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The Euler-Maclaurin summation formula for functions of class C^3

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Abstract. We give a self-contained proof of the Euler-Maclaurin summation formula for functions of class C^3. We derive a general result from which, as applications, we prove: the famous Stirling formula, the asymptotic evaluations for the Stieltjes constants, the asymptotic evaluation for the Glaisher-Kinkelin constant and an exercise in N. Bourbaki's book regarding the asymptotic evaluation for Vandermonde determinant associated to $(1, 2, \ldots, n)$.

Keywords: Euler-Maclaurin summation formula, asymptotic expansions, approximation to limiting values, Stirling formula, Stieltjes constants, Glaisher-Kinkelin constant.

MSC: Primary 65B15; Secondary 41A60, 40A25.

1. Introduction

The Euler-Maclaurin summation formula is one of the most powerful results to obtain various asymptotic evaluations for real sequences. From many books where this formula is proved we mention our favorites: N. Bourbaki's book [2, pages 282–283] and K. Knopp's book [3, Chapter XIV]. The aim of this paper is to give a self-contained proof of the Euler-Maclaurin summation formula only for functions of class C^3, Theorem 1. We also prove a general result, Corollary 4, which, as the author knows, in concrete applications gives, with minimum of calculations, strong enough asymptotic evaluations for a large class of real sequences. As applications, we prove: the famous Stirling formula, Theorem 5, the asymptotic evaluations for the Stieltjes constants, Theorem 6, the asymptotic evaluation for the Glaisher-Kinkelin constant, Theorem 7 and a proof for an exercise in N. Bourbaki's

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book regarding the asymptotic evaluation for the Vandermonde determinant associated to $(1, 2, \dots, n)$, Proposition 9. It is our hope that these results will be useful for students as a part of their mathematical culture and for other readers interested in this topic. The author used these notes for students at the master program Didactical mathematics at the Faculty of Mathematics and Informatics, Ovidius University of Constanţa. Let us mention that the notation and notion used in this paper are standard, in particular $\mathbb{N} = \{1, 2, \dots\}$ is the set of all natural numbers and $[x]$ is the integer part of $x \in \mathbb{R}$.

2. The main results

Let B_1, B_2, $B_3 : \mathbb{R} \to \mathbb{R}$ be the polynomials $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$, $B_3(x) = x^3 - \frac{3x^2}{2} + \frac{x}{2}$. These polynomials are called Bernoulli polynomials. The Euler-Maclaurin summation formula for functions of class C^3 is given by the following theorem.

Theorem 1. *Let $k \in \mathbb{N} \cup \{0\}$ and let $f : [k, \infty) \to \mathbb{R}$ be a function of class C^3. For every natural number $n \geq k$ the following equality holds*

$$f(k) + f(k+1) + \dots + f(n) = \int_k^n f(x)\,\mathrm{d}x + \frac{f(n) + f(k)}{2} + \frac{f'(n) - f'(k)}{12} + \frac{1}{6}\int_k^n B_3(x - [x])\, f'''(x)\,\mathrm{d}x.$$

Proof. For $n = k$ the equality is trivial. Let $n \geq k+1$ and $\nu \in \mathbb{N} \cup \{0\}$. Since $B_3'(x) = 3B_2(x)$, $B_3(0) = B_3(1) = 0$, $B_2'(x) = 2B_1(x)$, $B_2(0) = B_2(1) = \frac{1}{6}$, $B_1(0) = -\frac{1}{2}$, $B_1(1) = \frac{1}{2}$, integrating by parts, we have

$$\int_\nu^{\nu+1} B_3(x-\nu)\, f'''(x)\,\mathrm{d}x = B_3(x-\nu)\, f''(x)\Big|_\nu^{\nu+1} - \int_\nu^{\nu+1} B_3'(x-\nu)\, f''(x)\,\mathrm{d}x = -3\int_\nu^{\nu+1} B_2(x-\nu)\, f''(x)\,\mathrm{d}x =$$

$$= -3B_2(x-\nu)\, f'(x)\Big|_\nu^{\nu+1} + 6\int_\nu^{\nu+1} B_1(x-\nu)\, f'(x)\,\mathrm{d}x =$$

$$= -\frac{f'(\nu+1) - f'(\nu)}{2} + 6B_1(x-\nu)\, f(x)\Big|_\nu^{\nu+1} - 6\int_\nu^{\nu+1} f(x)\,\mathrm{d}x =$$

$$= -\frac{f'(\nu+1) - f'(\nu)}{2} + 3\left[f(\nu+1) + f(\nu)\right] - 6\int_\nu^{\nu+1} f(x)\,\mathrm{d}x,$$

that is,

$$\int_\nu^{\nu+1} f(x)\,\mathrm{d}x = \frac{f(\nu+1) + f(\nu)}{2} - \frac{f'(\nu+1) - f'(\nu)}{12} -$$

$$-\frac{1}{6}\int_{\nu}^{\nu+1} B_3\left(x-\nu\right) f'''\left(x\right)\mathrm{d}x.$$

Since $[x]=\nu$ for $x\in[\nu,\nu+1)$ we have $B_3\left(x-\nu\right) f'''\left(x\right)=B_3\left(x-[x]\right) f'''\left(x\right)$ for $x\in[\nu,\nu+1)$ and since f''' is continuous

$$\int_{\nu}^{\nu+1} B_3\left(x-\nu\right) f'''\left(x\right)\mathrm{d}x=\int_{\nu}^{\nu+1} B_3\left(x-[x]\right) f'''\left(x\right)\mathrm{d}x$$

(see [1, Observaţie, pag. 61]). Thus

$$\int_{\nu}^{\nu+1} f\left(x\right)\mathrm{d}x=\frac{f\left(\nu+1\right)+f\left(\nu\right)}{2}-\frac{f'\left(\nu+1\right)-f'\left(\nu\right)}{12}-$$

$$-\frac{1}{6}\int_{\nu}^{\nu+1} B_3\left(x-[x]\right) f'''\left(x\right)\mathrm{d}x. \tag{1}$$

Taking in (1), $\nu=k$, $\nu=k+1$, $\dots$, $\nu=n-1$ we have

$$\begin{aligned}
\int_{k}^{k+1} f\left(x\right)\mathrm{d}x &= \frac{f\left(k+1\right)+f\left(k\right)}{2}-\frac{f'\left(k+1\right)-f'\left(k\right)}{12}-\\
&\quad -\frac{1}{6}\int_{k}^{k+1} B_3\left(x-[x]\right) f'''\left(x\right)\mathrm{d}x\\
\int_{k+1}^{k+2} f\left(x\right)\mathrm{d}x &= \frac{f\left(k+2\right)+f\left(k+1\right)}{2}-\frac{f'\left(k+2\right)-f'\left(k+1\right)}{12}-\\
&\quad -\frac{1}{6}\int_{k+1}^{k+2} B_3\left(x-[x]\right) f'''\left(x\right)\mathrm{d}x\\
&\dots\\
\int_{n-1}^{n} f\left(x\right)\mathrm{d}x &= \frac{f\left(n\right)+f\left(n-1\right)}{2}-\frac{f'\left(n\right)-f'\left(n-1\right)}{12}-\\
&\quad -\frac{1}{6}\int_{n-1}^{n} B_3\left(x-[x]\right) f'''\left(x\right)\mathrm{d}x
\end{aligned}$$

and summing up it follows

$$\begin{aligned}
\int_{k}^{n} f\left(x\right)\mathrm{d}x &= f\left(k\right)+f\left(k+1\right)+\cdots+f\left(n\right)-\frac{f\left(n\right)+f\left(k\right)}{2}-\\
&\quad -\frac{f'\left(n\right)-f'\left(k\right)}{12}-\frac{1}{6}\int_{k}^{n} B_3\left(x-[x]\right) f'''\left(x\right)\mathrm{d}x
\end{aligned}$$

that is the Euler-Maclaurin summation formula for functions of class C^3. $\square$

We need in the sequel the following evaluation for the Bernoulli polynomial B_3.

Proposition 2. *For the Bernoulli polynomial $B_3:\mathbb{R}\to\mathbb{R}$, $B_3\left(x\right)=x^3-\frac{3x^2}{2}+\frac{x}{2}$ we have $\left|B_3\left(x\right)\right|\le\frac{\sqrt{3}}{36}, \forall\, x\in[0,1]$ and $\left|B_3\left(x-[x]\right)\right|\le\frac{\sqrt{3}}{36}$, $\forall\, x\in\mathbb{R}$.*

Proof. Indeed, $B_3'(x) = 3\left(x^2 - x + \frac{1}{6}\right)$ and $B_3'(x) = 0$ if and only if $x = \frac{3-\sqrt{3}}{6} \in [0,1]$ or $x = \frac{3+\sqrt{3}}{6} \in [0,1]$. Also

$$\begin{array}{cccccccccc} x & 0 & & \frac{3-\sqrt{3}}{6} & & \frac{3+\sqrt{3}}{6} & & 1 \\ B_3'(x) & & + & 0 & - & 0 & + & \\ B_3(x) & 0 & \nearrow & \frac{\sqrt{3}}{36} & \searrow & -\frac{\sqrt{3}}{36} & \nearrow & 0 \end{array}$$

We deduce that $|B_3(x)| \le \frac{\sqrt{3}}{36}, \forall\, x \in [0,1]$. If $x \in \mathbb{R}$, $x - [x] \in [0,1)$ and thus $|B_3(x - [x])| \le \frac{\sqrt{3}}{36}$. □

Proposition 3. *Let $k \in \mathbb{N} \cup \{0\}$ and let $\varphi : [k, \infty) \to \mathbb{R}$ be a function of class C^3. Suppose that the improper integral $\int_k^\infty |\varphi'''(x)|\,\mathrm{d}x$ is convergent. Then (i) the following limit exists*

$$L(\varphi) = \lim_{n\to\infty}\left(\varphi(k) + \varphi(k+1) + \cdots + \varphi(n) - \int_k^n \varphi(t)\mathrm{d}t - \frac{\varphi(n)}{2} - \frac{\varphi'(n)}{12}\right) \in \mathbb{R}$$

and

$$L(\varphi) = \frac{\varphi(k)}{2} - \frac{\varphi'(k)}{12} + \frac{1}{6}\int_k^\infty B_3(x - [x])\,\varphi'''(x)\,\mathrm{d}x;$$

(ii) for every natural number $n \ge k$ the following asymptotic evaluation holds

$$\varphi(k) + \varphi(k+1) + \cdots + \varphi(n) = \int_k^n \varphi(t)\mathrm{d}t + \frac{\varphi(n)}{2} + \frac{\varphi'(n)}{12} + L(\varphi) + R_n(\varphi)$$

$$|R_n(\varphi)| \le \frac{\sqrt{3}}{216}\int_n^\infty |\varphi'''(x)|\mathrm{d}x.$$

Proof. From Theorem 1 we have

$$\varphi(k) + \varphi(k+1) + \cdots + \varphi(n) - \int_k^n \varphi(t)\,\mathrm{d}t - \frac{\varphi(n)}{2} - \frac{\varphi'(n)}{12} =$$
$$= \frac{\varphi(k)}{2} - \frac{\varphi'(k)}{12} + \frac{1}{6}\int_k^n B_3(x - [x])\,\varphi'''(x)\,\mathrm{d}x,\ \forall\, n \ge k. \tag{2}$$

From Proposition 2, $|B_3(x - [x])\,\varphi'''(x)| \le \frac{\sqrt{3}}{36}|\varphi'''(x)|$, $\forall\, x \in \mathbb{R}$. Since $\int_k^\infty |\varphi'''(x)|\,\mathrm{d}x$ is convergent, by the comparison test for improper integrals of positive functions it follows that $\int_k^\infty |B_3(x - [x])\,\varphi'''(x)|\,\mathrm{d}x$ is convergent (see [5, page 60]), that is, the improper integral $\int_k^\infty B_3(x - [x])\,\varphi'''(x)\,\mathrm{d}x$ is absolutely convergent, hence convergent (see [5, Teorema 2, pg. 59]). From (2) passing to the limit as $n \to \infty$ we deduce the existence of the limit

$$L(\varphi) = \lim_{n\to\infty}\left(\varphi(k) + \varphi(k+1) + \cdots + \varphi(n) - \int_k^n \varphi(t)\mathrm{d}t - \frac{\varphi(n)}{2} - \frac{\varphi'(n)}{12}\right) \in \mathbb{R}$$

and

$$L(\varphi) = \frac{\varphi(k)}{2} - \frac{\varphi'(k)}{12} + \frac{1}{6}\int_k^\infty B_3(x - [x])\,\varphi'''(x)\,\mathrm{d}x. \tag{3}$$

From (2) and (3) we deduce

$$\begin{aligned}&\varphi(k)+\varphi(k+1)+\cdots+\varphi(n)-\int_k^n\varphi(t)\,\mathrm{d}t-\frac{\varphi(n)}{2}-\frac{\varphi'(n)}{12}\\=\;&L(\varphi)-\frac{1}{6}\int_n^\infty B_3(x-[x])\,\varphi'''(x)\,\mathrm{d}x,\forall\, n\geq k.\end{aligned}$$

Let us denote $R_n(\varphi)=-\frac{1}{6}\int_n^\infty B_3(x-[x])\,\varphi'''(x)\,\mathrm{d}x$. Then, by Proposition 2, we get

$$\begin{aligned}|R_n(\varphi)|&=\frac{1}{6}\left|\int_n^\infty B_3(x-[x])\,\varphi'''(x)\,\mathrm{d}x\right|\leq\frac{1}{6}\int_n^\infty\left|B_3(x-[x])\,\varphi'''(x)\right|\mathrm{d}x\leq\\&\leq\frac{\sqrt{3}}{216}\int_n^\infty\left|\varphi'''(x)\right|\mathrm{d}x.\end{aligned}$$

□

From Proposition 3 and hence, from the Euler-Maclaurin summation formula Theorem 1, we derive the next Corollary. As the author knows, this is the result which, in concrete applications, gives, with minimum of calculations, strong asymptotic evaluations for a large class of sequences.

Corollary 4. *Let $k\in\mathbb{N}\cup\{0\}$, $\varphi:[k,\infty)\to\mathbb{R}$ be a function of the class C^3. Suppose that $\lim\limits_{x\to\infty}\varphi''(x)=0$ and there exists $a\geq k$ such that φ''' has constant sign on the interval $[a,\infty)$. The following limit exists*

$$L(\varphi)=\lim_{n\to\infty}\left(\varphi(k)+\varphi(k+1)+\cdots+\varphi(n)-\int_k^n\varphi(t)\mathrm{d}t-\frac{\varphi(n)}{2}-\frac{\varphi'(n)}{12}\right)\in\mathbb{R}$$

and for every natural number $n\geq a$ the following asymptotic evaluation holds

$$\varphi(k)+\varphi(k+1)+\cdots+\varphi(n)=\int_k^n\varphi(t)\mathrm{d}t+\frac{\varphi(n)}{2}+\frac{\varphi'(n)}{12}+L(\varphi)+R_n(\varphi)$$

$$|R_n(\varphi)|\leq\frac{\sqrt{3}}{216}\left|\varphi''(n)\right|.$$

Proof. Since φ''' has constant sign on the interval $[a,\infty)$ we have two situations. The case $\varphi'''(x)\leq 0$, $\forall\, x\in[a,\infty)$. In this case for every $t\geq a$ we have

$$\int_t^\infty\left|\varphi'''(x)\right|\mathrm{d}x=-\int_t^\infty\varphi'''(x)\,\mathrm{d}x=-\varphi''(x)\mid_t^\infty=\varphi''(t)$$

($\lim\limits_{x\to\infty}\varphi''(x)=0$) and thus $\varphi''(t)\geq 0$, $\forall\, t\geq a$. From Proposition 3 we get the evaluation from the statement and

$$|R_n(\varphi)|\leq\frac{\sqrt{3}}{216}\int_n^\infty\left|\varphi'''(x)\right|\mathrm{d}x=\frac{\sqrt{3}}{216}\varphi''(n)=\frac{\sqrt{3}}{216}\left|\varphi''(n)\right|.$$

The case $\varphi'''(x) \geq 0$, $\forall\, x \in [a, \infty)$. In this case for every $t \geq a$ we have

$$\int_t^{\infty} \left|\varphi'''(x)\right| \mathrm{d}x = \int_t^{\infty} \varphi'''(x)\, \mathrm{d}x = \varphi''(x)\ |_t^{\infty} = -\varphi''(t)$$

$(\lim\limits_{x\to\infty} \varphi''(x) = 0)$ and thus $\varphi''(t) \leq 0$, $\forall\, t \geq a$. From Proposition 3 we get the evaluation from the statement and

$$|R_n(\varphi)| \leq \frac{\sqrt{3}}{216} \int_n^{\infty} \left|\varphi'''(x)\right| \mathrm{d}x = -\frac{\sqrt{3}}{216} \varphi''(n) = \frac{\sqrt{3}}{216} \left|\varphi''(n)\right|.$$

□

3. Applications

The first application is the famous Stirling formula.

Theorem 5. *The following asymptotic evaluation holds*

$$\ln n! = \left(n + \frac{1}{2}\right) \ln n - n + \ln \sqrt{2\pi} + \frac{1}{12n} + R_n;\ |R_n| \leq \frac{\sqrt{3}}{216n^2},\ \forall\, n \in \mathbb{N}.$$

In particular, $n! \sim \sqrt{2\pi} \cdot \frac{n^n}{e^n} \cdot \sqrt{n}$, *for large values of* n.

Proof. Let $\varphi : (0, \infty) \to \mathbb{R}$, $\varphi(x) = \ln x$. Then $\varphi'(x) = \frac{1}{x}$, $\varphi''(x) = -\frac{1}{x^2}$, $\varphi'''(x) = \frac{2}{x^3}$. From Corollary 4 there exists $B \in \mathbb{R}$ such that

$$\ln 1 + \ln 2 + \cdots + \ln n = \left(n + \frac{1}{2}\right) \ln n - n + B + \frac{1}{12n} + R_n, \tag{4}$$

$$|R_n| \leq \frac{\sqrt{3}}{216n^2},\ \forall\, n \in \mathbb{N}.$$

We prove that $B = \ln \sqrt{2\pi}$ which ends the proof. Let us denote $\beta_n = \frac{1}{12n} + R_n$ and note that $\beta_n \to 0$. Then from (4) we have

$$\ln n! = \left(n + \frac{1}{2}\right) \ln n - n + B + \beta_n, \forall\, n \in \mathbb{N}. \tag{5}$$

In the sequel we follow K. Knopp, see [3, page 528]. Let $n \in \mathbb{N}$. From (5) we deduce

$$\ln\left[(2n+1)!\right] = \left(2n + \frac{3}{2}\right) \ln(2n+1) - 2n - 1 + B + \beta_{2n+1}. \tag{6}$$

Also by (5)

$$\begin{aligned} 2\ln(2 \cdot 4 \cdot \cdots \cdot (2n)) &= 2\left(n \ln 2 + \ln n!\right) \\ &= 2n \ln 2 + (2n+1) \ln n - 2n + 2B + 2\beta_n. \end{aligned} \tag{7}$$

By subtracting (6) from (7) we obtain

$$\ln \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot \cdots \cdot (2n)} + \ln(2n+1) = \ln\left[(2n+1)!\right] - 2\ln(2 \cdot 4 \cdot \cdots \cdot (2n)) =$$

$$= \left(2n + \frac{3}{2}\right) \ln (2n+1) - 2n \ln 2 - (2n+1) \ln n - 1 - B + \beta_{2n+1} - 2\beta_n,$$

that is,

$$\begin{aligned} & \ln \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot \cdots \cdot (2n)} \cdot \sqrt{2n+1} = \\ = \; & \ln \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot \cdots \cdot (2n)} + \frac{1}{2} \ln (2n+1) = \\ = \; & (2n+1) \ln (2n+1) - 2n \ln 2 - (2n+1) \ln n - 1 - B + \beta_{2n+1} - 2\beta_n = \\ = \; & \left(2n \ln \frac{2n+1}{2n} - 1\right) + \ln \frac{2n+1}{n} - B + \beta_{2n+1} - 2\beta_n. \end{aligned} \tag{8}$$

Now, from the Wallis formula

$$\lim_{n\to\infty} \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \cdot \sqrt{2n+1} = \sqrt{\frac{2}{\pi}},$$

see for example [6], $\lim\limits_{n\to\infty} \beta_n = 0$, $\lim\limits_{n\to\infty} 2n \ln \frac{2n+1}{2n} = 1$ and (8) we deduce $\ln \sqrt{\frac{2}{\pi}} = \ln 2 - B$, $B = \ln \sqrt{2\pi}$. □

The second application is related to the Stieltjes constants.

Theorem 6. *(i) Let $\alpha \in \mathbb{R} \setminus \{-1\}$. Then there exists*

$$\gamma_\alpha := \lim_{n\to\infty} \left(\frac{\ln^\alpha 1}{1} + \frac{\ln^\alpha 2}{2} + \cdots + \frac{\ln^\alpha n}{n} - \frac{\ln^{\alpha+1} n}{\alpha+1} \right) \in \mathbb{R}$$

called the constant α of Stieltjes, and the following asymptotic evaluation holds

$$\begin{aligned} \frac{\ln^\alpha 1}{1} + \frac{\ln^\alpha 2}{2} + \cdots + \frac{\ln^\alpha n}{n} &= \frac{\ln^{\alpha+1} n}{\alpha+1} + \gamma_\alpha + \frac{\ln^\alpha n}{2n} - \frac{(\ln n)^\alpha}{12n^2} + \\ &+ \frac{\alpha (\ln n)^{\alpha-1}}{12n^2} + R_n(\alpha), \\ |R_n(\alpha)| &\leq \frac{C_\alpha (\ln n)^\alpha}{n^3}, \ \forall\, n \geq x_\alpha. \end{aligned}$$

(ii) There exists $\gamma_{-1} := \lim\limits_{n\to\infty} \left(\frac{1}{2\ln 2} + \cdots + \frac{1}{n \ln n} - \ln(\ln n) \right) \in \mathbb{R}$, called the constant -1 of Stieltjes, and the following asymptotic evaluation holds

$$\begin{aligned} \frac{1}{2\ln 2} + \cdots + \frac{1}{n\ln n} &= \ln(\ln n) + \gamma_{-1} + \frac{1}{2n \ln n} - \frac{1}{12n^2 \ln n} - \\ &- \frac{1}{12n^2 \ln^2 n} + R_n; \\ |R_n| &\leq \frac{7\sqrt{3}}{216 n^3 \ln n}, \ \forall\, n \in \mathbb{N}. \end{aligned}$$

Proof. (i) Let $\varphi_\alpha : (0,\infty) \to \mathbb{R}$, $\varphi_\alpha(x) = \frac{\ln^\alpha x}{x}$. By calculations we have

$$\begin{aligned}
\varphi'_\alpha(x) &= -\frac{(\ln x)^{\alpha-1}(\ln x - \alpha)}{x^2},\\
\varphi''_\alpha(x) &= \frac{(\ln x)^{\alpha-2}\left(2(\ln x)^2 - 3\alpha\ln x + \alpha(\alpha-1)\right)}{x^3},\\
\varphi'''_\alpha(x) &= -\frac{(\ln x)^{\alpha-3}\left[6\ln^3 x - 11\alpha\ln^2 x + 6\alpha(\alpha-1)\ln x - \alpha(\alpha-1)(\alpha-2)\right]}{x^4}.
\end{aligned}$$

Since

$$\lim_{x\to\infty}\left[6\ln^3 x - 11\alpha\ln^2 x + 6\alpha(\alpha-1)\ln x - \alpha(\alpha-1)(\alpha-2)\right] = \infty$$

there exists $x_\alpha \geq 2$ such that $\varphi'''_\alpha(x) < 0$, $\forall\, x \geq x_\alpha$.

