m}}h(K_j,H_j)\big),\ \forall k=\overline{1,n}.$$

We also have:

$$\begin{aligned}&h(\mathcal{F}_s(K_1,\dots,K_m),\mathcal{F}_s(H_1,\dots,H_m))\\&=h\big(\bigcup_{k=1}^{n}f_k(K_1,\dots,K_m),\bigcup_{k=1}^{n}f_k(H_1,\dots,H_m)\big)\\&\le\max_{j=\overline{1,m}}h\big(f_k(K_1,\dots,K_m),f_k(H_1,\dots,H_m)\big)\\&\le\varphi\big(\max_{j=\overline{1,m}}h(H_j,K_j)\big),\forall K_1,\dots,K_m\in\mathcal{K}(X),\ \forall H_1,\dots,H_m\in\mathcal{K}(X).\end{aligned}$$

□

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An estimate for the difference between the p-norm and the q-norm

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Abstract. Let $0 < q < p < \infty$ and $p \geq 1$. We prove that:
(i) If $p < 2q$ then for all $a, b \in [0, \infty)$ the following relation holds

$$0 \leq (a^q + b^q)^{\frac{1}{q}} - (a^p + b^p)^{\frac{1}{p}} \leq \left(\frac{p}{q}\right)^{\frac{1}{p}} [\max(a, b)]^{\frac{2q}{p}-1} (ab)^{1-\frac{q}{p}}.$$

(ii) If $2q \leq p$ then for all $a, b \in [0, \infty)$ the following relation holds

$$0 \leq (a^q + b^q)^{\frac{1}{q}} - (a^p + b^p)^{\frac{1}{p}} \leq \left(2^{\frac{p}{q}} - 2\right)^{\frac{1}{p}} [\max(a, b)]^{1-\frac{2q}{p}} (ab)^{\frac{1}{p}}.$$

We give similar estimates for $(a^s + b^s)^{\frac{1}{s}} - (a + b)$ when $0 < s < 1$. These estimates are used to give a solution of a problem given at the Cambridge mathematical tripos examination and which asserts that

$$\lim_{n\to\infty} \frac{n}{2^n} \int_{-1}^{1} \sqrt{a^2 (1 + x)^{2n} + b^2 (1 - x)^{2n}} \mathrm{d}x = 2 (a + b), \ a > 0, \ b > 0.$$

Keywords: Inequalities for sums, series and integrals; the limit of sequences of integrals
MSC: Primary 26D15; Secondary 28A20.

Introduction and notation

The main purpose of this paper is to prove the estimates stated in the Abstract. We obtained these estimates in an attempt to solve the following problem given at the Cambridge mathematical tripos: prove that

$$\lim_{n\to\infty} \frac{n}{2^n} \int_{-1}^{1} \sqrt{a^2 (1 + x)^{2n} + b^2 (1 - x)^{2n}} dx = 2 (a + b), \ a > 0, \ b > 0;$$

see [2, Problem 1891, page 323]. Latter we have discovered that G. Bennett in [3, Lemma 1] has proved some different type of inequalities. Our notation and notions are standard, see [1].

1. The results

We need the following well-known result. For the sake of completeness we include its proof.

Lemma 1. *Let $f : [0, \infty) \to \mathbb{R}$ be a differentiable function on $[0, \infty)$ and twice differentiable on $(0, \infty)$.*
(i) *If $f''(x) \geq 0$, $\forall x \in (0, \infty)$, then for all $x \in [0, 1]$*

$$f(x) - f(0) \leq (f(1) - f(0)) x.$$

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(ii) *If* $f''(x) \le 0$, $\forall x \in (0,\infty)$, *then for all* $x \in [0,1]$

$$f(x) - f(0) \le f'(0)\, x.$$

Proof. Let $g : [0,1] \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} \frac{f(x)-f(0)}{x}, & x \in (0,1], \\ f'(0), & x = 0. \end{cases}$$

Clearly, g is continuous. Further, for all $x \in (0,1)$, $g'(x) = \frac{h(x)}{x^2}$, where $h : [0,\infty) \to \mathbb{R}$, $h(x) = xf'(x) - [f(x) - f(0)]$. Also $h'(x) = xf''(x)$, $\forall x \in (0,\infty)$.

(i) Since $f''(x) \ge 0$, $\forall x \in (0,\infty)$, it follows that $h'(x) \ge 0$, $\forall x \in (0,\infty)$, that is $g'(x) \ge 0$, $\forall x \in (0,1]$, and thus g is increasing on $[0,1]$. Then $g(x) \le g(1)$, $\forall x \in (0,1]$, which gives us $f(x) - f(0) \le (f(1) - f(0))\, x$, $\forall x \in (0,1]$. Note that for $x = 0$ the inequality holds trivially.

(ii) Since $f''(x) \le 0$, $\forall x \in (0,\infty)$, it follows that $h'(x) \le 0$, $\forall x \in (0,\infty)$, that is $g'(x) \le 0$, $\forall x \in (0,1)$, and thus g is decreasing on $[0,1]$. Then $g(x) \le g(0)$, $\forall x \in (0,1]$, which gives us $f(x) - f(0) \le f'(0)\, x$, $\forall x \in (0,1]$. Note that for $x = 0$ the inequality holds trivially. □

Proposition 2. (i) *If* $1 \le p \le 2$ *then for all* $x \in [0,1]$ *it holds*

$$0 \le (1+x)^p - (1+x^p) \le px.$$

(ii) *If* $2 \le p < \infty$ *then for all* $x \in [0,1]$ *the following relation holds*

$$0 \le (1+x)^p - (1+x^p) \le (2^p - 2)\, x.$$

Proof. For $1 < p < \infty$ let $f_p : [0,\infty) \to \mathbb{R}$ be defined by $f_p(x) = (1+x)^p - (1+x^p)$. Then f_p is differentiable on $(0,\infty)$ and by l'Hôpital rule,

$$f_p'(0) = \lim_{x\to 0,\ x>0} \frac{(1+x)^p - (1+x^p)}{x} = \lim_{x\to 0,\ x>0} \left(p(1+x)^{p-1} - px^{p-1}\right) = p.$$

Let us note that since $1 < p < \infty$, $(1+x)^p - (1+x^p) > 0$, $\forall x > 0$. Also $f_p''(x) = p(p-1)\left[(1+x)^{p-2} - x^{p-2}\right]$, $\forall x > 0$. Apply now Lemma 1. □

Proposition 3. (i) *If* $1 < p < 2$ *then, for all* $a, b \in [0,\infty)$ *the following relation holds*

$$0 \le (a+b)^p - (a^p + b^p) \le p\,[\max(a,b)]^{2-p}\,(ab)^{p-1}.$$

(ii) *If* $2 \le p < \infty$ *then, for all* $a, b \in [0,\infty)$ *the following relation holds*

$$0 \le (a+b)^p - (a^p + b^p) \le (2^p - 2)\,[\max(a,b)]^{p-2}\,(ab).$$

Proof. If $a = 0$ or $b = 0$ the inequalities are trivial. So, let us suppose that $a > 0$ and $b > 0$. By symmetry reasons, in both (i) and (ii) it suffices to consider only the case when $a \le b$.

(i) If $a \leq b$ then, for $x = \frac{a}{b} \in [0,1]$, by Proposition 2(i), we have $0 \leq \left(1 + \frac{a}{b}\right)^p - \left(1 + \frac{a^p}{b^b}\right) \leq p \cdot \frac{a}{b}$, that is, since $p < 2$,

$$(a+b)^p - (a^p + b^p) \leq pb^{p-1}a = p\,(ab)^{p-1}\,a^{2-p} \leq p\,[\max(a,b)]^{2-p}\,(ab)^{p-1}\,.$$

(ii) If $a \leq b$ then, for $x = \frac{a}{b} \in [0,1]$, by Proposition 2(ii), we have

$$0 \leq \left(1 + \frac{a}{b}\right)^p - \left(1 + \frac{a^p}{b^b}\right) \leq (2^p - 2) \cdot \frac{a}{b},$$

that is, since $p \geq 2$,

$$(a+b)^p - (a^p + b^p) \leq (2^p - 2)\,b^{p-2}ab = (2^p - 2)\,[\max(a,b)]^{p-2}\,(ab)\,.$$

□

Proposition 4. *Let* $0 < q < p < \infty$ *and* $p \geq 1$.
(i) *If* $q < p < 2q$ *then, for all* $a, b \in [0, \infty)$ *the following relation holds*

$$0 \leq (a^q + b^q)^{\frac{1}{q}} - (a^p + b^p)^{\frac{1}{p}} \leq \left(\frac{p}{q}\right)^{\frac{1}{p}} [\max(a,b)]^{\frac{2q}{p}-1}\,(ab)^{1-\frac{q}{p}}\,.$$

(ii) *If* $2q \leq p$ *then, for all* $a, b \in [0, \infty)$ *the following relation holds*

$$0 \leq (a^q + b^q)^{\frac{1}{q}} - (a^p + b^p)^{\frac{1}{p}} \leq \left(2^{\frac{p}{q}} - 2\right)^{\frac{1}{p}} [\max(a,b)]^{1-\frac{2q}{p}}\,(ab)^{\frac{q}{p}}\,.$$

Proof. (i) Since $1 < \frac{p}{q} < 2$, from Proposition 3(i) applied for $\frac{p}{q}$, a^q and b^q instead of p, a and b we have

$$0 \leq (a^q + b^q)^{\frac{p}{q}} - (a^p + b^p) \leq \frac{p}{q}\,[\max(a,b)]^{2q-p}\,(ab)^{p-q}\,.$$

Then, since for $p \geq 1$, $(x+y)^{\frac{1}{p}} \leq x^{\frac{1}{p}} + y^{\frac{1}{p}}$, $\forall x \geq 0$, $y \geq 0$, we deduce

$$\begin{aligned}(a^q + b^q)^{\frac{1}{q}} &\leq \left(a^p + b^p + \frac{p}{q}\,[\max(a,b)]^{2q-p}\,(ab)^{p-q}\right)^{\frac{1}{p}} \\ &\leq (a^p + b^p)^{\frac{1}{p}} + \left(\frac{p}{q}\right)^{\frac{1}{p}} [\max(a,b)]^{\frac{2q}{p}-1}\,(ab)^{1-\frac{q}{p}}\,.\end{aligned}$$

(ii) Since $2 \leq \frac{p}{q} < \infty$, from Proposition 3(ii) applied for $\frac{p}{q}$, a^q and b^q instead of p, a and b we have

$$0 \leq (a^q + b^q)^{\frac{p}{q}} - (a^p + b^p) \leq \left(2^{\frac{p}{q}} - 2\right) [\max(a,b)]^{p-2q}\,(ab)^q\,.$$

Then, since for $p > 1$, $(x+y)^{\frac{1}{p}} \leq x^{\frac{1}{p}} + y^{\frac{1}{p}}$, $\forall x \geq 0$, $y \geq 0$, we deduce

$$\begin{aligned}(a^q + b^q)^{\frac{1}{q}} &\leq \left((a^p + b^p) + \left(2^{\frac{p}{q}} - 2\right) [\max(a,b)]^{p-2q}\,(ab)^q\right)^{\frac{1}{p}} \\ &\leq (a^p + b^p)^{\frac{1}{p}} + \left(2^{\frac{p}{q}} - 2\right)^{\frac{1}{p}} [\max(a,b)]^{1-\frac{2q}{p}}\,(ab)^{\frac{q}{p}}\,.\end{aligned}$$

□

Taking $q=1$ in Proposition 4 we get the following proposition.

Proposition 5. (i) *If* $1<p<2$ *then, for all* $a,b\in[0,\infty)$*, the following relation holds*

$$0\le a+b-(a^p+b^p)^{\frac{1}{p}}\le p^{\frac{1}{p}}(ab)^{\frac{p-1}{p}}\left[\max(a,b)\right]^{\frac{2-p}{p}}.$$

(ii) *If* $2\le p<\infty$ *then, for all* $a,b\in[0,\infty)$*, the following relation holds*

$$0\le a+b-(a^p+b^p)^{\frac{1}{p}}\le (2^p-2)^{\frac{1}{p}}(ab)^{\frac{1}{p}}\left[\max(a,b)\right]^{\frac{p-2}{p}}.$$

Next we find similar estimates for the difference $(a^s+b^s)^{\frac{1}{s}}-(a+b)$ when $0<s<1$.

Proposition 6. (i) *If* $0<s\le\frac{1}{2}$ *then, for all* $a,b\in[0,\infty)$*, the following relation holds*

$$0\le(a^s+b^s)^{\frac{1}{s}}-(a+b)\le(2^{\frac{1}{s}}-2)\left[\max(a,b)\right]^{1-2s}(ab)^s.$$

(ii) *If* $\frac{1}{2}<s<1$ *then, for all* $a,b\in[0,\infty)$*, the following relation holds*

$$0\le(a^s+b^s)^{\frac{1}{s}}-(a+b)\le\frac{1}{s}\left[\max(a,b)\right]^{2s-1}(ab)^{1-s}.$$

Proof. Write $p=\frac{1}{s}$.

(i) Since $0<s\le\frac{1}{2}$ we have $2\le p<\infty$. Replacing in Proposition 3(ii) a with $a^{\frac{1}{p}}$ and b with $b^{\frac{1}{p}}$ we get

$$\begin{aligned}0\le(a^s+b^s)^{\frac{1}{s}}-(a+b)&\le(2^{\frac{1}{s}}-2)\left[\max(a,b)\right]^{\frac{p-2}{p}}(ab)^{\frac{1}{p}}\\&=(2^{\frac{1}{s}}-2)\left[\max(a,b)\right]^{1-2s}(ab)^s.\end{aligned}$$

(ii) Since $\frac{1}{2}<s<1$, we have $1<p<2$. Replacing in Proposition 3(i) a with $a^{\frac{1}{p}}$ and b with $b^{\frac{1}{p}}$ we get

$$\begin{aligned}0\le(a^s+b^s)^{\frac{1}{s}}-(a+b)&\le\frac{1}{s}\left[\max(a,b)\right]^{\frac{2-p}{p}}(ab)^{\frac{p-1}{p}}\\&=\frac{1}{s}\left[\max(a,b)\right]^{2s-1}(ab)^{1-s}.\end{aligned}$$

□

The next result is the missing part in Proposition 3.

Proposition 7. (i) *If* $0<s\le\frac{1}{2}$ *then, for all* $a,b\in[0,\infty)$*, the following relation holds*

$$0\le a^s+b^s-(a+b)^s\le\left(2^{\frac{1}{s}}-2\right)^s\left[\max(a,b)\right]^{s(1-2s)}(ab)^{s^2}.$$

(ii) *If* $\frac{1}{2}<s<1$ *then, for all* $a,b\in[0,\infty)$*, the following relation holds*

$$0\le a^s+b^s-(a+b)^s\le\frac{1}{s^s}\left[\max(a,b)\right]^{s(2s-1)}(ab)^{s(1-s)}.$$

Proof. Write $p=\frac{1}{s}$.

