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On a two parameter class of quadratic Diophantine equations

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Abstract. We characterize the solution set of a two parameter class of quadratic Diophantine equations. Our proof relies on the solvability of the positive Pell equation.

Keywords: Diophantine equation, quadratic equation, Pell equation

MSC: Primary 11D09; Secondary 40-01.

In this note we will be interested in exploring the solutions of quadratic Diophantine equations of the form

$$ax^2 + by^2 + cx + dy + f = 0,$$

where $a, b, c, d, f \in \mathbb{Z}$. Naturally, there is no hope in solving such a problem for arbitrary coefficients, and simple examples show that there are such equations which have zero, finitely many or infinitely many solutions $(x, y) \in \mathbb{Z}\times\mathbb{Z}$. Equivalently, we can see this to be the case by inspecting the lattice points sitting on the algebraic curve defined by the quadratic equation above, but this geometric intuition is not the one we will be using here. Our motivation for this note stems from the following two problems proposed at the 7–8 grade level:

[3]. *Let $a, b \in \mathbb{N}$ be such that $4a^2 + 9a + 1 = 3b^2 + 7b$. Show that $3a + 3b + 7$ and $4a + 4b + 9$ are perfect squares.*

[4]. *Let $x, y \in \mathbb{N}$ such that $x > y$ and $x + 4x^2 = y + 5y^2$. Show that $x - y$ is a perfect square.*

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We wish to show that the intuition behind these problems lends itself nicely to completely solving a two parameter class of quadratic Diophantine equations. Specifically, let $\alpha, \beta \in \mathbb{N} \cup \{0\}$ and consider the following equation in variables $(x, y) \in \mathbb{Z} \times \mathbb{Z}$:

$$\alpha x^2 - (\alpha+1)y^2 + (2\alpha\beta+1)x - (2\alpha\beta+2\beta+1)y - \beta^2 = 0. \tag{1}$$

Clearly, for $\alpha = 3$ and $\beta = 1$ we obtain the equation in [3], while for $\alpha = 4$, $\beta = 0$ we get the equation in [4].

Let us remark right away that when $\alpha = 0$, the equation (1) reduces to $x = y^2 + (2\beta+1)y + \beta^2$, which already produces the parametrization for all the infinitely many integer solutions of the equation. In fact, slightly re-writing, we have in this case $x = (y+\beta)^2 + y$, $y \in \mathbb{Z}$, which suggests that understanding the solutions of the two parameter Diophantine equation (1) could potentially be reduced to the solutions of just a one parameter Diophantine equation by appropriately substituting for x and y. We show first that this is indeed the case.

Reduction to a one parameter Diophantine equation. Let us make the change of variables

$$X = x + \beta \text{ and } Y = y + \beta.$$

Thus, replacing $x = X - \beta$ and $y = Y - \beta$ in (1) we obtain

$$\alpha(X-\beta)^2 - (\alpha+1)(Y-\beta)^2 + (2\alpha\beta+1)(X-\beta) - (2\alpha\beta+2\beta+1)(Y-\beta) = \beta^2,$$

which after some straightforward algebra simplifies to

$$\alpha X^2 - (\alpha+1)Y^2 + X - Y = 0.$$

This simply means that in the two parameter Diophantine equation (1) we can assume without loss of generality $\beta = 0$. Moreover, if a solution of (1) exists for $\beta = 0$, the general solution for $\beta \neq 0$ is obtained from that one by translating by $-\beta$ in both unknowns.

Note also that if we let $\alpha = 0$ and $\beta = 0$ in (1), we already have the parametrization of all the integer solutions: $x = y^2 - y$, with $y \in \mathbb{Z}$. With these considerations, we will assume in the remainder of this note that $\alpha \in \mathbb{N}$ and $\beta = 0$ in (1), that is we will investigate the integer solutions of

$$\alpha x^2 - (\alpha+1)y^2 + x - y = 0. \tag{2}$$

Our main result is the following.

Theorem 1. *Let $\alpha \in \mathbb{N}$. The non-trivial solutions $(x_n, y_n)_{n\geq 1}$ of the quadratic Diophantine equation (2) are given by*

$$x_n = (\alpha+1)v_n^2 \pm u_n v_n, \quad y_n = \alpha v_n^2 \pm u_n v_n,$$

where

$$u_n = \frac{(\sqrt{\alpha+1}+\sqrt{\alpha})^{2n} + (\sqrt{\alpha+1}-\sqrt{\alpha})^{2n}}{2},$$

$$v_n = \frac{(\sqrt{\alpha+1}+\sqrt{\alpha})^{2n} - (\sqrt{\alpha+1}-\sqrt{\alpha})^{2n}}{2\sqrt{\alpha(\alpha+1)}}.$$

Moreover, the solution set of (1) *is given by*

$$\{(x_n-\beta, y_n-\beta) : n \in \mathbb{N}\} \cup \{(-\beta,-\beta)\},$$

with x_n, y_n *as above.*

The word "trivial" above refers to the pair $(0,0)$ which obviously satisfies the given equation. We point out also that the $\pm$ signs in the expressions of x_n and y_n, respectively, coincide. We have the following immediate consequence of our main result.

Corollary 2. *Let* $(x,y) \in \mathbb{Z}$ *be a solution of* (2)*. Then* $x-y$ *is a perfect square and the values of* $x-y$ *belong precisely to the set*

$$\{0\} \cup \{v_n^2 : n \in \mathbb{N}\},$$

with v_n *defined in Theorem 1.*

Note that, in particular, we recover the statement in [4]. In fact, the idea of proving Theorem 1 is guided by the intuition contained in the elementary problems [3] and [4], combined with the well-understood theory of positive Pell equations. As we shall see, the statement in [3] is also a by-product of our arguments showing the main result. Before proceeding with the proof of Theorem 1, we take a brief excursion into the theory of Pell equations.

The general positive Pell equation. Let D be a positive integer that is not a perfect square. It is a known fact that the *positive Pell equation*

$$u^2 - Dv^2 = 1 \tag{3}$$

has infinitely many solutions in $\mathbb{Z}\times\mathbb{Z}$. Clearly, if $(u,v) \in \mathbb{N}\times\mathbb{N}$ is a solution to the Pell equation, then $(u,-v), (-u,v)$ and $(-u,-v)$ are also solutions. Thus, without loss of generality, it suffices to consider only its solutions in the natural lattice. Any equation of the form (3) has the *trivial* solution $(1,0)$. Besides the trivial solution, the other solutions in the natural lattice can be obtained from the *fundamental solution* (u_0, v_0), which is the least positive integer solution to (3) different from $(1,0)$ — the so-called *fundamental solution* of (3) for which the expression $u + v\sqrt{D}$ is minimal, via the following formulas:

$$u_n = \frac{(u_0+v_0\sqrt{D})^n + (u_0-v_0\sqrt{D})^n}{2}$$

$$v_n = \frac{(u_0+v_0\sqrt{D})^n - (u_0-v_0\sqrt{D})^n}{2\sqrt{D}}, n \in \mathbb{N}.$$

For a brief introduction to Pell equations, the interested reader can consult, for example, [1] and the references therein.

With these prerequisites we are ready to proceed with the proof of our main result.

Proof of Theorem 1. In what follow, we assume that both unknowns, x and y, are non-zero. In particular, this means that $x \neq y$.

Our first claim, following the statement in [4], is that if x, y satisfy (2), then $x - y$ must be a perfect square. Let us start by re-writing the equation (2) in two ways:

$$(x-y)[1+\alpha(x+y)] = y^2,$$
$$(x-y)[1+(\alpha+1)(x+y) = x^2.$$

Denoting $A = 1+\alpha(x+y)$ and $B = 1+(\alpha+1)(x+y)$ and multiplying these two equations we obtain $(x-y)^2AB = (xy)^2$, thus showing that AB must be a perfect square. In particular, we find that A and B must be either both positive or both negative.

Let us assume for the moment that *both A and B are positive.* Note now that $(\alpha+1)A - \alpha B = 1$. Therefore, A and B are relatively prime natural numbers. Combining this with the fact that AB is a perfect square shows that A and B have to be perfect squares as well; this is a simple consequence of the Fundamental Theorem of Arithmetic. Incidentally, the statement that A and B must be perfect squares is precisely the content of the problem [3] if we take into account also the natural change of variables reducing (1) to (2). Now, since A is a perfect square and $(x-y)A$ is a perfect square, we conclude that $x - y > 0$ must also be a perfect square. This proves our first claim.

With this information in hand, let us write then $x - y = v^2$ for some $v \in \mathbb{N}$. Our next claim is that v must be of the form v_n, $n \geq 1$, with v_n as in the statement of our Theorem 1. Substituting $x = y + v^2$ in (2) we find that

$$\alpha(y+v^2)^2 - (\alpha+1)y^2 + v^2 = 0 \Leftrightarrow y^2 - 2\alpha v^2 y - (\alpha v^4 + v^2) = 0.$$

We are interested in the integer solutions of the quadratic equation in y, which are given by

$$y = \alpha v^2 \pm v\sqrt{\alpha(\alpha+1)v^2+1}. \tag{4}$$

Clearly, for y to be an integer, we need now

$$\alpha(\alpha+1)v^2 + 1 = u^2 \Leftrightarrow u^2 - \alpha(\alpha+1)v^2 = 1, \tag{5}$$

for some $u \in \mathbb{Z}$. But this is exactly a positive Pell equation of the form (3), with $D = \alpha(\alpha+1)$ obviously not being a perfect square. It is equally easy to see that the fundamental solution of (5) is given by

$$u_0 = 2\alpha+1, \;\; v_0 = 2.$$

Thus, the general solution of (5) in $\mathbb{N} \times \mathbb{N}$ can be expressed as

$$u_n = \frac{(2\alpha+1+2\sqrt{\alpha(\alpha+1)})^n + (2\alpha+1-2\sqrt{\alpha(\alpha+1)})^n}{2},$$

$$v_n = \frac{(2\alpha+1+2\sqrt{\alpha(\alpha+1)})^n - (2\alpha+1-2\sqrt{\alpha(\alpha+1)})^n}{2\sqrt{\alpha(\alpha+1)}}.$$

It is straightforward to see that these expressions are precisely the ones stated in Theorem 1, since

$$2\alpha+1 \pm 2\sqrt{\alpha(\alpha+1)} = (\sqrt{\alpha+1} \pm \sqrt{\alpha})^2.$$

Returning now to the formula (4), we thus find that the integer solutions are precisely those of the form $y_n = \alpha v_n^2 \pm v_n u_n$, and consequently

$$x_n = y_n^2 + v_n^2 = (\alpha+1)v_n^2 \pm v_n u_n.$$

In the argument above, we assumed that both A and B are strictly positive. The remainder of our proof will show that the scenario in which *A and B are both negative cannot happen.* Note that if A, B would be negative, then $-A$ and $-B$ would be positive and since A and B are relatively prime, so are $-A$ and $-B$, thus implying that $-A$ and $-B$ are perfect squares. Hence, since $(y-x)(-A) = y^2$, we conclude now that $y - x > 0$ must be a perfect square.

It is worth noting that if $y > x$ and $x > 0$, the Diophantine equation (2) has no solutions. Indeed, in this case

$$\alpha x^2 - (\alpha+1)y^2 + x - y < \alpha y^2 - (\alpha+1)y^2 = -y^2 < 0.$$

Thus, the only possible solutions would have to have $x < 0$. Substituting $y = x + w^2$ in (2) we arrive to

$$\alpha x^2 - (\alpha+1)(x+w^2)^2 - w^2 = 0.$$

This is now equivalent to

$$x^2 + 2(\alpha+1)xw^2 + (\alpha+1)w^4 + w^2 = 0,$$

and solving for x gives

$$\begin{aligned} x &= -(\alpha+1)w^2 \pm \sqrt{(\alpha+1)^2w^4 - (\alpha+1)w^4 - w^2} = \\ &= -(\alpha+1)w^2 \pm w\sqrt{\alpha(\alpha+1)w^2 - 1}. \end{aligned}$$

Therefore, clearly, if $x \in \mathbb{Z}$, it must be negative. We stumble here across a similar issue as in the first part of our argument, except that now me must require

$$\alpha(\alpha+1)w^2 - 1 = z^2 \Leftrightarrow z^2 - \alpha(\alpha+1)w^2 = -1$$

for some $z \in \mathbb{Z}$. This is an example of a *negative Pell equation*

$$z^2 - Dw^2 = -1, \tag{6}$$

with $D = \alpha(\alpha+1)$ not a perfect square. Now, it is well known [2, Theorem 3.3.4] that the negative Pell equation (6) is solvable if and only if the period of the continued fraction expansion of $\sqrt{D}$ is odd. For the fact that, for D

not a perfect square, the continued fraction expansion of $\sqrt{D}$ is periodic, see [2, Theorem 2.1.21]. Notice next that we have the following identity

$$\sqrt{\alpha(\alpha+1)} - \alpha = \frac{\alpha}{\alpha + \sqrt{\alpha(\alpha+1)}}.$$

Equivalently, the continued fraction expansion of $\sqrt{\alpha(\alpha+1)}$ is given by

$$\sqrt{\alpha(\alpha+1)} = \alpha + \cfrac{\alpha}{2\alpha + \cfrac{\alpha}{2\alpha + \cfrac{\alpha}{\ddots}}} = [\alpha; \overline{\alpha, 2\alpha}].$$

We have thus obtained that the period of the continued fraction expansion of $\sqrt{D} = \sqrt{\alpha(\alpha+1)}$ is even, effectively showing that (6) above has no integer solutions. We conclude that we cannot have solutions (x, y) of (2) with $x < y$, thus finishing the proof of our main theorem.