Also $\int_1^n \varphi_\alpha(x)\,\mathrm{d}x = \frac{\ln^{\alpha+1} n}{\alpha+1}$. From Corollary 4 there exists

$$\gamma_\alpha := \lim_{n\to\infty}\left(\frac{\ln^\alpha 1}{1} + \frac{\ln^\alpha 2}{2} + \cdots + \frac{\ln^\alpha n}{n} - \frac{\ln^{\alpha+1} n}{\alpha+1}\right) \in \mathbb{R}$$

and moreover, the following asymptotic evaluation holds

$$\begin{aligned}
\frac{\ln^\alpha 1}{1} + \frac{\ln^\alpha 2}{2} + \cdots + \frac{\ln^\alpha n}{n} &= \frac{\ln^{\alpha+1} n}{\alpha+1} + \gamma_\alpha + \frac{\ln^\alpha n}{2n} - \frac{(\ln n)^\alpha}{12n^2} +\\
&+ \frac{\alpha(\ln n)^{\alpha-1}}{12n^2} + R_n,\\
|R_n| &\leq \tfrac{\sqrt{3}}{216}\left|\varphi''_\alpha(n)\right| \leq \tfrac{C_\alpha(\ln n)^\alpha}{n^3},\ \forall\, n \geq x_\alpha.
\end{aligned}$$

(ii) Let $\varphi : (1,\infty) \to \mathbb{R}$, $\varphi(x) = \frac{1}{x\ln x}$. We have

$$\int_2^x \varphi(t)\,\mathrm{d}t = \ln(\ln x) - \ln(\ln 2),$$

$$\begin{aligned}
\varphi'(x) &= -\frac{\ln x + 1}{(x\ln x)^2},\ \varphi''(x) = \frac{2\ln^2 x + 3\ln x + 2}{(x\ln x)^3},\\
\varphi'''(x) &= -\frac{6(\ln x)^3 + 11(\ln x)^2 + 12\ln x + 6}{(x\ln x)^4}.
\end{aligned}$$

From Corollary 4 there exists

$$\gamma_{-1} := \lim_{n\to\infty}\left(\frac{1}{2\ln 2} + \cdots + \frac{1}{n\ln n} - \ln(\ln n)\right) \in \mathbb{R}$$

and moreover, the following asymptotic evaluation holds

$$\begin{aligned}
\frac{1}{2\ln 2} + \cdots + \frac{1}{n\ln n} &= \ln(\ln n) + \gamma_{-1} + \frac{1}{2n\ln n} - \frac{1}{12}\cdot\frac{\ln n + 1}{(n\ln n)^2} + R_n\\
|R_n| &\leq \frac{\sqrt{3}}{216}\left|\varphi''(n)\right| \leq \frac{7\sqrt{3}}{216 n^3\ln n},\ \forall\, n \in \mathbb{N}.
\end{aligned}$$

$\square$

Let us note that γ_0 is the Euler constant. The next application is related to the so called Glaisher-Kinkelin constant.

Theorem 7. *(i) There exists a real number $L \in \mathbb{R}$ such that the following asymptotic evaluation holds*

$$\sum_{k=1}^{n} k \ln k = \frac{1}{2}\left(n^2 + n + \frac{1}{6}\right) \ln n - \frac{n^2}{4} + L + R_n,\ \ |R_n| \le \frac{\sqrt{3}}{216n},\ \forall\, n \in \mathbb{N}.$$

(ii) There exists

$$A := \lim_{n\to\infty} \frac{1^1 2^2 \cdots n^n}{n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}}} \in (0,\infty)$$

called the Glaisher-Kinkelin constant, and the following asymptotic evaluation holds

$$\sum_{k=1}^{n} k \ln k = \frac{1}{2}\left(n^2 + n + \frac{1}{6}\right) \ln n - \frac{n^2}{4} + \ln A + R_n,\ \ |R_n| \le \frac{\sqrt{3}}{216n}, \forall\, n \in \mathbb{N}.$$

Proof. Let $\varphi : (0,\infty) \to \mathbb{R}$, $\varphi(x) = x \ln x$. We have $\varphi'(x) = \ln x + 1$, $\varphi''(x) = \frac{1}{x}$, $\varphi'''(x) = -\frac{1}{x^2}$. From Corollary 4 there exists $D \in \mathbb{R}$ such that

$$\begin{aligned} \sum_{k=1}^{n} k \ln k &= \int_1^n t \ln t \mathrm{d}t + \frac{n \ln n}{2} + \frac{\ln n}{12} + \frac{1}{12} + D + R_n, \\ |R_n| &\le \frac{\sqrt{3}}{216} \left|\varphi''(n)\right| = \frac{\sqrt{3}}{216n},\ \forall\, n \in \mathbb{N}. \end{aligned}$$

Since $\int_1^n t \ln t \mathrm{d}t = \frac{n^2 \ln n}{2} - \frac{n^2}{4} + \frac{1}{4}$ we deduce that there exists $L = D + \frac{1}{3} \in \mathbb{R}$ such that

$$\sum_{k=1}^{n} k \ln k = \frac{1}{2}\left(n^2 + n + \frac{1}{6}\right) \ln n - \frac{n^2}{4} + L + R_n, |R_n| \le \frac{\sqrt{3}}{216n},\ \forall\, n \in \mathbb{N}.$$

This means that $\lim\limits_{n\to\infty} \frac{1^1 2^2 \cdots n^n}{n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}}} = e^L \lim\limits_{n\to\infty} e^{R_n} = e^L = A \in (0,\infty)$, that is, $L = \ln A$ which ends the proof of theorem. □

We need the next evaluation in which appear the Glaisher-Kinkelin constant. A less precise evaluation can be found in [4, Problem 2.14(b), page 21].

Proposition 8. *The following asymptotic evaluation holds*

$$\begin{aligned} \sum_{k=1}^{n} \ln k! &= \left(\frac{n^2}{2} + n + \frac{5}{12}\right) \ln n - \frac{3n^2}{4} + n\left(\ln \sqrt{2\pi} - 1\right) + \ln \sqrt{2\pi} + \frac{1}{12} \\ &- \ln A + R_n, \\ |R_n| &\le \frac{\sqrt{3}+6}{72n},\ \forall\, n \in \mathbb{N}, \end{aligned}$$

where A is the Glaisher-Kinkelin constant.

Proof. Let us note the following equality

$$\begin{aligned}\sum_{k=1}^{n}(a_1+\cdots+a_k) &= a_1+(a_1+a_2)+(a_1+a_2+a_3)+\cdots+(a_1+\cdots+a_n)=\\ &= na_1+(n-1)\,a_2+(n-2)\,a_3+\cdots+a_n=\\ &= \sum_{k=1}^{n}(n+1-k)\,a_k=(n+1)\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}ka_k\end{aligned}$$

which gives us

$$\sum_{k=1}^{n}\ln k! = (n+1)\ln n! - \sum_{k=1}^{n}k\ln k. \tag{9}$$

Now by Stirling's formula (Theorem 5)

$$\ln n! = \left(n+\frac{1}{2}\right)\ln n - n + \ln\sqrt{2\pi} + \frac{1}{12n} + R_n^1, \quad \left|R_n^1\right| \le \frac{\sqrt{3}}{216n^2}, \ \forall\, n\in\mathbb{N} \tag{10}$$

and by Glaisher-Kinkelin evaluation (Theorem 7)

$$\sum_{k=1}^{n}k\ln k = \frac{1}{2}\left(n^2+n+\frac{1}{6}\right)\ln n - \frac{n^2}{4} + \ln A + R_n^2, \tag{11}$$

$$\left|R_n^2\right| \le \frac{\sqrt{3}}{216n}, \ \forall n\in\mathbb{N}.$$

From (9), (10) and (11) we get that

$$\sum_{k=1}^{n}\ln k! = \left(\frac{n^2}{2}+n+\frac{5}{12}\right)\ln n - \frac{3n^2}{4} + n\left(\ln\sqrt{2\pi}-1\right) + \ln\sqrt{2\pi} + \frac{1}{12} - \ln A + R_n$$

where $R_n = (n+1)\,R_n^1 - R_n^2 + \frac{1}{12n}$. Now we note that

$$\begin{aligned}|R_n| \le (n+1)\left|R_n^1\right| + \left|R_n^2\right| + \frac{1}{12n} \quad &\le \frac{\sqrt{3}\,(n+1)}{216n^2} + \frac{\sqrt{3}}{216n} + \frac{1}{12n} \le\\ &\le \frac{2n\sqrt{3}}{216n^2} + \frac{\sqrt{3}+18}{216n} = \frac{\sqrt{3}+6}{72n}.\end{aligned}$$

□

In our last application we give a proof for an exercise in N. Bourbaki's book regarding the asymptotic evaluation for the Vandermonde determinant associated to $(1,2,\ldots,n)$ (see [2, Exercise 6, page 328]).

Proposition 9. *For every natural number $n \geq 2$ we denote*

$$V_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n \\ 1^2 & 2^2 & 3^2 & \cdots & n^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1^{n-1} & 2^{n-1} & 3^{n-1} & \cdots & n^{n-1} \end{vmatrix}$$

the Vandermonde determinant corresponding to $(1, 2, \ldots, n)$. Then, the following asymptotic evaluation holds

$$\begin{aligned} \ln V_n &= \frac{n^2}{2} \ln n - \frac{3n^2}{4} + n \ln \sqrt{2\pi} - \frac{1}{12} \ln n + \frac{1}{12} - \ln A + R_n, \\ |R_n| &\leq \frac{\sqrt{3}}{108n}, \ \forall\, n \in \mathbb{N}, \end{aligned}$$

where A is the Glaisher-Kinkelin constant. In particular, $V_n \sim \frac{\sqrt[12]{e}}{A} \cdot n^{\frac{n^2}{2}} \cdot e^{-\frac{3n^2}{4}} \cdot (2\pi)^{\frac{n}{2}} \cdot n^{-\frac{1}{12}}$, for large values of n.

Proof. As is it well-known we have

$$\begin{aligned} V_n &= \prod_{1 \leq i < j \leq n} (j - i) = \prod_{j=2}^{n} (j-1) \prod_{j=3}^{n} (j-2) \cdots \prod_{j=n}^{n} (j - (n-1)) = \\ &= (n-1)!\,(n-2)! \cdots 2! \cdot 1! \end{aligned}$$

Then, see the equality shown in the proof of Proposition 8, we have

$$\ln V_n = \sum_{k=1}^{n-1} \ln k! = \sum_{k=1}^{n} \ln k! - \ln n! = n \ln n! - \sum_{k=1}^{n} k \ln k.$$

Now, by Stirling evaluation (Theorem 5) and Glaisher-Kinkelin evaluation (Theorem 7) we have

$$\ln n! = \left(n + \frac{1}{2}\right) \ln n - n + \ln \sqrt{2\pi} + \frac{1}{12n} + R_n^1, \ \left|R_n^1\right| \leq \frac{\sqrt{3}}{216n^2}, \ \forall\, n \in \mathbb{N}$$

$$\sum_{k=1}^{n} k \ln k = \frac{1}{2}\left(n^2 + n + \frac{1}{6}\right) \ln n - \frac{n^2}{4} + \ln A + R_n^2, \ \left|R_n^2\right| \leq \frac{\sqrt{3}}{216n}, \ \forall n \in \mathbb{R}.$$

We deduce

$$\begin{aligned} \ln V_n &= n\left(n + \frac{1}{2}\right) \ln n - n^2 + n \ln \sqrt{2\pi} + \frac{1}{12} - \frac{1}{2}\left(n^2 + n + \frac{1}{6}\right) \ln n + \\ &+ \frac{1}{4} n^2 - \ln A + R_n \\ &= \frac{n^2}{2} \ln n - \frac{3n^2}{4} + n \ln \sqrt{2\pi} - \frac{1}{12} \ln n + \frac{1}{12} - \ln A + R_n, \end{aligned}$$

where $R_n = nR_n^1 - R_n^2$ and $|R_n| \leq n \left|R_n^1\right| + \left|R_n^2\right| \leq \frac{\sqrt{3}}{108n}, \forall\, n \in \mathbb{N}$. $\square$

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On some perturbed Bullen inequalities

Vlad Ciobotariu-Boer[1)]

Abstract. In this paper we establish some new general integral inequalities by perturbed Bullen type for twice differentiable functions. Then we apply these inequalities to obtain some inequalities for special means of real numbers and some new general quadrature rules of trapezoidal type.

Keywords: convex function, integral inequalities, special means
MSC: Primary 26D10; Secondary 26D15, 41A55

Introduction

In [2], Minculete, Dicu and Raţiu gave the following perturbed Bullen inequality:

$$\left|\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(x)\mathrm{d}x-\frac{(b-a)[f'(b)-f'(a)]}{24}\right| \leq \frac{(M-m)(b-a)^2}{64}, \tag{1}$$

for all twice differentiable functions $f:[a,b]\to\mathbb{R}$ with the property there exist real constants m and M such that

$$m \leq f''(x) \leq M \text{ for all } x\in[a,b].$$

In this article, we first find an integral identity for twice differentiable functions. Then we use this identity to obtain new integral inequalities similar to and improving inequality (1). Finally, we give some applications for special means of real numbers and some numerical quadrature rules.

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1. Main results

In order to prove our main results, we need the following lemmas.

Lemma 1. *Let $f : [0,1] \to \mathbb{R}$ be a function defined by*

$$f(t) = |1 - nt + nt^2|, \ \ with\ n \in (0, +\infty).$$

Then we have

$$\int_0^1 f(t)\mathrm{d}t = v(n),$$

where

$$v(n) = \begin{cases} \dfrac{6-n}{6}, & if\ \ n \in (0,4], \\ \dfrac{6n - n^2 + 2(n-4)\sqrt{n^2-4n}}{6n}, & if\ \ n \in (4, +\infty). \end{cases}$$

Proof. See [1]. □

Lemma 2. *Let $g : (0, +\infty) \to \mathbb{R}$ be a mapping defined by*

$$g(x) = \begin{cases} \dfrac{6-x}{6x}, & for\ x \in (0,4], \\ \dfrac{6x - x^2 + 2(x-4)\sqrt{x^2-4x}}{6x^2}, & for\ x \in (4, +\infty). \end{cases}$$

Then we have

$$\min_{x \in (0,+\infty)} g(x) = g\left(\frac{16}{3}\right) = \frac{1}{16}.$$

Proof. See [1]. □

For the rest of the paper, I will denote a real interval and I° its interior. For p a positive real number and $a < b$, let $L_p(a,b)$ denote the class of real functions defined on $[a,b]$ for which the pth power of the absolute value is Lebesgue integrable. Moreover, $L_\infty(a,b)$ denotes the class of real functions defined on $[a,b]$ which are essentially bounded.

Lemma 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° with $f'' \in L_1[a,b]$, where $a, b \in I^\circ$, $a < b$. Then*

$$\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{b-a}{8n}[f'(b) - f'(a)] =$$
$$= \frac{(b-a)^2}{16n}\int_0^1 (1 - nt + nt^2)\left[f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) + f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right]\mathrm{d}t$$

holds for all $n \in (0, +\infty)$.

Proof. By integration by parts, we get

$$\int_0^1 (1-nt+nt^2)f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\mathrm{d}t =$$
$$= \frac{2}{b-a}\left[(1-nt+nt^2)f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\Big|_0^1 + \right.$$
$$\left. +n\int_0^1 (1-2t)f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\mathrm{d}t\right] =$$
$$= \frac{2}{b-a}\left[f'(b)-f'\left(\frac{a+b}{2}\right)\right] + \frac{4n}{(b-a)^2}\left[(1-2t)f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\Big|_0^1 + \right.$$
$$\left. +2\int_0^1 f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\mathrm{d}t\right]$$

and then, using the change of variables

$$u = \frac{1-t}{2}a+\frac{1+t}{2}b \quad \text{for } t\in[0,1],$$

we obtain

$$\int_0^1 (1-nt+nt^2)f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\mathrm{d}t$$
$$= \frac{2}{b-a}\left[f'(b)-f'\left(\frac{a+b}{2}\right)\right] - \frac{4n}{(b-a)^2}\left[f(b)+f\left(\frac{a+b}{2}\right)\right]$$
$$+ \frac{16n}{(b-a)^3}\int_{\frac{a+b}{2}}^b f(u)\mathrm{d}u. \tag{2}$$

Then, multiplying both sides of (2) by $\frac{(b-a)^2}{16n}$, we get

$$\frac{1}{b-a}\int_{\frac{a+b}{2}}^b f(u)\mathrm{d}u - \frac{1}{4}\left[f(b)+f\left(\frac{a+b}{2}\right)\right] + \frac{b-a}{8n}\left[f'(b)-f'\left(\frac{a+b}{2}\right)\right] =$$
$$= \frac{(b-a)^2}{16n}\int_0^1 (1-nt+nt^2)f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\mathrm{d}t. \tag{3}$$

Similarly, we find

$$\frac{1}{b-a}\int_a^{\frac{a+b}{2}} f(u)\mathrm{d}u - \frac{1}{4}\left[f(a)+f\left(\frac{a+b}{2}\right)\right] + \frac{b-a}{8n}\left[f'\left(\frac{a+b}{2}\right)-f'(a)\right] =$$
$$= \frac{(b-a)^2}{16n}\int_0^1 (1-nt+nt^2)f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\mathrm{d}t. \tag{4}$$

Summing the above equalities (3) and (4), we get the desired equality. □

Now, by using Lemma 3, we prove our main theorem. Recall that for $g\in L_\infty[a,b]$ one puts $\|g\|_\infty = \sup\limits_{u\in(a,b)}|g(u)| < \infty$.

Theorem 4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_\infty[a,b]$, where $a, b \in I$, $a < b$. Then the following inequality*

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{b-a}{8n}[f'(b)-f'(a)]\right| \leq \frac{1}{8}g(n)(b-a)^2\|f''\|_\infty$$

holds for all $n \in (0, +\infty)$.

Proof. From Lemma 3, using the properties of modulus, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{b-a}{8n}[f'(b)-f'(a)]\right| \leq \frac{\|f''\|_\infty}{8n}(b-a)^2\int_0^1 |1-nt+nt^2|\mathrm{d}t.$$

We obtain the desired inequality from Lemma 1 and the above inequality. □

Corollary 5. *Under the assumptions of Theorem 4, for $f'(a) = f'(b)$ we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{1}{8}g(n)(b-a)^2\|f''\|_\infty$$

for all $n \in (0, +\infty)$.

Remark 6. By using Lemma 2, we deduce that the equation $g(n) = \frac{1}{12}$ has the solutions $n = 4$ and $n = \frac{64}{7}$ on the interval $(0, +\infty)$. We also obtain that $g(n) \in \left[\frac{1}{16}, \frac{1}{12}\right)$ if $n \in \left(4, \frac{64}{7}\right)$.

For $n = \frac{16}{3}$, we find the best inequality of this type:

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{3(b-a)}{128}[f'(b)-f'(a)]\right| \leq \leq \frac{1}{128}(b-a)^2\|f''\|_\infty. \tag{5}$$

The inequality (5) improves inequality (1) if $0 \leq 2m \leq M$.

For $n = 4$ we obtain the inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{b-a}{32}[f'(b)-f'(a)]\right| \leq \leq \frac{(b-a)^2}{96}\|f''\|_\infty, \tag{6}$$

and for $n = \frac{64}{7}$ we obtain a new inequality as good as (6):

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{7(b-a)}{512}[f'(b)-f'(a)]\right| \leq$$

$$\leq \frac{(b-a)^2}{96}\|f''\|_\infty. \tag{7}$$

For $n \in \left(4, \frac{64}{7}\right) \cap \mathbb{N}$, we find some important integral inequalities of perturbed Bullen type which are better than (6):

1) if $n = 5$, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{b-a}{40}[f'(b)-f'(a)]\right| \leq$$
$$\leq \frac{5+2\sqrt{5}}{1200}(b-a)^2\|f''\|_\infty < \frac{(b-a)^2}{96}\|f''\|_\infty;$$

2) if $n = 6$, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{b-a}{48}[f'(b)-f'(a)]\right| \leq$$
$$\leq \frac{\sqrt{3}}{216}(b-a)^2\|f''\|_\infty < \frac{(b-a)^2}{96}\|f''\|_\infty;$$

3) if $n = 7$, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{b-a}{56}[f'(b)-f'(a)]\right| \leq$$
$$\leq \frac{6\sqrt{21}-7}{2352}(b-a)^2\|f''\|_\infty < \frac{(b-a)^2}{96}\|f''\|_\infty;$$

4) if $n = 8$, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{b-a}{64}[f'(b)-f'(a)]\right| \leq$$
$$\leq \frac{2\sqrt{2}-1}{192}(b-a)^2\|f''\|_\infty < \frac{(b-a)^2}{96}\|f''\|_\infty;$$

5) if $n = 9$, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{b-a}{72}[f'(b)-f'(a)]\right| \leq$$
$$\leq \frac{10\sqrt{5}-9}{1296}(b-a)^2\|f''\|_\infty < \frac{(b-a)^2}{96}\|f''\|_\infty.$$

Theorem 7. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a,b]$, where $a, b \in I$, $a < b$.*

If $|f''|$ is convex on $[a,b]$, then the following inequality

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{b-a}{8n}[f'(b)-f'(a)]\right| \leq$$
$$\leq \frac{1}{16}g(n)(b-a)^2(|f''(a)| + |f''(b)|) \tag{8}$$

holds for all $n \in (0, +\infty)$.

Proof. From Lemma 3, using the properties of modulus, by the convexity of $|f''|$, we obtain

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{b-a}{8n}[f'(b)-f'(a)]\right| \leq$$
$$\leq \frac{(b-a)^2}{16n}(|f''(a)|+|f''(b)|)\int_0^1 |1-nt+nt^2|\mathrm{d}t. \tag{9}$$

Applying Lemma 2 in the inequality (9), we find (8). □

Corollary 8. *Under the assumptions of Theorem 7, for $f'(a) = f'(b)$ we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \leq$$
$$\leq \frac{1}{16}g(n)(b-a)^2(|f''(a)|+|f''(b)|)$$

for all $n \in (0, +\infty)$.

Remark 9. *By using similar considerations as in Remark 6, we see that the inequality* (8) *provides interesting inequalities of perturbed Bullen type.*

For $n = \frac{16}{3}$ we get the best inequality of this type:

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{3(b-a)}{128}[f'(b)-f'(a)]\right| \leq$$
$$\leq \frac{1}{256}(b-a)^2(|f''(a)|+|f''(b)|). \tag{10}$$

This inequality improves the inequality (1) if $0 \leq 2m \leq M$.