(i) Since $0<s\le\frac{1}{2}$, $2\le p<\infty$. Replacing in Proposition 5(ii) a with $a^{\frac{1}{p}}$ and b with $b^{\frac{1}{p}}$ we get

$$\begin{aligned}0\le a^s+b^s-(a+b)^s &\le \left(2^{\frac{1}{s}}-2\right)^s (ab)^{\frac{1}{p^2}}\left[\max(a,b)\right]^{\frac{p-2}{p\cdot p}}\\ &=\left(2^{\frac{1}{s}}-2\right)^s (ab)^{s^2}\left[\max(a,b)\right]^{s(1-2s)}.\end{aligned}$$

(ii) Since $\frac{1}{2}<s<1$, $1<p<2$. Replacing in Proposition 5(i) a with $a^{\frac{1}{p}}$ and b with $b^{\frac{1}{p}}$ we get

$$\begin{aligned}0\le a^s+b^s-(a+b)^s &\le \frac{1}{s^s}(ab)^{\frac{p-1}{p\cdot p}}\left[\max(a,b)\right]^{\frac{2-p}{p\cdot p}}\\ &=\frac{1}{s^s}(ab)^{s(1-s)}\left[\max(a,b)\right]^{s(2s-1)}.\end{aligned}$$

□

2. The limit of a sequence of integrals

Proposition 8. *Let $0<p<\infty$ and $f,g:[0,1]\to[0,\infty)$ be continuous functions. Then*

$$\lim_{n\to\infty} n\int_0^1\left(f^p(t)\,t^{np}+g^p(t)\,(1-t)^{np}\right)^{\frac{1}{p}}\mathrm{d}t=f(1)+g(0).$$

Proof. The case $1<p<\infty$. Let $n\in\mathbb{N}$. From Proposition 5 for all $t\in[0,1]$ we have

$$\begin{aligned}0 &\le t^n f(t)+(1-t)^n g(t)-\left(f^p(t)\,t^{np}+g^p(t)\,(1-t)^{np}\right)^{\frac{1}{p}}\\ &\le p^{\frac{1}{p}}\left(f(t)\,g(t)\,t^n(1-t)^n\right)^{\frac{p-1}{p}}\left[\max\left(t^nf(t),(1-t)^ng(t)\right)\right]^{\frac{2-p}{p}}\\ &\le p^{\frac{1}{p}}\left(M_{fg}\right)^{\frac{p-1}{p}}\cdot\frac{1}{4^{\frac{n(p-1)}{p}}}\left[\max\left(M_f,M_g\right)\right]^{\frac{2-p}{p}},\quad \text{if } 1<p<2,\end{aligned}$$

respectively

$$\begin{aligned}0 &\le t^n f(t)+(1-t)^n g(t)-\left(f^p(t)\,t^{np}+g^p(t)\,(1-t)^{np}\right)^{\frac{1}{p}}\\ &\le \left(2^p-2\right)^{\frac{1}{p}}\left(f(t)\,g(t)\,t^n(1-t)^n\right)^{\frac{1}{p}}\left[\max\left(t^nf(t),(1-t)^ng(t)\right)\right]^{\frac{p-2}{p}}\\ &\le \left(2^p-2\right)^{\frac{1}{p}}\left(M_{fg}\right)^{\frac{1}{p}}\cdot\frac{1}{4^{\frac{n}{p}}}\left[\max\left(M_f,M_g\right)\right]^{\frac{p-2}{p}},\quad \text{if } 2\le p<\infty.\end{aligned}$$

We have used that $t(1-t)\le\frac{1}{4}$, $\forall t\in\mathbb{R}$. Above and in the sequel, if $h:[0,1]\to[0,\infty)$ is a continuous function, then $M_h:=\sup_{x\in[0,1]}h(x)\in[0,\infty)$. By integration, we deduce

$$0\le n\int_0^1 t^nf(t)\,\mathrm{d}t+n\int_0^1(1-t)^ng(t)\,\mathrm{d}t-$$

$$-n\int_0^1 \left(f^p(t)\,t^{np}+g^p(t)\,(1-t)^{np}\right)^{\frac{1}{p}}\mathrm{d}t$$

$$\le p^{\frac{1}{p}}\left(M_{fg}\right)^{\frac{p-1}{p}}\cdot\frac{n}{4^{\frac{n(p-1)}{p}}}\left[\max\left(M_f,M_g\right)\right]^{\frac{2-p}{p}},\quad \text{if } 1<p<2,$$

respectively

$$0\le n\int_0^1 t^n f(t)\,\mathrm{d}t+n\int_0^1(1-t)^n g(t)\,\mathrm{d}t$$

$$-\,n\int_0^1 \left(f^p(t)\,t^{np}+g^p(t)\,(1-t)^{np}\right)^{\frac{1}{p}}\mathrm{d}t$$

$$\le\left(2^p-2\right)^{\frac{1}{p}}\left(M_{fg}\right)^{\frac{1}{p}}\cdot\frac{n}{4^{\frac{n}{p}}}\left[\max\left(M_f,M_g\right)\right]^{\frac{p-2}{p}},\quad \text{if } 2\le p<\infty.$$

Now using the well-known result that if $\varphi:[0,1]\to\mathbb{R}$ is a continuous function, $\lim\limits_{n\to\infty} n\int_0^1 t^n\varphi(t)\,\mathrm{d}t=\varphi(1)$, passing to the limit for $n\to\infty$ in the above relations, we get the desired result.

The case $0<p<1$. Let $n\in\mathbb{N}$. From Proposition 6, for all $t\in[0,1]$ we have

$$\begin{aligned}
0 &\le \left(f^p(t)\,t^{np}+g^p(t)\,(1-t)^{np}\right)^{\frac{1}{p}}-\left(t^n f(t)+(1-t)^n g(t)\right)\\
&\le \left(2^{\frac{1}{p}}-2\right)\left(f(t)\,g(t)\,t^n(1-t)^n\right)^p\left[\max\left(t^n f(t),(1-t)^n g(t)\right)\right]^{1-2p}\\
&\le \left(2^{\frac{1}{p}}-2\right)\left(M_{fg}\right)^p\cdot\frac{1}{4^{np}}\left[\max\left(M_f,M_g\right)\right]^{1-2p},\quad \text{if } 0<p\le\frac{1}{2},
\end{aligned}$$

respectively

$$\begin{aligned}
0 &\le \left(f^p(t)\,t^{np}+g^p(t)\,(1-t)^{np}\right)^{\frac{1}{p}}-\left(t^n f(t)+(1-t)^n g(t)\right)\\
&\le \frac{1}{p}\left(f(t)\,g(t)\,t^n(1-t)^n\right)^{1-p}\left[\max\left(t^n f(t),(1-t)^n g(t)\right)\right]^{2p-1}\\
&\le \frac{1}{p}\left(M_{fg}\right)^{1-p}\cdot\frac{1}{4^{n(1-p)}}\left[\max\left(M_f,M_g\right)\right]^{1-2p},\quad \text{if } \frac{1}{2}<p<1.
\end{aligned}$$

By integration we deduce

$$0\le n\int_0^1\left(f^p(t)\,t^{np}+g^p(t)\,(1-t)^{np}\right)^{\frac{1}{p}}\mathrm{d}t-n\int_0^1\left(t^n f(t)+(1-t)^n g(t)\right)\mathrm{d}t$$

$$\le\left(2^{\frac{1}{p}}-2\right)\left(M_{fg}\right)^p\cdot\frac{n}{4^{np}}\left[\max\left(M_f,M_g\right)\right]^{1-2p},\quad \text{if } 0<p\le\frac{1}{2},$$

respectively

$$0\le n\int_0^1\left(f^p(t)\,t^{np}+g^p(t)\,(1-t)^{np}\right)^{\frac{1}{p}}\mathrm{d}t-n\int_0^1\left(t^n f(t)+(1-t)^n g(t)\right)\mathrm{d}t$$

$$\le\frac{1}{p}\left(M_{fg}\right)^{1-p}\cdot\frac{n}{4^{n(1-p)}}\left[\max\left(M_f,M_g\right)\right]^{1-2p},\quad \text{if } \frac{1}{2}<p<1.$$

Passing to the limit for $n \to \infty$ in the above relations we get the statement. $\square$

To obtain the problem at the Cambridge mathematical tripos, note that

$$\begin{aligned}
&\frac{n}{2^n}\int_{-1}^{1}\sqrt{a^2(1+x)^{2n}+b^2(1-x)^{2n}}\mathrm{d}x\\
&\quad = n\int_{-1}^{1}\sqrt{a^2\left(\frac{1+x}{2}\right)^{2n}+b^2\left(1-\frac{1+x}{2}\right)^{2n}}\mathrm{d}x\\
&\quad = 2n\int_{0}^{1}\sqrt{a^2t^{2n}+b^2(1-t)^{2n}}\mathrm{d}t.
\end{aligned}$$

We have made the change of variable $\frac{1+x}{2}=t$. Apply now Proposition 8.

In the end we reproduce the following nice elementary proof of the problem at the Cambridge mathematical tripos shown to us by the reviewer of this paper. From the well known equality

$$\int_{-1}^{1}h(x)\,\mathrm{d}x=\int_{0}^{1}h(x)\,\mathrm{d}x+\int_{0}^{1}h(-x)\,\mathrm{d}x$$

we obtain

$$\begin{aligned}
&\frac{n}{2^n}\int_{-1}^{1}\sqrt{a^2(1+x)^{2n}+b^2(1-x)^{2n}}\mathrm{d}x\\
&\quad = \frac{n}{2^n}\int_{0}^{1}\sqrt{a^2(1+x)^{2n}+b^2(1-x)^{2n}}\mathrm{d}x\\
&\quad + \frac{n}{2^n}\int_{0}^{1}\sqrt{b^2(1+x)^{2n}+a^2(1-x)^{2n}}\mathrm{d}x.
\end{aligned}$$

Now from the obvious inequalities $u\le\sqrt{u^2+v^2}\le u+v$, $u,v\ge 0$, we deduce $a(1+x)^n\le\sqrt{a^2(1+x)^{2n}+b^2(1-x)^{2n}}\le a(1+x)^n+b(1-x)^n$, $\forall x\in[0,1]$, and by integration we obtain that for every natural number n we have

$$\begin{aligned}
\frac{an}{2^n}\cdot\frac{2^{n+1}-1}{n+1} &\le n2^n\int_{0}^{1}\sqrt{a^2(1+x)^{2n}+b^2(1-x)^{2n}}\mathrm{d}x\\
&\le \frac{an}{2^n}\cdot\frac{2^{n+1}-1}{n+1}+\frac{b}{n+1}\cdot\frac{n}{2^n}.
\end{aligned}$$

Passing to the limit we obtain $\lim\limits_{n\to\infty}\frac{n}{2^n}\int_0^1\sqrt{a^2(1+x)^{2n}+b^2(1-x)^{2n}}\mathrm{d}x=2a$ and similarly $\lim\limits_{n\to\infty}\frac{n}{2^n}\int_0^1\sqrt{b^2(1+x)^{2n}+a^2(1-x)^{2n}}\mathrm{d}x=2b$.

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About Kubik mean value theorem and its stability

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Abstract. In this paper we investigate Kubik Theorem. We present a new proof, without using Weierstrass and Fermat Theorem. Also, we establish some conditions for the stability of the intermediary point arising from this theorem.

Keywords: Kubik Theorem, continuous function, both-sided differentiable function, stability.
MSC: Primary 39B82; Secondary 37C25.

Introduction

The mean value theorems represent some of the most useful mathematical analysis tools. Starting from Rolle or Lagrange Theorem, we can find more results, generalizations or extensions. One of the less known result is due to Kubik [6].

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ and both-sided differentiable, for any $x \in (a, b)$. If $f(a) = f(b)$ then there exists $c \in (a, b)$ such that*

$$f'_-(c) \cdot f'_+(c) \leq 0, \tag{1}$$

where f'_- and f'_+ denote the respective one-sided derivatives of the function f.

The previous theorem represents a generalization of Rolle Theorem. Indeed, if f is differentiable, the relation (1) leads to $(f'(c))^2 \leq 0$. Since $f'(c) \in \mathbb{R}$, we obtain $f'(c) = 0$. As a consequences of this result, Kubik obtained some generalizations for Lagrange and Cauchy Theorems. We can find similar theorems in [5] and [10]. Moreover, in [11] it is proven the equivalence between Kubik's result and Karamata or Vučković results.

The aim this paper is to investigate Kubik Theorem. In the following section we present a new proof of this theorem. We mention that this proof does not use Weierstrass and Fermat theorem. Also, we establish conditions

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for the stability of the intermediary point arising from this mean value theorem (see Theorem 4 from Section 2). We provide an example which explains why the hypothesis from Theorem 4 cannot be weakened.

1. A new proof for Kubik's theorem

It is known the following chain of implications in the proof of some classical theorems: Weierstrass $\Longrightarrow$ Rolle $\Longrightarrow$ Lagrange.

Several mathematicians considered this way too long and tried to shorten it. For example, D. Pompeiu (see [8]) proved Lagrange theorem by using convergent sequences. Motivated by Pompeiu's proof and using an idea from [1], we present a new proof of Kubik Theorem. First, we need the following lemma.

Lemma 2. *Let $a, b \in \mathbb{R}$, $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) = f(b)$. Then there exist $x \in (a, b)$ and two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ with the following properties:*

(i) *$a \le a_n < b_n \le b$ for any $n \in \mathbb{N}$;*

(ii) *the sequence $(a_n)_{n\in\mathbb{N}}$ is increasing and the sequence $(b_n)_{n\in\mathbb{N}}$ is decreasing;*

(iii) *$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = x$;*

(iv) *$f(a_n) = f(b_n)$, for any $n \in \mathbb{N}$.*

Proof. We claim that for any $\varepsilon > 0$ there exists $c \in [a, b]$ such that f is noninjective on $(c - \varepsilon, c + \varepsilon) \cap [a, b]$. We prove this claim by contradiction. Suppose to the contrary that there exists $\varepsilon > 0$ such that, for any $c \in [a, b]$, the function f is injective on $(c - \varepsilon, c + \varepsilon) \cap [a, b]$. Due to its continuity, f is strictly monotone on $(c - \varepsilon, c + \varepsilon) \cap [a, b]$.

The interval $[a, b]$ is compact and $[a, b] \subset \bigcup\limits_{c\in[a,b]} (c - \varepsilon, c + \varepsilon)$. Then, there exist $n \in \mathbb{N}^*$ and $c_1, c_2, \ldots, c_n \in [a, b]$, $c_1 < c_2 < \cdots < c_n$, such that $[a, b] \subset \bigcup\limits_{1\le i\le n} (c_i - \varepsilon, c_i + \varepsilon)$. Hence $(c_i - \varepsilon, c_i + \varepsilon) \cap (c_{i+1} - \varepsilon, c_{i+1} + \varepsilon)$ is an open nonempty interval, f has the same type of monotonicity on $(c_i - \varepsilon, c_i + \varepsilon) \cap [a, b]$ and $(c_{i+1} - \varepsilon, c_{i+1} + \varepsilon) \cap [a, b]$, where $i \in \{1, 2, \ldots, n-1\}$. We obtain that f is strictly monotone on $[a, b]$. This contradicts the equality $f(a) = f(b)$ and our assumption fails.