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Limits of integrals of functions over various domains

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Abstract. We prove that, under natural hypotheses, the following equality holds

$$\lim_{n\to\infty} n^k \int_0^1 v_n(x)\,\varphi(x) \prod_{i=1}^{k} \left(\sqrt[n]{f_i(x)} - 1\right) \mathrm{d}x$$
$$= \lim_{n\to\infty} \int_0^1 v_n(x)\,\varphi(x) \prod_{i=1}^{k} \ln f_i(x)\,\mathrm{d}x.$$

We also show that the main ideas of the proof of this result can be adapted to obtain a similar result on the unit square. Various applications are given.

Keywords: Riemann integral, multiple Riemann integral, the limit of sequences of multiple integrals

MSC: Primary 26B15; Secondary 28A35.

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1. Introduction and a preliminary result

The starting point for this paper was the following question: how can we obtain some analogous results to the following limit

$$\lim_{n\to\infty} n^2 \int_0^1 \left(\sqrt[n]{1+x^n}-1\right) \mathrm{d}x = \frac{\pi^2}{12} \tag{1}$$

given at Vojtěch Jarník International Mathematical Competition 2002, and proposed by Sofia University St. Kliment Ohridski; see [4]. The result of this investigation is the present paper. Let us mention that the notation and concepts used and not defined are standard, see [1]. We need the following well-known result whose proof is left to the reader.

Proposition 1. *If $M>0$, then*

$$\begin{aligned} 0 \ &\leq\ \mathrm{e}^a - 1 \leq \frac{\mathrm{e}^M-1}{M}\cdot a, \quad \forall\ 0\leq a\leq M;\\ 0 \ &\leq\ \mathrm{e}^a - 1 - a \leq \frac{\mathrm{e}^M-1-M}{M^2}\cdot a^2, \quad \forall\ 0\leq a\leq M. \end{aligned}$$

2. An illustrative particular case

We begin by proving the first result which the author has obtained motivated by the limit (1). Since, shortly after proving this particular result we have observed that this can be extended to very general situations we think that it is instructive for the readers to present first this particular case.

Proposition 2. *The following equality holds*

$$\lim_{n\to\infty} n^2 \int_0^1 x^n \left(\sqrt[n]{1+x}-1\right) \mathrm{d}x = \ln 2.$$

Proof. The idea is to observe that $\sqrt[n]{1+x}=\mathrm{e}^{\frac{1}{n}\ln(1+x)}$ and to use Proposition 1, which was expanded to the following reasoning. Let $n\in\mathbb{N}$ and $x\in[0,1]$. Then $0\leq a=\frac{1}{n}\ln(1+x)\leq\frac{1}{n}\ln 2\leq\frac{1}{n}\leq 1$ and by Proposition 1 we deduce

$$\begin{aligned} 0 \ &\leq\ \sqrt[n]{1+x}-1-\frac{1}{n}\ln(1+x) = \mathrm{e}^{\frac{1}{n}\ln(1+x)}-1-\frac{1}{n}\ln(1+x)\\ &\leq\ \frac{\mathrm{e}-2}{n^2}\ln^2(1+x) \leq \frac{(\mathrm{e}-2)\ln^2 2}{n^2}. \end{aligned}$$

By multiplication with n^2x^n we obtain

$$0\leq n^2x^n\left(\sqrt[n]{1+x}-1-\frac{1}{n}\ln(1+x)\right)\leq(\mathrm{e}-2)\,x^n\ln^2 2$$

and

$$0\leq n^2\int_0^1 x^n\left(\sqrt[n]{1+x}-1-\frac{1}{n}\ln(1+x)\right)\mathrm{d}x\leq(\mathrm{e}-2)\ln^2 2\int_0^1 x^n\mathrm{d}x\to 0.$$

From the equality

$$n^2x^n\left(\sqrt[n]{1+x}-1\right)=n^2x^n\left(\sqrt[n]{1+x}-1-\frac{1}{n}\ln\left(1+x\right)\right)+nx^n\ln\left(1+x\right)$$

we deduce

$$\begin{aligned}&n^2\int_0^1 x^n\left(\sqrt[n]{1+x}-1\right)\mathrm{d}x\\&=n^2\int_0^1 x^n\left(\sqrt[n]{1+x}-1-\frac{1}{n}\ln\left(1+x\right)\right)\mathrm{d}x+n\int_0^1 x^n\ln\left(1+x\right)\mathrm{d}x.\end{aligned}$$

Since, as it is well known, $n\int_0^1 x^n\ln\left(1+x\right)\mathrm{d}x\to\ln 2$, the proof is finished.

Second proof. The following proof has been shown to us by the reviewer. Integrating by parts we have

$$\begin{aligned}\int_0^1 x^n\left(\sqrt[n]{1+x}-1\right)\mathrm{d}x &= \frac{x^{n+1}\left(\sqrt[n]{1+x}-1\right)}{n+1}\Big|_0^1-\\&\quad-\frac{1}{n\left(n+1\right)}\int_0^1\left(1+x\right)^{\frac{1}{n}-1}x^{n+1}\mathrm{d}x\\&=\frac{\sqrt[n]{2}-1}{n+1}-\frac{1}{n\left(n+1\right)}\int_0^1\left(1+x\right)^{\frac{1}{n}-1}x^{n+1}\mathrm{d}x.\end{aligned}$$

It follows that

$$n^2\int_0^1 x^n\left(\sqrt[n]{1+x}-1\right)\mathrm{d}x=\frac{n^2\left(\sqrt[n]{2}-1\right)}{n+1}-\frac{n}{n+1}\int_0^1\left(1+x\right)^{\frac{1}{n}-1}x^{n+1}\mathrm{d}x.$$

The limit follows since $n\left(\sqrt[n]{2}-1\right)\to\ln 2$ and

$$0<\int_0^1\left(1+x\right)^{\frac{1}{n}-1}x^{n+1}\mathrm{d}x\le\frac{\sqrt[n]{2}}{n+2}.$$

□

3. The case of Riemann integrable functions on the unit interval: the first step

The main feature of the proof of Proposition 2 is that it can be extended to more general context.

Theorem 3. *Let $v_n:[0,1]\to\mathbb{R}$ be a sequence of Riemann integrable functions such that the sequence $\left(\int_0^1\left|v_n\left(x\right)\right|\mathrm{d}x\right)_{n\in\mathbb{N}}$ is bounded. Then for each Riemann integrable function $\varphi:[0,1]\to\mathbb{R}$ and each continuous function $f:[0,1]\to[1,\infty)$ the following equality holds*

$$\lim_{n\to\infty}\left[n\int_0^1 v_n\left(x\right)\left(\sqrt[n]{f\left(x\right)}-1\right)\varphi\left(x\right)\mathrm{d}x-\int_0^1 v_n\left(x\right)\varphi\left(x\right)\ln f\left(x\right)\mathrm{d}x\right]=0.$$

In particular, if $\lim\limits_{n\to\infty}\int_0^1 v_n\left(x\right)\varphi\left(x\right)\ln f\left(x\right)\mathrm{d}x\in\mathbb{R}$, then

$$\lim_{n\to\infty} n\int_0^1 v_n(x)\left(\sqrt[n]{f(x)}-1\right)\varphi(x)\,\mathrm{d}x=\lim_{n\to\infty}\int_0^1 v_n(x)\,\varphi(x)\ln f(x)\,\mathrm{d}x.$$

Proof. Since $[0,1]$ is compact and f is continuous, by the Weierstrass theorem, there exists $x_0\in[0,1]$ such that $M=\sup\limits_{x\in[0,1]} f(x)=f(x_0)\in[1,\infty)$. Let $n\in\mathbb{N}$, $n\geq \ln M$. For each $x\in[0,1]$ we have $0\leq a=\frac{1}{n}\ln f(x)\leq\frac{1}{n}\ln M\leq 1$ and by Proposition 1 we deduce

$$\begin{aligned}0&\leq \sqrt[n]{f(x)}-1-\frac{1}{n}\ln f(x)=\mathrm{e}^{\frac{1}{n}\ln f(x)}-1-\frac{1}{n}\ln f(x)\\&\leq\frac{1}{n^2}(\mathrm{e}-2)\ln^2 f(x)\leq\frac{1}{n^2}(\mathrm{e}-2)\ln^2 M\end{aligned}$$

and thus

$$0\leq|v_n(x)|\,|\varphi(x)|\left(\sqrt[n]{f(x)}-1-\frac{1}{n}\ln f(x)\right)\leq\frac{1}{n^2}(\mathrm{e}-2)\left(\ln^2 M\right)A\,|v_n(x)|$$

where $A=\sup\limits_{x\in[0,1]}|\varphi(x)|$. We have

$$\begin{aligned}0&\leq\left|\int_0^1 nv_n(x)\left(\sqrt[n]{f(x)}-1-\frac{1}{n}\ln f(x)\right)\varphi(x)\right|\mathrm{d}x\\&\leq n\int_0^1|v_n(x)|\,|\varphi(x)|\left(\sqrt[n]{f(x)}-1-\frac{1}{n}\ln f(x)\right)\mathrm{d}x\\&\leq\frac{1}{n}(e-2)\left(\ln^2 M\right)A\int_0^1|v_n(x)|\,\mathrm{d}x\to 0\end{aligned}$$

since the sequence $\left(\int_0^1|v_n(x)|\,\mathrm{d}x\right)_{n\in\mathbb{N}}$ is bounded. From

$$\begin{aligned}&nv_n(x)\left(\sqrt[n]{f(x)}-1\right)\varphi(x)-v_n(x)\,\varphi(x)\ln f(x)\\&\qquad=nv_n(x)\,\varphi(x)\left(\sqrt[n]{f(x)}-1-\frac{1}{n}\ln f(x)\right)\end{aligned}$$

we deduce

$$\begin{aligned}&n\int_0^1 v_n(x)\left(\sqrt[n]{f(x)}-1\right)\varphi(x)\,\mathrm{d}x-\int_0^1 v_n(x)\,\varphi(x)\ln f(x)\,\mathrm{d}x\\&=n\int_0^1 v_n(x)\left(\sqrt[n]{f(x)}-1-\frac{1}{n}\ln f(x)\right)\varphi(x)\,\mathrm{d}x\to 0,\end{aligned}$$

which proves the first part of the theorem. The second assertion follows simply from the first one. □

Corollary 4. (i) *Let $f:[0,1]\to[1,\infty)$ be a continuous function. Then*

$$\lim_{n\to\infty} n^2\int_0^1 x^n\left(\sqrt[n]{f(x)}-1\right)\mathrm{d}x=\ln f(1).$$

(ii) *Let* $g : [0,1] \to \mathbb{R}$ *and* $f : [0,1] \to [1,\infty)$ *be continuous functions. Then*

$$\lim_{n\to\infty} n^2 \int_0^1 x^n g\left(x^n\right) \left(\sqrt[n]{f\left(x\right)} - 1 \right) \mathrm{d}x = \left(\int_0^1 g\left(x\right) \mathrm{d}x \right) \ln f\left(1\right).$$

(iii) *Let* $f : [0,1] \to [1,\infty)$*be a continuous function. Then*

$$\lim_{n\to\infty} n^3 \int_0^1 \frac{x^n}{1+x+x^2+\cdots+x^n} \left(\sqrt[n]{f\left(x\right)} - 1 \right) \mathrm{d}x = \frac{\pi^2}{6} \ln f\left(1\right).$$

Proof. (i) Take in Theorem 3, $\varphi\left(x\right) = 1$, $v_n : [0,1] \to \mathbb{R}$, $v_n\left(x\right) = nx^n$ and use that $\lim\limits_{n\to\infty} n \int_0^1 x^n \ln f(x) \mathrm{d}x = \ln f(1)$.

(ii) Take in Theorem 3, $\varphi(x) = 1$, $v_n : [0,1] \to \mathbb{R}$, $v_n(x) = x^n g(x^n)$ and use that $\lim\limits_{n\to\infty} n \int_0^1 x^n g(x^n) \ln f(x) \mathrm{d}x = \left(\int_0^1 g\left(x\right) \mathrm{d}x \right) \ln f(1)$, see [2].