For $n = 4$ we obtain the inequality:

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{b-a}{32}[f'(b)-f'(a)]\right| \leq$$
$$\leq \frac{(b-a)^2}{192}(|f''(a)|+|f''(b)|) \tag{11}$$

and for $n = \frac{64}{7}$ we obtain a new inequality as good as (11):

$$\left|\frac{1}{b-a}\int_a^b f(u)\ du - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{7(b-a)}{512}[f'(b)-f'(a)]\right| \leq$$
$$\leq \frac{(b-a)^2}{192}(|f''(a)|+|f''(b)|).$$

For $n \in \left(4, \frac{64}{7}\right) \cap \mathbb{N}$, we find some important integral inequalities of this type which are better than (11):

1) if $n = 5$, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{b-a}{40}[f'(b)-f'(a)]\right| \leq$$

$$\leq \frac{5+2\sqrt{5}}{2400}(b-a)^2(|f''(a)|+|f''(b)|) < \frac{(b-a)^2}{192}(|f''(a)|+|f''(b)|);$$

2) if $n=6$, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]+\frac{b-a}{48}[f'(b)-f'(a)]\right| \leq$$

$$\leq \frac{\sqrt{3}}{432}(b-a)^2(|f''(a)|+|f''(b)|) < \frac{(b-a)^2}{192}(|f''(a)|+|f''(b)|);$$

3) if $n=7$, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]+\frac{b-a}{56}[f'(b)-f'(a)]\right| \leq$$

$$\leq \frac{6\sqrt{21}-7}{4704}(b-a)^2(|f''(a)|+|f''(b)|) < \frac{(b-a)^2}{192}(|f''(a)|+|f''(b)|);$$

4) if $n=8$, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]+\frac{b-a}{64}[f'(b)-f'(a)]\right| \leq$$

$$\leq \frac{2\sqrt{2}-1}{384}(b-a)^2(|f''(a)|+|f''(b)|) < \frac{(b-a)^2}{192}(|f''(a)|+|f''(b)|);$$

5) if $n=9$, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u - \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]+\frac{b-a}{72}[f'(b)-f'(a)]\right| \leq$$

$$\leq \frac{10\sqrt{5}-9}{2592}(b-a)^2(|f''(a)|+|f''(b)|) < \frac{(b-a)^2}{192}(|f''(a)|+|f''(b)|).$$

2. Applications for special means

Recall the following means:

(a) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2},\ \ a,b \geq 0;$$

(b) The geometric mean

$$G = G(a,b) := \sqrt{ab},\ \ a,b \geq 0;$$

(c) The harmonic mean

$$H = H(a,b) := \frac{2ab}{a+b},\ \ a,b > 0;$$

(d) The logarithmic mean

$$L = L(a,b) := \begin{cases} a & \text{if} \quad a=b>0, \\ \dfrac{b-a}{\ln b - \ln a} & \text{if} \quad a \neq b,\ a,b>0; \end{cases}$$

(e) The identric mean

$$I = I(a,b) := \begin{cases} a & \text{if} \quad a = b > 0, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)} & \text{if} \quad a \neq b, \ a, b > 0; \end{cases}$$

(f) The p-logarithmic mean, defined for $p \in \mathbb{R} \setminus \{-1, 0\}$ by

$$L_p = L_p(a,b) := \begin{cases} a & \text{if} \quad a = b > 0, \\ \dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} & \text{if} \quad a \neq b, \ a, b > 0, \end{cases}$$

together with $L_{-1} := L$ and $L_0 := I$.

It is known that L_p is monotonically nondecreasing in $p \in \mathbb{R}$. The following simple relationship are also known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

Now, using the results of Section 2, some new inequalities are derived for the above means.

Proposition 10. *Let $p \geq 2$ and $0 < a < b$. Then we have the inequality*

$$\left| L_p(a,b)^p - A(A(a^p, b^p), A(a,b)^p) + \frac{p(p-1)}{8n}(b-a)^2 L_{p-2}(a,b)^{p-2} \right| \leq$$

$$\leq \frac{p(p-1)}{8} g(n)(b-a)^2 A(a^{p-2}, b^{p-2}) \tag{12}$$

for all $n \in (0, +\infty)$.

Proof. The assertion follows from (8) applied for $f(x) = x^p$, $x \in [a,b]$. □

Remark 11. *Letting $n = \frac{16}{3}$ in (12), we obtain the best inequality of this type:*

$$\left| L_p(a,b)^p - A(A(a^p, b^p), A(a,b)^p) + \frac{3p(p-1)}{128}(b-a)^2 L_{p-2}(a,b)^{p-2} \right| \leq$$

$$\leq \frac{p(p-1)}{128}(b-a)^2 A(a^{p-2}, b^{p-2}).$$

Proposition 12. *Let $0 < a < b$. Then we have the inequality*

$$\left| L(a,b)^{-1} - A(H(a,b)^{-1}, A(a,b)^{-1}) + \frac{(b-a)^2 A(a,b)}{4nG(a,b)^4} \right| \leq$$

$$\leq \frac{1}{4} g(n)(b-a)^2 A(a^{-3}, b^{-3}) \tag{13}$$

for all $n \in (0, +\infty)$.

Proof. The assertion follows from (8) applied for $f(x) = \frac{1}{x}$, $x \in [a,b]$. □

Remark 13. *Letting $n = \frac{16}{3}$ in (13) we find the best inequality of this type:*

$$\left| L(a,b)^{-1} - A(H(a,b)^{-1}, A(a,b)^{-1}) + \frac{3(b-a)^2 A(a,b)}{64G(a,b)^4} \right| \leq$$

$$\leq \frac{(b-a)^2}{64} A(a^{-3}, b^{-3}).$$

Proposition 14. *Let $0 < a < b$. Then we have the inequality:*

$$\left| \ln I(a,b) + \ln G(G(a,b), A(a,b)) + \frac{(b-a)^2}{8nG(a,b)^2} \right| \leq$$

$$\leq \frac{1}{8} g(n)(b-a)^2 A(a^{-2}, b^{-2}) \tag{14}$$

for all $n \in (0, +\infty)$.

Proof. The assertion follows from (8) applied for $f(x) = -\ln x$, $x \in [a,b]$. □

Remark 15. *Letting $n = \frac{16}{3}$ in (14) we find the best inequality of this type:*

$$\left| \ln I(a,b) + \ln G(G(a,b), A(a,b)) + \frac{3(b-a)^2}{128G(a,b)^2} \right| \leq$$

$$\leq \frac{(b-a)^2}{128} A(a^{-2}, b^{-2}).$$

3. Applications for composite quadrature formula

Let Δ be a division $a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$ of the interval $[a,b]$. Then the following results hold.

Theorem 16. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_\infty[a,b]$, where $a, b \in I$, $a < b$. Then we have*

$$\int_a^b f(u)\mathrm{d}u = A(f, f', \Delta) + R(f, f', \Delta),$$

where

$$A(f, f', \Delta) := \frac{1}{2} \sum_{i=1}^m \frac{h_i}{2} \left[\frac{f(x_{i-1}) + f(x_i)}{2} + f\left(\frac{x_{i-1} + x_i}{2} \right) \right] -$$

$$- \frac{1}{8n} \sum_{i=1}^m h_i^2 [f'(x_i) - f'(x_{i-1})]$$

and $h_i := x_i - x_{i-1}$.

The remainder $R(f, f', \Delta)$ satisfies the estimation

$$|R(f, f', \Delta)| \leq \frac{1}{8} g(n) \|f''\|_\infty \sum_{i=1}^m h_i^3, \tag{15}$$

for all $n \in (0, +\infty)$ and for all $m \in \mathbb{N}^*$.

Proof. Apply Theorem 4 on the interval $[x_{i-1}, x_i]$, $i = \overline{1,m}$, to get

$$\Big| \int_{x_{i-1}}^{x_i} f(u)\mathrm{d}u - \frac{1}{2} \cdot \frac{h_i}{2} \left[\frac{f(x_{i-1}) + f(x_i)}{2} + f\left(\frac{x_{i-1}+x_i}{2} \right) \right] + \\ + \frac{h_i^2}{8n}[f'(x_i) - f'(x_{i-1})] \Big| \le \frac{1}{8} g(n) h_i^3 \|f''\|_\infty.$$

Summing the above inequalities over i from 1 to m and using the generalized triangle inequality, we get the desired estimation (15). □

Theorem 17. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a,b]$, where $a, b \in I$, $a < b$.*

If $|f''|$ is convex on $[a,b]$, then we have

$$\int_a^b f(u)\mathrm{d}u = A(f, f', \Delta) + R(f, f', \Delta).$$

The remainder $R(f, f', \Delta)$ satisfies the estimation

$$|R(f, f', \Delta)| \le \frac{1}{16} g(n) \sum_{i=1}^m h_i^3 [|f''(x_{i-1})| + |f''(x_i)|] \tag{16}$$

for all $n \in (0, +\infty)$ and for all $m \in \mathbb{N}^*$.

Proof. Applying Theorem 7 on the interval $[x_{i-1}, x_i]$, $i = \overline{1,m}$, we find

$$\Big| \int_{x_{i-1}}^{x_i} f(u)\mathrm{d}u - \frac{h_i}{2} \left[\frac{f(x_{i-1}) + f(x_i)}{2} + f\left(\frac{x_{i-1}+x_i}{2} \right) \right] + \\ + \frac{h_i^2}{8n}[f'(x_i) - f'(x_{i-1})] \Big| \le \frac{1}{16} g(n) h_i^3 (|f''(x_{i-1})| + |f''(x_i)|).$$

Summing the above inequalities over i from 1 to m and using the generalized triangle inequality, we get the desired estimation (16). □

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Traian Lalescu mathematics contest for university students

GABRIEL MINCU[1)], RADU STRUGARIU[2)]

Abstract. This paper deals with the problems proposed at the 2016 edition of the Traian Lalescu national mathematics competition for university students, hosted in Sibiu, between May 19th and May 21st 2016, by the 'Lucian Blaga' University and the 'Nicolae Bălcescu' Land Forces Academy.

Keywords: congruence, eigenvalue, eigenvector, idempotent matrix, integral, limit, packing problems, quaternions, rank of a matrix, series, subgroup, vector

MSC: Primary 97U40; Secondary 97D50.

INTRODUCTION

The ninth edition of the 'Traian Lalescu' national mathematics contest for university students was hosted between May 19th and May 21st 2016, in Sibiu, by the 'Lucian Blaga' University and the 'Nicolae Bălcescu' Land Forces Academy.

The contest saw a participation of 83 contestants representing 13 universities from Bucureşti, Cluj, Constanţa, Craiova, Iaşi, Sibiu and Timişoara, and was organised in four sections:

(1) Section A: theoretical, first and second year of study — for students from the faculties of mathematics and informatics (12 students);
(2) Section B: electrical, first year of study — for students from the faculties of automatics and computer science, physics, electronics, energetics, including departments teaching in foreign languages having these profiles (30 students);
(3) Section C: nonelectrical — for students from the faculties of applied sciences, mechanics, chemistry, transports, metallurgy, constructions, economics (all sections), biology, including departments teaching in foreign languages having these profiles (22 students);
(4) Section D: engineering, second year of study — for students from all faculties teaching engineering (18 students).

We present in the sequel the problems proposed in Sections A and B of the contest, along with their solutions. More details concerning the competition can be found at the address *http://stiinte.ulbsibiu.ro/concurs-traian-lalescu/*.

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Problems and solutions

Section A

Problem 1. Find the integers x for which

$$2017 \mid x^{2016} + x^{2015} + \cdots + x + 1.$$

Gabriel Mincu

The jury considered this problem to be easy. The contestants confirmed this opinion, more than half of them managing to fully solve the problem. The solutions they gave went along the lines of Solution 1 below.

Solution 1. Let us notice that 2017 is a prime number. Let $x \in \mathbb{Z}$ be such that $2017 \mid x^{2016} + x^{2015} + \cdots + x + 1$. Then $2017 \mid x^{2017} - 1$, so

$$x^{2017} \equiv 1 \pmod{2017}. \tag{1}$$

This is obviously impossible if x divides 2017. If $2017 \nmid x$, we derive form Fermat's theorem that $x^{2016} \equiv 1 \pmod{2017}$, so $x^{2017} \equiv x \pmod{2017}$. From this and relation (1) we get $x \equiv 1 \pmod{2017}$, and since all these values actually satisfy the given condition, they form the solution of the problem. $\square$

Solution 2. Let p be a prime number. Then $p \mid \binom{p}{k}$ for all $k \in \{1, 2, \ldots, p-1\}$, so, working in $\mathbb{Z}_p[X]$, $X^p - 1 = (X-1)^p$, whence $X^{p-1} + X^{p-2} + \cdots + X + 1 = (X-1)^{p-1}$. Therefore, the only root of $X^{p-1} + X^{p-2} + \cdots + X + 1$ in $\mathbb{Z}_p$ is 1, so for $x \in \mathbb{Z}$ we have $p \mid x^{p-1} + x^{p-2} + \cdots + x + 1$ if and only if $x \equiv 1 \pmod{p}$.

Taking $p = 2017$, we get that $2017 \mid x^{2016} + x^{2015} + \cdots + x + 1$ if and only if $x \equiv 1 \pmod{2017}$. $\square$

Problem 2. Compute $\displaystyle \lim_{h \searrow 0} \sum_{n=1}^{\infty} \frac{h}{1 + n^2 h^2}$.

Ovidiu Furdui

Although the jury considered this problem to be easy-medium, only two contestants managed to give full solutions. We present the ideas of these solutions below.

Solution 1. For all $n \in \mathbb{N}^*$ and $h > 0$ we have

$$\frac{h}{1 + (n+1)^2 h^2} < \int_n^{n+1} \frac{h}{1 + x^2 h^2}\, \mathrm{d}x < \frac{h}{1 + n^2 h^2},$$

so that

$$\sum_{n=1}^{\infty} \frac{h}{1 + n^2 h^2} - \frac{h}{1 + h^2} < \int_1^{\infty} \frac{h}{1 + x^2 h^2}\, \mathrm{d}x < \sum_{n=1}^{\infty} \frac{h}{1 + n^2 h^2}.$$

Since $\displaystyle\int_1^\infty \frac{h}{1+x^2h^2}\,\mathrm{d}x = \arctan(xh)\Big|_1^\infty = \frac{\pi}{2} - \arctan h$, we get

$$\frac{\pi}{2} - \arctan h \le \sum_{n=1}^{\infty} \frac{h}{1+n^2h^2} \le \frac{\pi}{2} - \arctan h + \frac{h}{1+h^2}.$$

Taking limits for $h \searrow 0$, we obtain $\lim\limits_{h\searrow 0} \sum\limits_{n=1}^{\infty} \frac{h}{1+n^2h^2} = \frac{\pi}{2}$. $\square$

Solution 2. Let $\varepsilon > 0$. Since $\lim\limits_{x\to\infty} \arctan x = \frac{\pi}{2}$, there is $A > 1$ such that

$$\frac{\pi}{2} - \varepsilon < \arctan A < \frac{\pi}{2}, \tag{2}$$

and we have $\lim\limits_{h\searrow 0} \sum\limits_{n=1}^{\infty} \frac{h}{1+n^2h^2} = \lim\limits_{h\searrow 0} \sum\limits_{1\le n\le \frac{A}{h}} \frac{h}{1+n^2h^2} + \lim\limits_{h\searrow 0} \sum\limits_{n>\frac{A}{h}} \frac{h}{1+n^2h^2}$. The first term of the right hand side is a limit of Riemann sums of the function $\frac{1}{1+x^2}$ over the interval $[0, A]$, while the second is nonnegative and, since $\frac{h}{1+n^2h^2} <$ $< \int\limits_{(n-1)h}^{nh} \frac{\mathrm{d}x}{1+x^2}$ for all $n \in \mathbb{N}^*$, is less than $\lim\limits_{h\searrow 0} \left(\int\limits_{h\left[\frac{A}{h}\right]}^{\infty} \frac{\mathrm{d}x}{1+x^2} \right)$. Therefore,

$$\arctan A \le \lim_{h\searrow 0} \sum_{n=1}^{\infty} \frac{h}{1+n^2h^2} \le \arctan A + \lim_{h\searrow 0} \left(\frac{\pi}{2} - \arctan h \left[\frac{A}{h}\right] \right) = \frac{\pi}{2},$$

so, in view of relation (2), we get $\frac{\pi}{2} - \varepsilon \le \lim\limits_{h\searrow 0} \sum\limits_{n=1}^{\infty} \frac{h}{1+n^2h^2} \le \frac{\pi}{2}$.

Since ε may be taken arbitrarily small, $\lim\limits_{h\searrow 0} \sum\limits_{n=1}^{\infty} \frac{h}{1+n^2h^2} = \frac{\pi}{2}$. $\square$

Problem 3. We denote by K the subfield $\{a+bi+cj+dk : a, b, c, d \in \mathbb{Q}\}$ of the field $\mathbb{H}$ of the quaternions. Prove that:
a) The order of any finite subgroup of $(K \setminus \{0\}, \cdot)$ is of the form $2^\alpha \cdot 3^\beta$, $\alpha, \beta \in \mathbb{N}^*$.
b) Any commutative finite subgroup of $(K \setminus \{0\}, \cdot)$ has order 1, 2, 3, 4 or 6.

Victor Alexandru

The jury considered this problem medium-hard. This opinion has been confirmed by the contestants, only two of whom found suitable approaches, but none managed to solve the problem entirely.

Solution. For each $x = a_x + b_x i + c_x j + d_x k \in K$, we will call a_x the scalar part of x, and $V_x = b_x i + c_x j + d_x k$ the vector part of x.

The following computational properties are immediate and will be used throughout the solution:

• If $x, x' \in K$, then $xx' = a_x a_{x'} - b_x b_{x'} - c_x c_{x'} - d_x d_{x'} + (a_x b_{x'} + b_x a_{x'} + c_x d_{x'} - d_x c_{x'})i + (a_x c_{x'} - b_x d_{x'} + c_x a_{x'} + d_x b_{x'})j + (a_x d_{x'} + b_x c_{x'} - c_x b_{x'} + d_x a_{x'})k$.

• Denoting $\overline{x} = a_x - b_x i - c_x j - d_x k$, we have $x\overline{x} = a_x^2 + b_x^2 + c_x^2 + d_x^2 \in \mathbb{Q}$.
• $x^2 = a_x^2 - b_x^2 - c_x^2 - d_x^2 + 2a_x V_x = 2a_x x - x\overline{x}$,
• $x, x' \in K$ commute if and only if V_x and $V_{x'}$ are $\mathbb{Q}$-linearly dependent.

a) Let H be a finite, nontrivial subgroup of the multiplicative group $G = K \setminus \{0\}$ and let p be a prime divisor of the order of H. According to Cauchy's theorem, there is $x \in H$ of order p. Since $x^p = 1 \neq x$, we derive that x, seen as an element of the commutative field $L = \mathbb{Q}(x)$, is a root of the irreducible polynomial $f = X^{p-1} + X^{p-2} + \cdots + X + 1 \in \mathbb{Q}[X]$.

On the other hand, $x^2 - 2a_x x + x\overline{x} = 0$, so x is a root of the polynomial $g = X^2 - 2a_x X + x\overline{x} \in \mathbb{Q}[X]$. Since f is irreducible, we derive that f divides g, so $p = 1 + \deg(f) \leq 1 + \deg(g) = 3$.

Consequently, the order of H is of the form $2^\alpha \cdot 3^\beta$, $\alpha, \beta \in \mathbb{N}$.

b) Let $x \in G$ be an element of finite order n. We then have

$$x^n = (a_x + V_x)^n = \sum_{0 \leq 2k \leq n} \binom{n}{2k} a_x^{n-2k} V_x^{2k} + \left(\sum_{0 \leq 2k+1 \leq n} \binom{n}{2k+1} a_x^{n-2k-1} V_x^{2k} \right) V_x.$$

Putting $T_x = \sqrt{b_x^2 + c_x^2 + d_x^2}$, we have $V_x^2 = -T_x^2$, so the relation above yields

$$\begin{aligned} x^n &= \sum_{0 \leq 2k \leq n} \binom{n}{2k} a_x^{n-2k} (-T_x^2)^k + \left(\sum_{0 \leq 2k+1 \leq n} \binom{n}{2k+1} a_x^{n-2k-1} (-T_x^2)^k \right) V_x = \\ &= \sum_{0 \leq 2k \leq n} \binom{n}{2k} a_x^{n-2k} (iT_x)^{2k} + \left(\sum_{0 \leq 2k+1 \leq n} \binom{n}{2k+1} a_x^{n-2k-1} (iT_x)^{2k} \right) V_x = \\ &= \frac{(a_x + iT_x)^n + (a_x - iT_x)^n}{2} + \frac{(a_x + iT_x)^n - (a_x - iT_x)^n}{2} V_x. \end{aligned}$$

Thus, the condition $x^n = 1$ is equivalent to the system

$$\begin{cases} (a_x + iT_x)^n = (a_x - iT_x)^n \\ (a_x + iT_x)^n + (a_x - iT_x)^n = 2, \end{cases}$$

and therefore to $(a_x + iT_x)^n = (a_x - iT_x)^n = 1$. Since the order of x is n, there is $k \in \{1, 2, \ldots, n-1\}$, coprime to n, such that $a_x = \cos \frac{2k\pi}{n}$ and $T_x = \sin \frac{2k\pi}{n}$. But $a_x \in \mathbb{Q}$, so $a_x \in \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}$.

Let now H be a finite commutative subgroup of G. If $H = \{1\}$, then $|H| = 1$. If $H = \{-1, 1\}$, then $|H| = 2$. If $H \not\subseteq \{-1, 1\}$, let $x \in H \setminus \{-1, 1\}$ and $x' \in H$. Since x and x' commute, there are $\mu, \mu' \in \mathbb{Q}$, $\mu^2 + \mu'^2 \neq 0$, such that $\mu V_x + \mu' V_{x'} = 0$.

(I) If $a_x = a_{x'} = 0$, then $T_x^2 = T_{x'}^2 = 1$, so $V_x^2 = V_{x'}^2 = -1$. We get $\mu^2 = \mu'^2$, so $x' = \pm x$. But $x^2 = V_x^2 = -1$, so $x^3 = -x$. Thus, $-x \in \langle x \rangle$, so $H = \langle x \rangle$ and $|H| = 4$.

(II) If $a_x = 0$ and $a_{x'} = \pm\frac{1}{2}$, then $T_x^2 = 1$, $T_{x'}^2 = \frac{3}{4}$, $V_x^2 = -1$ and $V_{x'}^2 = -\frac{3}{4}$. We obtain $\mu'^2 = \dfrac{4}{3}\mu^2$, contradiction.

(III) If $a_x = 0$ and $a_{x'} = \pm 1$, then $T_x^2 = 1$, $T_{x'}^2 = 0$, $V_x^2 = -1$, $V_{x'}^2 = 0$, so $x' = \pm 1$. Since $x^2 = -1$ and $x^4 = 1$, we get $x' \in \langle x \rangle$. As in Case (I), we obtain $H = \langle x \rangle$ and $|H| = 4$.

(IV) If $a_x = \pm\frac{1}{2}$ and $a_{x'} = \pm\frac{1}{2}$, then $T_x^2 = T_{x'}^2 = \frac{3}{4}$, $V_x^2 = V_{x'}^2 = -\frac{3}{4}$ and we get $\mu^2 = \mu'^2$ and $x' \in \{x, -x, \overline{x}, -\overline{x}\}$. Since $x = \pm\frac{1}{2} + V_x$, we obtain $x^2 = -\frac{1}{2} \pm V_x = \mp\overline{x}$ and $x^3 = \pm 1$. If $x^3 = -1$, then $x^4 = -x$, $x^5 = -x^2$, $x^6 = 1$. We conclude that $H = \langle x \rangle$ or $H = \langle x, -x \rangle$ and $|H| \in \{3, 6\}$.

(V) If $a_x = \pm\frac{1}{2}$ and $a_{x'} = \pm 1$, then $T_x^2 = \frac{3}{4}$, $T_{x'}^2 = 0$, $V_x^2 = -\frac{3}{4}$, $V_{x'}^2 = 0$. We get $x' = \pm 1$ and, reasoning as in Case (IV), $|H| \in \{3, 6\}$.

(VI) If $a_x = \pm 1$ and $a_{x'} = \pm 1$, then $H \subseteq \{-1, 1\}$, contradiction.

The other cases reduce to the ones above by swapping the roles of a_x and $a_{x'}$. □

Problem 4. Let $(D_n)_{n\in\mathbb{N}^*}$ be a sequence of discs such that for each $n \in \mathbb{N}^*$ the radius of D_n is $\frac{1}{n}$. Find the minimum side of a square inside which one may arrange without superpositions all the discs D_n, $n \in \mathbb{N}^*$.

Gabriel Mincu

The jury considered this problem to be hard. Only one of the contestants approached the problem, but his idea was not effective enough to yield a complete solution.