Now, we choose $\varepsilon = \frac{b-a}{4}$. There exists $c \in [a, b]$ such that f is noninjective on $(c - \varepsilon, c + \varepsilon) \cap [a, b]$. We find $x_1, x_2 \in (c - \varepsilon, c + \varepsilon) \cap [a, b]$, $x_1 < x_2$, such that $f(x_1) = f(x_2)$. If we suppose that f is injective on (x_1, x_2), we obtain that f is strictly monotone on (x_1, x_2). Using its continuity, we have $f(x_1) \ne f(x_2)$, which is not true. Thus, we find $x_3, x_4 \in (x_1, x_2)$, $x_3 < x_4$, with $f(x_3) = f(x_4)$.

Now, we choose $a_0 = a$, $b_0 = b$, $a_1 = x_3$ and $b_1 = x_4$. We have $a_0 < a_1 < b_1 < b_0$, $f(a_0) = f(b_0)$, $f(a_1) = f(b_1)$ and $b_1 - a_1 < 2 \cdot \dfrac{b-a}{4} = \dfrac{b_0 - a_0}{2}$. We repeat the same reasoning, but starting from the interval $[a_1, b_1]$.

We obtain two sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ such that $f(a_n) = f(b_n)$ and $a \le a_n < b_n \le b$, for any $n \in \mathbb{N}$. Moreover, the sequence $(a_n)_{n\in\mathbb{N}}$ is increasing, the sequence $(b_n)_{n\in\mathbb{N}}$ is decreasing, and

$$0 < b_n - a_n < \frac{b_{n-1} - a_{n-1}}{2} < \cdots < \frac{b_0 - a_0}{2^n}$$

for any $n \in \mathbb{N}$. Then, these sequences are convergent and have the same limit. There exists $x \in (a, b)$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$ and the proof is complete. □

Now, we are in position to present a new proof of Kubik Theorem.

We suppose that $f_-(c) \cdot f_+(c) > 0$ for any $c \in (a, b)$. The function f is continuous on $[a.b]$, so we consider $x \in (a, b)$ and the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ given by the previous Lemma. We have

$$f'_-(x) = \lim_{n\to\infty} \frac{f(a_n) - f(x)}{a_n - x}$$

and

$$f'_+(x) = \lim_{n\to\infty} \frac{f(b_n) - f(x)}{b_n - x}.$$

Since $a_n < x < b_n$ and $f(a_n) - f(x) = f(b_n) - f(x)$, we obtain $f'_-(x) \cdot f'_+(x) \le 0$. This contradicts our supposition and concludes the proof of Theorem 1.

In the next section we study the stability of Kubik points, defined as any $c \in (a, b)$ which is satisfying (1).

2. The stability of Kubik points

The parents of the stability concept are considered the mathematicians S. Ulam and D. Hyers (see [9] and [4]). This notion is associated now with functional equations, differential equations or linear recurrences. Starting with the reference [2], a new direction was created. It is about the stability of the point arising from the mean value theorems. The references [3] or [7] are relevant in this respect.

In this section, we establish some conditions for stability of Kubik points. We shall use the following result (see Theorem 4 from [7]).

Proposition 3. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function which admits a unique minimum point $\alpha \in (a, b)$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$*

with the following property: Any continuous function $g : [a, b] \to \mathbb{R}$ *satisfying the following relation for every* $x \in [a, b]$

$$|g(x) - f(x)| < \delta,$$

admits a minimum point $\beta \in (a, b)$ *such that* $|\beta - \alpha| < \varepsilon$.

To avoid repetitions, we denote by K the set of all functions $f : [a, b] \to \mathbb{R}$, continuous on $[a, b]$ and both-sided differentiable for any $x \in (a, b)$.

Theorem 4. *Let* $f \in K$ *with* $f(a) = f(b)$ *and having a unique Kubik point* $c \in (a, b)$. *Then for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *with the following property: Every function* $g \in K$ *with* $g(a) = g(b)$ *and satisfying*

$$|g(x) - f(x)| < \delta \quad \textit{for every} \quad x \in [a, b]$$

admits a Kubik point $d \in (a, b)$ *such that* $|d - c| < \varepsilon$.

Proof. The function f is continuous, so there exists $u \in (a, b)$ an extremum point. Suppose that it is a minimum point. Then

$$f'_-(u) = \lim_{x \nearrow u} \frac{f(x) - f(u)}{x - u} \leq 0$$

and

$$f'_+(u) = \lim_{x \searrow u} \frac{f(x) - f(u)}{x - u} \geq 0.$$

We obtain $f'_-(u) \cdot f'_+(u) \leq 0$, so u is a Kubik point for f. The uniqueness of Kubik point leads to the conclusion $c = u$.

Let $\varepsilon > 0$. We choose $\delta > 0$ defined by Proposition 3. Let $g \in K$ with $g(a) = g(b)$ be such that $|g(x) - f(x)| < \delta$ for any $x \in [a, b]$. Proposition 3 gives us a point $d \in (a, b)$ which is a minimum point of g and $|d - c| < \varepsilon$. Hence

$$g'_-(u) = \lim_{x \nearrow u} \frac{g(x) - g(u)}{x - u} \leq 0$$

and

$$g'_+(u) = \lim_{x \searrow u} \frac{g(x) - g(u)}{x - u} \geq 0,$$

whence we obtain $g'_-(u) \cdot g'_+(u) \leq 0$, so d is a Kubik point for g and the proof is complete. □

In fact, the conclusion of the previous theorem remains valid if the Kubik point of f is unique on a neighborhood.

We conclude this paper with the following example which shows that the hypothesis of Theorem 4 cannot be weakened.

First, we consider the function $f : [0, 3] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 - x, & x \in [0, 1], \\ 0, & x \in (1, 2), \\ x - 2, & x \in [2, 3]. \end{cases}$$

We have $f(0) = f(3) = 1$. Further, $\lim\limits_{x \nearrow 1} f(x) = 0$, $\lim\limits_{x \searrow 1} f(x) = 0$ and $f(1) = 0$, so f is continuous in $x = 1$. Similarly, $\lim\limits_{x \nearrow 2} f(x) = 0$, $\lim\limits_{x \searrow 2} f(x) = 0$ and $f(2) = 0$, so f is continuous in $x = 2$ too. By its definition, f is continuous on $[0, 3]$. On the other hand, $f'_-(1) = -1$, $f'_+(1) = 0$, $f'_-(2) = 0$ and $f'_+(2) = 1$. Hence f is differentiable on $(0, 3) \setminus \{1, 2\}$, so that $f \in K$ and f is satisfying the hypothesis of Kubik Theorem. For any $x \in (0, 1)$ we have $f'(x) = -1$ and, for $x \in (2, 3)$, we have $f'(x) = 1$. The previous reasoning and the fact that $f'(0) = 0$ for any $x \in (1, 2)$ leads to the conclusion that the Kubik point set of f is $[1, 2]$.

For example, let $\delta \in (0, 2)$. We define a function $g : [0, 3] \to \mathbb{R}$ by

$$g(x) = \begin{cases} \dfrac{\delta - 2}{2}x + 1, & x \in [0, 1], \\ -\dfrac{\delta}{2}x + \delta, & x \in (1, 2), \\ x - 2, & x \in [2, 3]. \end{cases}$$

This function is continuous on $[0, 3] \setminus \{1, 2\}$ and differentiable on $(0, 3) \setminus \{1, 2\}$. But, $\lim\limits_{x \nearrow 1} g(x) = \dfrac{\delta}{2}$, $\lim\limits_{x \searrow 1} g(x) = \dfrac{\delta}{2}$ and $g(1) = \dfrac{\delta}{2}$, so g is continuous in $x = 1$. On the other hand, $\lim\limits_{x \nearrow 2} g(x) = 0$, $\lim\limits_{x \searrow 2} g(x) = 0$ and $g(2) = 0$, so g is continuous in $x = 2$, so g is continuous on $[0, 3]$. Further, $g'_-(1) = \dfrac{\delta - 2}{2}$ and $g'_+(1) = -\dfrac{\delta}{2}$, but $g'_-(1) \cdot g'_+(1) > 0$ and $x = 1$ is not a Kubik point of g. Since $g'_-(2) = -\dfrac{\delta}{2}$ and $g'_+(2) = 1$, we obtain $g'_-(2) \cdot g'_+(2) \le 0$ and $x = 2$ is a Kubik point of g. We observe that $g'(x) < 0$ for any $x \in (0, 1) \cup (1, 2)$ and $g'(x) > 0$ for any $x \in (2, 3)$. On the other hand, $g(0) = g(3) = 1$, so g is satisfying the hypothesis of Kubik theorem. Previous reasoning shows that $x = 2$ is the only Kubik point of g.

Now, let $x \in [0, 1]$. Then $|g(x) - f(x)| = \left|\dfrac{\delta}{2}x\right| \le \dfrac{\delta}{2} < \delta$. For any $x \in (1, 2)$, we have $|g(x) - f(x)| = \left|-\dfrac{\delta}{2}x + \delta\right| \le \dfrac{\delta}{2} < \delta$. Since $g(x) = f(x)$, for any $x \in [2, 3]$ we obtain

$$|g(x) - f(x)| < \delta,$$

so this inequality holds for any $x \in [0, 3]$.

Finally, we observe that $x = 1$ is a Kubik point of f, but if we choose $\varepsilon = \dfrac{1}{2}$, we cannot find any Kubik point of g on the interval $(1 - \varepsilon, 1 + \varepsilon)$.

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Olimpiada de matematică a studenţilor din sud-estul Europei, SEEMOUS 2017[1)]

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Abstract. The 11^{th} South Eastern European Mathematical Olympiad for University Students, SEEMOUS 2017, was hosted by the Union of Mathematicians of Macedonia in Ohrid, Republic of Macedonia, between February 28 and March 5. We present the competition problems and their solutions as given by the corresponding authors. Solutions to Problem 1 provided by some of the competing students, as well as a solution to Problem 2 given by a member of the jury, are also included here.

Keywords: diagonalizable matrix, rank, change of variable, integrals, series

MSC: Primary 15A03; Secondary 15A21, 26D15..

INTRODUCTION

SEEMOUS (South Eastern European Mathematical Olympiad for University Students) este o competiţie anuală de matematică, adresată studenţilor din anii I şi II ai universităţilor din sud-estul Europei. A 11-a ediţie a acestei competiţii a fost găzduită între 28 februarie şi 5 martie 2017 în Ohrid, FYR Macedonia, de către Uniunea Matematicienilor din Macedonia. Au participat 84 de studenţi de la 18 universităţi din Argentina, Bulgaria, FYR Macedonia, Grecia, România, Turkmenistan.

A existat o singură probă de concurs, cu 5 ore ca timp de lucru pentru rezolvarea a patru probleme (problemele 1–4 de mai jos). Acestea au fost selectate de juriu dintre cele 35 de probleme propuse şi au fost considerate ca având diverse grade de dificultate: Problema 1 – grad redus de dificultate, Problemele 2, 3 – dificultate medie, Problema 4 – grad ridicat de dificultate. Pentru studenţi, însă, Problema 3 s-a dovedit a fi cea cu grad ridicat de dificultate.

Au fost acordate 9 medalii de aur, 18 medalii de argint, 29 de medalii de bronz şi o menţiune. Trei studenţi medaliaţi cu aur au obţinut punctajul maxim, doi dintre aceştia fiind din România: *Bogdan Daniel Moldovan* – Universitatea Babeş-Bolyai din Cluj-Napoca şi *Emanuel Necula*[4)] — Universitatea Politehnica din Bucureşti (UPB). Cei 6 studenţi care au reprezentat UPB au obţinut 3 medalii de aur şi 3 de argint, cu acestea clasând UPB pe primul loc între universităţile participante la competiţie.

[1)] http://www.seemous2017.smm.com.mk/index.php/welcome

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[4)] Emanuel Necula a realizat aceeaşi performanţă la SEEMOUS 2016.

Prezentăm, în continuare, problemele de concurs şi soluţiile acestora, aşa cum au fost indicate de autorii lor. Având acces la lucrările studenţilor pe care i-am însoţit la competiţie, pentru Problema 1 prezentăm şi soluţiile date de trei dintre aceşti studenţi, diferite de soluţia autorului. De asemenea, includem în prezentare rezolvarea dată Problemei 2 de profesorul Cornel Băeţica, membru în juriul competiţiei.

Problema 1. *Considerăm* $A \in \mathcal{M}_2(\mathbb{R})$,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Presupunem că elementele matricei A *satisfac inegalitatea* $a^2+b^2+c^2+d^2 < \frac{1}{5}$. *Să se arate că matricea* $I_2 + A$ *este inversabilă.*

Cornel Băeţica, Universitatea din Bucureşti, România

Observaţia autorului. Rezultatul din această problemă este caz particular al unui rezultat cunoscut din teoria matricelor: dacă $\|\cdot\|$ este o normă submultiplicativă de matrice şi $\|A\| < 1$, unde $A \in \mathcal{M}_n(\mathbb{R})$, atunci matricea $I_n + A$ este inversabilă.

Soluţia 1 (a autorului). Avem

$$\det(I_2 + A) = (a+1)(d+1) - bc = ad + a + d + 1 - bc\,.$$

Folosim inegalitatea

$$2|xy| \le x^2 + y^2, \quad \forall\, x, y \in \mathbb{R}\,,$$

sub forma echivalentă

$$\pm\, xy \ge -\frac{1}{2}\left(x^2+y^2\right), \quad \forall\, x, y \in \mathbb{R}\,,$$

şi deducem

$$\det(I_2 + A) \ge 1 + a + d - \frac{1}{2}\left(a^2+b^2+c^2+d^2\right) > 1 + a + d - \frac{1}{10}.$$

Apoi, deoarece a^2, $d^2 < 1/5$ implică $|a|$, $|d| < 1/\sqrt{5}$, obţinem

$$\det(I_2 + A) > 1 - \frac{2\sqrt{5}}{5} - \frac{1}{10} > 0\,,$$

deci matricea $I_2 + A$ este inversabilă.

Soluţia 2 (**Iustin Sîrbu**, student, Universitatea Politehnica din Bucureşti). Matricea $I_2{+}A$ este inversabilă dacă şi numai dacă $\operatorname{rang}(I_2{+}A) = 2$. Evident, $\operatorname{rang}(I_2 + A) \le 2$. Vom arăta că rangul matricei $I_2 + A$ nu poate fi 0 sau 1, deci $\operatorname{rang}(I_2 + A) = 2$.