(iii) Take in Theorem 3, $\varphi(x) = 1$, $v_n : [0,1] \to \mathbb{R}$, $v_n(x) = \frac{n^2 x^n}{1+x+x^2+\cdots+x^n}$ and use that $\lim\limits_{n\to\infty} n^2 \int_0^1 \frac{x^n}{1+x+x^2+\cdots+x^n} \ln f\left(x\right) \mathrm{d}x = \frac{\pi^2}{6} \ln f\left(1\right)$, see [2].

Let us mention here the following proof for the boundedness of the sequence $\int_0^1 v_n\left(x\right) \mathrm{d}x$ indicated to us by the reviewer. By applying AM-GM inequality in the denominator we have $1 + x + x^2 + \cdots + x^n \geq (n+1)\, x^{\frac{n}{2}}$, which gives us $\int_0^1 v_n\left(x\right) \mathrm{d}x \leq \frac{n^2}{n+1} \int_0^1 x^{\frac{n}{2}} \mathrm{d}x = \frac{2n^2}{(n+1)(n+2)} < 2$. $\square$

4. The case of Riemann integrable functions on the unit interval: the second step

The next idea which we have taken into account was: what happen if we iterate the process in Theorem 3? In the sequel we present some possible results in this direction.

Theorem 5. *Let* $v_n : [0,1] \to \mathbb{R}$ *be a sequence of Riemann integrable functions such that the sequence* $\left(\int_0^1 \left| v_n\left(x\right) \right| \mathrm{d}x \right)_{n\in\mathbb{N}}$ *is bounded. Let also* k *be a positive integer,* $\varphi : [0,1] \to \mathbb{R}$ *be a Riemann integrable function and* $f_1, \ldots, f_k : [0,1] \to [1,\infty)$ *be continuous functions. If there exists*

$$\lim_{n\to\infty} \int_0^1 v_n\left(x\right) \varphi\left(x\right) \prod_{i=1}^{k} \ln f_i(x) \mathrm{d}x \in \mathbb{R}$$

then the following equality holds

$$\lim_{n\to\infty} n^k \int_0^1 v_n\left(x\right) \varphi\left(x\right) \prod_{i=1}^{k} \left(\sqrt[n]{f_i\left(x\right)} - 1 \right) \mathrm{d}x =$$

$$= \lim_{n\to\infty} \int_0^1 v_n\left(x\right) \varphi\left(x\right) \prod_{i=1}^{k} \ln f_i\left(x\right) \mathrm{d}x.$$

Proof. We argue by induction on k. The base case $k=1$ was shown in Theorem 3. Let us suppose the assertion is true for k, i.e., suppose that for each Riemann integrable function $g:[0,1]\to\mathbb{R}$ and continuous functions $f_1,\ldots,f_k:[0,1]\to[1,\infty)$ with the property that there exists

$$\lim_{n\to\infty}\int_0^1 v_n(x)\ \ g(x)\prod_{i=1}^{k}\ln f_i(x)\mathrm{d}x\in\mathbb{R},$$

the following equality holds

$$\lim_{n\to\infty} n^k\int_0^1 v_n(x)\,g(x)\prod_{i=1}^{k}\left(\sqrt[n]{f_i(x)}-1\right)\mathrm{d}x=$$

$$=\lim_{n\to\infty}\int_0^1 v_n(x)\,g(x)\prod_{i=1}^{k}\ln f_i(x)\,\mathrm{d}x.$$

Now, let $\varphi:[0,1]\to\mathbb{R}$ be Riemann integrable, $f_1,\ldots,f_{k+1}:[0,1]\to[1,\infty)$ continuous functions with the property that there exists

$$L=\lim_{n\to\infty}\int_0^1 v_n(x)\,\varphi(x)\prod_{i=1}^{k+1}\ln f_i(x)\mathrm{d}x\in\mathbb{R}.$$

Let us define $g:[0,1]\to\mathbb{R}$ by putting $g(x)=\varphi(x)\ln f_{k+1}(x)$ and note that g is Riemann integrable. Then

$$L=\lim_{n\to\infty}\int_0^1 v_n(x)\,g(x)\prod_{i=1}^{k}\ln f_i(x)\mathrm{d}x\in\mathbb{R}$$

and by the inductive hypothesis we get

$$\lim_{n\to\infty} n^k\int_0^1 v_n(x)\,g(x)\prod_{i=1}^{k}\left(\sqrt[n]{f_i(x)}-1\right)\mathrm{d}x=$$

$$=\lim_{n\to\infty}\int_0^1 v_n(x)\,g(x)\prod_{i=1}^{k}\ln f_i(x)\,\mathrm{d}x,$$

that is, by replacing the expression of $g(x)$ in the left member,

$$\lim_{n\to\infty} n^k\int_0^1 v_n(x)\,\varphi(x)\ln f_{k+1}(x)\prod_{i=1}^{k}\left(\sqrt[n]{f_i(x)}-1\right)\mathrm{d}x=L.$$

Let us define $h_n:[0,1]\to\mathbb{R}$ by $h_n(x)=n^k v_n(x)\,\varphi(x)\prod\limits_{i=1}^{k}\left(\sqrt[n]{f_i(x)}-1\right)$. Let $n\in\mathbb{N}$, $n\geq\max\limits_{1\leq i\leq k}\ln M_i$ ($M_i=\sup\limits_{x\in[0,1]} f_i(x)\in[1,\infty)$). For each $x\in[0,1]$

we have $0 \leq a = \frac{1}{n} \ln f_i(x) \leq \frac{1}{n} \ln M_i \leq 1$. Then from Proposition 1 we deduce

$$0 \leq \sqrt[n]{f_i(x)} - 1 = \mathrm{e}^{\frac{1}{n} \ln f_i(x)} - 1 \leq \frac{\mathrm{e}-1}{n} \ln f_i(x) \leq \frac{\mathrm{e}-1}{n} \ln M_i,$$

whence

$$|h_n(x)| \leq n^k |v_n(x)||\varphi(x)| \prod_{i=1}^{k} \left(\sqrt[n]{f_i(x)} - 1 \right) \leq (\mathrm{e}-1)^k |v_n(x)| M \prod_{i=1}^{k} \ln M_i,$$

where $M = \sup_{x \in [0,1]} |\varphi(x)|$. Since $\left(\int_0^1 |v_n(x)| \, \mathrm{d}x \right)_{n \in \mathbb{N}}$ is bounded, it follows that $\left(\int_0^1 |h_n(x)| \, \mathrm{d}x \right)_{n \geq \max\limits_{1 \leq i \leq k} \ln M_i}$ is bounded, hence $\left(\int_0^1 |h_n(x)| \, \mathrm{d}x \right)_{n \in \mathbb{N}}$ is bounded. Since

$$\begin{aligned} &\lim_{n \to \infty} \int_0^1 h_n(x) \ln f_{k+1}(x) \, \mathrm{d}x \\ &\qquad = \lim_{n \to \infty} n^k \int_0^1 v_n(x) \varphi(x) \prod_{i=1}^{k} \left(\sqrt[n]{f_i(x)} - 1 \right) \ln f_{k+1}(x) \, \mathrm{d}x = L, \end{aligned}$$

from Theorem 3 we deduce

$$\lim_{n \to \infty} n \int_0^1 h_n(x) \left(\sqrt[n]{f_{k+1}(x)} - 1 \right) \mathrm{d}x = \lim_{n \to \infty} \int_0^1 h_n(x) \ln f_{k+1}(x) \, \mathrm{d}x,$$

that is, by replacing in the left member the expression of $h_n(x)$,

$$\begin{aligned} &\lim_{n \to \infty} n^{k+1} \int_0^1 v_n(x) \varphi(x) \prod_{i=1}^{k+1} \left(\sqrt[n]{f_i(x)} - 1 \right) \mathrm{d}x \\ &\qquad = \lim_{n \to \infty} n^k \int_0^1 v_n(x) \varphi(x) \prod_{i=1}^{k} \left(\sqrt[n]{f_i(x)} - 1 \right) \ln f_{k+1}(x) \, \mathrm{d}x = L. \end{aligned}$$

This means that the statement is true for $k+1$. $\square$

Taking in Theorem 5, $\varphi(x) = 1$ for all $x \in [0,1]$, we get the following corollary.

Corollary 6. *Let $v_n : [0,1] \to \mathbb{R}$ be a sequence of Riemann integrable functions such that the sequence $\left(\int_0^1 |v_n(x)| \, \mathrm{d}x \right)_{n \in \mathbb{N}}$ is bounded. Let also k be a positive integer and $f_1, \ldots, f_k : [0,1] \to [1, \infty)$ be continuous functions with the property that there exists*

$$\lim_{n \to \infty} \int_0^1 v_n(x) \prod_{i=1}^{k} \ln f_i(x) \mathrm{d}x \in \mathbb{R}.$$

Then the following equality holds

$$\lim_{n\to\infty} n^k \int_0^1 v_n(x) \prod_{i=1}^{k} \left(\sqrt[n]{f_i(x)} - 1\right) \mathrm{d}x = \lim_{n\to\infty} \int_0^1 v_n(x) \prod_{i=1}^{k} \ln f_i(x) \mathrm{d}x.$$

We recall a well known result, see [2, Ex. 3.13, pag. 53–54].

Lemma 7. *Let $v_n : [0,1] \to \mathbb{R}$ be a sequence of Riemann integrable functions such that the sequence $\left(\int_0^1 |v_n(x)|\, dx\right)_{n\in\mathbb{N}}$ is bounded, an for all $0 < u < 1$ we have $\lim\limits_{n\to\infty} \int_0^u |v_n(x)|\, \mathrm{d}x = 0$ and $\lim\limits_{n\to\infty} \int_0^1 v_n(x)\, \mathrm{d}x \in \mathbb{R}$. Then for each continuous function $f : [0,1] \to \mathbb{R}$ the following equality holds*

$$\lim_{n\to\infty} \int_0^1 v_n(x) f(x)\, \mathrm{d}x = f(1) \lim_{n\to\infty} \int_0^1 v_n(x)\, \mathrm{d}x.$$

From Corollary 6 and Lemma 7 we get the following result which is an extension of Proposition 2 and Corollary 4.

Corollary 8. *Let $v_n : [0,1] \to \mathbb{R}$ be a sequence of Riemann integrable functions such that the sequence $\left(\int_0^1 |v_n(x)|\, \mathrm{d}x\right)_{n\in\mathbb{N}}$ is bounded, and for all $0 < u < 1$ we have $\lim\limits_{n\to\infty} \int_0^u |v_n(x)|\, \mathrm{d}x = 0$ and $\lim\limits_{n\to\infty} \int_0^1 v_n(x)\, \mathrm{d}x \in \mathbb{R}$. Let also k be a positive integer and $f_1, \ldots, f_k : [0,1] \to [1,\infty)$ be continuous functions. Then the following equality holds*

$$\lim_{n\to\infty} n^k \int_0^1 v_n(x) \prod_{i=1}^{k} \left(\sqrt[n]{f_i(x)} - 1\right) \mathrm{d}x = \left[\prod_{i=1}^{k} \ln f_i(1)\right] \lim_{n\to\infty} \int_0^1 v_n(x)\, \mathrm{d}x.$$

Taking in Theorem 5, $v_n(x) = 1$ for all $x \in [0,1]$ and $n \in \mathbb{N}$, we get the following corollary.

Corollary 9. *Let k be a positive integer, $\varphi : [0,1] \to \mathbb{R}$ a Riemann integrable function and $f_1, \ldots, f_k : [0,1] \to [1,\infty)$ continuous functions. Then the following equality holds*

$$\lim_{n\to\infty} n^k \int_0^1 \varphi(x) \prod_{i=1}^{k} \left(\sqrt[n]{f_i(x)} - 1\right) \mathrm{d}x = \int_0^1 \varphi(x) \prod_{i=1}^{k} \ln f_i(x)\, \mathrm{d}x.$$

Corollary 10. *For each positive integer k the following equality holds*

$$\lim_{n\to\infty} n^k \int_0^1 \left(\sqrt[n]{1+x} - 1\right)^k \mathrm{d}x = I_k,$$

where $I_1 = 2\ln 2 - 1$, $I_k = 2\ln^k 2 - kI_{k-1}$, for $k \geq 2$, and I_k are as above.

Proof. From Corollary 9 we have

$$\lim_{n\to\infty} n^k \int_0^1 \left(\sqrt[n]{1+x} - 1\right)^k \mathrm{d}x = \int_0^1 [\ln(1+x)]^k\, \mathrm{d}x = I_k.$$

It is a standard exercise to prove that I_k has the stated properties. $\square$

5. The case of Riemann integrable functions on the unit square

In the sequel we show that, following similar ideas, we can prove that Theorems 3 and 5 can be extended to Riemann integrable functions on the unit square.