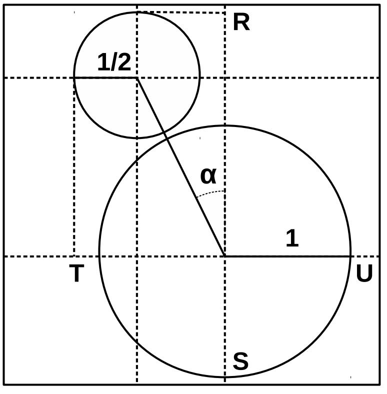

Figure 1

Solution. Let us first notice that if D_1 and D_2 lie inside a square of side L, and the line of their centers makes an angle α with one of the sides of the square, then, according to Figure 1,

$L \geq RS \geq \dfrac{3}{2}(1 + \cos\alpha)$ and

$L \geq TU \geq \dfrac{3}{2}(1 + \sin\alpha)$, so

$$L \geq \frac{3}{2}(1 + \max\{\sin\alpha, \cos\alpha\}) \geq \frac{3}{2}\left(1 + \frac{\sqrt{2}}{2}\right) = \frac{3(2+\sqrt{2})}{4},$$

equality holding only for $\alpha = 45^\circ$.

We will prove that $\frac{3(2+\sqrt{2})}{4}$ is the minimum we are looking for. To this end, we arrange the discs inside a square of side $l = \frac{3(2+\sqrt{2})}{4}$ as shown in Figure 2.

We will certify this is a valid configuration by proving the following claims:

1) D_3 fits in the upper right corner
2) D_6 fits in the lower right corner.
3) Between D_2, D_1, AD and CD we may fit a rectangle $MNPD$ of sides $\frac{1}{2}$ and $l-1$.
4) We may fit D_4 and D_7 between D_2, D_1, D_3 and AB.
5) We may fit all D_n, $n \geq 8$, in the interior of $MNPD$ and outside D_5, $MNPD$ being the rectangle from 3).

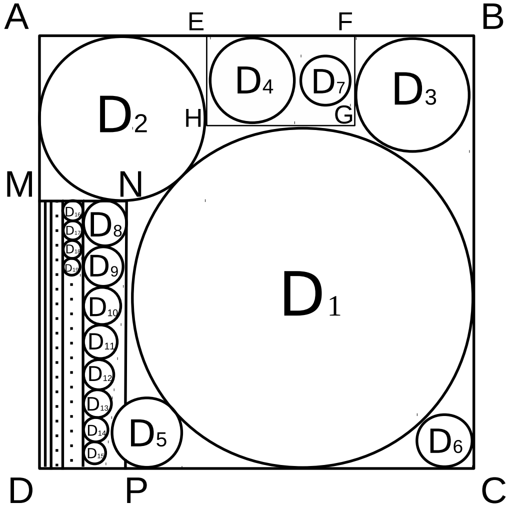

Figure 2

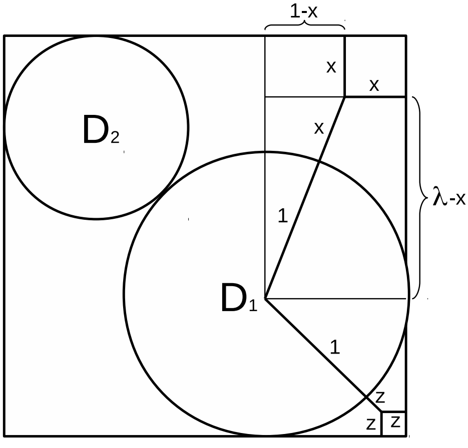

Figure 3

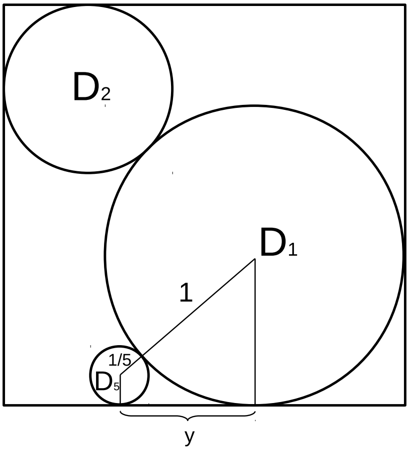

Figure 4

To prove 1), denote $\lambda = l - 1$. Using the notations from Figure 3, we have $(1+x)^2 = (1-x)^2 + (\lambda - x)^2$, relation that also writes as $(\lambda - x)^2 - 4x = 0$. We want to prove that $x \geq \frac{1}{3}$; since the product of the roots of the last equation is $\lambda^2 > 1$, we derive that x, which is smaller than unity, has to be the small root of that equation.

Therefore, in order for us to have $x \geq \frac{1}{3}$ it suffices to prove that $\left(\lambda - \frac{1}{3}\right)^2 - \frac{4}{3} \geq 0$. This relation is equivalent to $\frac{2+3\sqrt{2}}{4} - \frac{1}{3} \geq \frac{2}{\sqrt{3}}$ or, after computation, to $8\sqrt{3} - 9\sqrt{2} \leq 2$. But this relation is true, since $8\sqrt{3} - 9\sqrt{2} < 8 \cdot 1.8 - 9 \cdot 1.4 = 14.4 - 12.6 < 2$, so claim 1) is proved.

We see in Figure 3 that $1 + z + z\sqrt{2} = \sqrt{2}$, so $z = \frac{\sqrt{2}-1}{\sqrt{2}+1} = 3 - 2\sqrt{2} > 3 - 2 \cdot 1.4143 > 0.17 > \frac{1}{6}$. This proves 2).

To prove 3), we notice that $l - 2 = \frac{3\sqrt{2}-2}{4} > \frac{3 \cdot 1.4 - 2}{4} > \frac{1}{2}$. Since $MNPD$ is tangent to both D_2 and CD, its length is $l - 1$.

We have $2\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{7}\right) = 2\left(\frac{3}{4} + \frac{10}{21}\right) < \frac{5}{2}$. On the other hand, $l = \frac{3(2+\sqrt{2})}{4} > \frac{10.2}{4} = \frac{5.1}{2}$. Consequently, $2\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{7}\right) < l$. We derive that the length of the rectangle $EFGH$ in Figure 2 is greater than $2\left(\frac{1}{4} + \frac{1}{7}\right)$ and, according to claim 3), its width is larger than $\frac{1}{2}$, so D_4 and D_7 fit inside it. This proves claim 4).

To prove Claim 5), we divide the interior of $MNPD$ into vertical strips that we denote, from right to left, by $\mathcal{F}_3, \mathcal{F}_4, \ldots$, such as the width of $\mathcal{F}_k$ is $\frac{1}{2^{k-1}}$ (see Figure 2). Let $f : \mathbb{N}^* \to \mathbb{R}$, $f(n) = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}$, and $n \in \mathbb{N}^*$. Then $f(n+1) - f(n) = \frac{1}{2n} + \frac{1}{2n+1} - \frac{1}{n} < 0$, so f is strictly decreasing. Consequently, for all $k \geq 3$ we have $f(2^k) \leq f(8) = \sum\limits_{j=8}^{15} \frac{1}{j} < 0.728$. Thus, the sum of the diameters of $D_{2^k}, D_{2^k+1}, \ldots, D_{2^{k+1}-1}$ is less than 1.456, and therefore less than the length of $\mathcal{F}_k$. Consequently, we may arrange D_{2^k}, $D_{2^k+1}, \ldots, D_{2^{k+1}-1}$ inside $\mathcal{F}_k$ vertically.

We will place them in this order from top to bottom and tangent to the left side of $\mathcal{F}_k$, assigning to each D_n a horizontal strip of height $\frac{2}{n}$ of the corresponding $\mathcal{F}_k$, as suggested in Figure 2.

The disc D_5 has a region in common with $\mathcal{F}_3$. Since $\frac{1}{13} + \frac{1}{14} + \frac{1}{15} > \frac{1}{5}$, the discs D_8, D_9, D_{10}, D_{11} and D_{12} are above the upper horizontal tangent to D_5. Consequently, there is only left to prove that the distance d between AD and the left vertical tangent to D_5 is larger than $\frac{1}{4} + \frac{2}{13}$.

But, as seen on Figure 4, $y^2 = \left(\frac{6}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{4}{5}$, so $y = \frac{2}{\sqrt{5}}$. Therefore, $d = l - \frac{6+2\sqrt{5}}{5}$, and $d - \frac{1}{4} = \frac{1+15\sqrt{2}-8\sqrt{5}}{20} > \frac{4.314}{20} > \frac{2}{9} > \frac{2}{13}$. $\square$

Section B

Problem 1. a) Knowing that $\int\limits_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}$, compute the improper integral $I_n = \int\limits_0^\infty \frac{\sin^2 \frac{x}{n}}{x^2}\,\mathrm{d}x$.
b) Determine the sum of the series $\sum\limits_{n=1}^\infty I_{f(n)}$, where $f(n) = \frac{1}{\arctan \frac{2}{n^2}}$.

Florian Munteanu

Solution. a) We notice that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,\mathrm{d}x = -\frac{\sin^2 x}{x}\bigg|_0^\infty \int_0^\infty \frac{2\sin x\ \cos x}{x}\,\mathrm{d}x = \int_0^\infty \frac{\sin 2x}{x}\,\mathrm{d}x.$$

Via the change of variable $2x = t$, we get $\int\limits_0^\infty \frac{\sin 2x}{x}\,\mathrm{d}x = \int\limits_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \frac{\pi}{2}$. Now, using the change of variable $\frac{x}{n} = t$, we get $I_n = \int\limits_0^\infty \frac{\sin^2 t}{nt^2}\,\mathrm{d}t = \frac{\pi}{2n}$.
b) Note firstly that $I_{f(n)} = \frac{\pi}{2f(n)} = \frac{\pi}{2}\arctan\frac{2}{n^2}$. Therefore, it is enough to compute the sum of the series $\sum\limits_{n=1}^\infty I_{f(n)} = \frac{\pi}{2}\sum\limits_{n=1}^\infty \arctan\frac{2}{n^2}$ (the convergence of which follows easily from a comparison test). To this end, we compute the sequence $(S_n)_n$ of the partial sums of the series $\sum\limits_{n=1}^\infty \arctan\frac{2}{n^2}$:

$$\begin{aligned} S_n &= \sum_{k=1}^n \arctan\frac{2}{k^2} = \sum_{k=1}^n (\arctan(k+1) - \arctan(k-1)) \\ &= \arctan(n+1) + \arctan n - \arctan 1 - \arctan 0 \xrightarrow[n\to\infty]{} \frac{3\pi}{4}. \end{aligned}$$

It follows that $\sum\limits_{n=1}^\infty I_{f(n)} = \frac{3\pi^2}{8}$. □

Problem 2. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a matrix having the distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, with $-1 < \lambda_i \leq 1$, $i = \overline{1,n}$. Suppose that the sum of the elements in each column is 1, and the elements in each line of the matrix sum to the same value.
a) Prove that 1 is an eigenvalue for A, and $(1, 1, \ldots, 1)^T$ is a corresponding eigenvector.
b) For an arbitrary fixed $(x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ compute

$$\lim_{k\to\infty} A^k\left((x_1, x_2, \ldots, x_n)^T\right).$$

Ariadna Lucia Pletea

Solution. a) We prove first that the sum of the elements in each line is equal to 1. We will denote by b the common value of $\sum\limits_{j=1}^{n} a_{ij}$, $i=\overline{1,n}$. Since the sum of all the elements of the matrix A equals n, and is n times the sum of the elements in one column, we have $n=\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} a_{ij}\right)=nb$, and therefore $b=1$.

Furthermore, if in $\det(A-I_n)$ we add all the columns to the first one, and we use the fact that the sum of the elements in each line of the matrix A is 1, the first column will only consist of zeros. It follows that 1 is an eigenvalue for A. Since the sum of the elements in each line of A equals unity, $v=(1,1,\ldots,1)^T$ is an eigenvector corresponding to the eigenvalue 1.

b) Suppose, without loss of generality, that $\lambda_1=1$. The absolute values of the other eigenvalues will then be strictly smaller than 1.

Since the eigenvalues of A are pairwise distinct, it follows that the matrix A is diagonalizable and hence there exists a basis consisting only of eigenvectors of A. We denote this basis by $\mathcal{B}'=\{v_1',v_2,\ldots,v_n\}$; rearranging its vectors if necessary, we may suppose that v_1' is the eigenvector corresponding to the eigenvalue $\lambda_1=1$, and, in fact, since every such vector is a scalar multiple of $v=(1,1,\ldots,1)^T$, we may replace in $\mathcal{B}'$ the vector v_1' by $v_1=v$, obtaining the basis $\mathcal{B}=\{v_1,v_2,\ldots,v_n\}$ which still consists only of eigenvectors of A. Now, each $x\in\mathbb{R}^n$ may be written in the form

$$x=(x_1,x_2,\ldots,x_n)^T=\alpha_1v_1+\alpha_2v_2+\cdots+\alpha_nv_n,$$

and hence

$$\begin{aligned}A^k\left((x_1,x_2,\ldots,x_n)^T\right)&=\alpha_1A^kv_1+\alpha_2A^kv_2+\cdots+\alpha_nA^kv_n=\\&=\alpha_1\lambda_1^kv_1+\alpha_2\lambda_2^kv_2+\cdots+\alpha_n\lambda_n^kv_n.\end{aligned}\tag{3}$$

For each $i=\overline{2,n}$ we have $\lambda_i\in(-1,1)$, so $\lim_{k\to\infty}\lambda_i^k=0$, and thus

$$\lim_{k\to\infty}A^k\left((x_1,x_2,\ldots,x_n)^T\right)=$$
$$=\lim_{k\to\infty}\left(\alpha_1\left(1,1,\ldots,1\right)^T+\alpha_2\lambda_2^kv_2+\cdots+\alpha_n\lambda_n^kv_n\right)=\alpha_1\left(1,1,\ldots,1\right)^T.$$

The sum of the coordinates of the vector

$$Ax=A\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}=\begin{pmatrix}a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n\\a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n\\\vdots\\a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n\end{pmatrix}=\begin{pmatrix}y_1\\y_2\\\vdots\\y_n\end{pmatrix}$$

is $y_1+y_2+\cdots+y_n=x_1+x_2+\cdots+x_n$. Inductively, the sum of the coordinates of the vector A^kx is $x_1+x_2+\cdots+x_n$. On the other hand, using (3), this sum equals $n\alpha_1+\alpha_2\lambda_2^k(v_{21}+\cdots+v_{2n})+\cdots+\alpha_n\lambda_n^k(v_{n1}+\cdots+v_{nn})$, where by v_{ij} we denoted the j-th coordinate of the eigenvector v_i, $i=\overline{2,n}$, $j=\overline{1,n}$.

Computing the limit of the above expression for $k \to \infty$, it follows that $x_1 + x_2 + \cdots + x_n = n\alpha_1$. Consequently, one has

$$\lim_{k\to\infty} A^k\left((x_1, x_2, \ldots, x_n)^T\right) = \frac{x_1 + x_2 + \cdots + x_n}{n}(1, 1, \ldots, 1)^T. \qquad \square$$

Remark. For the first two problems, the solutions of the candidates were, broadly speaking, close to the official solutions presented before.

Problem 3. Consider the sequence $(x_n)_{n\geq 1}$ given by the recurrence $x_{n+1} = x_n(1 - x_n)$ for all $n \geq 1$, $x_1 \in (0, 1)$.
a) Prove that $(n+1)x_n < 1$ for all $n \geq 2$.
b) Study the convergence of the series $\sum\limits_{n=1}^{\infty} x_n^{\alpha}$, where α is a real parameter.

Alexandru Negrescu

Solution. a) Applying the GM-AM inequality, one obtains

$$x_2 = x_1(1 - x_1) \leq \left(\frac{x_1 + 1 - x_1}{2}\right)^2 = \frac{1}{4} < \frac{1}{2}.$$

Since the function $f(x) = x(1 - x)$ is strictly increasing on $\left(-\infty, \frac{1}{2}\right)$, it follows that for any $n \geq 2$, if $x_n < \frac{1}{n+1}$ one has

$$x_{n+1} = x_n(1 - x_n) < \frac{1}{n+1}\left(1 - \frac{1}{n+1}\right) = \frac{n}{(n+1)^2} < \frac{1}{n+2}.$$

b) Since $0 < x_n < \frac{1}{n+1}$ for every $n \geq 2$, it follows that $x_n \to 0$. Moreover, the recurrence from the definition of the sequence shows that $(x_n)_n$ is strictly decreasing. Since

$$\lim_{n\to\infty} \frac{n + 1 - n}{\dfrac{1}{x_{n+1}} - \dfrac{1}{x_n}} = \lim_{n\to\infty} \frac{x_n x_{n+1}}{x_n - x_{n+1}} = \lim_{n\to\infty} \frac{x_n^2(1 - x_n)}{x_n^2} = 1,$$

the Stolz-Cesáro Lemma yields $\lim\limits_{n\to\infty} \frac{n}{\frac{1}{x_n}} = 1$. Therefore, the series $\sum\limits_{n=1}^{\infty} x_n^{\alpha}$ converges if and only if $\sum\limits_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ does.

Consequently, the series $\sum\limits_{n=1}^{\infty} x_n^{\alpha}$ is convergent for $\alpha > 1$, and divergent for $\alpha \leq 1$. $\square$

Remark. Some students had a slightly different approach for item b). For instance, Emanuel Necula from Politehnica University of Bucharest studied the convergence of the series $\sum\limits_{n=1}^{\infty} x_n^{\alpha}$ as shown below.

We notice that, for $\alpha > 1$, since

$$x_n^{\alpha} < \frac{1}{(n+1)^{\alpha}}, \quad \forall n \geq 2,$$

and the series $\sum\limits_{n=2}^{\infty} \frac{1}{(n+1)^{\alpha}}$ is convergent, it follows that $\sum\limits_{n=1}^{\infty} x_n^{\alpha}$ is convergent.
If $\alpha < 0$, then by denoting $\beta = -\alpha > 0$, one has

$$x_n^{\alpha} = \frac{1}{x_n^{\beta}} > (n+1)^{\beta} \quad \text{for all } n \geq 2,$$

so $\sum\limits_{n=1}^{\infty} x_n^{\alpha}$ is divergent. Since for $\alpha = 0$ the series obviously diverges, the only situation left to examine is $\alpha \in (0,1]$. But in this case

$$\frac{x_{n+1}}{x_n} = 1 - x_n > \frac{n}{n+1},$$

$$\text{hence} \quad \frac{x_{n+1}^{\alpha}}{x_n^{\alpha}} > \frac{\frac{1}{(n+1)^{\alpha}}}{\frac{1}{n^{\alpha}}} \quad \text{for all } n \geq 2.$$

Since for $\alpha \in (0,1]$ the series $\sum\limits_{n=2}^{\infty} \frac{1}{n^{\alpha}}$ is divergent, it follows from a comparison test that $\sum\limits_{n=1}^{\infty} x_n^{\alpha}$ is divergent. The conclusion follows. $\square$

Problem 4. Prove that, for a matrix $A \in \mathcal{M}_n(\mathbb{C})$, the next assertions are equivalent:

a) $A^2 = A$;

b) $\operatorname{rank} A + \operatorname{rank}(I_n - A) = n$.

Vasile Pop

Solution. Consider the subspaces

$$V_1 = \{X \in \mathbb{C}^n |\ AX = \mathcal{O}\},$$
$$V_2 = \{X \in \mathbb{C}^n |\ AX = X\}$$

of $\mathbb{C}^n = \mathcal{M}_{n,1}(\mathbb{C})$. Here $\mathcal{O} = (0,\ldots,0)^T \in \mathbb{C}^n$.

a) $\Rightarrow$ *b)* For $X \in \mathbb{C}^n$, we have $X = (X - AX) + AX$. Since $X - AX \in V_1$ and $AX \in V_2$, we obtain $\mathbb{C}^n = V_1 + V_2$. Obviously, $V_1 \cap V_2 = \{\mathcal{O}\}$. Consequently, $\mathbb{C}^n = V_1 \dot{+} V_2$

Therefore, $\dim V_1 + \dim V_2 = n$. Since $\operatorname{rank}(I_n - A) = n - \dim V_2$ and $\operatorname{rank} A = n - \dim V_1$, the conclusion follows.

b) $\Rightarrow$ *a)* We have $V_1 \cap V_2 = \{\mathcal{O}\}$, $\dim V_1 = n - \operatorname{rank} A$, and $\dim V_2 = n - \operatorname{rank}(I_n - A)$. From *b)*, it follows that

$$\dim V_1 + \dim V_2 = n,$$

hence $\mathbb{C}^n = V_1 \dot{+} V_2$.

Let $X \in \mathbb{C}^n$ be arbitrary. Then there are $X_1 \in V_1$ and $X_2 \in V_2$ such that $X = X_1 + X_2$. One has

$$(A^2 - A)X = A\,(AX_1 + AX_2) - AX_1 - AX_2 = A\,(\mathcal{O} + X_2) - \mathcal{O} - X_2 = \mathcal{O}.$$

Since $X \in \mathbb{C}^n$ was arbitrarily chosen, it follows that $A^2 - A$ is the zero matrix. $\square$

Remark. The student Emanuel Necula found an alternative proof for $a) \Rightarrow b)$ based on Sylvester's inequality. We present his proof in the sequel.

It is known that for any two matrices $M, N \in \mathcal{M}_n(\mathbb{C})$,

$$\operatorname{rank} M + \operatorname{rank} N \geq \operatorname{rank}(M + N).$$

Therefore, $\operatorname{rank} A + \operatorname{rank}(I_n - A) \geq \operatorname{rank}(A + (I_n - A)) = \operatorname{rank} I_n = n$.
On the other hand, applying Sylvester's inequality one gets

$$\operatorname{rank} A + \operatorname{rank}(I_n - A) \leq \operatorname{rank} A(I_n - A) + n = n.$$

It follows that $\operatorname{rank} A + \operatorname{rank}(I_n - A) = n$. $\square$

NOTE MATEMATICE

A note on a UTCN SEEMOUS selection test problem

Ovidiu Furdui[1)], Mircea Ivan[2)], Alina Sîntămărian[3)]

Abstract. In this note we generalize a problem proposed by Mircea Ivan to the test for the selection of Technical University of Cluj-Napoca team for SEEMOUS 2016.

Keywords: limits, Riemann integrals, single variable calculus

MSC: Primary 26A06; Secondary 26A42.

Introduction and the main result

The goal of this note is problem 4 proposed by Mircea Ivan to the test for the selection of Technical University of Cluj-Napoca team for SEEMOUS 2016. The problem asks to calculate the limit

$$\lim_{n\to\infty}\int_0^n \frac{\mathrm{d}x}{1+n^2\cos^2 x}.$$

In this note we prove the following theorem.

Theorem 1. *If $f:[0,1]\to\mathbb{R}$ is a continuous function, then*

$$\lim_{n\to\infty}\int_0^n \frac{f\left(\frac{x}{n}\right)}{1+n^2\cos^2 x}\mathrm{d}x=\int_0^1 f(x)\mathrm{d}x.$$

In particular, we have that the following limits hold:

(a) $\displaystyle\lim_{n\to\infty}\int_0^n \frac{\mathrm{d}x}{1+n^2\cos^2 x}=1.$

(b) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\int_0^n \frac{x}{1+n^2\cos^2 x}=\frac{1}{2}.$

(c) $\displaystyle\lim_{n\to\infty}\frac{1}{n^2}\int_0^n \frac{x^2}{1+n^2\cos^2 x}=\frac{1}{3}.$

We mention that Theorem 1 also holds when cos is replaced by sin, i.e., if $f:[0,1]\to\mathbb{R}$ is a continuous function, then

$$\lim_{n\to\infty}\int_0^n \frac{f\left(\frac{x}{n}\right)}{1+n^2\sin^2 x}\mathrm{d}x=\int_0^1 f(x)\mathrm{d}x.$$

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We give two proofs of Theorem 1. One proof, which is natural, is based on calculating the limit for the special case when f is a monomial and then proving the general case using an approximation process. The second proof is based on an application of the Mean Value Theorem for the Integral. Before we prove Theorem 1 we collect some lemmas we need in our analysis.