Dacă $\operatorname{rang}(I_2 + A) = 0$, rezultă $I_2 + A = O_2$, deci

$$A = -I_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

însă elementele matricei A nu verifică inegalitatea din enunţ. Acum, dacă $\operatorname{rang}(I_2 + A) = 1$, există matricele inversabile $P, Q \in \mathcal{M}_2(\mathbb{R})$ astfel încât

$$I_2 + A = P \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Q\,.$$

Cu

$$P = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \quad \text{şi} \quad Q = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}$$

rezultă

$$A = \begin{pmatrix} p_1q_1 - 1 & p_1q_2 \\ p_3q_1 & p_3q_2 - 1 \end{pmatrix}.$$

Astfel, inegalitatea pe care trebuie să o verifice elementele lui A se rescrie ca

$$(p_1q_1 - 1)^2 + (p_1q_2)^2 + (p_3q_1)^2 + (p_3q_2 - 1)^2 < \frac{1}{5},$$

echivalent,

$$\left(p_1^2 + p_3^2\right)\left(q_1^2 + q_2^2\right) - 2\left(p_1q_1 + p_3q_2\right) + \frac{9}{5} < 0.$$

Însă, conform inegalităţii Cauchy-Buniakowski-Schwarz avem

$$(p_1q_1 + p_3q_2)^2 \le \left(p_1^2 + p_3^2\right)\left(q_1^2 + q_2^2\right),$$

şi astfel deducem

$$(p_1q_1 + p_3q_2)^2 - 2\left(p_1q_1 + p_3q_2\right) + \frac{9}{5} < 0,$$

echivalent,

$$(p_1q_1 + p_3q_2 - 1)^2 + \frac{4}{5} < 0,$$

ceea ce este absurd, deci $\operatorname{rang}(I_2 + A) \neq 1$.

Soluţia 3 (**Alexandru Brătei**, student, Universitatea Politehnica din Bucureşti). Matricea $I_2 + A$ este inversabilă dacă arătăm că toate valorile sale proprii sunt nenule. Vom folosi următorul rezultat din algebra liniară: dacă $\lambda \in \mathbb{C}$ este valoare proprie pentru matricea $A \in \mathcal{M}_n(\mathbb{C})$, iar p este un polinom cu coeficienţi complecşi, atunci $p(\lambda)$ este valoare proprie pentru matricea $p(A)$. Avem $I_2 + A = p(A)$, unde $p = 1 + X$. Astfel, dacă $\lambda \in \mathbb{C}$ este valoare proprie pentru A, $1 + \lambda$ este valoare proprie pentru $I_2 + A$. Vom arăta că $1 + \lambda \neq 0$, adică $\lambda \neq -1$. Pentru aceasta considerăm un vector propriu $v \equiv (v_1, v_2)^T \in \mathcal{M}_{2,1}(\mathbb{C})$ al matricei A, corespunzător valorii proprii λ. Din egalitatea $Av = \lambda v$ rezultă

$$|\lambda v_1| = |av_1 + bv_2|\,, \qquad |\lambda v_2| = |cv_1 + dv_2|\,,$$

de unde

$$|\lambda|\,|v_1| \le |a|\,|v_1| + |b|\,|v_2|\,, \qquad |\lambda|\,|v_2| \le |c|\,|v_1| + |d|\,|v_2|.$$

Cum $|a|, |b|, |c|, |d| < \sqrt{5}/5$, obţinem

$$|\lambda|\,|v_1| < \frac{\sqrt{5}}{5}\left(|v_1| + |v_2|\right), \quad |\lambda|\,|v_2| < \frac{\sqrt{5}}{5}\left(|v_1| + |v_2|\right),$$

ceea ce, prin însumare, conduce la

$$|\lambda|\left(|v_1| + |v_2|\right) < \frac{2\sqrt{5}}{5}\left(|v_1| + |v_2|\right).$$

Dar, ca vector propriu, v este diferit de vectorul nul, deci $|v_1| + |v_2| \neq 0$. Astfel,

$$|\lambda| < \frac{2\sqrt{5}}{5} < 1,$$

de unde rezultă $\lambda \neq -1$.

Soluţia 4 (**Cristian Pavel**, student, Universitatea Politehnica din Bucureşti). Pe baza raţionamentului din soluţia precedentă, matricea $I_2 + A$ este inversabilă dacă valorile proprii λ_1, λ_2 ale lui A sunt diferite de -1. Presupunem, prin absurd, că cel puţin o valoare proprie, fie aceasta λ_1, este egală cu -1. Atunci, $\lambda_2 \in \mathbb{R}$, iar urma şi determinantul lui A sunt daţi de

$$\operatorname{Tr} A = -1 + \lambda_2\,, \quad \det A = -\lambda_2.$$

Acum,

$$|\operatorname{Tr} A| = |a + d| \leq |a| + |d| < \frac{2\sqrt{5}}{5},$$

deci

$$|\lambda_2 - 1| < \frac{2\sqrt{5}}{5},$$

ceea ce implică $\lambda_2 > \frac{5-2\sqrt{5}}{5}$. Pe de altă parte, dacă $\alpha_1, \alpha_2 \in \mathbb{R}$ sunt valorile proprii ale matricei (simetrice) AA^T, avem

$$\alpha_1 + \alpha_2 = \operatorname{Tr}\left(AA^T\right) = a^2 + b^2 + c^2 + d^2 < \frac{1}{5}\,,$$
$$\alpha_1\alpha_2 = \det(AA^T) = (\det A)^2 = \lambda_2^2\,.$$

De aici[1)] rezultă $\alpha_1, \alpha_2 \geq 0$ şi, mai mult, utilizând inegalitatea mediilor, $\sqrt{\alpha_1\alpha_2} \leq (\alpha_1 + \alpha_2)/2$, conchidem că

$$|\lambda_2| < \frac{1}{10} < \frac{5 - 2\sqrt{5}}{5},$$

în contradicţie cu $\lambda_2 > \frac{5-2\sqrt{5}}{5}$. Aceasta arată că presupunerea $\lambda_1 = -1$ este falsă.

Observaţie. La această problemă 51 de studenţi au obţinut punctaj maxim.

[1)] Orice matrice simetrică pozitiv semidefinită din $\mathcal{M}_n(\mathbb{R})$ are toate valorile proprii reale, mai mari sau egale decât zero. Matricea AA^T este o astfel de matrice, deci $\alpha_1, \alpha_2 \geq 0$.

Problema 2. *Se consideră matricele* A, $B \in \mathcal{M}_n(\mathbb{R})$.

a) *Să se arate că există* $a > 0$, *aşa încât,* $\forall \varepsilon \in (-a, a)$, $\varepsilon \neq 0$, *ecuaţia matriceală*

$$AX + \varepsilon X = B\,, \quad X \in \mathcal{M}_n(\mathbb{R}),$$

are soluţie unică, $X(\varepsilon) \in \mathcal{M}_n(\mathbb{R})$.

b) *Dacă* $B^2 = I_n$ *şi* A *este matrice diagonalizabilă, să se arate că*

$$\lim_{\varepsilon \to 0} \varepsilon \operatorname{Tr}(B\,X(\varepsilon)) = n - \operatorname{rang} A,$$

unde Tr *reprezintă aplicaţia urmă a unei matrice.*

Radu Strugariu, Universitatea Tehnică „Gh. Asachi", Iaşi, România

Soluţia 1 (a autorului). a) Ecuaţia matriceală dată se scrie $(A + \varepsilon I_n)X = B$. Ea are soluţie unică, $X \equiv X(\varepsilon)$, dacă şi numai dacă matricea $A + \varepsilon I_n$ este inversabilă. Notăm cu $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ valorile proprii ale lui A. Atunci, $\lambda_1 + \varepsilon, \ldots, \lambda_n + \varepsilon$ sunt valorile proprii ale lui $A + \varepsilon I_n$. Alegând

$$a \equiv \min\{\,|\lambda| \;\vdots\; \lambda \in \{\lambda_1, \ldots, \lambda_n\} \setminus \{0\}\,\},$$

pentru orice $\varepsilon \neq 0$, $\varepsilon \in (-a, a)$, avem

$$\det(A + \varepsilon I_n) = (\lambda_1 + \varepsilon) \cdots (\lambda_n + \varepsilon) \neq 0.$$

b) Deoarece matricea A este diagonalizabilă, $A + \varepsilon I_n$ este diagonalizabilă. Prin urmare, există o matrice inversabilă $P \in \mathcal{M}_n(\mathbb{R})$ astfel încât $A + \varepsilon I_n = PD_\varepsilon P^{-1}$, unde

$$D_\varepsilon = \begin{pmatrix} \lambda_1 + \varepsilon & 0 & \ldots & 0 \\ 0 & \lambda_2 + \varepsilon & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_n + \varepsilon \end{pmatrix}.$$

Pentru $\varepsilon \neq 0$, $\varepsilon \in (-a, a)$, cu constanta a indicată la punctul a) de mai sus, matricea $A + \varepsilon I_n$ este inversabilă, inversa ei fiind dată de

$$(A + \varepsilon I_n)^{-1} = P \begin{pmatrix} \dfrac{1}{\lambda_1 + \varepsilon} & 0 & \ldots & 0 \\ 0 & \dfrac{1}{\lambda_2 + \varepsilon} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \dfrac{1}{\lambda_n + \varepsilon} \end{pmatrix} P^{-1}.$$

Deoarece $X(\varepsilon) = (A + \varepsilon I_n)^{-1}B$, rezultă

$$\varepsilon X(\varepsilon) = P \begin{pmatrix} \dfrac{\varepsilon}{\lambda_1+\varepsilon} & 0 & \ldots & 0 \\ 0 & \dfrac{\varepsilon}{\lambda_2+\varepsilon} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \dfrac{\varepsilon}{\lambda_n+\varepsilon} \end{pmatrix} P^{-1}B.$$

Astfel, cu $B^2 = I_n$ şi proprietatea $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ a aplicaţiei urmă, obţinem

$$\varepsilon \mathrm{Tr}\,(B\,X(\varepsilon)) = \mathrm{Tr}\,(\varepsilon X(\varepsilon)\,B) = \mathrm{Tr}\left(\varepsilon(A+\varepsilon I_n)^{-1}\right) = \frac{\varepsilon}{\lambda_1+\varepsilon} + \cdots + \frac{\varepsilon}{\lambda_n+\varepsilon},$$

ceea ce implică

$$\lim_{\varepsilon\to 0} \varepsilon\,\mathrm{Tr}(B\,X(\varepsilon)) = \text{ numărul de valori proprii ale lui } A \text{ egale cu zero.}$$

Dacă acest număr este egal cu zero (adică, toate valorile proprii ale lui A sunt nenule), matricea A este inversabilă. Astfel, $\mathrm{rang}\, A = n$ şi cerinţa problemei este justificată. Dacă $\lambda = 0$ este valoare proprie pentru A, limita de mai sus este egală cu multiplicitatea algebrică a acestei valori proprii; cu A matrice diagonalizabilă, multiplicitatea algebrică şi cea geometrică ale lui $\lambda = 0$ coincid. Cum multiplicitatea geometrică a valorii proprii $\lambda = 0$ este $\dim \mathrm{Ker} A$, Teorema dimensiunii ($\dim \mathrm{Ker} A + \mathrm{rang}\, A = n$) justifică cerinţa problemei.

Soluţia 2 (**Cornel Băeţica**, Universitatea din Bucureşti). b) Cu $\varepsilon \neq 0$, $\varepsilon \in (-a, a)$, avem $X(\varepsilon) = (A + \varepsilon I_n)^{-1}B$. Folosind $B^2 = I_n$ şi proprietatea $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ deducem

$$\begin{aligned} \varepsilon\,\mathrm{Tr}\,(B\,X(\varepsilon)) &= \varepsilon\,\mathrm{Tr}\,(X(\varepsilon)\,B) = \varepsilon\,\mathrm{Tr}\left((A+\varepsilon I_n)^{-1} =\right) \\ &= \varepsilon\left(\frac{1}{\lambda_1+\varepsilon} + \cdots + \frac{1}{\lambda_n+\varepsilon}\right). \end{aligned}$$

În continuare urmează raţionamentul din cazul soluţiei autorului.

Observaţii. (1) În enunţul problemei condiţia ca matricea A să fie diagonalizabilă poate fi înlocuită cu cerinţa, mai puţin restrictivă, ca valoarea proprie $\lambda = 0$ (dacă există) a matricei A să aibă multiplicitatea algebrică egală cu cea geometrică.
(2) La această problemă 31 de studenţi au obţinut punctaj maxim (10 puncte), un student a obţinut 9 puncte, un altul 8 puncte, iar restul studenţilor au obţinut între 0 şi 6 puncte.

Problema 3. *Fie $f:\mathbb{R}\to\mathbb{R}$ o funcţie continuă. Să se arate că*

$$\int_0^4 f\big(x(x-3)^2\big)\,\mathrm{d}x = 2\int_1^3 f\big(x(x-3)^2\big)\,\mathrm{d}x.$$

Vasile Pop, Universitatea Tehnică din Cluj-Napoca, România
Ioan Şerdean, Colegiul Naţional „Aurel Vlaicu", Orăştie, România

Soluţia autorilor. Definim funcţia

$$g:[0,4]\to\mathbb{R}\,,\quad g(x)=x(x-3)^2.$$

Avem $g'(x)=3(x-1)(x-3)$ şi comportarea funcţiei g redată în următorul tablou:

x	0		1		3		4
$g'(x)$	$+$	$+$	0	$-$	0	$+$	$+$
$g(x)$	0	$\nearrow$	4	$\searrow$	0	$\nearrow$	4

Notăm cu g_1, g_2, g_3 restricţiile lui g la intervalele $(0,1)$, $(1,3)$ şi, respectiv, $(3,4)$, şi fie h_1, h_2, h_3 inversele acestor restricţii. Astfel,

$$h_1:(0,4)\to(0,1)\,,\quad h_2:(0,4)\to(1,3)\,,\quad h_3:(0,4)\to(3,4)$$

şi, pentru orice t în $(0,4)$,

$x_1=h_1(t)$ este soluţia din $(0,1)$ a ecuaţiei $x(x-3)^2=t$,

$x_2=h_2(t)$ este soluţia din $(1,3)$ a ecuaţiei $x(x-3)^2=t$,

$x_3=h_3(t)$ este soluţia din $(3,4)$ a ecuaţiei $x(x-3)^2=t$.

Utilizând schimbările de variabile $x=h_i(t)$, $i=1,2,3$, obţinem

$$\int_0^4 f\big(x(x-3)^2\big)\ \mathrm{d}x - 2\int_1^3 f\big(x(x-3)^2\big)\ \mathrm{d}x =$$

$$= \int_0^1 f(g(x))\,\mathrm{d}x - \int_1^3 f(g(x))\,\mathrm{d}x + \int_3^4 f(g(x))\,\mathrm{d}x =$$
$$= \int_0^4 f(t)h_1'(t)\,\mathrm{d}t - \int_4^0 f(t)h_2'(t)\,\mathrm{d}t + \int_0^4 f(t)h_3'(t)\,\mathrm{d}t =$$
$$= \int_0^4 f(t)\big(h_1'(t)+h_2'(t)+h_3'(t)\big)\ \mathrm{d}t.$$

Deoarece suma rădăcinilor ecuaţiei $x(x-3)^2=t$ este 6, rezultă

$$h_1(t)+h_2(t)+h_3(t)=6 \quad \text{pentru orice } t\in(0,4),$$

deci

$$h_1'(t)+h_2'(t)+h_3'(t)=0 \quad \text{pentru orice } t\in(0,4),$$

ceea ce justifică egalitatea din enunţul problemei.