Theorem 11. *Let $h_n : [0,1]^2 \to \mathbb{R}$ be a sequence of Riemann integrable functions such that the sequence* $\left(\iint\limits_{[0,1]^2} |h_n(x,y)| \, \mathrm{d}x\mathrm{d}y \right)_{n\in\mathbb{N}}$ *is bounded. Then for each continuous function $f : [0,1]^2 \to [1,\infty)$ the following equality holds*

$$\lim_{n\to\infty} \left(n \iint\limits_{[0,1]^2} h_n(x,y)\big(\sqrt[n]{f(x,y)} - 1\big)\mathrm{d}x\mathrm{d}y - \iint\limits_{[0,1]^2} h_n(x,y) \ln f(x,y\ )\mathrm{d}x\mathrm{d}y \right) = 0.$$

In particular, if there exists $\lim\limits_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y) \ln f(x,y) \, \mathrm{d}x\mathrm{d}y \in \mathbb{R}$, *then*

$$\lim_{n\to\infty} n \iint\limits_{[0,1]^2} h_n(x,y)\big(\sqrt[n]{f(x,y)} - 1\big)\mathrm{d}x\mathrm{d}y = \lim_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y) \ln f(x,y)\mathrm{d}x\mathrm{d}y.$$

Proof. The proof of Theorem 3 uses the Weierstrass theorem: continuous functions on compact sets are bounded and attain theirs bounds (the set $[0,1]$ is compact); here $[0,1]^2$ is compact and again we use the Weierstrass theorem. Now the reader will have no difficulty to mimic the proof of Theorem 3 in this new context. □

Theorem 12. *Let $h_n : [0,1]^2 \to \mathbb{R}$ be a sequence of Riemann integrable functions such that the sequence* $\left(\iint\limits_{[0,1]^2} |h_n(x,y)| \, \mathrm{d}x\mathrm{d}y \right)_{n\in\mathbb{N}}$ *is bounded. Let also k be a positive integer, $\varphi : [0,1]^2 \to \mathbb{R}$ be a Riemann integrable function and $f_1, \ldots, f_k : [0,1]^2 \to [1,\infty)$ be continuous functions. If there exists*

$$\lim_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y) \, \varphi(x,y) \prod_{i=1}^{k} \ln f_i(x,y) \, \mathrm{d}x\mathrm{d}y \in \mathbb{R}$$

then the following equality holds

$$\lim_{n\to\infty} n^k \iint\limits_{[0,1]^2} h_n(x,y) \, \varphi(x,y) \prod_{i=1}^{k} \left(\sqrt[n]{f_i(x,y)} - 1 \right) \mathrm{d}x\mathrm{d}y$$

$$= \lim_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y) \, \varphi(x,y) \prod_{i=1}^{k} \ln f_i(x,y)\mathrm{d}x\mathrm{d}y.$$

Proof. The idea of the proof of Theorem 5 was to use the induction and Theorem 3. In this new context we apply again the induction and now we use Theorem 11 instead of Theorem 3. We leave the details to the reader. □

Taking in Theorem 12, $h_n(x,y)=1$ for all x, $y\in[0,1]$ and $n\in\mathbb{N}$, we get the following corollary.

Corollary 13. *For each positive integer k, each Riemann integrable function $\varphi:[0,1]^2\to\mathbb{R}$ and arbitrary continuous functions $f_1,\ldots,f_k:[0,1]^2\to[1,\infty)$ the following equality holds*

$$\lim_{n\to\infty} n^k \iint\limits_{[0,1]^2} \varphi(x,y)\prod_{i=1}^{k}\left(\sqrt[n]{f_i(x,y)}-1\right)\mathrm{d}x\mathrm{d}y=\iint\limits_{[0,1]^2}\varphi(x,y)\prod_{i=1}^{k}\ln f_i(x,y)\mathrm{d}x\mathrm{d}y.$$

Corollary 14. *For each positive integer k the following equality holds*

$$\lim_{n\to\infty} n^k \iint\limits_{[0,1]^2}\left(\sqrt[n]{(1+x)(1+y)}-1\right)^k\mathrm{d}x\mathrm{d}y=A_k,$$

where $A_1=4\ln 2-2$, $A_k=2I_k+\sum\limits_{i=1}^{k-1}\binom{k}{i}I_{k-i}I_i$, for $k\geq 2$, and I_k are as in Corollary 10.

Proof. From Corollary 13 we get $\lim\limits_{n\to\infty} n^k \iint\limits_{[0,1]^2}\left(\sqrt[n]{(1+x)(1+y)}-1\right)^k\mathrm{d}x\mathrm{d}y=A_k$, where $A_k=\iint\limits_{[0,1]^2}[\ln(1+x)(1+y)]^k\,\mathrm{d}x\mathrm{d}y$. Further, by Fubini's theorem,

$$A_1=\iint\limits_{[0,1]^2}\big[\ln(1+x)+\ln(1+y)\big]\mathrm{d}x\mathrm{d}y=2I_1=4\ln 2-2,$$

where I_k $(k\geq 1)$ are defined in Corollary 10. For $k\geq 2$, by the Newton binomial formula and Fubini's Theorem we have

$$\begin{aligned}A_k &= \iint\limits_{[0,1]^2}[\ln(1+x)+\ln(1+y)]^k\,\mathrm{d}x\mathrm{d}y\\ &= 2\int_0^1[\ln(1+x)]^k\,\mathrm{d}x+\sum_{i=1}^{k-1}\binom{k}{i}\iint\limits_{[0,1]^2}[\ln(1+x)]^{k-i}[\ln(1+y)]^i\,\mathrm{d}x\mathrm{d}y\\ &= 2I_k+\sum_{i=1}^{k-1}\binom{k}{i}I_{k-i}I_i.\end{aligned}$$

□

Now we recall the following result, see [3, Theorem 5].

Theorem 15. *Let $h_n : [0,1]^2 \to \mathbb{R}$ be a sequence of continuous functions such that the sequence* $\left(\iint\limits_{[0,1]^2} |h_n(x,y)| \, \mathrm{d}x\mathrm{d}y \right)_{n\in\mathbb{N}}$ *is bounded and the following symmetric conditions are satisfied:*

$$\forall\, 0 < u < 1, \ \lim_{n\to\infty} \iint\limits_{[0,u]\times[0,1]} |h_n(x,y)| \, \mathrm{d}x\mathrm{d}y = 0;$$

$$\forall\, 0 < v < 1, \ \lim_{n\to\infty} \iint\limits_{[0,1]\times[0,v]} |h_n(x,y)| \, \mathrm{d}x\mathrm{d}y = 0.$$

If $\lim\limits_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y) \, \mathrm{d}x\mathrm{d}y \in \mathbb{R}$, *then for each continuous function $f : [0,1]^2 \to \mathbb{R}$ the following equality holds*

$$\lim_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y) f(x,y) \, \mathrm{d}x\mathrm{d}y = f(1,1) \lim_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y) \, \mathrm{d}x\mathrm{d}y.$$

The result shown in Theorem 12 can be made more precise, as we prove next.

Theorem 16. *Let $h_n : [0,1]^2 \to \mathbb{R}$ be a sequence of continuous functions such that the sequence* $\left(\iint\limits_{[0,1]^2} |h_n(x,y)| \, \mathrm{d}x\mathrm{d}y \right)_{n\in\mathbb{N}}$ *is bounded, the following symmetric conditions are satisfied:*

$$\forall\, 0 < u < 1, \ \lim_{n\to\infty} \iint\limits_{[0,u]\times[0,1]} |h_n(x,y)| \, \mathrm{d}x\mathrm{d}y = 0,$$

$$\forall\, 0 < v < 1, \ \lim_{n\to\infty} \iint\limits_{[0,1]\times[0,v]} |h_n(x,y)| \, \mathrm{d}x\mathrm{d}y = 0,$$

and there exists $\lim\limits_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y) \, \mathrm{d}x\mathrm{d}y \in \mathbb{R}$. *Then for each positive integer k and arbitrary continuous functions $f_1, \ldots, f_k : [0,1]^2 \to [1,\infty)$ the following equality holds*

$$\lim_{n\to\infty} n^k \iint\limits_{[0,1]^2} h_n(x,y) \prod_{i=1}^{k} \left(\sqrt[n]{f_i(x,y)} - 1 \right) \mathrm{d}x\mathrm{d}y$$
$$= \left(\lim_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y) \mathrm{d}x\mathrm{d}y \right) \prod_{i=1}^{k} \ln f_i(1,1).$$

Proof. Under our hypothesis and Theorem 15 it follows that

$$\lim_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y) \prod_{i=1}^{k} \ln f_i(x,y)\mathrm{d}x\mathrm{d}y = \prod_{i=1}^{k} \ln f_i(1,1) \in \mathbb{R}.$$

The conclusion now follows from Theorem 12. □

As applications of Theorem 16 we give the following corollary.

Corollary 17. (i) *For every $a,b>0$, each positive integer k and each continuous function $f:[0,1]^2 \to [1,\infty)$ the following equality holds*

$$\lim_{n\to\infty} n^{k+2} \iint\limits_{[0,1]^2} \left(\frac{ax+by}{a+b}\right)^n \left(\sqrt[n]{f(x,y)}-1\right)^k \mathrm{d}x\mathrm{d}y = \frac{(a+b)^2}{ab} \cdot [\ln f(1,1)]^k.$$

(ii) *For each continuous function $f:[0,1]^2 \to [1,\infty)$ and each positive integer k the limit*

$$\lim_{n\to\infty} n^{k+4} \iint\limits_{[0,1]^2} \frac{x^n y^n}{(1+x+x^2+\cdots+x^n)(1+y+y^2+\cdots+y^n)} \left(\sqrt[n]{f(x,y)}-1\right)^k \mathrm{d}x\mathrm{d}y$$

has the value $\frac{\pi^4}{36} \cdot [\ln f(1,1)]^k$.

Proof. (i) and (ii).

The sequence of continuous functions $h_n : [0,1]^2 \to \mathbb{R}$, $h_n(x,y) = n^2 \left(\frac{ax+by}{a+b}\right)^n$ (respectively $h_n(x,y) = \frac{n^4 x^n y^n}{(1+x+x^2+\cdots+x^n)(1+y+y^2+\cdots+y^n)}$) satisfies the hypotheses in Theorem 16 and $\lim\limits_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y)\,\mathrm{d}x\mathrm{d}y = \frac{(a+b)^2}{ab}$ (respectively $\lim\limits_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y)\,\mathrm{d}x\mathrm{d}y = \frac{\pi^4}{36}$), see [3, Prop. 6 and Cor. 7]. □

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Asupra funcţiilor convexe cu graficele tangente – cazul unei mulţimi oarecare de puncte

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Abstract. Examples of two real, strictly convex, indefinitely derivable functions whose difference is non-negative and vanishes on a set M and whose graphics are tangent on points of abscissa in M are thought for. It is shown that under the hypothesis of being increasing, such functions exist if and only if M is closed. A similar result is proved if instead of monotonicity it is required that both functions are unbounded.

Keywords: strict convex function, indefinitely derivable function, closed set

MSC: Primary 26A51; Secondary 26A48.

În [1], autorul a construit două exemple de funcţii cu graficele tangente într-un număr finit de puncte. Din păcate, tehnica respectivă nu se poate aplica pentru un număr infinit de puncte. În cele ce urmează, ne propunem să determinăm **toate mulţimile de puncte** M pentru care sunt posibile construcţii care să respecte cerinţele din [1], mai exact să rezolvăm următoarele probleme:

Problema 1. Să se studieze existenţa a două funcţii strict convexe $f, g : \mathbb{R} \to \mathbb{R}$ cu următoarele proprietăţi:

a) $f(x) \geqslant g(x)$, $\forall\, x \in \mathbb{R}$, cu egalitate dacă şi numai dacă $x \in M$;
b) f şi g sunt indefinit derivabile;
c) graficele funcţiilor f şi g sunt tangente în punctele de abscisă $x \in M$.

Problema 2. Notăm $\mathbb{R} \setminus M = \Delta$. Să se studieze existenţa a două funcţii strict convexe $f, g : \mathbb{R} \to \mathbb{R}$ şi a unei partiţii $\Delta = \Delta_1 \cup \Delta_2$, în care Δ_1 şi Δ_2 să fie cardinal echivalente, cu următoarele proprietăţi:

a) $f(x) = g(x)$, $\forall\, x \in M$;
b) $f - g\,|_{\Delta_1} > 0$ şi $f - g\,|_{\Delta_2} < 0$
c) f şi g sunt indefinit derivabile;
d) graficele funcţiilor f şi g sunt tangente în punctele de abscisă $x \in M$.

În plus, să se realizeze construcţiile *în fiecare* din **ipotezele suplimentare**:

i) $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} g(x) = +\infty$;
ii) f şi g strict crescătoare.

Reamintim în prealabil următoarele noţiuni de topologie

1) O mulţime $D \subseteq \mathbb{R}$ se numeşte *deschisă* dacă pentru orice $x \in D$ există $I_x \subset \mathbb{R}$, interval deschis, astfel încât $x \in I_x \subset D$

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2) O mulţime $F \subseteq \mathbb{R}$ se numeşte *închisă* dacă $\mathbb{R} \setminus F$ este deschisă.
3) (*Teorema de structură a mulţimilor deschise din* $\mathbb{R}$) Orice submulţime deschisă a lui $\mathbb{R}$ se scrie în mod unic ca o submulţime cel mult numărabilă de intervale deschise disjuncte.