Lemma 2. *The following equalities hold*:

(a) $\displaystyle\lim_{m\to\infty}\frac{1^j+2^j+\cdots+(m-1)^j}{m^{j+1}}=\frac{1}{j+1},\quad j\in\mathbb{N};$

(b) If $a\in\mathbb{R}$, *then* $\displaystyle\int_0^{\pi}\frac{\mathrm{d}x}{1+a^2\cos^2 x}=\frac{\pi}{\sqrt{1+a^2}}.$

Proof. (a) This part of the lemma follows by an application of Cesáro-Stolz lemma ([1, Appendix B, 263–266]) or by observing that the sequence is a Riemann sum.

(b) We have

$$\begin{aligned}\int_0^{\pi}\frac{\mathrm{d}x}{1+a^2\cos^2 x}&=2\int_0^{\pi/2}\frac{\mathrm{d}x}{1+a^2\cos^2 x}\overset{\tan x=t}{=}2\int_0^{\infty}\frac{\mathrm{d}t}{t^2+1+a^2}\\&=\frac{2}{\sqrt{1+a^2}}\arctan\frac{t}{\sqrt{1+a^2}}\Bigg|_0^{\infty}=\frac{\pi}{\sqrt{1+a^2}},\end{aligned}$$

and the lemma is proved. □

Lemma 3. *Let* $k\geq 0$ *be an integer. The following equality holds*

$$\lim_{n\to\infty}\frac{1}{n^k}\int_0^{n}\frac{x^k}{1+n^2\cos^2 x}\mathrm{d}x=\frac{1}{k+1}.$$

Proof. Let $m=\left\lfloor\frac{n}{\pi}\right\rfloor$ and observe that $m\pi<n<(m+1)\pi$. Let $J_m=\int\limits_0^{m\pi}\frac{x^k}{1+n^2\cos^2 x}\mathrm{d}x$. We have

$$\begin{aligned}J_m&=\int_0^{m\pi}\frac{x^k}{1+n^2\cos^2 x}\mathrm{d}x\\&=\sum_{l=0}^{m-1}\int_{l\pi}^{(l+1)\pi}\frac{x^k}{1+n^2\cos^2 x}\mathrm{d}x\\&=\sum_{l=0}^{m-1}\int_0^{\pi}\frac{(y+l\pi)^k}{1+n^2\cos^2 y}\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\pi} \frac{y^k}{1+n^2\cos^2 y}\mathrm{d}y + \sum_{l=1}^{m-1} \int_0^{\pi} \frac{(y+l\pi)^k}{1+n^2\cos^2 y}\mathrm{d}y \\
&= \int_0^{\pi} \frac{y^k}{1+n^2\cos^2 y}\mathrm{d}y + \sum_{l=1}^{m-1} \int_0^{\pi} \left(\sum_{j=0}^{k} \binom{k}{j} \pi^j l^j y^{k-j}\right) \frac{1}{1+n^2\cos^2 y}\mathrm{d}y \\
&= \int_0^{\pi} \frac{y^k}{1+n^2\cos^2 y}\mathrm{d}y + \sum_{j=0}^{k} \binom{k}{j} \pi^j \left(\sum_{l=1}^{m-1} l^j\right) \int_0^{\pi} \frac{y^{k-j}}{1+n^2\cos^2 y}\mathrm{d}y \\
&= \int_0^{\pi} \frac{y^k}{1+n^2\cos^2 y}\mathrm{d}y + \sum_{j=0}^{k} \binom{k}{j} \pi^j a_{m,j} \int_0^{\pi} \frac{y^{k-j}}{1+n^2\cos^2 y}\mathrm{d}y,
\end{aligned}
$$

where $a_{m,j} = 1^j + 2^j + \cdots + (m-1)^j$.

On the other hand, part (b) of Lemma 2 implies that for any $p \in \mathbb{N}$ one has

$$
0 < \int_0^{\pi} \frac{y^p}{1+n^2\cos^2 y}\mathrm{d}y < \pi^p \int_0^{\pi} \frac{1}{1+n^2\cos^2 y}\mathrm{d}y = \frac{\pi^{p+1}}{\sqrt{1+n^2}}. \tag{1}
$$

We have

$$
\begin{aligned}
\frac{J_m}{n^k} &= \frac{1}{n^k} \int_0^{\pi} \frac{y^k}{1+n^2\cos^2 y}\mathrm{d}y + \sum_{j=0}^{k-1} \binom{k}{j} \pi^j \frac{a_{m,j}}{n^k} \int_0^{\pi} \frac{y^{k-j}}{1+n^2\cos^2 y}\mathrm{d}y \\
&\quad + \pi^{k+1} \frac{a_{m,k}}{n^k\sqrt{1+n^2}}.
\end{aligned}
$$

Inequality (1) implies that

$$
\lim_{n\to\infty} \frac{1}{n^k} \int_0^{\pi} \frac{y^k}{1+n^2\cos^2 y}\mathrm{d}y = 0
$$

and observe that if $j = 0, 1, \ldots, k-1$ we have, based on part (a) of Lemma 2 and inequality (1), that

$$
\lim_{n\to\infty} \frac{a_{m,j}}{n^k} \int_0^{\pi} \frac{y^{k-j}}{1+n^2\cos^2 y}\mathrm{d}y = 0
$$

and

$$
\lim_{n\to\infty} \pi^{k+1} \frac{a_{m,k}}{n^k\sqrt{1+n^2}} = \frac{1}{k+1}.
$$

Combining these limits we get that $\lim\limits_{n\to\infty} \frac{J_m}{n^k} = \frac{1}{k+1}$. Similarly we have $\lim\limits_{n\to\infty} \frac{J_{m+1}}{n^k} = \frac{1}{k+1}$.

Let $I_n = \int_0^n \frac{x^k}{1+n^2\cos^2 x}\mathrm{d}x$. Since $J_m < I_n < J_{m+1}$, it follows, based on the previous limits, that $\lim\limits_{n\to\infty} \frac{I_n}{n^k} = \frac{1}{k+1}$ and the lemma is proved. □

Now we are ready to prove Theorem 1.

The first proof of Theorem 1. Let $\varepsilon > 0$ and let $P_\varepsilon = \sum_k a_k x^k$ be the polynomial that uniformly approximates f, i.e., $|f(x) - P_\varepsilon(x)| < \varepsilon$, for all $x \in [0,1]$. It follows that $|f\left(\frac{x}{n}\right) - P_\varepsilon\left(\frac{x}{n}\right)| < \varepsilon$, for all $x \in [0,n]$. This implies that

$$\int_0^n \frac{P_\varepsilon\left(\frac{x}{n}\right)}{1+n^2\cos^2 x}\mathrm{d}x - \varepsilon\int_0^n \frac{\mathrm{d}x}{1+n^2\cos^2 x} < \int_0^n \frac{f\left(\frac{x}{n}\right)}{1+n^2\cos^2 x}\mathrm{d}x$$
$$< \int_0^n \frac{P_\varepsilon\left(\frac{x}{n}\right)}{1+n^2\cos^2 x}\mathrm{d}x + \varepsilon\int_0^n \frac{\mathrm{d}x}{1+n^2\cos^2 x}.$$

Passing to the limit, as $n \to \infty$, in the preceding inequalities we get, since

$$\lim_{n\to\infty}\int_0^n \frac{\mathrm{d}x}{1+n^2\cos^2 x} = 1 \quad \text{(see Lemma 3 with } k=0),$$

that

$$\lim_{n\to\infty}\int_0^n \frac{P_\varepsilon\left(\frac{x}{n}\right)}{1+n^2\cos^2 x}\mathrm{d}x - \varepsilon \le \lim_{n\to\infty}\int_0^n \frac{f\left(\frac{x}{n}\right)}{1+n^2\cos^2 x}\mathrm{d}x$$
$$\le \lim_{n\to\infty}\int_0^n \frac{P_\varepsilon\left(\frac{x}{n}\right)}{1+n^2\cos^2 x}\mathrm{d}x + \varepsilon.$$

On the other hand, we have, based on Lemma 3, that

$$\lim_{n\to\infty}\int_0^n \frac{P_\varepsilon\left(\frac{x}{n}\right)}{1+n^2\cos^2 x}\mathrm{d}x = \sum_k \frac{a_k}{k+1} = \int_0^1 P_\varepsilon(x)\mathrm{d}x,$$

and it follows that

$$-\varepsilon \le \lim_{n\to\infty}\int_0^n \frac{f\left(\frac{x}{n}\right)}{1+n^2\cos^2 x}\mathrm{d}x - \int_0^1 P_\varepsilon(x)\mathrm{d}x \le \varepsilon.$$

Passing to the limit as $\varepsilon \to 0$ in the above inequalities we have, since $\lim\limits_{\varepsilon\to 0}\int\limits_0^1 P_\varepsilon(x)\mathrm{d}x = \int\limits_0^1 f(x)\mathrm{d}x$, that

$$\lim_{n\to\infty}\int_0^n \frac{f\left(\frac{x}{n}\right)}{1+n^2\cos^2 x}\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x.$$

The theorem is proved. □

Next we give another proof of Theorem 1 which is based on an application of The Mean Value Theorem for the Integral.

The second proof of Theorem 1. Let $m = \left\lfloor \frac{n}{\pi} \right\rfloor$ and let

$$A_m = \int_0^{m\pi} \frac{f\left(\frac{x}{n}\right)}{1+n^2\cos^2 x}\mathrm{d}x.$$

We have that

$$\int_0^n \frac{f\left(\frac{x}{n}\right)}{1+n^2\cos^2 x}\mathrm{d}x = A_m + \int_{m\pi}^n \frac{f\left(\frac{x}{n}\right)}{1+n^2\cos^2 x}\mathrm{d}x. \tag{2}$$

On the other hand

$$0 < \left| \int_{m\pi}^n \frac{f\left(\frac{x}{n}\right)}{1+n^2\cos^2 x}\mathrm{d}x \right| \le ||f||_\infty \int_{m\pi}^{(m+1)\pi} \frac{1}{1+n^2\cos^2 x}\mathrm{d}x = ||f||_\infty \frac{\pi}{\sqrt{1+n^2}},$$

which implies that

$$\lim_{n\to\infty} \int_{m\pi}^n \frac{f\left(\frac{x}{n}\right)}{1+n^2\cos^2 x}\mathrm{d}x = 0. \tag{3}$$

A calculation shows that

$$\begin{aligned} A_m &= \int_0^{m\pi} \frac{f\left(\frac{x}{n}\right)}{1+n^2\cos^2 x}\mathrm{d}x = \sum_{k=0}^{m-1} \int_{k\pi}^{(k+1)\pi} \frac{f\left(\frac{x}{n}\right)}{1+n^2\cos^2 x}\mathrm{d}x \\ &\overset{(*)}{=} \sum_{k=0}^{m-1} f\left(\frac{\theta_k}{n}\right) \int_{k\pi}^{(k+1)\pi} \frac{1}{1+n^2\cos^2 x}\mathrm{d}x \\ &= \sum_{k=0}^{m-1} f\left(\frac{\theta_k}{n}\right) \int_0^{\pi} \frac{1}{1+n^2\cos^2 x}\mathrm{d}x = \frac{\pi}{\sqrt{1+n^2}} \sum_{k=0}^{m-1} f\left(\frac{\theta_k}{n}\right), \end{aligned}$$

where $k\pi \le \theta_k \le (k+1)\pi$, $k = 0, \ldots, m-1$. We applied, at step (*), the Mean Value Theorem for the Integral (see, e.g., [3, Theorem 5, p. 352]).

Let $\mathcal{D} = \left\{0 < \frac{\pi}{n} < \frac{2\pi}{n} < \cdots < \frac{m\pi}{n} < 1\right\}$ be a division of the interval $[0,1]$ and let $\mathcal{I}$ be the system of intermediary points

$$\mathcal{I} = \left\{ \frac{\theta_0}{n} \le \frac{\theta_1}{n} \le \cdots \le \frac{\theta_{m-1}}{n} \le \frac{m\pi}{n} < 1 \right\},$$

where $\frac{k\pi}{n} \le \frac{\theta_k}{n} \le \frac{(k+1)\pi}{n}$, $k = 0, 1, \ldots, m-1$, and we considered 1 to be the intermediary point on the interval $\left[\frac{m\pi}{n}, 1\right]$. We have

$$A_m = \frac{n}{\sqrt{1+n^2}} \left[\frac{\pi}{n} \sum_{k=0}^{m-1} f\left(\frac{\theta_k}{n}\right) + \left(1 - \frac{m\pi}{n}\right) f(1) \right] - \frac{n}{\sqrt{1+n^2}} \left(1 - \frac{m\pi}{n}\right) f(1)$$

and observe that the sum between brackets is the Riemann sum of f associated to the division $\mathcal{D}$ and the system of intermediary points $\mathcal{I}$. Letting $n \to \infty$ we have, since f is integrable, that

$$\lim_{n\to\infty} A_m = \int_0^1 f(x)\mathrm{d}x. \tag{4}$$

Combining (2), (3) and (4), the theorem is proved. □

The reader may wish to prove that if $\alpha \neq 0$ is a real number and $f : [0,1] \to \mathbb{R}$ is a continuous function, then

$$\lim_{n\to\infty} \int_0^n \frac{f\left(\frac{x}{n}\right)}{\alpha^2 + n^2 \cos^2 x} \mathrm{d}x = \frac{1}{|\alpha|} \int_0^1 f(x) \mathrm{d}x.$$

Also, by using the techniques in this paper one may prove that the following theorem holds.

Theorem 4. *If $f : [0,1] \to \mathbb{R}$ is a continuous function, then*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \int_0^n \frac{f\left(\frac{x}{n}\right)}{1 + n \cos^2 x} \mathrm{d}x = \int_0^1 f(x) \mathrm{d}x.$$

Proof. We leave the details to the interested reader. □

We mention that other exercises involving limits of special integrals, similar to those discussed in this paper, as well as open problems can be found in [1, Chapter 1] and [2, Chapter 1].

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A property of unidimensional distributions which is lost in multidimensional case

Luigi-Ionuţ Catana[1)]

Abstract. Let $d \geq 1$ be an integer, μ be a probability distribution on $\left(\mathbb{R}^d, B\left(\mathbb{R}^d\right)\right)$ and $F(x) = \mu\left(\left\{y \in \mathbb{R}^d : y \leq x\right\}\right)$. Let also $F^*(x) = \mu\left(\left\{y \in \mathbb{R}^d : y > x\right\}\right)$. Here $y \leq x$ means that $y_j \leq x_j$ for all $1 \leq j \leq d$ and $y > x$ means $y \geq x, y \neq x$. If $d = 1$ then obviously $F + F^* \equiv 1$. We prove that if $d \geq 2$ the identity never holds.

Keywords: Probability, random vector, distribution

MSC: Primary 60A10; Secondary 60G07.

Let (Ω, K, P) be a probability space and let $X : \Omega \to \mathbb{R}^d$ be a random vector. Let $\mu(B) = P(X \in B)$ be its distribution and $F(x) = P(X \leq x)$ its distribution function. Let also $F^*(x) = P(X \geq x, X \neq x)$. Notice that if we denote by $L(x) = \left\{y \in \mathbb{R}^d : y \leq x\right\}$ and $U(x) = \left\{y \in \mathbb{R}^d : y > x\right\}$ then $L(x) \cup U(x) = \mathbb{R}^d$ for $d = 1$, but for $d \geq 2$ this fails to be true. However, this does not immediately imply that we cannot imagine probability distributions

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on $\mathbb{R}^d$ such that $\mu(L(x)) + \mu(U(x)) = 1$ for all $x \in \mathbb{R}^d$. Rather surprisingly, we found that for $d \geq 2$ the equality $\mu(L(x)) + \mu(U(x)) = 1$ for all $x \in \mathbb{R}^d$ doesn't hold.

Lemma 1. Let G and H be two unidimensional distribution functions. Define

$$F(x,y) = \frac{1}{2}(G(x) + H(y)).$$

Then F cannot be a distribution function.

Moreover there exists no distribution function F such that the equality $F(x,y) = \frac{1}{2}(G(x) + H(y))$ holds for $(x,y) \in \Gamma$ where $\Gamma \subset \mathbb{R}^2$ is dense.

Proof. Let $x = (a,b)$ and $x' = (a',b')$ be two points from $\mathbb{R}^2$ such that $a < a', b < b'$. Let μ be the corresponding probability distribution, if any. In the following we denote $R(a,b) = (a,\infty) \times (b,\infty)$. As

$$(a,a'] \times (b,b'] = \left(R\left(a',b'\right) \setminus R\left(a,b'\right)\right) \setminus \left(R\left(a',b\right) \setminus R(a,b)\right)$$

we can write

$$\begin{aligned}
&\mu((a,a'] \times (b,b']) = \mu((R(a',b') \setminus R(a,b')) \setminus (R(a',b) \setminus R(a,b)))\\
&= \mu(R(a',b') \setminus R(a,b')) - \mu(R(a',b) \setminus R(a,b))\\
&= \mu(R(a',b')) - \mu(R(a,b')) - \mu(R(a',b)) + \mu(R(a,b))
\end{aligned}$$

or

$$\mu((a,a'] \times (b,b']) = F(a,b) + F(a',b') - F(a,b') - F(a',b).$$

If we replace $F(x,y)$ by $\frac{G(x)+H(y)}{2}$ we get that $\mu((a,a'] \times (b,b']) = 0$ for all $x = (a,b)$ and $x' = (a',b')$. If we let $a, b \to -\infty$ and $b' \to \infty$ it follows that $F(a') = 0$ for all a', contradiction.

The same argument points out that there exists no probability distribution μ such that $\mu((a,a'] \times (b,b']) = 0$ for (a,b) and (a',b') from Γ. $\square$

Lemma 2. There exists no probability distribution μ on $\left(\mathbb{R}^2, B\left(\mathbb{R}^2\right)\right)$ such that $F + F^* \equiv 1$. Moreover, it is not possible for a random vector $Z = (X,Y)$ that the equality $P(X \leq x, Y \leq y) + P(X \geq x, Y \geq y) = 1$ for any $(x,y) \in \mathbb{R}^2$.

Proof. Note that

$$\begin{aligned}
&1 - F^*(x,y) = \mu(((-\infty,x] \times \mathbb{R}) \cup (\mathbb{R} \times (-\infty,y]) \setminus \{(x,y)\})\\
&= \mu((-\infty,x] \times \mathbb{R}) \cup (\mathbb{R} \times (-\infty,y]) - \mu(\{(x,y)\})\\
&= \mu((-\infty,x] \times \mathbb{R}) + \mu(\mathbb{R} \times (-\infty,y]) - \mu(((-\infty,x] \times \mathbb{R}) \cap (\mathbb{R} \times (-\infty,y]))\\
&- \mu(\{(x,y)\})\\
&= F(x,\infty) + F(\infty,y) - F(x,y) - \mu(\{(x,y)\}).
\end{aligned}$$

Let $A = \left\{(x,y) \in \mathbb{R}^2 : \mu(\{(x,y)\}) > 0\right\}$. This set is at most countable. Thus its complement A^c is dense in $\mathbb{R}^2$ and for any $(x,y) \in A^c$ it holds that $1 - F^*(x,y) = F(x,\infty) + F(\infty,y) - F(x,y)$

Now suppose $F \equiv 1 - F^*$.

For points $(x,y) \notin A$ we get $F(x,\infty) + F(\infty,y) - F(x,y) = F(x,y)$ or

$$F(x,y) = \frac{F(x,\infty) + F(\infty,y)}{2} := \frac{G(x) + H(y)}{2}.$$

But this is impossible according to Lemma 1.

The last assertion results from the fact that

$$P(X \geq x, Y \geq y) = P(X > x, Y > y)$$

on a dense subset of $\mathbb{R}^2$. □

Proposition 3. If $d > 2$ the identity $F + F^* \equiv 1$ never holds.

Proof. Suppose that $F(x_1, x_2, x_3, ..., x_d) + F^*(x_1, x_2, x_3..., x_d) = 1$ for any $x = (x_j)_{1 \leq j \leq d}$.

In terms of random vectors this implies

$$P(X_1 \leq x_1, X_2 \leq x_2, ..., X_d \leq x_d) + P(X_1 \geq x_1, X_2 \geq x_2, ..., X_d \geq x_d) = 1$$

for any $x \in \mathbb{R}^d$.

If one lets $x_j \to \infty$ for $j \geq 3$ one finds that $P(X_1 \leq x_1, X_2 \leq x_2) + P(X_1 \geq x_1, X_2 \geq x_2) = 1$ on a dense subset of $\mathbb{R}^2$. But this is not possible according to Lemma 2. □

A reciprocal of the Cayley–Hamilton theorem for square matrices of order two

Ovidiu Furdui[1)], Vasile Pop[2)]

Abstract. In this short note we give a reciprocal of the Cayley–Hamilton theorem for square matrices of order two.

Keywords: Matrices, Cayley–Hamilton theorem

MSC: Primary 15A15; Secondary 15A24.

Theorem. *Let $A \in \mathcal{M}_2(\mathbb{C})$ and let $a, b \in \mathbb{C}$ such that $A^2 - aA + bI_2 = O_2$. If $A \notin \{\alpha I_2 : \alpha \in \mathbb{C}\}$, then* $\mathrm{Tr}(A) = a$ *and* $\det A = b$.

Proof. We have, based on the Cayley–Hamilton Theorem, that

$$A^2 - aA + bI_2 = O_2$$
$$A^2 - \mathrm{Tr}(A)A + (\det A)I_2 = O_2,$$

and it follows that $[a - \mathrm{Tr}(A)]\, A = (b - \det A)I_2$.

If $a - \mathrm{Tr}(A) \neq 0$ we get that

$$A = \frac{b - \det A}{a - \mathrm{Tr}(A)} I_2,$$

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which is a contradiction to $A \notin \{\alpha I_2 : \alpha \in \mathbb{C}\}$.

If $a - \mathrm{Tr}(A) = 0$ we get $b - \det A = 0$ and the theorem is proved. $\square$

Remark. It is worth mentioning that there do exist matrices $A \in \mathcal{M}_2(\mathbb{C})$ such that $A^2 - aA + bI_2 = O_2$, with $a \neq \mathrm{Tr}(A)$ and $b \neq \det A$. To see this we let $A = \alpha I_2$, where $\alpha \in \mathbb{C}$ verifies the equation $\alpha^2 - a\alpha + b = 0$. Then, $\mathrm{Tr}(A) = 2\alpha$, $\det A = \alpha^2$ and if a and b are such that $a^2 - 4b \neq 0$ and $b \neq 0$ one has $a \neq \mathrm{Tr}(A)$ and $b \neq \det A$.

PROBLEMS

Authors should submit proposed problems to gmaproblems@rms.unibuc.ro. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of November 2017**.

PROPOSED PROBLEMS

447. Let $A \in \mathcal{M}_2(\mathbb{Z})$. Prove that $\mathrm{e}^A \in \mathcal{M}_2(\mathbb{Z})$ if and only if $A^2 = O_2$.

Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

448. Let $f : [0, \infty) \to (0, \infty)$ be a continuous function and let $F(x) = \int_0^x f(t)\mathrm{d}t$. Let $k > 0$.

(i) Prove that if $\alpha < ke$ then there are $\varepsilon, A > 0$ with the property that

$$\int_0^{2A} e^{-\varepsilon(x-A)^2}(F(x+k) - \alpha f(x))\mathrm{d}x > 0.$$

Moreover ε, A are independent of f. (Hint: take ε small and A large.)