Observaţia autorilor. Deoarece $g'(1) = g'(3) = 0$, rezultă că $h_1'(4)$, $h_2'(0)$, $h_2'(4)$ şi $h_3'(0)$ sunt infinite, deci integralele

$$\int_0^4 f(t)\left|h_1'(t)\right| \mathrm{d}t\,, \quad \int_0^4 f(t)\left|h_2'(t)\right| \mathrm{d}t\,, \quad \int_0^4 f(t)\left|h_3'(t)\right| \mathrm{d}t$$

sunt improprii, dar convergente pentru că au fost obţinute din integrale proprii prin schimbări de variabile.

Observaţie. La această problemă 5 studenţi au obţinut punctaj maxim (10 puncte), 2 studenţi au obţinut 9 puncte, 1 student — 6 puncte, iar restul studenţilor au obţinut între 0 şi 5 puncte. Conform baremului de corectare, un punct s-a acordat pentru remarca faptului că integralele în raport cu variabila t sunt improprii convergente.

Problema 4. a) *Să se calculeze* $\int_0^1 (1-t)^n \mathrm{e}^t\,\mathrm{d}t$, *unde* n *este un număr natural.*

b) *Fie* k *un număr natural fixat şi considerăm şirul* $(x_n)_{n\ge k}$ *definit de*

$$x_n = \sum_{i=k}^{n} \binom{i}{k}\left(\mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{i!}\right), \quad n \ge k.$$

Să se arate că şirul $(x_n)_{n\ge k}$ *este convergent şi să se afle limita sa.*

Ovidiu Furdui, Universitatea Tehnică din Cluj-Napoca, România

Soluţia autorului. a) Notăm cu I_n, $n \in \mathbb{N}$, integrala din enunţul problemei. Avem $I_0 = e - 1$, iar pentru $n \ge 1$, integrând prin părţi obţinem relaţia de recurenţă $I_n = -1 + nI_{n-1}$, care implică

$$\frac{I_n}{n!} = -\frac{1}{n!} + \frac{I_{n-1}}{(n-1)!}\,, \qquad n \ge 1.$$

De aici deducem

$$\frac{I_n}{n!} = I_0 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!}\,, \qquad n \ge 1,$$

şi astfel

$$I_n = n!\left(\mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!}\right), \quad \forall\, n \in \mathbb{N}.$$

b) Pentru a stabili convergenţa şirului $(x_n)_{n\ge k}$ arătăm că $(x_n)_{n\ge k}$ este monoton şi mărginit. Obţinem imediat

$$x_{n+1} - x_n = \binom{n+1}{k}\left(\mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{(n+1)!}\right) > 0,$$

deci şirul este strict crescător. Apoi, conform formulei Taylor cu restul Lagrange, avem

$$\mathrm{e} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{\mathrm{e}^{\theta}}{(n+1)!}, \qquad \theta \in (0,1),$$

ceea ce implică

$$0 < \mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!} < \frac{\mathrm{e}}{(n+1)!}.$$

Astfel,

$$\begin{aligned} x_n \le \sum_{i=k}^{n} \binom{i}{k} \frac{\mathrm{e}}{(i+1)!} &\le \frac{\mathrm{e}}{k!} \sum_{i=k}^{n} \frac{1}{(i-k)!} \\ &= \frac{\mathrm{e}}{k!}\left(\frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{(n-k)!}\right) \le \frac{\mathrm{e}^2}{k!}, \end{aligned}$$

deci şirul este mărginit. În consecinţă, $(x_n)_{n \ge k}$ este convergent. Pentru calculul limitei folosim rezultatul obţinut la punctul a), adică,

$$\mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{i!} = \frac{1}{i!} \int_0^1 (1-t)^i \mathrm{e}^t \, \mathrm{d}t.$$

Aşadar,

$$x_n = \sum_{i=k}^{n} \binom{i}{k} \frac{1}{i!} \int_0^1 (1-t)^i \mathrm{e}^t \, \mathrm{d}t = \frac{1}{k!} \int_0^1 (1-t)^k \mathrm{e}^t \left(\sum_{i=k}^{n} \frac{(1-t)^{i-k}}{(i-k)!} \right) \mathrm{d}t.$$

Cum

$$\lim_{n\to\infty} \sum_{i=k}^{n} \frac{(1-t)^{i-k}}{(i-k)!} = \mathrm{e}^{1-t}, \qquad \sum_{i=k}^{n} \frac{(1-t)^{i-k}}{(i-k)!} < \mathrm{e}^{1-t},$$

invocând teorema lui Lebesgue de convergenţă dominată obţinem

$$\lim_{n\to\infty} x_n = \frac{1}{k!} \int_0^1 (1-t)^k \mathrm{e}^t \mathrm{e}^{1-t} \, \mathrm{d}t = \frac{\mathrm{e}}{(k+1)!}.$$

Observaţii. (1) O formulare echivalentă a cerinţei punctului b) este următoarea: *Să se arate că*

$$\sum_{i=k}^{\infty} \binom{i}{k} \left(\mathrm{e} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{i!} \right) = \frac{\mathrm{e}}{(k+1)!}.$$

(2) Pentru Problema 4 punctajul maxim (10 puncte) a fost obţinut de 9 dintre studenţi, iar 6 studenţi au obţinut câte 9 puncte.

PROBLEMS

Authors should submit proposed problems to gmaproblems@rms.unibuc.ro. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of May 2018**.

PROPOSED PROBLEMS

455. Let $n \geq 2$ be an integer. Determine the largest number of real solutions the equation $a_1\sqrt{x+b_1}+\cdots+a_n\sqrt{x+b_n}=0$ can have. Here $a_1,\ldots,a_n$ are real numbers, not all zero, and $b_1,\ldots,b_n$ are mutually distinct numbers.

`Proposed by Marius Cavachi, Ovidius University, Constanţa, Romania.`

456. Let f, g, h be non-negative continuous functions on $[0,1]$ satisfying the inequality $f(tx+(1-t)y) \geq g^t(x)h^{1-t}(y)$ for all $x, y \in [0,1]$ and some (fixed) $t \in (0,1)$. If, in additon, we have $\int_0^1 g(x)\mathrm{d}x = \int_0^1 h(x)\mathrm{d}x = 1$, prove that $\int_0^1 f(x)\mathrm{d}x \geq 1$.

`Proposed by George Stoica, University of New Brunswick, Saint John, New Brunswick, Canada.`

457. Let $A, B \in \mathcal{M}_n(\mathbb{C})$ so that $A^2+B^2+A-B=2(AB+I_n)$. Prove the following equalities:

a) $\operatorname{Tr}((A-B)(A-B+I_n)) = 2n$,
b) $\det((A-B)(A-B+I_n)) = 2^n$.

`Proposed by Vasile Pop, Tehnical University of Cluj-Napoca, Cluj-Napoca, Romania.`

458. For a continuous and non-negative function f on $[0,1]$ we define the Hausdorff moments

$$\mu_n := \int_0^1 x^n f(x)\mathrm{d}x, \quad n = 0,1,2,\ldots.$$

Prove that

$$\mu_{n+2}\cdot\mu_0^2+\mu_{n+1}\cdot\mu_1^2 \geq 2\mu_{n+1}\cdot\mu_1\cdot\mu_0 \quad n=0,1,2,\ldots.$$

`Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA, USA.`

459. The faces of an icosahedron are colored with blue or white such that a blue face cannot be adjacent to more than two other blue faces. What is the largest number of blue faces that can be obtained following this rule?

(Two faces are considered adjacent if they share an edge.)

Proposed by Eugen J. Ionaşcu, Columbus State University, Columbus, GA, USA.

460. Let X be a set with at least two elements, and fix $a, b \in X$, $a \neq b$. We define the function $f : X^3 \to X$ by

$$f(x,y,z) = \begin{cases} a \text{ if } x,y,z \neq a, \\ b \text{ if } a \in \{x,y,z\}. \end{cases}$$

Is there a binary operation $* : X^2 \to X$ such that $f(x,y,z) = (x * y) * z$ for all $x, y, z \in X$?

Proposed by George Stoica, University of New Brunswick, Saint John, New Brunswick, Canada.

461. Let $A, B \in \mathcal{M}_n(\mathbb{R})$ so that $A^2 = A$, $B^2 = B$, and $\det(2A + B) = 0$. Prove that $\det(A + 2B) = 0$.

Proposed by Vasile Pop, Tehnical University of Cluj-Napoca, Cluj-Napoca, Romania.

462. If $f : [0,1] \to \mathbb{R}$ is a convex function with $f(0) = 0$ then prove that

$$\frac{1}{6}\left(\int_{1/2}^{1} f(x)\mathrm{d}x - \int_{0}^{1/2} f(x)\mathrm{d}x\right) \geq \int_{0}^{1/2} x f(x)\mathrm{d}x.$$

Proposed by Florin Stănescu, Şerban Cioculescu School, Găeşti, Dâmboviţa, Romania.

463. Prove that there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ the equation

$$\frac{1}{1+x} + \frac{1}{2+x} + \cdots + \frac{1}{n+x} = \ln n$$

has a unique solution in the interval $(0, \infty)$, denoted by x_n, and that

$$\lim_{n\to\infty} x_n = a,$$

where $a \in (0,1)$ is the unique solution in the interval $(0,1)$ of the equation $x \sum_{i=1}^{\infty} \frac{1}{i(i+x)} = \gamma$, where γ is the Euler constant.

Proposed by Dumitru Popa, Ovidius University, Constanţa, Romania.

SOLUTIONS

439. For every $n \geq 1$ let

$$S_n = \int_0^1 \frac{\mathrm{d}x_1}{x_1+1} + \int_{[0,1]^2} \frac{\mathrm{d}x_1\mathrm{d}x_2}{\sqrt{x_1x_2}+2} + \cdots + \int_{[0,1]^n} \frac{\mathrm{d}x_1\cdots\mathrm{d}x_n}{\sqrt[n]{x_1\cdots x_n}+n} - \ln n.$$

Prove that the sequence $(S_n)_{n\geq 1}$ is convergent and, if $S = \lim_{n\to\infty} S_n \in \mathbb{R}$, find the value of the limit $\lim_{n\to\infty} n\,(S_n - S)$.

Proposed by Dumitru Popa, Department of Mathematics, Ovidius University, Constanţa, Romania.

Solution by the author. Let $q \geq 0$. From the equality $\frac{1}{1+q} - (1-q) = \frac{q^2}{1+q}$ we deduce

$$0 \leq \frac{1}{1+q} - 1 + q \leq q^2. \tag{1}$$

Let $n \geq 1$. For every $(x_1,\ldots,x_n) \in [0,1]^n$ by (1) it follows that

$$0 \leq \frac{1}{1+\frac{\sqrt[n]{x_1\cdots x_n}}{n}} - 1 + \frac{\sqrt[n]{x_1\cdots x_n}}{n} \leq \frac{\left(\sqrt[n]{x_1\cdots x_n}\right)^2}{n^2},$$

and from $(x_1,...,x_n) \in [0,1]^n$ we deduce

$$0 \leq \frac{1}{1+\frac{\sqrt[n]{x_1\cdots x_n}}{n}} - 1 + \frac{\sqrt[n]{x_1\cdots x_n}}{n} \leq \frac{1}{n^2}.$$

Then, by integration,

$$0 \leq \int_{[0,1]^n} \frac{1}{1+\frac{\sqrt[n]{x_1\cdots x_n}}{n}}\mathrm{d}x_1\cdots\mathrm{d}x_n - 1 + \int_{[0,1]^n} \frac{\sqrt[n]{x_1\cdots x_n}}{n}\mathrm{d}x_1\cdots\mathrm{d}x_n \leq \frac{1}{n^2}$$

and, since by Fubini's theorem,

$$\int_{[0,1]^n} \sqrt[n]{x_1\cdots x_n}\mathrm{d}x_1\cdots\mathrm{d}x_n = \left(\int_0^1 \sqrt[n]{x}\mathrm{d}x\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n},$$

we obtain

$$0 \leq \int_{[0,1]^n} \frac{1}{\sqrt[n]{x_1\cdots x_n}+n}\mathrm{d}x_1\cdots\mathrm{d}x_n - \frac{1}{n} + \frac{1}{n^2\left(1+\frac{1}{n}\right)^n} \leq \frac{1}{n^3},$$

i.e.,

$$\int_{[0,1]^n} \frac{1}{\sqrt[n]{x_1\cdots x_n}+n}\mathrm{d}x_1\cdots\mathrm{d}x_n = \frac{1}{n} - \frac{1}{n^2\left(1+\frac{1}{n}\right)^n} + O\left(\frac{1}{n^3}\right).$$

Let $n \geq 2$. With $\ln\left(1-\frac{1}{n}\right) = -\frac{1}{n} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right)$ we get

$$\begin{aligned} S_{n-1} - S_n &= -\ln\left(1-\frac{1}{n}\right) - \int_{[0,1]^n} \frac{1}{\sqrt[n]{x_1\cdots x_n}+n}\mathrm{d}x_1\cdots\mathrm{d}x_n \\ &= \frac{1}{n} + \frac{1}{2n^2} - \frac{1}{n} + \frac{1}{n^2\left(1+\frac{1}{n}\right)^n} + O\left(\frac{1}{n^3}\right) \\ &= \frac{1}{n^2}\left(\frac{1}{2} + \frac{1}{\left(1+\frac{1}{n}\right)^n}\right) + O\left(\frac{1}{n^3}\right). \end{aligned}$$

Thus $S_{n-1} - S_n \backsim \frac{1}{n^2}\cdot\frac{\mathrm{e}+2}{2\mathrm{e}}$; we use here the well-known notation that $a_n \backsim b_n$ means $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$. We deduce that there exists $n_0 \in \mathbb{N}$ such that $S_{n-1} - S_n > 0$, $\forall n \geq n_0$. Since the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, it follows that the series $\sum\limits_{n=2}^{\infty} (S_{n-1} - S_n)$ is convergent, thus there exists $S = \lim\limits_{n\to\infty} S_n \in \mathbb{R}$. Let $x_n = S_n - S$. Now note that $\ln\left(1+\frac{1}{n}\right) = \frac{1}{n} + O\left(\frac{1}{n}\right)$, so

$$\frac{1}{\left(1+\frac{1}{n}\right)^n} = \mathrm{e}^{-n\ln\left(1+\frac{1}{n}\right)} = \mathrm{e}^{-1}\cdot\mathrm{e}^{O\left(\frac{1}{n}\right)} = \mathrm{e}^{-1}\cdot\left(1+O\left(\frac{1}{n}\right)\right)$$

and

$$\frac{1}{n^2}\left(\frac{1}{2} + \frac{1}{\left(1+\frac{1}{n}\right)^n}\right) = \frac{1}{n^2}\left(\frac{1}{2} + \frac{1}{\mathrm{e}} + O\left(\frac{1}{n}\right)\right) = \frac{1}{n^2}\left(\frac{1}{2}+\frac{1}{\mathrm{e}}\right) + O\left(\frac{1}{n^3}\right).$$

For all $n \geq 2$ we have

$$x_{n-1} - x_n = S_{n-1} - S_n = \frac{1}{n^2}\left(\frac{1}{2}+\frac{1}{\mathrm{e}}\right) + O\left(\frac{1}{n^3}\right),$$

from which we deduce that

$$\begin{aligned} x_n &= \sum_{k=n+1}^{\infty} (x_{k-1} - x_k) \\ &= \left(\frac{1}{2}+\frac{1}{\mathrm{e}}\right)\left(\sum_{k=n+1}^{\infty}\frac{1}{k^2}\right) + O\left(\sum_{k=n+1}^{\infty}\frac{1}{k^3}\right). \end{aligned}$$

But, as is well-known, $\sum\limits_{k=n+1}^{\infty} \frac{1}{k^2} = \frac{1}{n} + O\left(\frac{1}{n^2}\right)$, $\sum\limits_{k=n+1}^{\infty} \frac{1}{k^3} = O\left(\frac{1}{n^2}\right)$ and thus

$$x_n = \left(\frac{1}{2}+\frac{1}{\mathrm{e}}\right)\cdot\frac{1}{n} + O\left(\frac{1}{n^2}\right),$$

which gives us

$$\lim_{n\to\infty} n\left(S_n - S\right) = \frac{1}{2} + \frac{1}{\mathrm{e}} = \frac{\mathrm{e}+2}{2\mathrm{e}}.$$

□

440. Let V be a linear space with finite dimension and let $T : V \to V$ be an endomorphism.

a) Prove that there exist an endomorphism $P : V \to V$ and a positive integer k such that

$$P^2 = P, \quad \operatorname{Ker} P = \operatorname{Ker} T^k, \quad \operatorname{Im} P = \operatorname{Im} T^k.$$

b) Is the assertion a) true when V does not have finite dimension?

`Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania.`

Solution by Francisco Perdomo and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain. a) For any positive integer s one has $\operatorname{Im} T^s \supset \operatorname{Im} T^{s+1}$. As V has finite dimension, $\dim \operatorname{Im} T < +\infty$ too, and there exists k such that $\operatorname{Im} T^k = \operatorname{Im} T^{k+1} = \operatorname{Im} T^{k+2} = \cdots$. We claim that $\operatorname{Ker} T^k \oplus \operatorname{Im} T^k = V$. As $\dim V = \dim \operatorname{Ker} T^k + \dim \operatorname{Im} T^k$, we only need to prove that $\operatorname{Ker} T^k \cap \operatorname{Im} T^k = \{0\}$. Suppose there is a $v \neq 0$ in $\operatorname{Ker} T^k \cap \operatorname{Im} T^k$. Then there are $f_2, \ldots, f_n$ such that $\{v, f_2, \ldots, f_n\}$ is a basis of $\operatorname{Im} T^k$. Therefore, $\{T^k(v) = 0, T^k(f_2), \ldots, T^k(f_n)\}$ is a system of generators for $\operatorname{Im} T^{2k}$, but then $\dim \operatorname{Im} T^{2k} < \dim \operatorname{Im} T^k$, contradicting the choice of k.

Let P be the projection of $\operatorname{Im} T^k$ from $V = \operatorname{Ker} T^k \oplus \operatorname{Im} T^k$ onto itself. Then $P^2 = P$, $\operatorname{Im} P = \operatorname{Im} T^k$ and $\operatorname{Ker} P = \operatorname{Ker} T^k$.

b) Any endomorphism $P : V \to V$ with $P^2 = P$ has the property that $V = \operatorname{Ker} P \oplus \operatorname{Im} P$: in fact, if $v \in \operatorname{Ker} P \cap \operatorname{Im} P$, then $v = P(w)$ with $w \in V$, and $P(v) = 0 = P^2(w) = P(w) = v$; further any $v \in V$ satisfies $v = (v - P(v)) + P(v)$ with $v - P(v) \in \operatorname{Ker} P$ and $P(v) \in \operatorname{Im} P$.

We will prove that assertion a) can not be true when V does not have finite dimension. Let $V = \mathbb{R}^{\mathbb{N}}$ be the space of real sequences, and $T : V \to V$ defined by $(x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, \ldots)$. Then, for any positive integer k, $\operatorname{Ker} T^k = \{0\}$ and $\operatorname{Im} T^k = \{(x_n) \in \mathbb{R}^{\mathbb{N}} \mid x_1 = \cdots = x_k = 0\} \subsetneq V$. Therefore, there are not a positive integer k and an endomorphism P as proposed in a) because in that case, $V = \operatorname{Ker} P \oplus \operatorname{Im} P = \operatorname{Ker} T^k \oplus \operatorname{Im} T^k = \operatorname{Im} T^k$ which is not true, since $\operatorname{Im} T^k$ is strictly contained in V. □

Notes from the editor. 1. Part a) was also solved by *Moubinool Omarjee, Lycée Henri IV, Paris, France.* In order to prove that one has $V = \operatorname{Ker} T^k \oplus \operatorname{Im} T^k$, he uses a result known as Fitting decomposition theorem.

2. The author's proof is in a certain sense the "dual" of the proof by F. Perdomo and Á. Plaza. Instead of using the decreasing sequence of vector spaces $\operatorname{Im} T \supset \operatorname{Im} T^2 \supset \operatorname{Im} T^3 \supset \cdots$, he uses the increasing sequence $\operatorname{Ker} T \subset \operatorname{Ker} T^2 \subset \operatorname{Ker} T^3 \subset \cdots$, which eventually stabilizes, i.e., there is $k \in \mathbb{N}$ for which $\operatorname{Ker} T^k = \operatorname{Ker} T^{k+1} = \cdots$. If $v \in \operatorname{Ker} T^k \cap \operatorname{Ker} T^k$ then $T^k(v) = 0$ and $v = T^k(w)$ for some $w \in V$. It follows that $T^{2k}(w) = T^k(v) = 0$, so $w \in$

Ker $T^{2k} = \text{Ker } T^k$. Thus $v = T^k(w) = 0$ and $\text{Ker } T^k \cap \text{Im } T^k = \{0\}$. Together with $\dim \text{Ker } T^k + \dim \text{Im } T^k = \dim V$, this implies $V = \text{Ker } T^k \oplus \text{Im } T^k$. From here the proof goes the same.

For b) the author uses the example $V = \mathbb{R}[X]$ and $T : \mathbb{R}[X] \to \mathbb{R}[X]$, $T(f) = f'$ (the differentiation operator). Then for every $k \geq 1$ we have $\text{Ker } T^k = \mathbb{R}_{k-1}[X]$, the set of all polynomials of degree $\leq k-1$, and $\text{Im } T^k = V$. Hence $\text{Ker } T^k \cap \text{Im } T^k = \mathbb{R}_{k-1}[X] \neq \{0\}$, which makes $V = \text{Ker } T^k \oplus \text{Im } T^k$ impossible. (As opposed to Perdomo-Plaza proof, where $\text{Ker } T^k = \{0\}$ and $\text{Im } T^k \subsetneq V$. Hence $\text{Ker } T^k + \text{Im } T^k = \text{Im } T^k \subsetneq V$, which makes the equality $V = \text{Ker } T^k \oplus \text{Im } T^k$ impossible.)

3. One can actually determine explicitly all values of $k \geq 1$ that qualify and the endomorphism P with the required properties in terms of the minimal polynomial of T, $\mu_T \in K[X]$. (Here K is the base field.) We need the following result.

Lemma. Let K be a field, V a K-vector space with $\dim_K V < \infty$ and let $T : V \to V$ be an endomorphism. If $f_1, f_2 \in K[X]$ are coprime such that $f_1(T)f_2(T) = 0$ then:

(i) $\text{Im } f_1(T) = \text{Ker } f_2(T)$ and $\text{Im } f_2(T) = \text{Ker } f_1(T)$.

(ii) If $V_i = f_i(T)V$ then $V = V_1 \oplus V_2$. Equivalently, $\text{Ker } f_i(T) \oplus \text{Im } f_i(T) = V$ for $i = 1, 2$.

(iii) If moreover $\mu_T = f_1 f_2$ and $T_i : V_i \to V_i$ is the restriction of T to V_i then $\mu_{T_1} = f_2$ and $\mu_{T_2} = f_1$.

Proof. Since f_1 and f_2 are coprime, there are $g_1, g_2 \in K[X]$ such that $f_1 g_1 + f_2 g_2 = 1$.

(i) If $x \in \text{Im } f_1(T)$ then $x = f_1(T)y$ for some $y \in V$ and we have $f_2(T)x = f_2(T)f_1(T)y = 0$, so $x \in \text{Ker } f_2(T)$. Conversely, if $x \in \text{Ker } f_2(T)$ then

$$\begin{aligned} x &= (f_1(T)g_1(T) + f_2(T)g_2(T))x = f_1(T)(g_1(T)x) + g_2(T)(f_2(T)x) \\ &= f_1(T)(g_1(T)x) \in \text{Im } f_1(T). \end{aligned}$$

Hence $\text{Im } f_1(T) = \text{Ker } f_2(T)$. Similarly for $\text{Im } f_2(T) = \text{Ker } f_1(T)$.

(ii) Note that by (i) we have $V_1 = \text{Im } f_1(T) = \text{Ker } f_2(T)$ and $V_1 = \text{Im } f_2(T) = \text{Ker } f_1(T)$, so $V = V_1 \oplus V_2$ also writes as $\text{Ker } f_i(T) \oplus \text{Im } f_i(T) = V$ for $i = 1, 2$. For any $x \in V$ we have $x = f_1(T)(g_1(T)x) + f_2(T)(g_2(T)x) \in f_1(T)V + f_2(T)V = V_1 + V_2$. So $V = V_1 + V_2$. If $x \in V_1 \cap V_2 = \text{Ker } f_2(T) \cap \text{Ker } f_1(T)$ then $x = g_1(T)(f_1(T)x) + g_2(T)(f_2(T)x) = 0$, so $V_1 \cap V_2 = \{0\}$. Thus $V = V_1 \oplus V_2$, as claimed.

(iii) For any $h \in K[X]$ we have $h(T) = 0$ if and only if μ_T divides h. Now $h(T_1) = h(T)_{V_1}$, so $h(T_1) = 0$ means that $h(T)x = 0\ \forall x \in V_1 = f_1(T)V$, i.e., that $h(T)f_1(T)y = 0\ \forall y \in V$. But this means $h(T)f_1(T) = 0$ and is equivalent to $\mu_T = f_1 f_2 \mid h f_1$, i.e., to $f_2 \mid h$. Hence $\mu_{T_1} = f_2$. Similarly $\mu_{T_2} = f_1$. $\square$

Let now l be the largest power of X dividing μ_T, so that $\mu_T = X^l f$ for some $f \in K[X]$, $X \nmid f$. If $l = 0$ then $X \nmid \mu_T$, so T is invertible and for every k we have $\operatorname{Ker} T^k = \{0\}$ and $\operatorname{Im} T^k = V$. Then there is a unique $P : V \to V$ with $P^2 = P$, $\operatorname{Ker} P = \{0\}$ and $\operatorname{Im} P = V$, namely $P = 1$. So we may assume that $l \geq 1$.

First we note that if $P^2 = P$ then $P(P-1) = 0$. Since X and $X - 1$ are coprime, by Lemma (ii) we have $V = \operatorname{Ker} P \oplus \operatorname{Im} P$. This implies the necessity of the condition $\operatorname{Ker}(T^k) \oplus \operatorname{Im}(T^k) = V$ for P with the required properties to exist.

We have $T^l f(T) = \mu_T(T) = 0$ and $\gcd(f, X^l) = 1$, so we may apply the above Lemma with $f_1 = f$, $f_2 = X^l$. We have $V = V_1 \oplus V_2$, where $V_1 = \operatorname{Im} f(T) = \operatorname{Ker} T^l$ and $V_2 = \operatorname{Im} T^l = \operatorname{Ker} f(T)$. Also if $T_i : V_i \to V_i$ is the restriction of T to V_i then $\mu_{T_1} = X^l$ and $\mu_{T_2} = f$.

We prove that the only values of k that satisfy the required conditions are $k \geq l$. Indeed, assume that $1 \leq k \leq l-1$. Since $\mu_{T_1} = X^l$, we have $T_1^l = 0$ and $T_1^{l-1} \neq 0$. Let $x \in V_1$ with $T^{l-1}x = T_1^{l-1}x \neq 0$ (but $T^l x = T_1^l x = 0$). Since $k \leq l-1$, we have $T^{l-1}x \in \operatorname{Im} T^{l-1} \subseteq \operatorname{Im} T^k$ and since $k \geq 1$ we have $T^k(T^{l-1}x) = T^{k-1}(T^l x) = 0$, so $T^{l-1}x \in \operatorname{Ker} T^k$. This shows that $0 \neq T^{l-1}x \in \operatorname{Ker} T^k \cap \operatorname{Im} T^k$, which contradicts $V = \operatorname{Ker}(T^k) \oplus \operatorname{Im}(T^k)$. So we must have $k \geq l$.

Suppose now that $k \geq l$. An element $x \in V$ writes uniquely in the form $x = x_1 + x_2$ with $x_i \in V_i$. Then $T^k x = T^k x_1 + T^k x_2 = T_1^k x_1 + T_2^k x_2 = T_2^k x_2$. (We have $T_1^l = 0$ and $k \geq l$, so $T_1^k = 0$.) It follows that

$$\operatorname{Ker} T^k = \{x_1 + x_2 \mid x_1 \in V_1, x_2 \in \operatorname{Ker} T_2^k\} = V_1 + \operatorname{Ker} T_2^k$$

and $\operatorname{Im} T^k = \operatorname{Im} T_2^k$. But $X \nmid f = \mu_{T_2}$, so T_2 is invertible. Then T_2^k too is invertible and we therefore have $\operatorname{Ker} T_2^k = 0$ and $\operatorname{Im} T_2^k = V_2$. It follows that $\operatorname{Ker} T^k = V_1$ and $\operatorname{Im} T^k = V_2$.

Now the only endomorphism $P : V \to V$ with $P^2 = P$, $\operatorname{Ker} P = V_1$ and $\operatorname{Im} P = V_2$ is the projection map defined by the author. Indeed, if $x_1 \in V_1$, $x_2 \in V_2$ then $x_1 \in \operatorname{Ker} P$, so $Px_1 = 0$, and $x_2 \in \operatorname{Im} P$, so $x_2 = Py$ for some $y \in V$, and we have $Px_2 = P^2 y = Py = x_2$. Therefore $P(x_1 + x_2) = x_2$.

We may actually write explicitly P as a polynomial of T. Since f and X^k are coprime, there are $g, h \in K[X]$ with $fg + X^k h = 1$. Then let $Q = X^k h = 1 - fg$. If $x_1 \in V_1 = \operatorname{Ker} T^k$ then $Q(T)x_1 = h(T)T^k x_1 = 0$ and if $x_2 \in V_2 = \operatorname{Ker} f(T)$ then

$$Q(T)x_2 = (1 - g(T)f(T))x_2 = x_2 - g(T)f(T)x_2 = x_2.$$

Hence, $Q(T)(x_1 + x_2) = 0 + x_2 = x_2 = P(x_1 + x_2)$. So $P = Q(T)$.