4) (*Teorema de caracterizare cu şiruri a mulţimilor închise din* $\mathbb{R}$) O mulţime $M \subset \mathbb{R}$ este închisă dacă şi numai dacă pentru orice şir convergent $(x_n)_{n\in\mathbb{N}}$ de elemente din M avem $\lim\limits_{n\to\infty} x_n \in M$.

Prezentăm în continuare rezultatul principal al lucrării.

Teorema 1. *Problema 1 cu ipoteza i) admite soluţii dacă şi numai dacă mulţimea M este închisă.*

Demonstraţie. „$\Longrightarrow$ ” Fie $M \subseteq \mathbb{R}$ o mulţime şi $f, g : \mathbb{R} \to \mathbb{R}$ două funcţii cu proprietăţile căutate. Din convexitate, rezultă că f şi g sunt continue. Fie $(x_n)_{n\in\mathbb{N}} \subset M$ un şir convergent arbitrar şi $l = \lim\limits_{n\to\infty} x_n \in \mathbb{R}$. Cum $f(x_n) = g(x_n), \forall\ n \in \mathbb{N}$, prin trecere la limită obţinem $f(l) = g(l)$.
Aşadar, $l \in M$, ceea ce arată că M este o mulţime închisă.
„$\Longleftarrow$ ” Fie $M \subset \mathbb{R}$ o mulţime închisă. Cum $\mathbb{R}\setminus M$ este mulţime deschisă, conform teoremei de structură a mulţimilor deschise există două şiruri $(a_n)_{n\in I}$, $(b_n)_{n\in I} \subset \overline{\mathbb{R}}$ cu proprietăţile:

1) $a_n < b_n, \forall n \in I$, unde $I \subseteq \mathbb{N}$ este o mulţime nevidă de numere consecutive;
2) $(a_m, b_m) \cap (a_n, b_n) = \emptyset, \quad \forall m \neq n$;
3) $\mathbb{R} \setminus M = \bigcup\limits_{n\in I} (a_n, b_n)$.

Observăm că $\{a_n, b_n \mid n \in I\} \cap \mathbb{R} \subset M$.

Vom utiliza funcţiile $\varphi_m : \mathbb{R} \to \mathbb{R},\ m \in \mathbb{Z}$,

$$\varphi_m(x) = \begin{cases} x^m e^{-x^{-2}}, & \text{dacă } x \neq 0, \\ 0, & \text{dacă } x = 0, \end{cases}$$

care au proprietatea că sunt indefinit derivabile şi $\varphi_m^{(k)}(0) = 0,\ \forall k \in \mathbb{N}$. În plus, pentru $m < 0$, aceste funcţii sunt mărginite, deoarece $\lim\limits_{x\to\pm\infty} \varphi_m(x) = 0$. Totodată, derivatele lor de orice ordin au aceleaşi proprietăţi, fiind combinaţii liniare de funcţii de acelaşi tip.

În continuare vom considera $m = -2$ şi vom nota $\varphi(x) = \varphi_{-2}(x)$.

I) Alegem funcţia diferenţă $h = f - g$ astfel:

$$h(x) = \begin{cases} \varphi\left(\frac{(x-a_n)(x-b_n)}{b_n-a_n+1}\right), & x \in (a_n, b_n) \quad (\text{dacă } a_n, b_n \in \mathbb{R}), \\ \varphi\left(x - a_k\right), & x > a_k \quad (\text{dacă } b_k = +\infty), \\ \varphi\left(x - b_l\right), & x < b_l \quad (\text{dacă } a_l = -\infty), \\ 0, & x \in M. \end{cases}$$

Funcţia h este indefinit derivabilă pe $\mathbb{R}\setminus M$. Să observăm, de asemenea, că avem $h(x) \geq 0, \forall x \in \mathbb{R}$, cu egalitate dacă şi numai dacă $x \in M$. Dacă $\mathbb{R}\setminus M$

este o reuniune finită de intervale deschise disjuncte, atunci derivabilitatea lui h rezultă din cea a lui φ.

Analizăm în continuare cazul în care h are un număr infinit de ramuri.

Studiem derivabilitatea lui h într-un punct $\alpha \in M$. Mai precis, arătăm că $h'(\alpha) = 0, \forall \alpha \in M$.

Fie $\alpha \in M$, arbitrar, fixat. Dacă α este un punct izolat al mulţimii M, atunci $\alpha \in \{a_n, b_n \mid n \in I\} \cap \mathbb{R}$, deci h este derivabilă în α şi $h'(\alpha) = 0$.

Presupunem în continuare că α este un punct de acumulare al lui M. Vom arăta că $h'_s(\alpha) = 0$.

Dacă α nu este punct de acumulare la stânga al mulţimii $\mathbb{R} \setminus M$, atunci $h(x) = 0$ pe o vecinătate la stânga a lui α, deci $h'_s(\alpha) = 0$.

În caz contrar, presupunem în cele ce urmează şi că α este punct de acumulare la stânga al mulţimii $\mathbb{R} \setminus M$.

Fie $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}\setminus M$ un şir crescător astfel încât $x_n \longrightarrow \alpha$. În acest caz, pentru orice $n \in \mathbb{N}$, există şi este unic $k_n \in \mathbb{N}$ (şir crescător şi nestaţionar) astfel încât $x_n \in (a_{k_n}, b_{k_n})$. Vom avea nevoie de următoarea observaţie.

Lema 2. $a_{k_n}, b_{k_n} \longrightarrow \alpha$.

Demonstraţie. Şirurile (a_{k_n}) şi (b_{k_n}) sunt crescătoare şi mărginite superior de α, deci sunt convergente, având limitele $l_1 \leq l_2 \leq \alpha$. Deoarece $x_n < b_{k_n}$, prin trecere la limită obţinem $\alpha \leq l_2$, deci $\alpha = l_2$.

Presupunem prin absurd că $l_1 < \alpha$. Cum $b_{k_n} \longrightarrow \alpha$, există $m \in \mathbb{N}$ astfel încât $b_{k_n} > l_1, \forall n \geq m$. De aici rezultă că $l_1 \in (a_{k_n}, b_{k_n})$, $\forall n \geqslant m$, ceea ce contrazice faptul că şirul $(k_n)_{n\in\mathbb{N}}$ este nestaţionar (întrucât intervalele (a_n, b_n) sunt disjuncte două câte două).

Aşadar, $l_1 = \alpha$, ceea ce încheie demonstraţia lemei. □

Revenim la demonstraţia Teoremei 1. Avem

$$\frac{h(x_n) - h(\alpha)}{x_n - \alpha} = \frac{\varphi\left(\frac{(x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right)}{x_n - \alpha} = \frac{\mathrm{e}^{-\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right]^{-2}}}{\left[\frac{x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right]^2 (x_n - \alpha)}.$$

Dar $|x_n - a_{k_n}| \leq 1$, pentru n suficient de mare, deci

$$\frac{\mathrm{e}^{-\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right]^{-2}}}{\left[\frac{x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right]^2 (x_n - \alpha)} \leq \frac{\mathrm{e}^{-\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n}}{b_{k_n} - a_{k_n} + 1}\right]^{-2}}}{\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right]^2 |x_n - b_{k_n}|} \leq$$

$$\leq \frac{\mathrm{e}^{-\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n}}{b_{k_n} - a_{k_n} + 1}\right]^{-2}}}{\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right]^2 |x_n - b_{k_n}||x_n - a_{k_n}|} =$$

$$= \frac{1}{b_{k_n} - a_{k_n} + 1} \cdot \frac{e^{-\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n}}{b_{k_n} - a_{k_n} + 1}\right]^{-2}}}{\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right]^2} =$$

$$= \frac{1}{b_{k_n} - a_{k_n} + 1} \cdot \varphi_{-3}\left(\frac{|x_n - a_{k_n}||x_n - b_{k_n}|}{b_{k_n} - a_{k_n} + 1}\right)$$

şi ultima expresie tinde la 0 deoarece

$$\frac{|x_n - a_{k_n}||x_n - b_{k_n}|}{b_{k_n} - a_{k_n} + 1} \leq |x_n - a_{k_n}| \cdot \frac{b_{k_n} - a_{k_n}}{b_{k_n} - a_{k_n} + 1} \leq |x_n - a_{k_n}| \to 0.$$

Prin urmare, din criteriul majorării rezultă că

$$\lim_{n\to\infty} \frac{h(x_n) - h(\alpha)}{x_n - \alpha} = 0$$

pentru orice şir crescător $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R} \setminus M$ convergent la α.

De fapt, constatăm că afirmaţia este adevărată pentru orice şir $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R} \setminus M$, $x_n \longrightarrow \alpha$, $x_n < \alpha$ (putem permuta termenii unui şir convergent, obţinând un şir monoton, având aceeaşi limită ca şirul iniţial).

Considerăm acum un şir oarecare $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ convergent la α pe care îl partiţionăm, eventual, în două subşiruri $(y_{k_n})_{n\in\mathbb{N}} \subset \mathbb{R}\setminus M$ şi $(y_{l_n})_{n\in\mathbb{N}} \subset M$, $y_{k_n}, y_{l_n} \longrightarrow \alpha$. Cum $y_{l_n} \in M$, rezultă că $h(y_{l_n}) = 0$ şi, împreună cu lema, obţinem

$$\lim_{n\to\infty} \frac{h(y_n) - h(\alpha)}{y_n - \alpha} = 0,$$

ceea ce arată că $h'_s(\alpha) = 0$.

Analog se arată că $h'_d(\alpha) = 0$.

Mai mult, obţinem că h este indefinit derivabilă, întrucât derivatele sale de orice ordin sunt sume de funcţii de tipul $\varphi_m,\ m < 0$.

II) Determinăm acum funcţiile f şi g.

Mai întâi, pentru o funcţie oarecare $f : \mathbb{R} \to \mathbb{R}$, notăm $\|f\| = \sup\limits_{x\in\mathbb{R}} |f(x)|$.

Constatăm că

$$h'(x) = \begin{cases} \varphi'\left(\frac{(x-a_n)(x-b_n)}{b_n-a_n+1}\right) \cdot \frac{(2x-(a_n+b_n))}{b_n-a_n+1}, & x \in (a_n, b_n) \quad (\text{dacă } a_n, b_n \in \mathbb{R}), \\ \varphi'(x - a_k), & x > a_k \quad (\text{dacă } b_k = +\infty), \\ \varphi'(x - b_l), & x < b_l \quad (\text{dacă } a_l = -\infty), \\ 0, & x \in M, \end{cases}$$

de unde rezultă

$$\left\|h'\right\| \leq \max\left\{\left\|\varphi'\right\|,\ \left\|\varphi'\right\| \cdot \sup_n \frac{b_n - a_n}{b_n - a_n + 1}\right\} \leq \left\|\varphi'\right\| < \beta \in \mathbb{R}_+^*. \qquad (1)$$

De asemenea

$$h''(x)=\begin{cases}\varphi''\left(\frac{(x-a_n)(x-b_n)}{b_n-a_n+1}\right)\left[\frac{(2x-a_n-b_n)}{b_n-a_n+1}\right]^2 & \\ \quad+\varphi'\left(\frac{(x-a_n)(x-b_n)}{b_n-a_n+1}\right)\frac{2}{b_n-a_n+1}, & x\in(a_n,b_n) \quad (\text{dacă } a_n,b_n\in\mathbb{R}),\\ \varphi''\left(x-a_k\right), & x>a_k \quad (\text{dacă } b_k=+\infty),\\ \varphi''\left(x-b_l\right), & x<b_l \quad (\text{dacă } a_l=-\infty),\\ 0, & x\in M,\end{cases}$$

deci

$$\left\|h''\right\|\leq\left\|\varphi''\right\|+2\left\|\varphi'\right\|<\beta, \tag{2}$$

unde β este ales convenabil pentru a verifica relaţiile (1) şi (2).

Considerăm

$$f(x)=h(x)+\frac{\beta}{2}x^2 \ \text{ şi } g(x)=\frac{\beta}{2}x^2.$$

Se arată uşor că funcţiile f şi g sunt strict convexe. În plus, doar în punctele $\alpha\in M$ avem $h(\alpha)=h'(\alpha)$, de unde rezultă că $f(\alpha)=g(\alpha)$ şi $f'(\alpha)=g'(\alpha)$, deci graficele celor două funcţii f şi g sunt tangente în aceste puncte. □

Corolar 3. *Problema 2 cu ipoteza i) admite soluţii dacă şi numai dacă mulţimea M este închisă, iar $\mathbb{R}\setminus M$ nu este interval.*

Demonstraţie. Cu argumentul din demonstraţia teoremei 1 se arată că mulţimea M trebuie să fie închisă. Pentru a avea sens construcţia, trebuie ca $\mathbb{R}\setminus M$ să se scrie ca o reuniune de cel puţin două intervale deschise disjuncte. Procedăm ca la demonstraţia teoremei 1, singura deosebire fiind aceea că alegem $h=\pm\varphi$, de exemplu alternativ, pe intervalele din definiţie. Considerăm $\Delta_1=\{x\in\mathbb{R}\mid h(x)>0\}$ şi $\Delta_2=\{x\in\mathbb{R}\mid h(x)<0\}$. Egalitatea cardinalelor mulţimilor Δ_1 şi Δ_2 rezultă din faptul că ambele sunt de puterea continuului. □

Teorema 4. *Problema 1 cu ipoteza ii) admite soluţii dacă şi numai dacă mulţimea M este închisă.*

Demonstraţie. Considerăm o nouă funcţie diferenţă $q=f-g$, anume

$$q(x)=\mathrm{e}^x h(x),$$

unde h este funcţia din demonstraţia teoremei 1.