(ii) Give an example of f such that $F(x+k) < kef(x)\ \forall\, x > 0$.

Conclude that if $k > 0$ and $\alpha \in \mathbb{R}$ then there is $f : [0, \infty) \to (0, \infty)$ such that $\int_0^{x+k} f(t)\mathrm{d}t < \alpha f(x)\ \forall x > 0$ if and only if $\alpha \geq ke$.

Proposed by Constantin-Nicolae Beli, IMAR, Bucureşti, Romania.

449. Let m, n be positive integers and let $f, g_1, \ldots, g_{n+1} \in \mathbb{Z}[X]$, where f is monic with $\deg f = n$, such that all roots of f are real and $|g_i(\alpha)| < m$ for $1 \leq i \leq n+1$ and for every root α of f. Show that there exist $n+1$ integers, not all zero, each of absolute value at most $(2m(n+1))^n$, such that every root of f is also a root of $\sum\limits_{i=1}^{n+1} a_i g_i$.

Proposed by Marius Cavachi, Ovidius University, Constanţa, Romania.

450. Find all polynomials $P \in \mathbb{R}[x]$ with the property

$$P(\sin x) + P(\cos x) = 1,\ \forall\ x \in \mathbb{R}.$$

Proposed by Vasile Pop, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

451. Let V be the linear space of the polynomial functions with real coefficients, defined on $[0,1]$. Denote $e_n(x) = x^n$, $x \in [0,1]$, $n = 0,1,2,\dots$. Let $T : V \to V$ be a linear operator such that:

(1) $Te_0 = e_0$, $Te_1 = Te_2 = e_1$;

(2) If $p \in V$, $p(x) \geq 0$, $\forall x \in [0,1]$, then $(Tp)(x) \geq 0$, $\forall x \in [0,1]$.

Prove that T is a projection, and find the null space $\operatorname{Ker} T$ and the range $\operatorname{Im} T$.

`Proposed by Ioan Raşa, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.`

453. Let $f, g \in \mathbb{Q}[X]$ be monic and irreducible polynomials over $\mathbb{Q}$ with the property that there are $\alpha, \beta \in \mathbb{C}$ with $\alpha + \beta \in \mathbb{Q}$ and $f(\alpha) = g(\beta) = 0$. Prove that the polynomial $f^2 - g^2$ has a root in $\mathbb{Q}$.

Remark. A generalization of this problem is the following: Let K, L be two fields with the properties: $\operatorname{char}(K) \neq 2$, $K \subseteq L$ and $f, g \in K[X]$ two monic irreducible polynomials over K so that $\alpha, \beta \in L$ with $\alpha + \beta \in K$ and $f(\alpha) = g(\beta) = 0$.

Prove that the polynomial $f^2 - g^2$ has a root in K.

`Proposed by Bogdan Moldovan, student, Babeş-Bolyai University, Cluj-Napoca, Romania.`

452. Let $(T_n)_{n\in\mathbb{N}}$ be the sequence of Chebyshev's polynomials, which on the interval $[-1,1]$ are defined by $T_n(x) = \cos(n \arccos x)$, $n \in \mathbb{N}$, and let $(F_n)_{n\geq 0}$ be the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$, for all $n \geq 1$. Prove that

$$T_n\left(-\frac{3}{2}\right) = 1 + (-1)^n \frac{5}{2} F_n^2, \quad \forall n \in \mathbb{N}.$$

`Proposed by Vasile Pop, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.`

454. Let $A \in M_3(\mathbb{Z})$ with $A^{30} - 2A^{25} = A^5 - 2I_3$. Prove that $\operatorname{tr}(A) \not\equiv 2 \pmod 5$.

`Proposed by Luigi-Ionuţ Catana,student, University of Bucharest, Bucureşti, Romania.`

SOLUTIONS

429. Let f be a C^1-class real valued function on $[0,1]$, infinitely differentiable at $x = 0$. If $f^{(n)}(0) = 0$ for every $n \in \mathbb{N}$ and, for some $C > 0$ one has: $|xf'(x)| \leq C \cdot |f(x)|$ for every $x \in [0,1]$, then $f(x) = 0$ for every $x \in [0,1]$.

`Proposed by George Stoica, Department of Mathematical Sciences, University of New Brunswick, Canada.`

Solution by the author. Let $a \geq 1$. We have

$$\left(x^{-a-1}f^2(x)\right)' = -(a+1)x^{-a-2}f^2(x) + 2x^{-a-1}f(x)f'(x).$$

From $f^{(n)}(0) = 0$ for any $n \geq [a/2]+1$ and Taylor's formula, it follows that

$$f^2(x) = O(x^{2n+2}) = o(x^{a+1}),$$

so

$$\int_0^1 \left(x^{-a-1}f^2(x)\right)' \mathrm{d}x = f^2(1) - \lim_{x\to 0^+} \frac{f^2(x)}{x^{a+1}} = f^2(1) \geq 0.$$

Therefore

$$(a+1)\int_0^1 x^{-a-2}f^2(x)\mathrm{d}x \leq 2\int_0^1 x^{-a-1}f(x)f'(x)\mathrm{d}x$$
$$= 2\int_0^1 \left(\sqrt{\frac{a+1}{2}}x^{-(a+2)/2}f(x)\right)\left(\sqrt{\frac{2}{a+1}}x^{-a/2}f'(x)\right)\mathrm{d}x.$$

Using the inequality $2xy \leq x^2 + y^2$, the last expression is at most

$$\frac{a+1}{2}\int_0^1 x^{-a-2}f^2(x)\mathrm{d}x + \frac{2}{a+1}\int_0^1 x^{-a}f'^2(x)\mathrm{d}x,$$

hence

$$(*) \qquad \frac{(a+1)^2}{4}\int_0^1 x^{-a-2}f^2(x)\mathrm{d}x \leq \int_0^1 x^{-a}f'^2(x)\mathrm{d}x.$$

Now, by hypothesis we have that

$$(**) \qquad \int_0^1 x^{-a}f'^2(x)\mathrm{d}x \leq C^2\int_0^1 x^{-a-2}f^2(x)\mathrm{d}x,$$

so, if $f(x) \not\equiv 0$ on $[0,1]$, we would have $|f'(x)| > 0$ on some subinterval of $[0,1]$, so that the integrals (on that subinterval) in equations $(*)$ and $(**)$ above are > 0. This gives

$$\frac{(a+1)^2}{4} \leq C^2 \text{ for all } a \geq 1,$$

which is absurd. The problem is now solved. □

430. Suppose that $r \in \mathbb{Q}$ and $m \in \mathbb{Z}$, $m > 0$, such that $\cos^m r\pi \in \mathbb{Q}$. Determine all possible values of $\cos r\pi$.

`Proposed by Marius Cavachi, Ovidius University, Constanţa, Romania.`

Solution by the author. If $\varepsilon = \cos r\pi + i\sin r\pi$ for some $r \in \mathbb{Q}$ then ε is a primitive nth root of the unity for some integer $n \geq 1$. We have $(\varepsilon + \varepsilon^{-1})^n = (2\cos r\pi)^n =: a \in \mathbb{Q}$.

Any morphism $\sigma \in \mathrm{Gal}(\mathbb{Q}(\varepsilon)/\mathbb{Q})$ is given by $\varepsilon \mapsto \varepsilon^k$ for some $1 \leq k \leq n$, $(k,n) = 1$. Then $(2\cos r\pi)^n = a = \sigma(a) = \sigma((\varepsilon + \varepsilon^{-1})^n) = (\varepsilon^k + \varepsilon^{-k})^n =$

$(2\cos kr\pi)^n$, so $|\cos kr\pi| = |\cos r\pi|$. But if k runs over all $1 \le k \le n$ with $(k,n) = 1$ then $|\cos kr\pi|$ runs over all $|\cos \frac{2l\pi}{n}|$ with $1 \le l \le n$ with $(l,n) = 1$. Now if $A = \{\frac{2l\pi}{n} \mid 1 \le l \le n,\ (l,n) = 1\}$ then $|A| = \phi(n)$, $A \subset (0, 2\pi]$ and for every $x \in A$ we have that $|\cos x|$ is the same. But for every $y \in [0,1]$ the set $\{x \in (0,2\pi] \mid \cos x = y\}$ has at most 4 elements. It follows that $\phi(n) = |A| \le 4$. We get $n \in \{1,2,3,4,5,6,8,10,12\}$. But $n = 5$ or $n = 10$ do not qualify since $|\cos \frac{2\pi}{5}| \ne |\cos \frac{4\pi}{5}|$ and $|\cos \frac{2\pi}{10}| \ne |\cos \frac{6\pi}{10}|$. The remaining cases lead to $\cos r\pi \in \{0, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{2}, \pm 1\}$. □

Notes from the editor. The information that all conjugates of $\cos r\pi$ are $\pm\cos r\pi$ can be used to deduce that $\cos^2 r\pi \in \mathbb{Q}$. Since moreover $2\cos r\pi$ is an algebraic integer we have in fact $a := (2\cos r\pi)^2 \in \mathbb{Z}$, so $\cos r\pi = \pm\frac{\sqrt{a}}{2}$ for some integer $a \ge 0$. Since also $|\cos x| \le 1$ we have $a \le 4$ and we get the desired result.

We received another solution from our reader *Victor Makanin*, which is too long to reproduce here. He too proved that if $r = \frac{l}{n}$ with $(n,l) = 1$ and $a = \cos \pi r$ then $\cos \frac{2j\pi}{n} = \pm a$ for every j with $(n,j) = 1$. Then he proved that if $n \ge 3$ then

$$\prod_{1 \le k < n,\ (k,n)=1} \cos \frac{k\pi}{n} = \frac{\Phi_n(-1)}{(-1)^{\varphi(n)/2} 2^{\varphi(n)}}.$$

From here, by using a well known formula for $\Phi_n(-1)$, he deduced that

$$|a|^{\phi(n)} = \prod_{(k,n)=1} |\cos \frac{k\pi}{n}| = \begin{cases} \frac{2}{2^{\varphi(n)}}, & n = 2^s,\ s \ge 2, \\ \frac{p}{2^{\varphi(n)}}, & n = 2p^s,\ \text{with } p \text{ odd prime and } s \ge 1, \\ \frac{1}{2^{\varphi(n)}}, & \text{else.} \end{cases}$$

If n is not of the form 2^s, $s \ge 2$, or $2p^s$ we get $|a| = \frac{1}{2}$, i.e., $a = \pm\frac{1}{2}$. When $n = 4$ or 6 we get $|a| = \frac{\sqrt{2}}{2}$ or $\frac{\sqrt{3}}{2}$, so $a = \pm\frac{\sqrt{2}}{2}$ or $\pm\frac{\sqrt{3}}{2}$, respectively. In all the other cases he obtains a contradiction by proving that $\cos \frac{\pi}{n} > |a|$, which is impossible, as $\cos \pi n = \pm a$. There are three cases:

If $n = 2^s$, $s \ge 3$,, so $\phi(n) = 2^{s-1}$ we have:

$$\cos \frac{\pi}{2^s} \ge \cos \frac{\pi}{8} > \frac{2^{1/4}}{2} \ge \frac{2^{1/2^{s-1}}}{2} = |a|.$$

If $n = 2 \cdot 3^s$, $s \ge 2$, so $\phi(n) = 2 \cdot 3^{s-1}$ we have

$$\cos \frac{\pi}{2 \cdot 3^s} \ge \cos \frac{\pi}{18} > \frac{3^{1/6}}{2} \ge \frac{3^{1/(2 \cdot 3^{s-1})}}{2} = |a|.$$

If $n = 2p^s$, $s \ge 1$, $p \ge 5$,, so $\phi(n) = (p-1)p^{s-1}$ we have

$$\cos \frac{\pi}{2p^s} \ge \cos \frac{\pi}{10} > \frac{5^{1/4}}{2} \ge \frac{p^{1/(p-1)}}{2} \ge \frac{p^{1/(p^{s-1}(p-1))}}{2} = |a|.$$

(Here he used the fact that $x \mapsto x^{1/(x-1)}$ is decreasing for $x > 1$.)

431. Let $f:[0,1]^2\to\mathbb{R}$ be such that $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}:[0,1]^2\to\mathbb{R}$ are continuous. Find the value of the limit

$$\lim_{n\to\infty} n\left(n^2\iint\limits_{[0,1]^2} x^n y^n f(x,y)\,\mathrm{d}x\mathrm{d}y - f(1,1)\right).$$

Proposed by Dumitru Popa, Faculty of Mathematics, Ovidius University, Constanţa, Romania.

Solution by the author. It is well-known that if $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ are continuous then f is continuous, see [1, Exerciţiul 6, page 79]. Let $n\in\mathbb{N}$ and define

$$I_n=\iint\limits_{[0,1]^2} x^n y^n f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

By Fubini's theorem we have $I_n=\int_0^1 y^n\mathrm{d}y\int_0^1 x^n f(x,y)\,\mathrm{d}x$. Let $y\in[0,1]$. Integrating by parts (recall that $\frac{\partial f}{\partial x}$ is continuous), we have

$$\begin{aligned}\int_0^1 x^n f(x,y)\,\mathrm{d}x &= \frac{x^{n+1}}{n+1}f(x,y)\Big|_{x=0}^{x=1}-\frac{1}{n+1}\int_0^1 x^{n+1}\frac{\partial f}{\partial x}(x,y)\,\mathrm{d}x\\ &= \frac{f(1,y)}{n+1}-\frac{1}{n+1}\int_0^1 x^{n+1}\frac{\partial f}{\partial x}(x,y)\,\mathrm{d}x.\end{aligned}$$

Then

$$I_n=\frac{1}{n+1}\int_0^1 y^n f(1,y)\,\mathrm{d}y-\frac{1}{n+1}\iint\limits_{[0,1]^2} x^{n+1}y^n\frac{\partial f}{\partial x}(x,y)\,\mathrm{d}x\mathrm{d}y.$$

Again integrating by parts, as $\frac{\partial f}{\partial y}$ is also continuous, we deduce

$$\begin{aligned}\int_0^1 y^n f(1,y)\,\mathrm{d}y &= \frac{y^{n+1}}{n+1}f(1,y)\Big|_{y=0}^{y=1}-\frac{1}{n+1}\int_0^1 y^{n+1}\frac{\partial f}{\partial y}(1,y)\,\mathrm{d}y\\ &= \frac{f(1,1)}{n+1}-\frac{1}{n+1}\int_0^1 y^{n+1}\frac{\partial f}{\partial y}(1,y)\,\mathrm{d}y\end{aligned}$$

and thus

$$\begin{aligned}I_n &= \frac{f(1,1)}{(n+1)^2}-\frac{1}{(n+1)^2}\int_0^1 y^{n+1}\frac{\partial f}{\partial y}(1,y)\,\mathrm{d}y\\ &\quad -\frac{1}{n+1}\iint\limits_{[0,1]^2} x^{n+1}y^n\frac{\partial f}{\partial x}(x,y)\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Then

$$n\left(\left(n+1\right)^2 I_n - f\left(1,1\right)\right) = -\,n\int_0^1 y^{n+1}\frac{\partial f}{\partial y}\left(1,y\right)\mathrm{d}y$$
$$-\,n\left(n+1\right)\iint\limits_{[0,1]^2} x^n y^n x\frac{\partial f}{\partial x}\left(x,y\right)\mathrm{d}x\mathrm{d}y.$$

But it is well-known that, since $\frac{\partial f}{\partial x}$ is continuous, we have

$$\lim_{n\to\infty} n\int_0^1 y^{n+1}\frac{\partial f}{\partial y}\left(1,y\right)\mathrm{d}y = \frac{\partial f}{\partial y}\left(1,1\right),$$

see [2] or [3, Exerciţiul 3.4(i)], and, since $\frac{\partial f}{\partial x}$ is continuous,

$$\lim_{n\to\infty} n^2\iint\limits_{[0,1]^2} x^n y^n x\frac{\partial f}{\partial x}\left(x,y\right)\mathrm{d}x\mathrm{d}y = \frac{\partial f}{\partial x}\left(1,1\right),$$

see [4, Theorem 5 and Proposition 6]. We obtain

$$\lim_{n\to\infty} n\left(\left(n+1\right)^2 I_n - f\left(1,1\right)\right) = -\left[\frac{\partial f}{\partial x}\left(1,1\right) + \frac{\partial f}{\partial y}\left(1,1\right)\right]. \tag{1}$$

But $n^2 I_n - f\left(1,1\right) = \left(n+1\right)^2 I_n - f\left(1,1\right) - \left(2n+1\right) I_n$, so that

$$n\left(n^2 I_n - f\left(1,1\right)\right) = n\left(\left(n+1\right)^2 I_n - f\left(1,1\right)\right) - n\left(2n+1\right) I_n. \tag{2}$$

Now again from [4, Theorem 5 and Proposition 6] we have

$$\lim_{n\to\infty} n\left(2n+1\right) I_n = 2\lim_{n\to\infty} n^2 I_n = 2\lim_{n\to\infty} n^2\iint\limits_{[0,1]^2} x^n y^n f\left(x,y\right)\mathrm{d}x\mathrm{d}y$$
$$= 2f\left(1,1\right). \tag{3}$$

From (1)–(3) we get

$$\lim_{n\to\infty} n\left(n^2 I_n - f\left(1,1\right)\right) = -2f\left(1,1\right) - \left[\frac{\partial f}{\partial x}\left(1,1\right) + \frac{\partial f}{\partial y}\left(1,1\right)\right].$$

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[4] D. Popa, The limit of some sequences of double integrals on the unit square, *G.M.A.* **32** (2014), Nr. 1–2, 19–26.

A note form the editor. This problem and similar ones can be tackled by noting that for n large if x or y is not close to 1 the product $x^n y^n$ is small. Therefore only the behaviour of f around $(1,1)$ matters.

We denote as above $I_n = \iint\limits_{[0,1]^2} x^n y^n f(x,y)\mathrm{d}x\mathrm{d}y$, so the problem amounts to determine $\lim_{n\to\infty}(n^3 I_n - nf(1,1))$. For convenience, write f_x, f_y instead of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

If $0 < x \le 1 - \frac{4\log n}{n}$ then

$$\log x \le \log\left(1 - \frac{4\log n}{n}\right) < -\frac{4\log n}{n},$$

and

$$\log(1-t) = -t - \frac{t^2}{2} - \cdots < -t$$

when $0 < t < 1$. It follows that $x^n = e^{n\log x} < e^{-4\log n} = n^{-4}$. Similarly $y^n < n^{-4}$ if $0 \le y \le 1 - \frac{4\log n}{n}$. It follows that for $(x,y) \in D := [0,1]^2 \setminus [1-\frac{4\log n}{n}, 1]^2$ we have $|x^2y^2 f(x,y)| \le n^{-4}M$, where $M = \max_{(x,y)\in[0,1]^2} |f(x,y)|$.

We obviously have $I_n = I_n' + I_n''$, where $I_n' = \iint\limits_{[1-\frac{4\log n}{n},1]^2} x^n y^n f(x,y)\mathrm{d}x\mathrm{d}y$ and $I_n'' = \iint\limits_D x^n y^n f(x,y)\mathrm{d}x\mathrm{d}y$. But $|D| < 1$ and for $(x,y) \in D$ we have $|x^2y^2 f(x,y)| \le n^{-4}M$. It follows that $|I_n''| \le n^{-4}M$, whence $\lim_{n\to\infty} n^3 I_n'' = 0$. Hence,

$$\lim_{n\to\infty}(n^3 I_n - nf(1,1)) = \lim_{n\to\infty}(n^3 I_n' - nf(1,1)).$$

We make the change of variables $x = 1 - \frac{u}{n}$, $y = 1 - \frac{v}{n}$ and we get $I_n' = \frac{1}{n^2} J_n$, where

$$J_n = \iint\limits_{[0,4\log n]^2} \big(1 - \frac{u}{n}\big)^n \big(1 - \frac{v}{n}\big)^n f\big(1 - \frac{u}{n}, 1 - \frac{v}{n}\big)\mathrm{d}u\mathrm{d}v.$$

Let $(u,v) \in [0, 4\log n]$. We have

$$n\log\big(1 - \frac{u}{n}\big) = n\left(-\frac{u}{n} - \frac{u^2}{2n^2} + O\big(\frac{u^3}{n^3}\big)\right) = -u - \frac{u^2}{2n} + O\big(\frac{u^3}{n^2}\big).$$

Hence,

$$\begin{aligned}\big(1 - \frac{u}{n}\big)^n &= e^{-u}e^{-\frac{u^2}{2n}}\left(1 + O\big(\frac{u^3}{n^2}\big)\right) = e^{-u}\left(1 - \frac{u^2}{2n} + O\big(\frac{u^4}{n^2}\big)\right)\left(1 + O\big(\frac{u^3}{n^2}\big)\right)\\ &= e^{-u}\left(1 - \frac{u^2}{2n} + O\big(\frac{\log^4 n}{n^2}\big)\right).\end{aligned}$$

Similarly, for $(1 - \frac{v}{n})^n$ we get

$$\big(1 - \frac{u}{n}\big)^n\big(1 - \frac{v}{n}\big)^n = e^{-u-v}\left(1 - \frac{u^2}{2n} - \frac{v^2}{2n} + O\big(\frac{\log^4 n}{n^2}\big)\right).$$

(We used the fact that $u, v = O(\log n)$.)