441. Let F be a field of characteristic 2 and let $F^2 := \{a^2 \mid a \in A\}$. We denote by ε the image of X in the quotient ring $F[X]/(X^2)$. Prove that

$R := F^2 + F\varepsilon$ is a subring of $F[\varepsilon]$ and define an operation $\boxplus : R \times R \to R$ such that $(R, \boxplus, \cdot)$ is a ring which is not isomorphic to $(R, +, \cdot)$.

Proposed by Constantin-Nicolae Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

Solution by the author. The ring $F[\varepsilon]$ is known as the ring of dual numbers over F, where $\varepsilon^2 = 0$. It has the basis $1, \varepsilon$ over F.

If $\alpha, \beta \in R$ then $\alpha = a^2 + b\varepsilon$, $\beta = c^2 + d\varepsilon$ for some $a, b, c, d \in F$. Then $\alpha\beta = (ac)^2 + (a^2 d + bc^2)\varepsilon \in R$ and $\alpha \pm \beta = (a^2 \pm c^2) + (b \pm d)\varepsilon = (a \pm c)^2 + (b \pm d)\varepsilon \in R$. (Here we use the fact that $\mathrm{char} F = 2$.) Hence R is a subring of $F[\varepsilon]$.

Since we are in characteristic 2, the map $x \mapsto x^2$ is injective. (If $x^2 = y^2$ then $0 = x^2 - y^2 = (x - y)^2$, so $x - y = 0$.) It follows that $f : F \times F \to R$, $f(a, b) = a^2 + b\varepsilon$, is a bijection. Then if we identify $F \times F$ with R via f the operations on $F \times F \cong R$ write as $(a, b) \cdot (c, d) = (ac, a^2 d + bc^2)$ and $(a, b) + (c, d) = (a + c, b + d)$.

We now define $\boxplus$ on $F \times F \cong R$ by $(a, b) \boxplus (c, d) = (a + c, b + d + ac)$. The commutativity of $\boxplus$ is obvious. For the associativity one checks that, regardless where we put the brackets, it holds

$$(a, b) \boxplus (c, d) \boxplus (e, f) = (a + c + e, b + d + f + ac + ae + ce).$$

(More generally, $\boxplus_{i=1}^{n} (a_i, b_i) = (\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i + \sum_{1 \le i < j \le n} a_i a_j)$.) The zero element of $(F \times F, \boxplus)$ is $(0, 0)$ and the opposite of an element (a, b) is $\boxminus(a, b) = (a, b + a^2)$. (We have $(a, b) + (a, b + a^2) = (a + a, b + b + a^2 + aa) = (0, 0)$.) For the distributivity of $\cdot$ over $\boxplus$ one checks that $(a, b) \cdot ((c, d) \boxplus (e, f)) = (a, b) \cdot (c, d) \boxplus (a, b) \cdot (e, f) = (ac + ae, a^2 d + a^2 f + a^2 ce + bc^2 + be^2)$.

We have $(F^2, \boxplus, \cdot) \ncong (F \times F, +, \cdot)$ because $(F \times F, +, \cdot)$ is 2-torsion (we have $2(a, b) = (0, 0)\ \forall (a, b) \in F^2$) while $(F \times F, \boxplus, \cdot)$ is not. For instance, in $(F \times F, \boxplus, \cdot)$ we have $2(1, 0) = (1, 0) \boxplus (1, 0) = (0, 1)$. More precisely, $(F \times F, \boxplus, \cdot)$ is 4-torsion, but not 2-torsion. Indeed, in $(F \times F, \boxplus, \cdot)$ we have $2(a, b) = (0, a^2)$, $3(a, b) = (a, b + a^2) = \boxminus(a, b)$ and $4(a, b) = (0, 0)$.

Translating $\boxplus$ from $F \times F$ to R via f we get the operation explicitly:

$$(a^2 + b\varepsilon) \boxplus (c^2 + d\varepsilon) = (a + c)^2 + (b + d + ac)\varepsilon.$$

442. Let $f : [0, 1] \to \mathbb{R}$ be a function with the following properties:

a) f is convex.

b) $f'_r(0) > f(0)$ and $f'_l(1) < f(1)$.

c) For any $x \in (0, 1) \cap \mathbb{Q}$ we have $(f'_l(x) - f(x))(f'_r(x) - f(x)) > 0$.

Prove that there is $c \in (0, 1)$ such that f is differentiable at c and $f'(c) = f(c)$.

Here $f'_l(x)$ and $f'_r(x)$ denote the left and right derivatives of f at x.

Proposed by Florin Stănescu, Şerban Cioculescu School, Găeşti, Dâmboviţa, Romania.

Solution by the author. We use the following well-known properties of a convex function $f : I \to \mathbb{R}$ where $I = (a, b)$:

1) f is continuous on I and it has left and right derivatives at any point.

2) $f_l'(t) \leq f_r'(t)$ $\forall t \in I$ and $f_r'(t_1) \leq f_l'(t_2)$ for any $t_1, t_2 \in I$ with $t_1 < t_2$. In particular, both f_l' and f_r' are increasing.

If $I = [a, b]$ then we have additional conditions involving a and b. Namely, both limits $\lim_{t \searrow a} f(t)$ and $\lim_{t \nearrow b} f(t)$ exist and $\lim_{t \searrow a} f(t) \leq f(a)$ and $\lim_{t \nearrow b} f(t) \leq f(b)$. Moreover, if f is continuous at a then $f_r'(a)$ exists with value in $\mathbb{R} \cup \{-\infty\}$ and if f is continuous at b then $f_l'(b)$ exists with value in $\mathbb{R} \cup \{\infty\}$. For property 2) we have $f_r'(a) \leq f_l'(t)$ and $f_r'(t) \leq f_l'(b)$ $\forall t \in (a, b)$. (Of course, provided that f is continuous at a and b, respectively.)

In our case, $f_r'(0)$ and $f_l'(1)$ are defined by hypothesis so f is continuos at 0 and 1. Hence it is continuous everywhere.

We now prove by induction that there are some intervals $[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots$ with $a_i, b_i \in [0, 1] \cap \mathbb{Q}$ and $b_i - a_i = \frac{1}{2^n}$ such that $f_r'(a_n) > f(a_n)$ and $f_l'(b_n) < f(b_n)$. At $n = 0$ we take $a_0 = 0$, $b_0 = 1$. For the induction step $n \mapsto n + 1$ let x be the midpoint of $[a_n, b_n]$. Obviously, $x \in (0, 1) \cap \mathbb{Q}$, so $(f_l'(x) - f(x))(f_r'(x) - f(x)) > 0$. If $f_l'(x) > f(x)$ and $f_r'(x) > f(x)$ then we take $a_{n+1} = x$, $b_{n+1} = b_n$. If $f_l'(x) < f(x)$, $f_r'(x) < f(x)$ then we take $a_{n+1} = a_n$, $b_{n+1} = x$. In both cases a_{n+1}, b_{n+1} satisfy the required conditions.

Since f is continuous and $f_r'(0) > f(0)$, $f_l'(1) < f(1)$, there is $\delta > 0$ such that $f_r'(0) > f(x)$ $\forall x \in (0, \delta)$ and $f_l'(1) < f(x)$ $\forall x \in (1 - \delta, 1)$. If $b_n \in (0, \delta)$ then $f_l'(b_n) \geq f_r'(0) > f(b_n)$, contradiction. If $a_n \in (1 - \delta, 1)$ then $f_r'(a_n) \leq f_l'(1) < f(a_n)$, contradiction. It follows that $a_n \leq 1 - \delta$, $b_n \geq \delta$ $\forall n$.

Since (a_n) increases and is bounded from above by $1 - \delta$, we have $a_n \nearrow c$ for some $c \leq 1 - \delta$. Since (b_n) is decreasing and bounded from bellow by δ, we have $b_n \searrow c'$ for some $c' \geq \delta$. Since $b_n - a_n \to 0$, we have $c = c'$. Note that $\delta \leq c \leq 1 - \delta$ yields $c \in (0, 1)$.

For any n we have $a_n \leq c \leq b_n$. If $a_n < c$ then $f(a_n) < f_r'(a_n) \leq f_l'(c)$. If $a_n = c$ then $a_n \in (0, 1) \cap \mathbb{Q}$, so $f(a_n) < f_r'(a_n)$ implies that $f(a_n) < f_l'(a_n)$, so again $f(a_n) < f_l'(c)$. Similarly, if $c < b_n$ then $f_r'(c) \leq f_l(b_n) < f(b_n)$. If $c = b_n$ then $b_n \in (0, 1) \cap \mathbb{Q}$, so $f(b_n) > f_l'(a_n)$ implies that $f(b_n) > f_r'(b_n)$, so again $f_r'(c) < f(b_n)$.

We have $f(a_n) < f_l'(c) \leq f_r'(c) < f(b_n)$. But $a_n, b_n \to c$, so that $f(a_n), f(b_n) \to f(c)$ as $n \to \infty$. It follows that $f_l'(c) = f_r'(c) = f(c)$. Hence the conclusion. $\square$

443. Let $\mathcal{F}$ be the set of continuous functions on $[0, 2\pi]$ that satisfy the equalities

$$\int_0^{2\pi} f(x) \cos kx \,\mathrm{d}x = \int_0^{2\pi} f(x) \sin kx \,\mathrm{d}x = \pi, \quad \text{for all } k = 1, 2, \ldots, n.$$

Prove that there exists a function $f_0 \in \mathcal{F}$ such that

$$\int_0^{2\pi} f_0^2(x)\,\mathrm{d}x \le \int_0^{2\pi} f^2(x)\,\mathrm{d}x, \quad \text{for all } f \in \mathcal{F},$$

and find such a function.

Remark. The problem is true in a more general case, considering the equalities

$$\int_0^{2\pi} f(x)\cos kx\,\mathrm{d}x = a_k \text{ and } \int_0^{2\pi} f(x)\sin kx\,\mathrm{d}x = b_k, \text{ for all } k = 1, 2, \ldots, n,$$

with a_k, b_k arbitrary real numbers.

`Proposed by Vasile Pop, Technical University of Cluj-Napoca, Romania.`

Solution by the author. For $f \in \mathcal{F}$ arbitrary, let us consider the linear subspaces of $C[0, 2\pi]$:

$$V_0 = \operatorname{span}\{\cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx\}$$

and

$$V_1 = \operatorname{span}\{\cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, f\}.$$

Obviously, V_0 is a linear subspace of V_1.

We will show that f_0 exists and that $f_0 \in V_0$.

Consider on V_1 the inner product defined by

$$\langle g, h\rangle = \int_0^{2\pi} g(x)h(x)\,\mathrm{d}x, \qquad g, h \in V_1,$$

and the norm $\|\cdot\|$ induced by this inner product:

$$\|g\| = \left(\int_0^{2\pi} g^2(x)\,\mathrm{d}x\right)^{1/2}.$$

It is easy to check that the functions $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x, \ldots,$ $\cos nx$, $\sin nx$ are orthogonal of norm $\sqrt{\pi}$.

We have that $V_1 = V_0 \oplus V_0^{\perp}$, hence f can be written uniquely in the form $f = f_1 + g_1$ with $f_1 \in V_0$ and $g_1 \in V_0^{\perp}$. Since $f_1 \in V_0$, we have that

$$f_1(x) = \sum_{k=1}^{n} (\alpha_k \cos kx + \beta_k \sin kx), \qquad \alpha_k, \beta_k \in \mathbb{R}.$$

Next, for every $m = 1, 2, \ldots, n$, we obtain that

$$\begin{aligned}\pi &= \langle f, \cos mx\rangle = \langle f_1, \cos mx\rangle + \langle g_1, \cos mx\rangle = \langle f_1, \cos mx\rangle \\ &= \sum_{k=1}^{n} (\alpha_k \langle \cos kx, \cos mx\rangle + \beta_k \langle \sin kx, \cos mx\rangle) = \|\cos mx\|^2 \alpha_m = \pi\alpha_m\end{aligned}$$

and

$$\pi = \langle f, \sin mx \rangle = \langle f_1, \sin mx \rangle + \langle g_1, \sin mx \rangle = \langle f_1, \sin mx \rangle$$
$$= \sum_{k=1}^{n} \left(\alpha_k \langle \cos kx, \sin mx \rangle + \beta_k \langle \sin kx, \sin mx \rangle \right) = \|\sin mx\|^2 \beta_m = \pi \beta_m,$$

hence

$$\alpha_k = \beta_k = 1 \qquad \text{for all } k = 1, 2, \ldots, n$$

and

$$f_1(x) = \sum_{k=1}^{n} (\cos kx + \sin kx),$$

which does not depend of f. Leting $f_0 := f_1$, i.e.,

$$f_0(x) = \sum_{k=1}^{n} (\cos kx + \sin kx),$$

we have that

$$\|f\|^2 = \|f_0\|^2 + \|g_1\|^2 + 2 \langle f_0, g_1 \rangle = \|f_0\|^2 + \|g_1\|^2 \geq \|f_0\|^2,$$

with equality only when $g_1 = 0$, hence $f = f_0$. Also, the minimum is

$$\|f_0\|^2 = \sum_{k=1}^{n} \|\cos kx\|^2 + \sum_{k=1}^{n} \|\sin kx\|^2 = 2n\pi.$$

444. Let $A, B \in M_n(\mathbb{R})$ be such that $A^2 + B^2 = e(AB - BA)$.
Prove that $AB - BA \notin GL_n(\mathbb{R})$.

`Proposed by Luigi-Ionuţ Catana, University of Bucharest, Romania.`

A question from the editor. For a given $n \geq 2$ try to determine all values of $\alpha \in \mathbb{R}$ such that there are $A, B \in M_n(\mathbb{R})$ with the property that $A^2 + B^2 = \alpha(AB - BA)$ and $AB - BA \in GL_n(\mathbb{R})$.

Solution by the author. We have

$$(A + iB)(A - iB) = A^2 + B^2 - i(AB - BA) = (e - i)(AB - BA).$$

It follows that $|\det(A + iB)|^2 = (e - i)^n \det(AB - BA)$.