Constatăm următoarele:

1) $q\geq 0$,
2) $h,h^{'},h^{''}>-\gamma$ (unde $\gamma\geq\beta>0$ este convenabil ales, iar β este constanta determinată în cadrul demonstraţiei Teoremei 1),
3) $q'(x)=e^x(h(x)+h^{'}(x))$,
4) $q''(x)=e^x(h(x)+2h^{'}(x)+h^{''}(x))$.

Alegem

$$f(x) = q(x) + 4\gamma e^x \text{ şi } g(x) = 4\gamma e^x.$$

Funcţiile f şi g sunt strict crescătoare, strict convexe, cu $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} g(x) = +\infty$ şi $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} g(x) = 0$. □

Corolar 5. *Problema 2 cu ipoteza ii) admite soluţii dacă şi numai dacă mulţimea M este închisă, iar $\mathbb{R} \setminus M$ nu este interval.*

Demonstraţie. Procedăm în maniera anterioară, alegând funcţia diferenţă $\pm q$ pe intervale. □

Observaţia 6. Tehnica utilizată în această lucrare generează o nouă soluţie a problemei iniţiale din [1].

Observaţia 7. În acest articol, din cauza dificultăţii construcţiei, nu ne-am propus să tratăm toate cazurile posibile, ca în [2], mărginindu-ne doar la ipotezele i) şi ii), pe care le considerăm relevante.

Observaţia 8. Se poate înlocui funcţia φ cu orice altă funcţie care are proprietăţile:

1) *φ este de două ori derivabilă;*
2) *$\varphi(0) = \varphi'(0) = 0$ şi $\varphi(x) > 0$, $\forall x \neq 0$ (în locul lui 0 putem alege orice punct $a \in \mathbb{R}$);*
3) *φ, φ', φ'' sunt mărginite.*

Un astfel de exemplu, altul decât cel prezentat, este funcţia $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = \frac{x^2}{1+x^2}$. Totuşi, funcţiile φ_m au proprietatea că derivatele lor de orice ordin se anulează în 0.

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Olimpiada de Matematică a studenţilor din sud-estul Europei, SEEMOUS 2016

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Abstract. We discuss the problems of the 10th South Eastern European Mathematical Olympiad for University Students, SEEMOUS 2016, organized by The Mathematical Society of South Eastern Europe and Cyprus Mathematical Society, that took place in Protaras, Cyprus, between March 1 and March 6, 2016.

Keywords: Integrals, series, matrices, Jordan form, rank

MSC: Primary 15A03; Secondary 15A21, 26D15.

În perioada 1–6 martie 2016 s-a desfăşurat cea de a zecea ediţie a Olimpiadei de Matematică pentru studenţii din sud-estul Europei (South Eastern European Mathematical Olympiad for University Students), SEEMOUS 2016. Ediţia din acest an a avut loc în Protaras, Cipru, şi a fost organizată de Societatea de Matematică din Sud-Estul Europei (MASSEE) şi de Societatea de Matematică din Cipru (CYMS), sub auspiciile Ministerului Educaţiei şi Culturii din Cipru. Au participat 23 de echipe din Bulgaria, Cipru, Grecia, Iran, Fosta Republică Yugoslavă a Macedoniei, Marea Britanie, România, Turkmenistan, Uzbekistan.

Concursul a constat dintr-o singură probă desfăşurată pe durata a cinci ore, timp în care studenţii au avut de rezolvat patru probleme selectate de către juriu în funcţie de nivelul de dificultate: prima problemă a fost considerată de juriu ca fiind una uşoară, următoarele două probleme au fost considerate de dificultate medie, iar ultima problemă s-a încadrat la nivel de dificultate ridicat.

Fiecare problemă s-a punctat cu un număr întreg de la 0 la 10. La finalul concursului s-au acordat 10 medalii de aur, 20 medalii de argint şi 21 medalii de bronz, la un total de 90 participanţi. Patru concurenţi au obţinut punctajul maxim, dintre care doi din România, şi anume Vlad Mihai Mihaly de la Universitatea Tehnică din Cluj-Napoca şi Emanuel Necula de la Universitatea Politehnică din Bucureşti.

Mai multe detalii despre concurs se pot consulta pe pagina web oficială: `http://www.massee-org.eu/index.php/news/item/49-seemous-2016`.

Prezentăm, în continuare, problemele date la concurs, însoţite de soluţii şi comentarii pe marginea acestora.

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Problema 1. *Să se arate că pentru orice funcţie continuă şi descrescătoare* $f : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ *au loc inegalităţile:*

$$\int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} f(x)\,\mathrm{d}x \le \int_0^{\frac{\pi}{2}} f(x)\cos x\,\mathrm{d}x \le \int_0^1 f(x)\,\mathrm{d}x.$$

Să se precizeze când are loc fiecare dintre egalităţi.

Pirmyrat Gurbanov, Turkmenistan

Comentarii şi observaţii. Această problemă a primit diferite soluţii din partea participanţilor la concurs (23 de concurenţi au obţinut punctaj maxim) precum şi din partea membrilor juriului.

Notăm faptul că pentru fiecare inegalitate în parte, cazul de egalitate are loc doar pentru funcţiile constante, fapt ce rezultă imediat pentru fiecare dintre abordările pe care le vom prezenta în continuare. Astfel, vom omite să mai discutăm acest aspect.

De asemenea, au existat câteva abordări ale problemei (pe care nu le vom mai prezenta aici) în care s-a presupus că funcţia f este pozitivă. De notat că această presupunere nu reduce din generalitate problemei (este suficient să se adune la f o constantă suficient de mare pentru a o face pozitivă, deoarece pentru funcţii constante are loc egalitate iar funcţia f este mărginită).

Soluţia 1 (a autorului). Pentru a demonstra prima inegalitate, se minorează diferenţa dintre a doua şi prima integrală, ţinând cont de monotonia funcţiei f. Astfel:

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} f(x)\cos x\,\mathrm{d}x - \int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} f(x)\,\mathrm{d}x = \\
&\qquad = \int_0^{\frac{\pi}{2}-1} f(x)\cos x\,\mathrm{d}x - \int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} f(x)\cdot(1-\cos x)\,\mathrm{d}x \\
&\qquad \ge \int_0^{\frac{\pi}{2}-1} f(x)\cos x\,\mathrm{d}x - f\left(\frac{\pi}{2}-1\right)\int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} (1-\cos x)\,\mathrm{d}x \\
&\qquad = \int_0^{\frac{\pi}{2}-1} f(x)\cos x\,\mathrm{d}x - f\left(\frac{\pi}{2}-1\right)\left(1 - \int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} \cos x\,\mathrm{d}x\right) \\
&\qquad = \int_0^{\frac{\pi}{2}-1} f(x)\cos x\,\mathrm{d}x - f\left(\frac{\pi}{2}-1\right)\left(\int_0^{\frac{\pi}{2}} \cos x\,\mathrm{d}x - \int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} \cos x\,\mathrm{d}x\right) \\
&\qquad = \int_0^{\frac{\pi}{2}-1} f(x)\cos x\,\mathrm{d}x - f\left(\frac{\pi}{2}-1\right)\int_0^{\frac{\pi}{2}-1} \cos x\,\mathrm{d}x
\end{aligned}$$

$$= \int_0^{\frac{\pi}{2}-1} \left(f(x) - f\left(\frac{\pi}{2} - 1\right)\right) \cos x \,\mathrm{d}x \geq 0.$$

Pentru a doua inegalitate se procedează similar, astfel:

$$\int_0^1 f(x)\,\mathrm{d}x - \int_0^{\frac{\pi}{2}} f(x) \cos x \,\mathrm{d}x =$$
$$= \int_0^1 f(x) \cdot (1 - \cos x)\ \mathrm{d}x - \int_1^{\frac{\pi}{2}} f(x) \cos x \,\mathrm{d}x$$
$$\geq f(1) \int_0^1 (1 - \cos x)\ \mathrm{d}x - \int_1^{\frac{\pi}{2}} f(x) \cos x \,\mathrm{d}x$$
$$= f(1) \left(1 - \int_0^1 \cos x \,\mathrm{d}x\right) - \int_1^{\frac{\pi}{2}} f(x) \cos x \,\mathrm{d}x$$
$$= f(1) \left(\int_0^{\frac{\pi}{2}} \cos x \,\mathrm{d}x - \int_0^1 \cos x \,\mathrm{d}x\right) - \int_1^{\frac{\pi}{2}} f(x) \cos x \,\mathrm{d}x$$
$$= f(1) \int_1^{\frac{\pi}{2}} \cos x \,\mathrm{d}x - \int_1^{\frac{\pi}{2}} f(x) \cos x \,\mathrm{d}x$$
$$= \int_1^{\frac{\pi}{2}} (f(x) - f(1)) \cos x \,\mathrm{d}x \geq 0.$$

Soluţia 2. Făcând schimbarea de variabilă $t = x - \left(\frac{\pi}{2} - 1\right)$ în prima integrală, respectiv $t = \sin x$ în a doua integrală, rezultă că şirul de inegalităţi ce trebuie demonstrat se rescrie echivalent

$$\int_0^1 f\left(t + \frac{\pi}{2} - 1\right)\ \mathrm{d}t \leq \int_0^1 f(\arcsin t)\,\mathrm{d}t \leq \int_0^1 f(t)\,\mathrm{d}t.$$

Se arată uşor că

$$t + \frac{\pi}{2} - 1 \geq \arcsin t \geq t \quad \text{pentru orice } t \in [0, 1],$$

astfel că şirul de inegalităţi de mai sus este o consecinţă imediată a monotoniei funcţiei f şi a integralei.

Soluţia 3. Pentru a demonstra prima inegalitate, se foloseşte inegalitatea lui Cebâşev sub formă integrală pentru funcţiile f şi cos. Astfel,

$$\int_0^{\frac{\pi}{2}} f(x) \cos x \,\mathrm{d}x \geq \frac{2}{\pi} \left(\int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x\right) \left(\int_0^{\frac{\pi}{2}} \cos x \,\mathrm{d}x\right) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x.$$

Mai departe, făcând schimbarea de variabilă $x = \frac{\pi}{2}\left(t - \frac{\pi}{2} + 1\right)$, rezultă că

$$\frac{2}{\pi} \int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x = \int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} f\left(\frac{\pi}{2}\left(t - \frac{\pi}{2} + 1\right)\right)\ \mathrm{d}t \geq \int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} f(t)\,\mathrm{d}t,$$

unde ultima inegalitatea rezultă din monotonia funcţiei f şi din faptul că
$$\frac{\pi}{2}\left(t-\frac{\pi}{2}+1\right)\le t \quad \text{pentru orice } t\le\frac{\pi}{2}.$$
Combinând cele două inegalităţi obţinute, rezultă prima inegalitate din enunţ.

Pentru a obţine a doua inegalitate, se consideră substituţia $x=\sin t$ în integrala din membrul drept, astfel că
$$\int_0^1 f(x)\,\mathrm{d}x=\int_0^{\frac{\pi}{2}} f(\sin t)\cos t\,\mathrm{d}t\ge\int_0^{\frac{\pi}{2}} f(t)\cos t\,\mathrm{d}t,$$
unde s-a ţinut cont de monotonia funcţiei f şi de faptul că $\sin t\le t$ pentru orice $t\in\left[0,\frac{\pi}{2}\right]$.

Problema 2.

a) *Să se demonstreze că pentru orice matrice $X\in\mathcal{M}_2(\mathbb{C})$ există o matrice $Y\in\mathcal{M}_2(\mathbb{C})$ astfel ca $Y^3=X^2$.*

b) *Să se demonstreze că există o matrice $X\in\mathcal{M}_3(\mathbb{C})$ astfel încât pentru orice matrice $Y\in\mathcal{M}_3(\mathbb{C})$ să aibă loc $Y^3\ne X^2$.*

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Comentarii şi observaţii. Dacă pentru $n,k\ge 2$ notăm
$$\mathcal{P}(n,k)=\left\{X^k : X\in\mathcal{M}_n(\mathbb{C})\right\},$$
atunci punctul a) al problemei cere incluziunea $\mathcal{P}(2,2)\subseteq\mathcal{P}(2,3)$. Se poate arăta, urmând soluţia prezentată mai jos, că şi incluziunea inversă are loc, deci $\mathcal{P}(2,3)\subseteq\mathcal{P}(2,2)$. De asemenea, la punctul b), se arată că $\mathcal{P}(3,2)\nsubseteq\mathcal{P}(3,3)$. Reciproc, se poate arăta, însă, că $\mathcal{P}(3,3)\subseteq\mathcal{P}(3,2)$.