We also have

$$f\Big(1-\frac{u}{n},1-\frac{v}{n}\Big)=f(1,1)-\frac{u}{n}f_x(1,1)-\frac{v}{n}f_y(1,1)+o\Big(\frac{u}{n}\Big)+o\Big(\frac{v}{n}\Big).$$

In conclusion, if $u,v\in[0,4\log n]$ we have

$$\begin{aligned}\left(1-\frac{u}{n}\right)^n\left(1-\frac{v}{n}\right)^n f\left(1-\frac{u}{n},1-\frac{v}{n}\right)&=e^{-u-v}\left(f(1,1)-\frac{u^2}{2n}f(1,1)\right)\\&+e^{-u-v}\left(-\frac{v^2}{2n}f(1,1)-\frac{u}{n}f_x(1,1)-\frac{v}{n}f_y(1,1)\right)\\&+e^{-u-v}\left(O\left(\frac{\log^4 n}{n^2}\right)+o\left(\frac{u}{n}\right)+o\left(\frac{v}{n}\right)\right).\end{aligned}$$

Now $\int_0^{4\log n}e^{-u}\mathrm{d}u=1-\frac{1}{n^4}$, $\int_0^{4\log n}ue^{-u}\mathrm{d}u=-(u+1)e^{-u}\big|_0^{4\log n}=1-\frac{4\log n+1}{n^4}$ and $\int_0^{4\log n}\frac{u^2}{2}e^{-u}\mathrm{d}u=-(\frac{u^2}{2}+u+1)e^{-u}\big|_0^{4\log n}=1-\frac{8\log^2 n+4\log n+1}{n^4}$. Similarly with u replaced by v. Then we have

$$\begin{aligned}&\iint\limits_{[0,4\log n]^2}e^{-u-v}\left(f(1,1)+O\left(\frac{\log^4 n}{n^2}\right)\right)\mathrm{d}u\mathrm{d}v\\&\qquad=\left(f(1,1)+O\left(\frac{\log^4 n}{n^2}\right)\right)\int_0^{4\log n}e^{-u}\mathrm{d}u\int_0^{4\log n}e^{-v}\mathrm{d}v\\&\qquad=\left(f(1,1)+O\left(\frac{\log^4 n}{n^2}\right)\right)\left(1-\frac{1}{n^4}\right)^2=f(1,1)+O\left(\frac{\log^4 n}{n^2}\right),\end{aligned}$$

$$\begin{aligned}&\iint\limits_{[0,4\log n]^2}e^{-u-v}\left(-\frac{u}{n}f_x(1,1)+o\left(\frac{u}{n}\right)\right)\mathrm{d}u\mathrm{d}v\\&\qquad=\left(-\frac{1}{n}f_x(1,1)+o\left(\frac{1}{n}\right)\right)\int_0^{4\log n}ue^{-u}\mathrm{d}u\int_0^{4\log n}e^{-v}\mathrm{d}v\\&\qquad=\left(-\frac{1}{n}f_x(1,1)+o\left(\frac{1}{n}\right)\right)\left(1-\frac{\log n+1}{n^4}\right)\left(1-\frac{1}{n^4}\right)\\&\qquad=\frac{1}{n}\big(-f_x(1,1)+o(1)\big),\end{aligned}$$

and, similarly,

$$\iint\limits_{[0,4\log n]^2}e^{-u-v}\left(-\frac{v}{n}f_x(1,1)+o\left(\frac{v}{n}\right)\right)\mathrm{d}u\mathrm{d}v=\frac{1}{n}\big(-f_y(1,1)+o(1)\big),$$

$$\begin{aligned}&\iint\limits_{[0,4\log n]^2}-e^{-u-v}\frac{u^2}{2n}f(1,1)\mathrm{d}u\mathrm{d}v=-\frac{1}{n}f(1,1)\int_0^{4\log n}\frac{u^2}{2}e^{-u}\mathrm{d}u\int_0^{4\log n}e^{-v}\mathrm{d}v\\&\qquad=-\frac{1}{n}f(1,1)\left(1-\frac{8\log^2 n+4\log n+1}{n^4}\right)\left(1-\frac{1}{n^4}\right)=\end{aligned}$$

$$= \frac{1}{n}\left(-f(1,1) + O\big(\frac{\log^2 n}{n^4}\big)\right),$$

and, finally,

$$\iint\limits_{[0,4\log n]^2} -e^{-u-v}\frac{v^2}{2n}f(1,1)\mathrm{d}u\mathrm{d}v = \frac{1}{n}\left(-f(1,1) + O\left(\frac{\log^2 n}{n^4}\right)\right).$$

By adding the five relations above we get

$$J_n = f(1,1) + \frac{1}{n}\big(-2f(1,1) - f_x(1,1) - f_y(1,1) + o(1)\big),$$

so

$$n^3 I_n' - nf(1,1) = nJ_n - nf(1,1) = -2f(1,1) - f_x(1,1) - f_y(1,1) + o(1).$$

It follows that

$$\begin{aligned}\lim_{n\to\infty}(n^3 I_n - nf(1,1)) &= \lim_{n\to\infty}(n^3 I_n' - nf(1,1)) \\ &= -2f(1,1) - f_x(1,1) - f_y(1,1).\end{aligned}$$

432. (Corrected) Let $k \geq 1$ be an integer. Find all $\alpha \in \mathbb{R}$ with the property that there is a sequence $(a_n)_{n\geq 1}$ with $a_n > 0$ such that $a_1 + \cdots + a_{n+k} < \alpha a_n$ $\forall n \geq 1$.

`Proposed by Constantin-Nicolae Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucureşti, Romania.`

Solution by the author. Note that if α satisfies the desired condition then, so does α' when $\alpha' \geq \alpha$. Hence the set of α with the required property is an interval of the form $[\alpha_0, \infty)$ or (α_0, ∞).

First we are looking for the best, i.e., the minimal α we can obtain when we take (a_n) to be a geometric series, say, $a_n = r^n$. Of course we must have $r > 1$ since otherwise $a_1 + \cdots + a_{n+k} > a_1 + \cdots + a_n \geq na_n \geq \alpha a_n$ when $n \geq \alpha$. For $n \geq 1$ we must have

$$\alpha > \frac{a_1 + \cdots + a_{n+k}}{a_n} = \frac{r + \cdots + r^{n+k}}{r^n} = \frac{r^{n+k+1} - r}{(r-1)r^n} = \frac{r^{k+1}}{r-1} - \frac{1}{(r-1)r^{n-1}}.$$

Since $\frac{1}{(r-1)r^{n-1}} \searrow 0$ as $n \to \infty$ this is equivalent to $\alpha \geq \frac{r^{k+1}}{r-1}$. So the best α for this geometric series is $\frac{r^{k+1}}{r-1}$. The minimal value of $\frac{r^{k+1}}{r-1}$ when $r > 1$ is $\frac{(k+1)^{k+1}}{k^k}$ and it is obtained when $r = \frac{k+1}{k}$. This can be proved by taking derivatives or by arithmetic mean-geometric mean inequality. Indeed, we have

$$k = \frac{k(r-1)}{r} + k\cdot\frac{1}{r} \geq (k+1)\sqrt[k+1]{\frac{k(r-1)}{r}\cdot\left(\frac{1}{r}\right)^k} = (k+1)\sqrt[k+1]{k\frac{r-1}{r^{k+1}}},$$

so $\left(\frac{k}{k+1}\right)^{k+1} \geq k\frac{r-1}{r^{k+1}}$, i.e., $\frac{r^{k+1}}{r-1} \geq \frac{(k+1)^{k+1}}{k^k}$, with equality when $\frac{k(r-1)}{r} = \frac{1}{r} = \cdots = \frac{1}{r}$, i.e., when $r = 1 + \frac{1}{k}$.

We now prove that we cannot have better values of α when considering arbitrary sequences of positive numbers. The idea is to produce a sequence satisfying the required condition that has a long subsequence of consecutive terms that are "almost geometric" in the sense that for many consecutive values of n we have $r - \varepsilon < \frac{a_{n+1}}{a_n} < r + \varepsilon$ for some $r > 0$ and some small $\varepsilon > 0$.

Suppose that there is $\alpha < \frac{(k+1)^{k+1}}{k^k}$ such that the set

$$A = \{a = (a_n)_{n\geq 1} \mid a_n \geq 0,\, a_1 + \cdots + a_{n+k} < \alpha a_n \,\forall n \geq 1\}$$

is not empty.

If $a = (a_n)_{n\geq 1} \in A$ then for every $n \geq 1$ we have $a_{n+1} < a_1 + \cdots + a_{n+k} < \alpha a_n$, so that $\frac{a_{n+1}}{a_n} < \alpha$. When $n > 1$ we also have $a_{n-1} < \alpha a_n$, so $\frac{a_n}{a_{n-1}} > \alpha$. Hence

$$\frac{1}{\alpha} < \frac{a_{n+1}}{a_n} < \alpha \,\forall n > 1.$$

As a consequence, $\frac{a_m}{a_n} < \alpha^{|m-n|}$ whenever $m, n \geq 1$, $m \neq n$.

Let $r(a) = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$ and $r = \inf_{a\in A} r(a)$. For every $a \in A$ we have $\frac{1}{\alpha} \leq r(a) \leq \alpha$, so $\frac{1}{\alpha} \leq r \leq \alpha$.

We claim that for some large enough N we have $\sum_{i=-N}^{k} r^i > \alpha$. Indeed, if $r \leq 1$ we have $\sum_{i=-N}^{k} r^i > \sum_{i=-N}^{0} r^i \geq \sum_{i=-N}^{0} 1 = N + 1 > \alpha$ when $N = [\alpha]$. If $r > 1$ we use the fact that $\min_{r>1} \frac{r^{k+1}}{r-1} = \frac{(k+1)^{k+1}}{k^k}$ to obtain

$$\lim_{N\to\infty} \sum_{i=-N}^{k} r^i = \sum_{i=-\infty}^{k} r^i = \frac{r^{k+1}}{r-1} \geq \frac{(k+1)^{k+1}}{k^k} > \alpha.$$

We fix a small positive number ε. Since $\inf_{a\in A} r(a) = r$ there is $a \in A$ with $r(a) < r + \varepsilon$. Since $r(a) = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$ we have $\frac{a_{n+1}}{a_n} < r(a) + \varepsilon < r + 2\varepsilon$ when $n \geq N_1$, for some large enough $N_1 = N_1(\varepsilon)$.

Now since $a = (a_n)_{n\geq 1} \in A$ we have $a^{(i)} := (a_{n+i})_{n\geq 1} \in A$ for every $i \geq 0$. (If $n \geq 1$ then $\alpha a_{n+i} > a_1 + \cdots + a_{n+i+k} \geq a_{1+i} + \cdots + a_{n+k+i}$.) Also every linear combination with positive coefficients of elements of A is also in A. It follows that $b := a^{(0)} + \cdots + a^{(k-1)} \in A$. We have $b = (b_n)_{n\geq 1}$, where $b_n = a_n + \cdots + a_{n+k-1}$. Since $\limsup_{n\to\infty} \frac{b_{n+1}}{b_n} = r(b) \geq r$ there is $n \geq N + N_1$ such that $\frac{b_{n+1}}{b_n} > r - \varepsilon$. This means that $(r-\varepsilon)(a_n + \cdots + a_{n+k-1}) = (r-\varepsilon)b_n <$

$b_{n+1} = a_{n+1} + \cdots + a_{n+k}$, i.e.,

$$\sum_{i=0}^{k-1} \bigl(a_{n+i+1} - (r-\varepsilon)a_{n+i}\bigr) > 0.$$

We fix $0 \le j \le k-1$. Then for every $0 \le i \le k-1$ we have $\frac{a_{n+i}}{a_{n+j}} < \alpha^{|i-j|} \le \alpha^{k-1}$. Also $n+i \ge N+N_1 \ge N_1$, so $\frac{a_{n+i+1}}{a_{n+i}} \le r+2\varepsilon$. Hence

$$a_{n+i+1} - (r-\varepsilon)a_{n+i} < (r+2\varepsilon)a_{n+i} - (r-\varepsilon)a_{n+i} = 3\varepsilon a_{n+i} < 3\alpha^{k-1}\varepsilon a_{n+j}.$$

It follows that

$$\begin{aligned} a_{n+j+1} - (r-\varepsilon)a_{n+j} &> -\sum_{i\ne j}\bigl(a_{n+i+1} - (r-\varepsilon)a_{n+i}\bigr) > -\sum_{i\ne j} 3\alpha^{k-1}\varepsilon a_{n+j} \\ &= -3(k-1)\alpha^{k-1}\varepsilon a_{n+j}, \end{aligned}$$

so that $a_{n+j+1} > (r - (1+3(k-1)\alpha^{k-1})\varepsilon)a_{n+j}$. It follows that for $1 \le i \le k$ we have

$$\frac{a_{n+i}}{a_n} = \frac{a_{n+i}}{a_{n+i-1}} \cdots \frac{a_{n+1}}{a_n} > \bigl(r - (1+3(k-1)\alpha^{k-1})\varepsilon\bigr)^i.$$

If $1 \le j \le N$ we have $n-j \le (N+N_1) - N = N_1$ so $\frac{a_{n-j+1}}{a_{n-j}} < r+2\varepsilon$. Hence if $1 \le i \le N$ then $\frac{a_n}{a_{n-i}} = \frac{a_n}{a_{n-1}} \cdots \frac{a_{n-i+1}}{a_{n-i}} < (r+2\varepsilon)^i$, so $\frac{a_{n-i}}{a_n} > (r+2\varepsilon)^{-i}$.

Consequently, $\alpha > \frac{a_1+\cdots+a_{n+k}}{a_n} \ge \frac{a_{n-N}}{a_n} + \cdots + \frac{a_{n-1}}{a_n} + 1 + \frac{a_{n+1}}{a_n} + \cdots + \frac{a_{n+k}}{a_n} > (r+2\varepsilon)^{-N} + \cdots + (r+2\varepsilon) + 1 + \bigl(r - (1+3(k-1)\alpha^{k-1})\varepsilon\bigr) + \cdots + \bigl(r - (1+3(k-1)\alpha^{k-1})\varepsilon\bigr)^k$. But this inequality holds for every $\varepsilon > 0$ small enough. By taking $\lim_{\varepsilon\searrow 0}$ one gets $\alpha \ge \sum_{i=-N}^{k} r^i$. Contradiction.

In conclusion, there is a sequence of positive numbers $(a_n)_{n\ge1}$ with the property $a_1 + \cdots + a_{n+k} < \alpha a_n$ $\forall n \ge 1$ iff $\alpha \ge \frac{(k+1)^{k+1}}{k^k}$. $\square$

There is another approach using the following result, which is the problem 8 (SWE 3) on the shortlist for the 1976 IMO.

Lemma. Let P be a polynomial with real coefficients such that $P(x) > 0$ if $x > 0$. Prove that there exist polynomials Q and R with nonnegative coefficients such that $P(x) = \frac{Q(x)}{R(x)}$ if $x > 0$.

Of course, $P(x) = \frac{Q(x)}{R(x)}$ for all positive x is equivalent to $P(X) = \frac{Q(X)}{R(X)}$, i.e., $P(X)R(X) = Q(X)$.

As a consequence, we have:

Corollary. If $P(X) = a_0 + \cdots + a_k X^k \in \mathbb{R}[X]$ then there is a sequence $(x_n)_{n\geq 1}$ with $x_n > 0$ such that $\sum_{i=0}^{k} a_i x_{n+i} \leq 0$ for $n \geq 1$ if and only if there is $x > 0$ with $P(x) \leq 0$.

Proof. For the "if" part we assume that $x > 0$ and $P(x) \leq 0$ and we take $x_n = x^n$. Then for $n \geq 1$ we have $\sum_{i=0}^{k} a_i x_{n+i} = \sum_{i=0}^{k} a_i x^{n+i} = x^n P(x) \leq 0$.

For the "only if" part assume that $P(x) > 0$ if $x > 0$. Then by our Lemma there are $Q, R \in \mathbb{R}[X]$, $R \neq 0$, with nonnegative coefficients, such that $PR = Q$. If $R = \sum_{j=0}^{l} b_j X^j$ then $Q = \sum_{h=0}^{k+l} c_h X^h$, where $c_h = \sum_{i+j=h} a_i b_j$.

Suppose that there is a sequence $(x_n)_{n\geq 1}$ with $x_n > 0$ such that $y_n := \sum_{i=0}^{k} a_i x_{n+i} \leq 0$ for $n \geq 1$. Then $z_n := \sum_{j=0}^{l} b_j y_{n+j} \leq 0$ as $b_j \geq 0$, $y_{n+j} \leq 0$ $\forall j$. On the other hand we have $z_n = \sum_{j=0}^{l} b_j \sum_{i=0}^{k} a_i x_{n+j+i} = \sum_{h=0}^{k+l} \sum_{i+j=h} a_i b_j x_{n+h} = \sum_{h=0}^{k+l} c_h x_{n+h}$. But for every h we have $x_{n+h} > 0$ and $c_h \geq 0$ and not all c_h are 0. It follows that $z_n > 0$. Contradiction. $\square$

Remark. Note that the result above is in fact effective, in the sense that if $x_n > 0$ $\forall n$ then $y_n = \sum_{i=0}^{k} a_i x_{n+i} \leq 0$ cannot hold for $1 \leq n \leq l+1$. Indeed otherwise we would have that $z_1 = \sum_{j=0}^{l} b_j y_{1+j}$ is both ≤ 0 and > 0. (See the proof of the Corollary.) More generally, $y_n \leq 0$ cannot hold for $l+1$ consecutive values of n.

We now give a new solution to our problem, more precisely to the necessity that $\alpha \geq \frac{(k+1)^{k+1}}{k^k}$. Assume that $\alpha < \frac{(k+1)^{k+1}}{k^k}$. Recall that for $r > 1$ we have $\sum_{i=-\infty}^{k} r^i = \frac{r^{k+1}}{r-1} \geq \frac{(k+1)^{k+1}}{k^k} > \alpha$. We claim that for $N > 0$ large enough we have $\sum_{i=-N}^{k} r^i > \alpha$ $\forall r > 0$. Let $N_1 = [2\alpha - 1]$. Let $r_0 = 2^{\frac{1}{N_1}}$. If $0 < r \leq r_0$ then for $-N_1 \leq i \leq -1$ we have $r^i > r_0^{-N_1} = \frac{1}{2}$. It follows that if $N \geq N_1$ then $\sum_{i=-N}^{k} r^i > \sum_{i=-N_1}^{-1} r^i + 1 \geq N_1 \cdot \frac{1}{2} + 1 = \frac{1}{2}[2\alpha - 1] + 1 > \frac{1}{2}(2\alpha - 2) + 1 = \alpha$. For N_2 small enough we have $\frac{r_0^{-N_2-1}}{r_0-1} < \frac{(k+1)^{k+1}}{k^k} - \alpha$. Then if $N \geq N_2$ for every $r \geq r_0$ we have $\sum_{i=-\infty}^{-N-1} r^i \leq \sum_{i=-\infty}^{-N_1-1} r_0^i = \frac{r_0^{-N_1-1}}{r_0-1} < \frac{(k+1)^{k+1}}{k^k} - \alpha$

and $\sum_{i=-\infty}^{k} r^i \geq \frac{(k+1)^{k+1}}{k^k}$. By subtracting we get $\sum_{i=-N}^{k} r^i > \alpha$. In conclusion, if $N = \max\{N_1, N_2\}$ then $\sum_{i=-N}^{k} r^i > \alpha\ \forall r > 0$.

Suppose now that there is $(a_n)_{n\geq 1}$ with $a_n > 0$ and $\sum_{i=0}^{n+k} a_i < \alpha a_n\ \forall n \geq 1$. It follows that for $n \geq 1$ we have $\sum_{i=0}^{N+k} a_{n+i} - \alpha a_{n+N} \leq \sum_{i=0}^{n+N+k} a_i - \alpha a_{n+k} < 0$. By our Corollary this implies that there is $r > 0$ with $P_N(r) \leq 0$, where $P_N = 1 + \cdots + X^{N+k} - \alpha X^N$. It follows that $1 + \cdots + r^{N+k} \leq \alpha r^N$ so $\sum_{i=-N}^{k} r^i \leq \alpha$. Contradiction.

The advantage of this approach is that it produces an effective result. If $R \in \mathbb{R}[X]$ is a polynomial with $\deg R = l$ such that R and $Q = P_N R$ have nonnegative coefficients then, by the remark following the Corollary, $\sum_{i=0}^{N+k} a_{n+i} - \alpha a_{n+N} \leq 0$ cannot hold for every $1 \leq n \leq l+1$. Therefore $\sum_{i=1}^{n+k} a_i < \alpha a_{n+k}$ fails for a value of n no larger than $N + l + 1$.

Our Lemma is a particular case of a result by Pólya in [1] as a companion to Artin's theorem. It states that if $F \in \mathbb{R}[X_1, \ldots, X_n]$ is a homogenous polynomial such that $F(x_1, \ldots, x_n) > 0\ \forall x_1 \geq 0, \ldots, x_n \geq 0$ with $\sum x_j > 0$ then $F = G/H$, where G, H are homogenous polynomials with all coefficients positive (i.e., every monomial of degree $\deg G$ or $\deg H$ appears in G or H, respectively, with a positive coefficient). In Pólya's proof $H = (X_1 + \cdots + X_n)^m$ for some m large enough. Our Lemma follows by dehomogenization from the case $n = 2$ of Pólya's result. In particular, one may take $R = (X+1)^m$ for m large enough.

A further generalization of this result can be seen in [2].

References

[1] G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1952.

[2] C.N. Beli, Polynomials over ordered fields, *Journal of Pure and Applied Algebra* **209** (2007), No. 2, 507–516.

Editor's note. In our 1–2/2015 issue this problem appeared without the hypothesis '$a_n > 0$'. With this wrong statement our problem is, of course, trivial: every $\alpha \in \mathbb{R}$ qualifies. One can simply take $a_1, \ldots, a_k \in \mathbb{R}$ arbitrary and construct $a_{k+1}, a_{k+2}, \ldots$ recursively such that $a_{n+k} < \alpha a_n - (a_1 + \cdots + a_{n+k-1})\ \forall n \geq 1$.

433. Prove that

$$\sum_{k=1}^{\infty} k\left(k+1-\zeta(2)-\zeta(3)-\cdots-\zeta(k+1)\right)=\zeta(3),$$

where ζ denotes the Riemann zeta function.

Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, România.

Solution by the author. Let $S_k=\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)^k}$. Note that $S_1=1$. Since

$$\frac{1}{n(n+1)^{k+1}}=\frac{1}{n(n+1)^k}-\frac{1}{(n+1)^{k+1}},$$

we have, by summation, that $S_{k+1}=S_k-(\zeta(k+1)-1)$. This implies, since $S_1=1$, that

$$\begin{aligned}S_{k+1}&=S_1-(\zeta(2)+\zeta(3)+\cdots+\zeta(k+1)-k)\\&=k+1-\zeta(2)-\zeta(3)-\cdots-\zeta(k+1).\end{aligned}$$

We have, based on the relation above, that

$$\sum_{k=1}^{\infty} k\left(k+1-\zeta(2)-\zeta(3)-\cdots-\zeta(k+1)\right)=\sum_{k=1}^{\infty}k\sum_{n=1}^{\infty}\frac{1}{n(n+1)^{k+1}}$$

$$=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\sum_{k=1}^{\infty}\frac{k}{(n+1)^k}=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\cdot\frac{\frac{1}{n+1}}{\left(1-\frac{1}{n+1}\right)^2}=\sum_{n=1}^{\infty}\frac{1}{n^3}=\zeta(3).$$

We used in the preceding calculations the power series formula

$$\sum_{k=1}^{\infty}kx^k=\frac{x}{(1-x)^2},\quad x\in(-1,1).$$

The problem is solved. □

Editor's note. We received essentially the same solution from our reader Moubinool Omarjee from Lycée Henri IV, Paris, France.

434. If $-1\le\alpha<0$, show that the function

$$f(x)=(x+4)^{\alpha}-3(x+6)^{\alpha}+x(x+3)^{\alpha}-x(x+5)^{\alpha}$$

is strictly increasing for $x\ge 0$.

Proposed by Ioan Tomescu, Faculty of Mathematics and Computer Science, University of Bucharest, Bucureşti, Romania.

Solution by the author. We can write $g(x) = \eta(x) - \eta(x+2) + \psi(x)$, where $\eta(x) = x(x+3)^\alpha + 2(x+5)^\alpha + (x+4)^\alpha$ and $\psi(x) = 2(x+7)^\alpha - 2(x+6)^\alpha$. Then $\psi'(x) > 0$, so ψ is strictly increasing. Also, $\eta(x)$ is strictly concave since

$$\begin{aligned}\eta''(x) =& (x+3)^{\alpha-2}(\alpha(\alpha+1)x + 6\alpha) + 2\alpha(\alpha-1)(x+5)^{\alpha-2}\\ &+ \alpha(\alpha-1)(x+4)^{\alpha-2} < \alpha(\alpha+1)(x+3)^{\alpha-1}.\end{aligned}$$

(Here we used the fact that $(x+5)^{\alpha-2}, (x+4)^{\alpha-2} < (x+3)^{\alpha-2}$.) It follows that $\eta(x) - \eta(x+2)$ is also strictly increasing. Hence our result. □

435. (Corrected) Let $f(x) = \frac{e^x}{(a+be^x)^2}$, where a, b are two integers with $a+b = 1$.

(i) Prove that $f^{(k)}(0) \in \mathbb{Z}$ for any $k \geq 0$. (Here $f^{(0)} = f$ and for $k \geq 1$ $f^{(k)}$ is the kth derivative of f.)

(ii) Prove that if $a \neq 0$, p is a prime and $\alpha \geq 1$ then for any $k, l \geq \alpha - 1$ with $k \equiv l \pmod{p^{\alpha-1}(p-1)}$ we have $f^{(k)}(0) \equiv f^{(l)}(0) \pmod{p^\alpha}$.