Suppose that $AB - BA \in GL_n(\mathbb{R})$. This implies

$$(e - i)^n = |\det(A + iB)|^2 \det(AB - BA)^{-1} \in \mathbb{R}.$$

Then by identifying the imaginary part of $(e - i)^n$ one gets $0 = -\binom{n}{1}e^{n-1} + \binom{n}{3}e^{n-3} - \binom{n}{5}e^{n-5} + \cdots$. Hence e is a root of a polinomial with integer coefficients. But this is impossible since e is a transcendental number. □

Notes from the editor. 1. We received a solution with the same aproach from *Moubinool Omarjee, Lycée Henri IV, Paris, France.*

2. We can in fact determine all $\alpha \in \mathbb{R}$ such there exist $A, B \in M_n(\mathbb{R})$ with $A^2 + B^2 = \alpha(AB - BA)$ and $AB - BA \in GL_n(\mathbb{R})$. Following the author's proof, with e replaced by α, we get $(\alpha - i)^n \in \mathbb{R}$. We write $\alpha - i = \rho(\cos\theta + i\sin\theta)$, where $\rho = |\alpha - i| = \sqrt{\alpha^2+1}$. We have $\alpha = -\frac{\cos\theta}{\sin\theta} = -\cot\theta$. Then $\rho^n(\cos(n\theta) + i\sin(n\theta)) = (\alpha - i)^n \in \mathbb{R}$ implies that $\sin(n\theta) = 0$, so $n\theta = k\pi$ for some $k \in \mathbb{Z}$. Hence $\theta = \frac{k\pi}{n}$ and $\alpha = -\cot\frac{k\pi}{n} = \cot\frac{-k\pi}{n}$. Hence the set of all eligible values of α is $S = \{\cot\frac{k\pi}{n} \mid k \in \mathbb{Z},\ n \nmid k\}$. (If $n \mid k$ then $\cot\frac{k\pi}{n} = \infty$.) Since the cotangent function is periodic of period π, we have in fact $S = \{\cot\frac{k\pi}{n} \mid 1 \le k \le n-1\}$.

We now prove that every $\alpha \in S$ qualifies. Let $\alpha = \cot\frac{k\pi}{n}$ and let $u \in \mathbb{R}$ arbitrary, not of the form $\frac{m\pi}{n}$ with $m \in \mathbb{Z}$. Then for every $l \in \mathbb{Z}$ we have $\frac{kl\pi}{n} + u \notin \pi\mathbb{Z}$ and so $\cot(\frac{kl\pi}{n} + u) \in \mathbb{R}$. For convenience, we index the rows and columns of the $n \times n$ matrices, not by $1, 2, \ldots, n$, but by $\mathbb{Z}_n$. Let $\{e_{q,r} \mid q, r \in \mathbb{Z}_n\}$ be $n \times n$ matrices indexed this way, where $e_{q,r}$ has 1 on the (q, r) position and 0 everywhere else. We have $e_{q,r}e_{s,t} = \delta_{r,s}e_{q,t}$.

Then we define $A = \sum_{l\in\mathbb{Z}_n} \cot(\frac{kl\pi}{n} + u)e_{l,l+1}$ and $B = \sum_{l\in\mathbb{Z}_n} e_{l,l+1}$. Now A is well defined because cot has period π, so $\cot(\frac{kl\pi}{n} + u)$ depends only on the value of l modulo n. From the obvious formula

$$\Big(\sum_{l\in\mathbb{Z}_n} a_l e_{l,l+1}\Big)\Big(\sum_{l\in\mathbb{Z}_n} b_l e_{l,l+1}\Big) = \sum_{l\in\mathbb{Z}_n} a_l b_{l+1} e_{l,l+2}$$

we get

$$\begin{aligned}
A^2 &= \sum_{l\in\mathbb{Z}_n} \cot\left(\frac{kl\pi}{n} + u\right)\cot\left(\frac{k(l+1)\pi}{n} + u\right) e_{l,l+2},\\
B^2 &= \sum_{l\in\mathbb{Z}_n} e_{l,l+2},\\
AB &= \sum_{l\in\mathbb{Z}_n} \cot\left(\frac{kl\pi}{n} + u\right) e_{l,l+2},\\
BA &= \sum_{l\in\mathbb{Z}_n} \cot\left(\frac{k(l+1)\pi}{n} + u\right) e_{l,l+2}.
\end{aligned}$$

It follows that

$$A^2 + B^2 = \sum_{l\in\mathbb{Z}_n} \left(\cot\left(\frac{kl\pi}{n} + u\right)\cot\left(\frac{k(l+1)\pi}{n} + u\right) + 1\right) e_{l,l+2},$$

$$AB - BA = \sum_{l\in\mathbb{Z}_n} \left(\cot\left(\frac{kl\pi}{n} + u\right) - \cot\left(\frac{k(l+1)\pi}{n} + u\right)\right) e_{l,l+2}.$$

But we have the formula $\cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$ (which can be easily deduced by writing cot in terms of cos and sin.) Hence $\cot x \cot y + 1 =$

$\cot(x-y)(\cot y - \cot x)$. It follows that for every $l \in \mathbb{Z}_n$ we have

$$\cot(\frac{kl\pi}{n}+u)\cot(\frac{k(l+1)\pi}{n}+u)+1 = \cot\frac{k\pi}{n}(\cot(\frac{kl\pi}{n}+u)-\cot(\frac{k(l+1)\pi}{n}+u)).$$

This implies $A^2+B^2 = \cot\frac{k\pi}{n}(AB-BA)$.

If we write $AB-BA = (c_{q,r})_{q,r\in\mathbb{Z}_n}$ then $c_{l,l+2} = \cot\big(\frac{kl\pi}{n}+u\big) - \cot\big(\frac{k(l+1)\pi}{n}+u\big)$ and all other entries are zero. Since $1 \le k \le n-1$, we have $\frac{k(l+1)\pi}{n}+u \not\equiv \frac{kl\pi}{n}+u \mod \pi\mathbb{Z}$, so $c_{l,l+2} \ne 0$. Now the only non-zero term of the sum $\det(AB-BA) = \sum_\sigma \varepsilon(\sigma)c_{1,\sigma(1)}\cdots c_{n,\sigma(n)}$ is the one corresponding to σ given by $l \mapsto l+2$, which has $\varepsilon(\sigma)=1$. Hence $\det(AB-BA) = \prod_{l\in\mathbb{Z}_n} c_{l,l+2} \ne 0$. So A, B have all the required properties.

445. Prove that

$$\lim_{n\to\infty} n^3 \iint\limits_{[0,1]^2} \left(\frac{x+y}{2}\right)^n \left(\sqrt[n]{1+x^ny^n}-1\right) \mathrm{d}x\mathrm{d}y = 4\ln 2 - 16 + 2\pi + 8K,$$

where $K = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2}$ is the Catalan constant.

`Proposed by Dumitru Popa, Department of Mathematics, Ovidius University, Constanţa, Romania.`

Solution by the author. Put

$$I_n = n^3 \iint\limits_{[0,1]^2} \left(\frac{x+y}{2}\right)^n \left(\sqrt[n]{1+x^ny^n}-1\right) \mathrm{d}x\mathrm{d}y$$

and note that by Fubini's theorem

$$I_n = n^3 \int_0^1 \left(\int_0^1 \left(\frac{x+y}{2}\right)^n \left(\sqrt[n]{1+x^ny^n}-1\right) \mathrm{d}x\right) \mathrm{d}y.$$

Let $n \in \mathbb{N}$. For $y \in [0,1]$ we have

$$\begin{aligned}\int_0^1 &\left(\frac{x+y}{2}\right)^n \left(\sqrt[n]{1+x^ny^n}-1\right) \mathrm{d}x \\ &= \int_{0+0}^1 \left(\frac{x+y}{2}\right)^n \left(\sqrt[n]{1+x^ny^n}-1\right) \mathrm{d}x \\ &= \frac{1}{n}\int_{0+0}^1 \left(\frac{\sqrt[n]{u}+y}{2}\right)^n \frac{\sqrt[n]{1+uy^n}-1}{u}\sqrt[n]{u}\mathrm{d}u.\end{aligned}$$

This is obtained with the change of variable $x = \sqrt[n]{u}$ in the improper integral. Then, again by Fubini's theorem,

$$I_n = n^2 \int_{0+0}^1 \left(\frac{\sqrt[n]{u}}{u}\int_0^1 \left(\frac{\sqrt[n]{u}+y}{2}\right)^n \left(\sqrt[n]{1+uy^n}-1\right)\mathrm{d}y\right)\mathrm{d}u.$$

Now for $u \in (0,1]$ as above we deduce

$$\int_0^1 \left(\frac{\sqrt[n]{u}+y}{2}\right)^n \left(\sqrt[n]{1+uy^n}-1\right)\mathrm{d}y$$
$$= \int_{0+0}^1 \left(\frac{\sqrt[n]{u}+y}{2}\right)^n \left(\sqrt[n]{1+uy^n}-1\right)\mathrm{d}y$$
$$= \frac{1}{n}\int_{0+0}^1 \left(\frac{\sqrt[n]{u}+\sqrt[n]{v}}{2}\right)^n \left(\frac{\sqrt[n]{1+uv}-1}{v}\right)\sqrt[n]{v}\mathrm{d}v.$$

In this way we obtain

$$\begin{aligned} I_n &= n\int_{0+0}^1 \left(\int_{0+0}^1 \left(\frac{\sqrt[n]{u}+\sqrt[n]{v}}{2}\right)^n \left(\frac{\sqrt[n]{1+uv}-1}{uv}\right)\sqrt[n]{uv}\mathrm{d}v\right)\mathrm{d}u \\ &= n\iint\limits_{(0,1]^2} \left(\frac{\sqrt[n]{u}+\sqrt[n]{v}}{2}\right)^n \left(\frac{\sqrt[n]{1+uv}-1}{uv}\right)\sqrt[n]{uv}\mathrm{d}u\mathrm{d}v. \end{aligned}$$

Now for all $(u,v) \in (0,1]^2$ we have

$$\begin{aligned} \lim_{n\to\infty}\left(\frac{\sqrt[n]{u}+\sqrt[n]{v}}{2}\right)^n &= \sqrt{uv}, \quad \lim_{n\to\infty}\sqrt[n]{uv}=1, \\ \lim_{n\to\infty} n\left(\sqrt[n]{1+uv}-1\right) &= \ln(1+uv). \end{aligned}$$

Since $0 < (e^a-1)/a \le e-1$, $0 < a \le 1$ and $\frac{1}{n}\ln(1+uv) \le \ln 2 \le 1$ for all $(u,v) \in (0,1]^2$, we deduce $0 < n\left(\sqrt[n]{1+uv}-1\right) \le (e-1)\ln(1+uv)$ for all $(u,v) \in (0,1]^2$. Then

$$0 \le n\left(\frac{\sqrt[n]{u}+\sqrt[n]{v}}{2}\right)^n \left(\frac{\sqrt[n]{1+uv}-1}{uv}\right)\sqrt[n]{uv} \le \frac{(e-1)\ln(1+uv)}{uv}$$

for all $(u,v) \in (0,1]^2$.

By the Lebesgue dominated convergence theorem we deduce

$$\lim_{n\to\infty} I_n = \iint\limits_{(0,1]^2} \frac{\ln(1+uv)}{\sqrt{uv}}\mathrm{d}u\mathrm{d}v.$$

From $\ln(1+x) = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$, $0 \le x \le 1$, we obtain

$$\iint\limits_{(0,1]^2} \frac{\ln(1+uv)}{\sqrt{uv}}\mathrm{d}u\mathrm{d}v = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\iint\limits_{(0,1]^2}(uv)^{n-\frac{1}{2}}\mathrm{d}u\mathrm{d}v$$

and using

$$\iint\limits_{(0,1]^2}(uv)^{n-\frac{1}{2}}\mathrm{d}u\mathrm{d}v = \left(\int_0^1 u^{n-\frac{1}{2}}\mathrm{d}u\right)^2 = \frac{4}{(2n+1)^2}$$

we get

$$\lim_{n\to\infty} I_n = 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,(2n+1)^2}.$$

Since $\frac{1}{n(2n+1)^2} = \frac{1}{n} - \frac{2}{2n+1} - \frac{2}{(2n+1)^2}$, we finally obtain

$$\begin{aligned}\lim_{n\to\infty} I_n &= 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} - 8\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1} - 8\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n+1)^2} \\ &= 4\ln 2 - 8\left(1-\frac{\pi}{4}\right) - 8\,(1-K) = 4\ln 2 - 16 + 2\pi + 8K.\end{aligned}$$

446. Let $A \in \mathcal{M}_2(\mathbb{Z})$. Prove that $e^A \in \mathcal{M}_2(\mathbb{Z})$ if and only if $A^2 = O_2$.

`Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.`

Solution by Francisco Perdomo and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain. If $A^2 = O_2$ then $e^A = I_2 + A + \dfrac{A^2}{2} + \cdots = I_2 + A \in \mathcal{M}_2(\mathbb{Z})$.

Conversely, let us suppose that $e^A \in \mathcal{M}_2(\mathbb{Z})$. Let $\lambda_1,\ \lambda_2 \in \mathbb{C}$ be the eigenvalues of A. Then there is $B \in \mathcal{M}_2(\mathbb{C})$ such that

$$A = B^{-1} \cdot \begin{pmatrix} \lambda_1 & * \\ 0 & \lambda_2 \end{pmatrix} \cdot B.$$

Then $e^A = B^{-1} \cdot \begin{pmatrix} e^{\lambda_1} & * \\ 0 & e^{\lambda_2} \end{pmatrix} \cdot B$ and $\det e^A = e^{\lambda_1+\lambda_2} \in \mathbb{Z}$. Since e is a trascendental number, it follows that $\lambda_1 + \lambda_2 = 0$. The determinant of A is $\lambda_1\lambda_2 = -\lambda_1^2 \in \mathbb{Z}$, so that λ_1 is an algebraic number. The trace of e^A is $e^{\lambda_1} + e^{-\lambda_1} \in \mathbb{Z}$. Then e^{λ_1} is an algebraic number and, by the Lindemann-Weierstrass theorem, $\lambda_1 = 0$. As the two eigenvalues of A are equal to 0, its characteristic polynomial is $P_A = X^2$. Hence $A^2 = O_2$. □

Notes from the editor. 1. The author's proof goes on the same lines. We also received a similar proof from *Gabriel Prăjitură, State University of New York, USA*.

2. A similar result holds for square matrices of arbitrary size. If A is an arbitrary $n \times n$ matrix and the roots (counting multiplicities) of the characteristic polynomial P_A are $\lambda_1, \ldots, \lambda_n$ then the roots of P_{e^A} are $e^{\lambda_1}, \ldots, e^{\lambda_n}$. If both A and $e^A \in \mathcal{M}_n(\mathbb{Q})$ then $P_A, P_{e^A} \in \mathbb{Q}[X]$, so $\lambda_1, \ldots, \lambda_n$ and $e^{\lambda_1}, \ldots, e^{\lambda_n}$ are all algebraic. But this implies by Lindemann–Weierstrass Theorem that $\lambda_1 = \cdots = \lambda_n = 0$. It follows that $P_A = X^n$, so $A^n = 0$. Conversely, if

$A \in \mathcal{M}_n(\mathbb{Q})$ and $A^n = 0_n$ then

$$e^A = I_n + \frac{A}{1!} + \cdots + \frac{A^{n-1}}{(n-1)!} + \frac{A^n}{n!} + \cdots = I_n + \frac{A}{1!} + \cdots + \frac{A^{n-1}}{(n-1)!} \in \mathcal{M}_n(\mathbb{Q}).$$

If we consider the original problem, with $A, e^A \in \mathcal{M}_n(\mathbb{Z})$, then we need the additional condition that $I_n + \frac{A}{1!} + \cdots + \frac{A^{n-1}}{(n-1)!} \in \mathcal{M}_n(\mathbb{Z})$.