Pentru această problemă, 15 dintre concurenţi au primit punctajul maxim.

Soluţia autorului.

a) Fie J_X forma canonică Jordan a matricei X şi fie P matricea de pasaj, astfel că
$$X=P\cdot J_X\cdot P^{-1}.$$
Căutăm o matrice $Y\in\mathcal{M}_2(\mathbb{C})$ de forma
$$Y=P\cdot Y_1\cdot P^{-1}$$
astfel că egalitatea $Y^3=X^2$ devine $Y_1^3=J_X^2$.

Distingem două cazuri, după felul formei canonice Jordan.

Astfel, dacă $J_X=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$, atunci $J_X^2=\begin{bmatrix}\lambda_1^2 & 0\\ 0 & \lambda_2^2\end{bmatrix}$ şi putem considera $Y_1=\begin{bmatrix}\mu_1 & 0\\ 0 & \mu_2\end{bmatrix}$, unde $\mu_1,\mu_2\in\mathbb{C}$ sunt astfel încât $\mu_1^3=\lambda_1^2$ şi $\mu_2^3=\lambda_2^2$.

Dacă $J_X = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, atunci $J_X^2 = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix}$ şi căutăm Y_1 de forma $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$. Atunci $Y_1^3 = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$, de unde rezultă egalităţile $a^3 = \lambda^2$ şi $3a^2b = 2\lambda$. Dacă $\lambda = 0$, atunci se alege $a = 0$ şi $b \in \mathbb{C}$ oarecare, astfel că $Y_1 = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$. Dacă $\lambda \neq 0$, atunci se alege $a \in \mathbb{C}^*$ astfel încât $a^3 = \lambda^2$, de unde $b = \dfrac{2\lambda}{3a^2}$.

b) Este suficient să alegem $X \in \mathcal{M}_3(\mathbb{C})$ astfel încât $X^2 \neq O_3$ şi $X^3 = O_3$, spre exemplu, $X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Astfel, dacă există $Y \in \mathcal{M}_3(\mathbb{C})$ astfel încât $Y^3 = X^2$, rezultă că $Y^6 = X^4 = O_3$, deci Y este o matrice nilpotentă, de unde rezultă că $Y^3 = O_3$, ceea ce reprezintă o contradicţie cu $Y^3 = X^2 \neq O_3$.

Problema 3. *Fie $n, k \in \mathbb{N}^*$ şi matricele idempotente $A_1, \ldots, A_k \in \mathcal{M}_n(\mathbb{R})$. Să se arate că*

$$\sum_{i=1}^{k} (n - \operatorname{rang}(A_i)) \geq \operatorname{rang}\left(I_n - \prod_{i=1}^{k} A_i\right).$$

* * *

Comentarii şi observaţii. Problema a fost propusă de delegaţia din Turkmenistan, însă s-a dovedit ulterior concursului ca fiind o problemă relativ cunoscută. Mai exact, concluzia problemei se obţine prin simpla combinare a două proprietăţi legate de rangul matricelor, proprietăţi ce se regăsesc ca exerciţii în lucrarea Fuzhen Zhang, *Matrix Theory. Basic Results and Techniques*. Universitext. Springer-Verlag, New York, 2011, pg. 55.

Un număr de 18 concurenţi au primit punctajul maxim pentru această problemă.

Soluţie. Rezultatul se obţine prin combinarea următoarelor două rezultate auxiliare:

(1) Pentru orice matrice $A \in \mathcal{M}_n(\mathbb{R})$ idempotentă are loc relaţia

$$\operatorname{rang} A + \operatorname{rang}(I_n - A) = n.$$

(2) Pentru orice matrice $A, B \in \mathcal{M}_n(\mathbb{R})$ are loc relaţia

$$\operatorname{rang}(I_n - AB) \leq \operatorname{rang}(I_n - A) + \operatorname{rang}(I_n - B).$$

Rezultatul (1) este o consecinţă a proprietăţii de subaditivitate a rangului, respectiv a inegalităţii lui Sylvester. Astfel,

$$\operatorname{rang}(A + B) \leq \operatorname{rang}(A) + \operatorname{rang}(B) \leq n - \operatorname{rang}(AB), \quad A, B \in \mathcal{M}_n(\mathbb{R}),$$

de unde, pentru $B = I_n - A$, se obţine concluzia.

Rezultatul (2) se obţine folosind tot proprietatea de subaditivitate a rangului, astfel că

$$\begin{aligned}\operatorname{rang}(I_n - AB) &= \operatorname{rang}\left((I_n - A) + A(I_n - B)\right)\\ &\le \operatorname{rang}(I_n - A) + \operatorname{rang}\left(A(I_n - B)\right)\\ &\le \operatorname{rang}(I_n - A) + \operatorname{rang}(I_n - B),\end{aligned}$$

unde ultima inegalitate rezultă din aplicarea proprietăţii

$$\operatorname{rang}(AB) \le \min\{\operatorname{rang}(A), \operatorname{rang}(B)\}$$

pentru orice $A, B \in \mathcal{M}_n(\mathbb{R})$.

Rezultatul (2) se extinde natural, prin inducţie, la oricare k matrice $A_1, A_2, \ldots, A_k \in \mathcal{M}_n(\mathbb{R})$, astfel că are loc inegalitatea

$$\operatorname{rang}\left(I_n - \prod_{i=1}^{k} A_i\right) \le \sum_{i=1}^{k} \operatorname{rang}(I_n - A_i),$$

de unde, aplicând apoi rezultatul (1) se obţine direct concluzia problemei.

Problema 4. *Pentru orice $n \in \mathbb{N}^*$, fie*

$$I_n = \int_0^{\infty} \frac{\operatorname{arctg} x}{(1+x^2)^n}\,\mathrm{d}x.$$

Să se demonstreze că:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{I_n}{n} = \frac{\pi^2}{6}$;

b) $\displaystyle\int_0^{\infty} \operatorname{arctg} x \cdot \ln\left(1 + \frac{1}{x^2}\right) \mathrm{d}x = \frac{\pi^2}{6}$.

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Comentarii şi observaţii. Pe lângă soluţia autorului, prezentăm încă două soluţii: prima propusă de conf. univ. dr. Tiberiu Trif de la Universitatea „Babeş-Bolyai” din Cluj-Napoca, a doua de prof. univ. dr. Mircea Ivan de la Universitatea Tehnică din Cluj-Napoca.

Numărul concurenţilor ce au primit punctajul maxim pentru această problemă a fost 7.

Soluţia 1 (a autorului). a) Calculând I_n prin părţi, cu alegerile

$$\begin{aligned} f(x) &= \frac{\operatorname{arctg} x}{(1+x^2)^n}, & f'(x) &= \frac{1 - 2nx \operatorname{arctg} x}{(1+x^2)^{n+1}},\\ g'(x) &= 1, & g(x) &= x,\end{aligned}$$

se obţine că

$$\begin{aligned}
I_n &= \frac{x \operatorname{arctg} x}{(1+x^2)^{n+1}}\Bigg|_0^\infty - \int_0^\infty \frac{x - 2nx^2 \operatorname{arctg} x}{(1+x^2)^{n+1}}\,\mathrm{d}x = \\
&= -\int_0^\infty \frac{x}{(1+x^2)^{n+1}}\,\mathrm{d}x + 2n\int_0^\infty \frac{x^2}{(1+x^2)^{n+1}} \operatorname{arctg} x\,\mathrm{d}x = \\
&= \frac{1}{2n(1+x^2)^n}\Bigg|_0^\infty + 2n\int_0^\infty \left(\frac{\operatorname{arctg} x}{(1+x^2)^n} - \frac{\operatorname{arctg} x}{(1+x^2)^{n+1}}\right)\mathrm{d}x = \\
&= -\frac{1}{2n} + 2n(I_n - I_{n+1}).
\end{aligned}$$

Concluzionând,

$$\frac{I_n}{n} = -\frac{1}{2n^2} + 2(I_n - I_{n+1}) \quad \text{pentru orice } n \geq 1$$

de unde, mai departe,

$$\sum_{n=1}^\infty \frac{I_n}{n} = -\frac{1}{2}\sum_{n=1}^\infty \frac{1}{n^2} + 2\sum_{n=1}^\infty (I_n - I_{n+1}) = -\frac{1}{2}\cdot\frac{\pi^2}{6} + 2I_1 = \frac{\pi^2}{6},$$

deoarece

$$\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6},$$

$$\lim_{n\to\infty} I_n = 0$$

şi

$$I_1 = \int_0^\infty \frac{\operatorname{arctg} x}{1+x^2}\,\mathrm{d}x = \frac{\operatorname{arctg}^2 x}{2}\Bigg|_0^\infty = \frac{\pi^2}{8}.$$

b) Are loc şirul de egalităţi:

$$\begin{aligned}
\int_0^\infty \operatorname{arctg} x \cdot \ln\left(1 + \frac{1}{x^2}\right)\mathrm{d}x &= -\int_0^\infty \operatorname{arctg} x \cdot \ln\left(1 - \frac{1}{1+x^2}\right)\mathrm{d}x = \\
&= \int_0^\infty \operatorname{arctg} x \cdot \left(\sum_{n=1}^\infty \frac{1}{n}\left(\frac{1}{1+x^2}\right)^n\right)\mathrm{d}x = \\
&= \sum_{n=1}^\infty \frac{I_n}{n} = \frac{\pi^2}{6},
\end{aligned}$$

unde ultima egalitate rezultă de la punctul a). Permutarea integralei cu suma se poate justifica, spre exemplu, folosind teorema lui Tonelli, deoarece toţi termenii sunt pozitivi.

Soluţia 2 (Tiberiu Trif). Egalitatea dintre suma $\displaystyle\sum_{n=1}^\infty \frac{I_n}{n}$ de la punctul a) şi integrala $\displaystyle\int_0^\infty \operatorname{arctg} x \cdot \ln\left(1 + \frac{1}{x^2}\right)\mathrm{d}x$ de la punctul b) se obţine la fel ca în Soluţia 1 pentru punctul b), astfel că este suficient să se dea o deducere directă a identităţii de la punctul b).

Astfel, prin schimbarea de variabilă $t = \operatorname{arctg} x$ şi apoi folosind metoda de integrare prin părţi se obţine că

$$I := \int_0^\infty \operatorname{arctg} x \cdot \ln\left(1 + \frac{1}{x^2}\right) \mathrm{d}x = -2\int_0^{\frac{\pi}{2}} \frac{t}{\cos^2 t} \ln(\sin t)\,\mathrm{d}t =$$

$$= -2\int_0^{\frac{\pi}{2}} (t \cdot \operatorname{tg} t + \ln(\cos t))' \cdot \ln(\sin t)\,\mathrm{d}t =$$

$$= -2\,(t \cdot \operatorname{tg} t + \ln(\cos t)) \cdot \ln(\sin t)\,\Big|_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} (t \cdot \operatorname{tg} t + \ln(\cos t))\,\frac{\cos t}{\sin t}\,\mathrm{d}t =$$

$$= 2\int_0^{\frac{\pi}{2}} t\,\mathrm{d}t + 2\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t} \ln(\cos t)\,\mathrm{d}t = \frac{\pi^2}{4} + 2J,$$

unde

$$J := \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t} \ln(\cos t)\,\mathrm{d}t.$$

De notat că s-a ţinut cont în calculele intermediare de relaţiile:

$$\int \frac{t}{\cos^2 t}\,\mathrm{d}t = t \cdot \operatorname{tg} t + \ln(\cos t) + C,$$

$$(t \cdot \operatorname{tg} t + \ln(\cos t)) \cdot \ln(\sin t)\,\Big|_0^{\frac{\pi}{2}} = 0.$$

Mai departe,

$$J = \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{2\sin t\cos t}{\sin^2 t} \ln(1 - \sin^2 t)\,\mathrm{d}t,$$

astfel că prin schimbarea de variabilă $x = \sin^2 t$, se obţine

$$J = \frac{1}{4}\int_0^1 \frac{\ln(1-x)}{x}\,\mathrm{d}x.$$

În final,

$$\frac{\ln(1-x)}{x} = -\sum_{n=0}^{\infty} \frac{x^n}{n+1},$$

de unde

$$J = -\frac{1}{4}\sum_{n=0}^{\infty}\int_0^1 \frac{x^n}{n+1}\,\mathrm{d}x = -\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{\pi^2}{24}.$$

Revenind, rezultă

$$I = \frac{\pi^2}{4} + 2J = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6}.$$