`Proposed by Constantin-Nicolae Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucureşti, România.`

Solution by the author. We prove by induction on k that $f^{(k)}(x) = \frac{P_k(e^x)}{(a+be^x)^{k+2}}$, where $P_k \in \mathbb{Z}[X]$ with $X \mid P_k$ and $\deg P_k \leq k+1$. For $k = 0$ this statement is obvious, with $P_0 = X$. For the induction step $k \mapsto k+1$ note that

$$\begin{aligned}f^{(k+1)}(x) &= \left(\frac{P_k(e^x)}{(a+be^x)^{k+2}}\right)' = \frac{P_k'(e^x)e^x(a+be^x) - (k+2)be^x P_k(e^x)}{(a+be^x)^{k+3}}\\ &= \frac{P_{k+1}(e^x)}{(a+be^x)^{k+3}},\end{aligned}$$

where $P_{k+1}(X) = X(a+bX)P_k'(X) - (k+2)bXP_k(X)$. Obviously $X \mid P_{k+1}$ and, since $b \in \mathbb{Z}$, $P_k \in \mathbb{Z}[X]$ and $\deg P_k \leq k+1$, we have $P_{k+1} \in \mathbb{Z}[X]$ and $\deg P_{k+1} \leq k+1$.

We have

$$f^{(k)}(0) = \frac{P_k(e^0)}{(a+be^0)^{k+2}} = \frac{P_k(1)}{(a+b)^{k+2}} = P_k(1) \in \mathbb{Z},$$

which proves (i).

For (ii) we assume that $k > l$. We write $P_k(X) = XQ_k(X)$, with $Q_k \in \mathbb{Z}[X]$ and $\deg Q_k \leq k$. Then $f^{(k)}(x) = e^x g_k(e^x)$, where $g_k(X) = \frac{Q_k(X)}{(a+bX)^{k+2}}$. Since $\deg Q_k \leq k$ we have $g_k(X) = \sum_{i=2}^{k+2} \frac{a_i}{(a+bX)^i}$ for some $a_i \in \mathbb{Q}$. Then $f^{(k)}(x) = \sum_{i=2}^{k+2} \frac{a_i e^x}{(a+be^x)^i}$. Similarly $f^{(l)}(x) = \sum_{i=2}^{l+2} \frac{b_i e^x}{(a+be^x)^i}$ for some $b_i \in \mathbb{Q}$. It follows that $f^{(k)}(x)-f^{(l)}(x)=\sum_{i=2}^{k+2}\frac{c_ie^x}{(a+be^x)^i}$, where $c_i=a_i-b_i$ if $i\le l+2$ and $c_i=a_i$ otherwise. Since $a+b=0$, we have $f^{(k)}(0)-f^{(l)}(0)=\sum_{i=2}^{k+2}c_i$. Therefore it suffices to prove that $c_i\in\mathbb{Z}$ and $p^\alpha\mid c_i$ for $2\le i\le k+2$.

Now if $x<\log|a/b|$ then $|b/a\,e^x|<1$, so for every i we have the expansion

$$\begin{aligned}\frac{e^x}{(a+be^x)^i}&=a^{-i}e^x(1+b/a\,e^x)^{-i}=a^{-i}e^x\sum_{n=0}^{\infty}\binom{-i}{n}(b/a)^ne^{nx}\\&=a^{-i}e^x\sum_{n=0}^{\infty}\binom{n+i-1}{i-1}(b/a)^ne^{nx}\\&=a^{-i}\sum_{n=1}^{\infty}\binom{n+i-2}{i-1}(b/a)^{n-1}e^{nx}.\end{aligned}$$

Here we have used $\binom{-i}{n}=\frac{(-i)(-i-1)\cdots(-i-n+1)}{n!}=(-1)^n\binom{n+i-1}{n}=(-1)^n\binom{n+i-1}{i-1}$.

It follows that

$$f^{(k)}(x)-f^{(l)}(x)=\sum_{n=1}^{\infty}\sum_{i=2}^{k+2}c_ia^{-i}(-b/a)^{n-1}\binom{n+i-2}{i-1}e^{nx}.$$

On the other hand, when $i=2$ we get

$$f(x)=\frac{e^x}{(a+be^x)^2}=\sum_{n=1}^{\infty}a^{-2}(-b/a)^{n-1}ne^{nx}.$$

By taking successive derivatives we get $f^{(k)}(x)=\sum_{n=1}^{\infty}a^{-2}(-b/a)^{n-1}n^{k+1}e^{nx}$ and similarly for $f^{(l)}(x)$. We therefore get

$$f^{(k)}(x)-f^{(l)}(x)=\sum_{n=1}^{\infty}a^{-2}(-b/a)^{n-1}(n^{k+1}-n^{l+1})e^{nx}.$$

By identifying the coefficients of e^{nx} in the two formulas above for $f^{(k)}(x)-f^{(l)}(x)$ one gets $\sum_{n=1}^{\infty}\binom{n+i-2}{i-1}(b/a)^{n-1}=a_ia^{-2}(-b/a)^{n-1}(n^{k+1}-n^{l+1})e^{nx}$, i.e., $n^{k+1}-n^{l+1}=\sum_{i=2}^{k+2}c_ia^{2-i}\binom{n+i-2}{i-1}$ for $n\ge 1$. It follows that we have the polynomial equality $X^{k+1}-X^{l+1}=\sum_{i=2}^{k+2}c_ia^{2-i}\binom{X+i-2}{i-1}$.

For any $n\in\mathbb{Z}$ we have either $p\mid n$, so $p^\alpha\mid n^{l+1}$ (because $l\ge\alpha-1$), or $p\nmid n$, so $p^\alpha\mid n^{k-l}-1$ (since $\phi(p^\alpha)=p^{\alpha-1}(p-1)\mid k-l$). In both cases $p^\alpha\mid n^{l+1}(n^{k-l}-1)=n^{k+1}-n^{l+1}$. Hence $P(X):=p^{-\alpha}(X^{k+1}-X^{l+1})$

has the property that $P(n) \in \mathbb{Z}\ \forall n \in \mathbb{Z}$. Since also $\deg P = k+1$, we have $P(X) = \sum_{j=0}^{k+1} d_j \binom{X+j-1}{j}$ for some $d_j \in \mathbb{Z}$. (We have the following general result: If $P_j \in \mathbb{Q}[X]$ with $P_j(\mathbb{Z}) \subseteq \mathbb{Z}$, $\deg P_j = j$ and the dominant coefficient of P_j is $\frac{1}{j!}$ then for every $m \geq 0$ $P_0, \dots, P_m$ is a basis of the $\mathbb{Z}$-module $M_m := \{P \in \mathbb{Q}[X] \mid P[\mathbb{Z}] \subseteq \mathbb{Z}, \deg P \leq m\}$. In our case $P \in M_{k+1}$ and we take $P_j = \binom{X+j-1}{j}$.)

Then

$$\sum_{i=2}^{k+2} c_i a^{2-i} \binom{X+i-2}{i-1} = X^{k+1} - X^{l+1} = p^\alpha P(X) = \sum_{j=0}^{k+1} p^\alpha d_j \binom{X+j-1}{j}.$$

It follows that $d_0 = 0$ and for $2 \leq i \leq k+2$ we have $c_i a^{2-i} = p^\alpha d_{i-1}$ and therefore $c_i = p^\alpha a^{i-2} d_{i-1}$, which is an integer divisible by p^α. □

For the part (i) of the problem we received essentially the same solution from *Daniel Văcaru, Piteşti, Romania.*

Editor's note. In our 1–2/2015 issue this problem appeared with $k, l \geq 0$ instead of $k, l \geq \alpha - 1$. With this old statement the problem is wrong. For example, when $a = 2$, $b = -1$, $p = 2$, $\alpha = 3$, $k = 4$, $l = 0$ we have $f(0) = 1$ and $f^{(4)}(0) = 541$, but $541 \not\equiv 1 \pmod 8$.

436. Calculate

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln \sqrt{n(n+1)} - \gamma\right),$$

where γ denotes the Euler–Mascheroni constant.

`Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.`

Solution by the author. The series equals $\frac{1}{2} - \ln \sqrt{2\pi} + \gamma$. We need the following lemma.

Lemma *(Abel's summation by parts formula)* [1, pag. 55], [2, p. 258]

Let $(a_n)_{n\geq 1}$ and $(b_n)_{n \geq 1}$ be two sequences of real numbers and let $A_n = \sum_{k=1}^{n} a_k$. Then,

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k \left(b_k - b_{k+1}\right).$$

The lemma can be proved by elementary calculations. Also, we will be using in our calculations the infinite version of the preceding lemma:

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} (A_n b_{n+1}) + \sum_{k=1}^{\infty} A_k (b_k - b_{k+1}) . \tag{1}$$

Now we are ready to calculate our series. We apply formula (1) with $a_k = 1$ and $b_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k} - \ln\sqrt{k(k+1)} - \gamma$, and we get that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln\sqrt{n(n+1)} - \gamma\right) \\
&= \lim_{n\to\infty} n \left(1 + \frac{1}{2} + \cdots + \frac{1}{n+1} - \ln\sqrt{(n+1)(n+2)} - \gamma\right) \\
&+ \sum_{n=1}^{\infty} n \left(\ln\sqrt{\frac{n+2}{n}} - \frac{1}{n+1}\right) \\
&= \sum_{n=1}^{\infty} \left(\frac{n}{2}\ln\frac{n+2}{n} - 1 + \frac{1}{n+1}\right),
\end{aligned}$$

since $\lim_{n\to\infty} n\left(1 + \frac{1}{2} + \cdots + \frac{1}{n+1} - \ln\sqrt{(n+1)(n+2)} - \gamma\right) = 0$.

Let S_n be the nth partial sum of the preceding series. We have,

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} \left(\frac{k}{2}\ln\frac{k+2}{k} - 1 + \frac{1}{k+1}\right) \\
&= -\ln n! + \frac{n-1}{2}\ln(n+1) + \frac{n}{2}\ln(n+2) + H_{n+1} - n - 1,
\end{aligned}$$

where $H_n = 1 + 1/2 + \cdots + 1/n$ denotes the nth harmonic number.

An application of Stirling's formula [2, pag. 259]

$$\ln n! = \frac{1}{2}\ln(2\pi) + \frac{2n+1}{2}\ln n - n + O\left(\frac{1}{n}\right)$$

shows that

$$\begin{aligned}
S_n =& -1 - \frac{1}{2}\ln(2\pi) + H_{n+1} - \ln(n+1) + \frac{1}{2}\ln\frac{n+1}{n} + \frac{n}{2}\ln\frac{n+1}{n} \\
&+ \frac{n}{2}\ln\frac{n+2}{n} - O\left(\frac{1}{n}\right),
\end{aligned}$$

which implies that $\lim_{n\to\infty} S_n = \frac{1}{2} - \ln\sqrt{2\pi} + \gamma$ and the problem is solved. □

References

[1] Daniel D. Bonar, Michael J. Khoury, *Real Infinite Series*, Classroom Resource Materials, The Mathematical Association of America, Washington DC, 2006.

[2] Ovidiu Furdui, *Limits, Series and Fractional Part Integrals*, Problems in Mathematical Analysis, Springer, London, 2013.

Solution by Moubinool Omarjee, Lycée Henri IV, Paris, France. We have

$$\sum_{n=1}^{N} H_n = \sum_{n=1}^{N}\sum_{k=1}^{n}\frac{1}{k} = \sum_{k=1}^{N}\sum_{n=k}^{N}\frac{1}{k} = \sum_{k=1}^{N}\frac{N-k+1}{k} = (N+1)H_n - N.$$

It follows that

$$\sum_{n=1}^{N}\left(H_n - \ln\sqrt{n(n+1)} - \gamma\right) = (N+1)H_N - N - \ln(N!) - \frac{1}{2}\ln(N+1) - N\gamma$$
$$= (N+1)\left(\ln N + \gamma + \frac{1}{2N} + o\left(\frac{1}{N}\right)\right) - N - \left(\frac{\ln N}{2} + o(1)\right) - N\gamma$$
$$- \left(\frac{\ln(2\pi)}{2} - N + \left(N + \frac{1}{2}\right)\ln N + o(1)\right) = \frac{1}{2} + \gamma - \frac{\ln(2\pi)}{2} + o(1).$$

Hence

$$\sum_{n=1}^{\infty}(H_n - \ln\sqrt{n(n+1)} - \gamma) = \frac{1}{2} + \gamma - \frac{\ln(2\pi)}{2}.$$

□

437. Let $f : [0,1] \to \mathbb{R}$ with $f(0) \in \mathbb{Q}$ and, for every $x \in [0,1]$, there exists $\varepsilon_x > 0$ such that $f(x) - f(y) \in \mathbb{Q}$ for $|x - y| < \varepsilon_x$ and $y \in [0,1]$. Prove that $f(x) \in \mathbb{Q}$ for every $x \in [0,1]$.

`Proposed by George Stoica, Department of Mathematical Sciences, University of New Brunswick, Canada.`

Solution by Victor Makanin, Sankt Petersburg, Russia. We see that for every $x \in [0,1]$ there exists an interval $I_x = [x - \epsilon_x, x + \epsilon_x]$ such that $f(x) - f(y) \in \mathbb{Q}$ for every $y \in I_x \cap [0,1]$. It follows that $f(z) - f(y) = (f(x) - f(y)) - (f(x) - f(z))$ belongs to $\mathbb{Q}$ for every $y, z \in I_x \cap [0,1]$ (and for every $x \in [0,1]$); thus, if I_x contains a point z such that $f(z) \in \mathbb{Q}$, then $f(y)$ is rational for all $y \in I_x \cap [0,1]$.

The family of the intervals $J_x = (x - \epsilon_x, x + \epsilon_x)$ with $x \in [0,1]$ is an open cover of the compact $[0,1]$, hence, by the Heine-Borel theorem, a finite subcover must exist: there are $x_1, \ldots, x_n \in [0,1]$ such that $[0,1] \subseteq \cup_{i=1}^{n} J_{x_i} \subseteq \cup_{i=1}^{n} I_{x_i}$. We denote $I_{x_i} \cap [0,1] = [a_i, b_i]$, with $a_i < x_i < b_i$ for every $1 \leq i \leq n$. ($I_{x_i} \cap [0,1]$ contains $J_{x_i} \cap (0,1)$, which ia a non-empty open interval. Hence it is an interval of positive length.) Note that we can (and we do) chose the indexation such that $a_1 \leq a_2 \leq \cdots \leq a_n$.

Then, necessarily, $a_0 = 0$ (otherwise $0 < a_1 \leq a_2 \leq \cdots \leq a_n$, and 0 would be in none of the intervals I_{x_i}); due to the above observation and to the fact that $f(0)$ is rational, we have that $f(t)$ is rational for all $t \in I_{x_1}$. Now if a_2, the leftmost extremity of I_{x_2} lied to the right of b_1, the same would be true for all a_i with $i \geq 2$, thus between b_1 and a_2 some points would remain uncovered by any of the intervals I_{x_i} — which is not possible. So $a_2 \leq b_1$, and $I_{x_1} \cap I_{x_2}$ is nonempty. This means that f takes at least one rational value in I_{x_2} — consequently, $f(t)$ is rational for all $t \in I_{x_2}$.

Clearly, this reasoning can be repeated for I_{x_3} (by observing that one has $a_3 \leq \max\{b_1, b_2\}$), and, inductively, for each I_{x_i}, in order to infer that $f(t) \in \mathbb{Q}$ for each $t \in I_{x_i}$ and for every $1 \leq i \leq n$, that is, for all $t \in [0,1]$. So the problem is solved. $\square$

438. Let $f_1, \ldots, f_n$ be polynomials with real coefficients with positive dominant coefficients. Suppose that there is $M \in \mathbb{R}$ and $f \in \mathbb{R}[X]$ such that $f_i(x) > 0$ for $1 \leq i \leq n$ and $\sqrt{f_1(x)} + \cdots + \sqrt{f_n(x)} = f(x)\ \forall x > M$. Prove that for $1 \leq i \leq n$ we have $f_i = g_i^2$ for some $g_i \in \mathbb{R}[X]$.

`Proposed by Marius Cavachi, Ovidius University, Constanţa, Romania.`

Solution by the author. For convenience, for every polynomial u we write $\sqrt{u}$ instead of $\sqrt{u(x)}$. When necessary, we assume that x is large enough.

We will prove by induction on n the following stronger result:

If $f, f_1, \ldots, f_n \in \mathbb{R}[X]$, where f_is have positive dominant coefficients and are not squares of polynomials, and there are $e_1, \ldots, e_n \in \{\pm 1\}$ such that $e_1\sqrt{f_1} + \cdots + e_n\sqrt{f_n} = f(x)$ then $f = 0$.

Take first $n = 2$. If $f \neq 0$ then $(e_1\sqrt{f_1})^2 = (f - e_2\sqrt{f_2})^2$ implies $\sqrt{f_2} = \frac{f^2 - f_1 + f_2}{2e_2 f}$, so $f_2 = \frac{P^2}{Q^2}$, where $P = f^2 - f_1 + f_2$ and $Q = 2e_2 f$. But we have $Q^2 \mid P^2$, which implies $Q \mid P$. Hence f_2 is the square of the polynomial $\frac{P}{Q}$, in contradiction with the hypothesis.

Assume now that $n \geq 3$. Then there are two of $e_1, \ldots, e_n$ which are equal. Say, $e_1 = e_2$. We may assume that $e_1 = e_2 = 1$ since otherwise we multiply everything with -1, i.e., we replace every e_i by $-e_i$ and f by $-f$. We differentiate the formula $\sum\limits_{i=1}^{n} e_i\sqrt{f_i} = f$ and obtain $\sum\limits_{i=1}^{n} e_1\frac{\sqrt{f_1}}{2f_i} = f'$. We get a system of linear equations

$$\sqrt{f_1}\frac{f_1'}{f_1} + \sqrt{f_2}\frac{f_2'}{f_2} = 2f' - \sum_{k=3}^{n} \sqrt{f_k}\frac{e_k f_k'}{f_k},$$

$$\sqrt{f_1} + \sqrt{f_2} = f - \sum_{k=3}^{n} e_k\sqrt{f_k},$$

where the unknowns are $\sqrt{f_1}, \sqrt{f_2}$. The determinant d of this system is $\frac{f_1'}{f_1} - \frac{f_2'}{f_2}$.

If $d = 0$ then $\left(\frac{f_1}{f_2}\right)' = 0$, whence $f_1 = kf_2$ for some $k \in \mathbb{R}$, $k > 0$. It follows that $\sqrt{f_1} + \sqrt{f_2} = \sqrt{f_2(1+\sqrt{k})^2}$, so $\sqrt{f_2(1+\sqrt{k})^2} + \sum\limits_{k=3}^{n} e_k\sqrt{f_k} = f$. By the induction step $n-1$ we have $f = 0$.

If $d \neq 0$ we solve the system and get

$$d\sqrt{f_1} = \left(2f' - \sum_{k=3}^{n} \sqrt{f_k}\frac{e_k f_k'}{f_k}\right) - \frac{f_2'}{f_2}\left(f - \sum_{k=3}^{n} \sqrt{f_k}\right).$$

After chasing denominators by multiplication with a suitable polynomial $h \neq 0$, we end up with a relation of the form $u_1\sqrt{f_1} = \big(2f' - \frac{f_2'}{f_2}f\big)h + \sum\limits_{k=3}^{n} u_k\sqrt{f_k}$ for some polynomials u_k. But, when $u_k \neq 0$, $u_k\sqrt{f_k} = e_k'\sqrt{u_k^2 f_k}$, where $e_k' \in \{\pm 1\}$ is the sign of u_k, so we rewrite the last equality as

$$e_1'\sqrt{g_1} - \sum_{k=3}^{n}{}' e_k'\sqrt{g_k} = g,$$

where $\sum'$ means the sum restricted to the terms with $u_k \neq 0$, $g_k = u_k^2 f_k$ and $g = \big(2f' - \frac{f_2'}{f_2}f\big)h$. By the induction hypothesis we get $g = 0$. If $f \neq 0$ this implies $\left(\frac{f_2}{f^2}\right)' = \frac{f_2'f - 2f_2f'}{f^3} = 0$, so $f_2 = kf^2$ for some $k \in \mathbb{R}$. Since the dominant coefficient of f_2 is positive we have $k > 0$, so $f_2 = (\sqrt{k}f)^2$, contradiction. Hence $f = 0$. $\square$

Solution by Victor Makanin, Sankt Petersburg, Russia. First note that if polynomials $u, v, w \in \mathbb{R}[X]$ are such that $u(x) + v(x)\sqrt{w(x)} = 0$ for all sufficiently large x (assuming, of course, that $w(x)$ is positive for such values of x), then we have either $u = v = 0$, or that w is the square of a real polynomial. This follows from the fact that $\mathbb{R}[X]$ is a unique factorization domain (because the given relation yields $u^2(x) = v^2(x)w(x)$ for infinitely many x, which implies $u^2 = v^2w$, and, if u and v are nonzero, this equality leads to the conclusion that every irreducible factor of w has even exponent in its factorization).

Second, let us consider the polynomial

$$P(t, t_1, t_2, \ldots, t_k) = \prod(t - t_1 \pm t_2 \pm \cdots \pm t_k),$$

where the product is over all 2^{k-1} possibilities of choosing the signs plus and minus. One immediately sees that P is even in each of the variables $t_2, \ldots, t_k$, so

$$P(t) = Q(t, t_1^2, t_2^2, \ldots, t_k^2) + t_1R(t, t_1^2, t_2^2, \ldots, t_k^2),$$

where Q and R are real polynomials in $k+1$ variables.

Now we solve the problem, by inducting on n. Of course, we may assume that f is not identically zero. For $n=1$ everything is clear. We assume the enounce is true for $n \leq k-1$, and we prove it for $n=k \geq 2$. We thus have $f(x) = \sqrt{f_1(x)} + \sqrt{f_2(x)} + \cdots + \sqrt{f_k(x)}$ for the real polynomials $f, f_1, \ldots, f_k$ and for big enough x and we want to prove that each of $f_1, \ldots, f_k$ is the square of a polynomial. We obviously have

$$P(f(x), \sqrt{f_1(x)}, \sqrt{f_2(x)}, \ldots, \sqrt{f_k(x)}) = 0$$

for every sufficiently big x, therefore

$$Q(f(x), f_1(x), f_2(x), \ldots, f_k(x)) + \sqrt{f_1(x)} R(f(x), f_1(x), f_2(x), \ldots, f_k(x)) = 0$$

for all such x. By the first observation above, we have either

$$Q(f(x), f_1(x), f_2(x), \ldots, f_k(x)) = R(f(x), f_1(x), f_2(x), \ldots, f_k(x)) = 0$$

for big enough x, or f_1 is the square of a polynomial — and we are done in the second case, by using the inductive hypothesis. Nevertheless, if we are in the first case, we get

$$Q(f(x), f_1(x), f_2(x), \ldots, f_k(x)) - \sqrt{f_1(x)} R(f(x), f_1(x), f_2(x), \ldots, f_k(x)) = 0,$$

that is

$$\begin{aligned} &P(f(x), -\sqrt{f_1(x)}, \sqrt{f_2(x)}, \ldots, \sqrt{f_k(x)}) = \\ &= \prod (f(x) + \sqrt{f_1(x)} \pm \sqrt{f_2(x)} \pm \cdots \pm \sqrt{f_k(x)}) = 0 \end{aligned}$$

for big enough x (as before, the product is over all possible choices of the plus/minus signs). This means that for each x there is a choice of the $\pm$ signs such that

$$\sqrt{f_1(x)} + \cdots + \sqrt{f_n(x)} = f(x) = -\sqrt{f_1(x)} \pm \sqrt{f_2(x)} \pm \cdots \pm \sqrt{f_k(x)}.$$

But if we take x large enough such that $f_1(x) > 0$ (not merely $f_1(x) \geq 0$) then the left side of the equality above is strictly larger than the right side. Contradiction. Hence we have our result. $\square$