Soluţia 3 (Mircea Ivan). Diferenţa faţă de soluţia anterioară este dată de modul prin care este calculată integrala de la punctul b). Integrând prin părţi, se obţine că

$$\begin{aligned}
I &:= \int_0^\infty \operatorname{arctg} x \cdot \ln\left(1+\frac{1}{x^2}\right) \mathrm{d}x = x \cdot \operatorname{arctg} x \cdot \ln\left(1+\frac{1}{x^2}\right) \Bigg|_0^\infty - \\
&- \int_0^\infty x \left(\frac{1}{1+x^2} \ln\left(1+\frac{1}{x^2}\right) + \operatorname{arctg} x \cdot \left(-\frac{2}{x(x^2+1)}\right)\right) \mathrm{d}x = \\
&= 0 - \int_0^\infty \frac{x}{1+x^2} \ln\left(1+\frac{1}{x^2}\right) \mathrm{d}x + 2\int_0^\infty \frac{\operatorname{arctg} x}{1+x^2} \mathrm{d}x = \\
&= (\operatorname{arctg} x)^2 \Bigg|_0^\infty - \frac{1}{2}\int_0^\infty (1+x^2) \ln\left(1-\frac{1}{1+x^2}\right) \cdot \left(\frac{1}{1+x^2}\right)' \mathrm{d}x,
\end{aligned}$$

iar cu schimbarea de variabilă $u = \dfrac{1}{1+x^2}$ se obţine

$$I = \frac{\pi^2}{4} + \frac{1}{2}\int_0^1 \frac{\ln(1-u)}{u} \mathrm{d}u.$$

Folosind formula dedusă la soluţia anterioară

$$\int_0^1 \frac{\ln(1-u)}{u} \mathrm{d}u = -\frac{\pi^2}{6},$$

se obţine în final că

$$I = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6}.$$

Left and Right Isoscelizers and S-Triangles

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Abstract. In *Gazeta Matematică, Seria A*, **22(101)** (2004), 222–231 we defined the left and right isoscelizers and showed some related bicentric pairs of points similarly with the Yff's points. The aim of this paper is to link the left and right isoscelizers with the S-triangles (orthopolar triangles). The definition and properties of S-triangles have been introduced by Traian Lalescu in *Gazeta Matematică* volume XX, in February 1915, p. 213.

Keywords: Left and right isoscelizers, S-triangles, sine-triple-angle-circle

MSC: Primary 51M04.

1. LEFT AND RIGHT ISOSCELIZERS

Definition. Given a triangle ABC with acute angles, we consider the points $A_1, A_2 \in BC$, $B_1, B_2 \in CA$ and $C_1, C_2 \in AB$, such that trC_2B_1A, trA_2C_1B and trB_2A_1C are isosceles triangles: $\widehat{C_2B_1A} = \widehat{A}$, $\widehat{A_2C_1B} = \widehat{B}$ and $\widehat{B_2A_1C} = \widehat{C}$.

We call the line segments C_2B_1, A_2C_1, B_2A_1, *A-left isoscelizer, B-left isoscelizer* and *C-left isoscelizer* (see Fig. 1.ℓ).

If the points $A_1, A_2 \in BC$, $B_1, B_2 \in CA$ and $C_1, C_2 \in AB$, are such that $\widehat{B_1C_2A} = \widehat{A}$, $\widehat{C_1A_2B} = \widehat{B}$ and $\widehat{A_1B_2C} = \widehat{C}$, then the line segments B_1C_2, C_1A_2 and A_1B_2 are *A-right isoscelizer, B-right isoscelizer* and *C-right isoscelizer* (see Fig. 1.r).

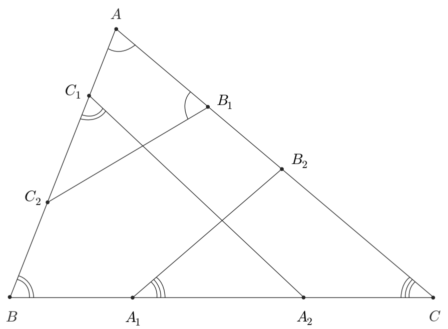

Figure 1.ℓ

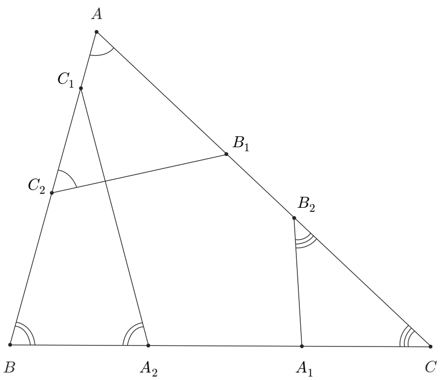

Figure 1.r

Denote the left-isoscelizer lengths as follows:

$$u = C_2B_1 = C_2A,\ v = A_2C_1 = A_2B,\ w = B_2A_1 = B_2C.$$

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Theorem 1.ℓ. *The three lines C_2B_1, A_2C_1 and B_2A_1 concur (see Fig. 2.ℓ) if and only if*

$$\begin{aligned}
&8\cos A\cos B\cos C(ab\cdot u+bc\cdot v+ca\cdot w)\\
&\qquad -4(a^2\cos A\cos B\cdot u+b^2\cos B\cos C\cdot v+c^2\cos C\cos A\cdot w)\\
&\qquad +2(ca\cos A\cdot u+ab\cos B\cdot v+bc\cos C\cdot w)\\
&\qquad =abc(8\cos A\cos B\cos C+1). \qquad (1.\ell)
\end{aligned}$$

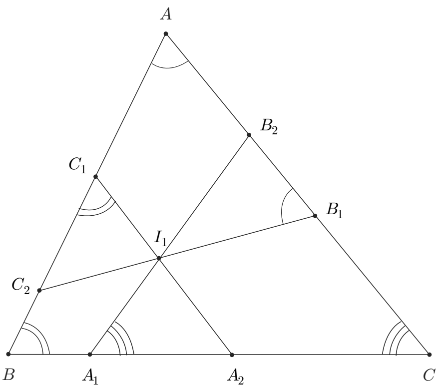

Figure 2.ℓ

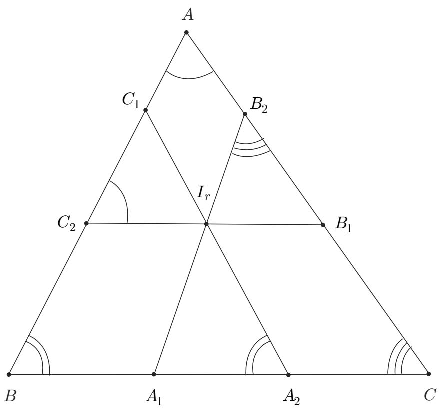

Figure 2.r

Proof. The barycentric coordinates of A_1 and B_2 are:

$$A_1=0:2w\cos C:a-2w\cos C,$$
$$B_2=w:0:b-w.$$

The line A_1B_2 is then given by

$$A_1B_2:\begin{vmatrix}\alpha & \beta & \gamma\\ 0 & 2w\cos C & a-2w\cos C\\ w & 0 & b-w\end{vmatrix}=0,$$

that is

$$A_1B_2:2(b-w)\cos C\cdot\alpha+(a-2w\cos C)\cdot\beta-2w\cos C-\gamma=0,$$

and similarly for B_1C_2 and C_1A_2:

$$B_1C_2:\mathbb{Q}-2u\cos A\cdot\alpha+2(c-u)\cos A\cdot\beta+(b-2u\cos A)\cdot\gamma=0,$$
$$C_1A_2:\mathbb{Q}(c-2v\cos B)\cdot\alpha-2v\cos B\cdot\beta+2(a-v)\cos B\cdot\gamma=0.$$

The three lines concur if and only if

$$\begin{vmatrix}2(b-w)\cos C & a-2w\cos C & -2w\cos C\\ -2u\cos A & 2(c-u)\cos A & b-2u\cos A\\ c-2v\cos B & -2v\cos B & 2(a-v)\cos B\end{vmatrix}=0,$$

which is equivalent with $(1.\ell)$. □

Let I_ℓ be the point of intersection of left-isoscelizers C_2B_1, A_2C_1 and B_2A_1. The trilinears of I_ℓ are:

$$\alpha_\ell = 4\cos A\cos B\sin 2C\cdot w - 2\cos B\sin 2A\cdot u + \sin 2B\cdot v,$$

$$\beta_\ell = 4\cos B\cos C\sin 2A\cdot u - 2\cos C\sin 2B\cdot v + \sin 2C\cdot w,$$

$$\gamma_\ell = 4\cos C\cos A\sin 2B\cdot v - 2\cos A\sin 2C\cdot w + \sin 2A\cdot u.$$

Similarly, for the right-isoscelizers

$$B_1C_2 = B_1A = u, \quad C_1A_2 = C_1B = v, \quad A_1B_2 = A_1C = w,$$

the following theorem holds.

Theorem 1.*r*. *The three lines C_2B_1, A_2C_1, B_2A_1 concur in a point I_r (see Fig. 2.r) if and only if*

$$\begin{aligned}
&8\cos A\cos B\cos C(ca\cdot u + ab\cdot v + bc\cdot w)\\
&\quad - 4(a^2\cos C\cos A\cdot u + b^2\cos A\cos B\cdot v + c^2\cos B\cos C\cdot w)\\
&\quad + 2(ab\cos A\cdot u + bc\cos B\cdot v + ca\cos C\cdot w)\\
&\quad = abc(8\cos A\cos B\cos C + 1)
\end{aligned}\tag{1.r}$$

The trilinear coordinates of I_r are

$$I_r = \alpha_r : \beta_r : \gamma_r,$$

where

$$\alpha_r = 4\cos C\cos A\sin 2B\cdot v - 2\cos C\sin 2A\cdot u + \sin 2C\cdot w,$$

$$\beta_r = 4\cos A\cos B\sin 2C\cdot w - 2\cos A\sin 2B\cdot v + \sin 2A\cdot u,$$

$$\gamma_r = 4\cos B\cos C\sin 2A\cdot u - 2\cos B\sin 2C\cdot w + \sin 2B\cdot v.$$

Choosing in Theorems $1.\ell$ and $1.r$ appropriate values for u, v, w such that triangles C_2B_1A, A_2C_1B, B_2A_1C have certain properties, we obtain several pairs of bicentric points, (I_ℓ, I_r), similar to the centers X_{173} (= congruent isoscelizers point), X_{362} (= congruent circumcircles isoscelizer point), X_{363} (= equal perimeters isoscelizer point), X_{364} (= Wabash center = equal areas isoscelizer point), X_{164} (= congruent altitudes), X_{258} (= congruent incircles) discovered by Peter Yff (see [4], [5]).

2. S-Triangles (Orthopolar Triangles)

In a note [8] published in *Gazeta Matematică* in 1915, Traian Lalescu wrote:

"Given a triangle ABC and two points B' and C' on its circumcircle. Let A' be the point on the same circle for which the Simson line is perpendicular on $B'C'$. It is known that this point is obtained as follows:

From A we draw a straight line perpendicular on $B'C'$ which intersect the circle in A'' and from A'' we draw the straight line perpendicular on BC which intersect the circle in the searched point A'. We say that the triangle $A'B'C'$ is a S-triangle with respect to the triangle ABC."

Next, Traian Lalescu proved the following properties:

1) The algebraic sum of arcs $\overset{\frown}{AA'}$, $\overset{\frown}{BB'}$ and $\overset{\frown}{CC'}$ considered with the same orientation is zero:

$$\mathrm{m}(\overset{\frown}{AA'}) + \mathrm{m}(\overset{\frown}{BB'}) + \mathrm{m}(\overset{\frown}{CC'}) \equiv 0 \pmod{2\pi}.$$

2) The Simson line of each vertex of the triangle $A'B'C'$ is perpendicular on the opposite side of this triangle.

3) The Simson lines of A', B', C' with respect to triangle ABC are concurrent.

4) The triangle ABC is an S-triangle with respect to the triangle $A'B'C'$.

5) The intersection point of the six Simson lines is the midpoint of the line segment joining the orthocenters of triangles ABC and $A'B'C'$ (see [8], [9], [10]).

Other authors showed that the point of intersection of the six Simson lines is the common orthopole of each side of the triangle $A'B'C'$ with respect to the triangle ABC and of each side of the triangle ABC with respect to the triangle $A'B'C'$, and for this reason the S-triangles are also named *orthopolar triangles*.

A classical example of S-triangles is the pair formed by the orthic triangle and the medial triangle, inscribed in the nine-point circle. The common orthopole is the midpoint of the line segment joining X_3 and X_{52}, which is X_{389}, the center of the Taylor's circle = the Spiecker point (incenter of the medial triangle) of the orthic triangle (notations from [5]).

A natural generalization of the S-triangles is obtained by replacing the Wallace-Simson lines with the Carnot lines.

Theorem 2.1. (Carnot) *Given a triangle ABC and M a point on its circumcircle. Let $P \in BC$, $Q \in CA$ and $R \in AB$ such that*

$$\mathrm{m}(\widehat{MP, BC}) = \mathrm{m}(\widehat{MQ, CA}) = \mathrm{m}(\widehat{MR, AB}) = \varphi.$$

Then the points P, Q and R are collinear, and the straight line PQR is called the φ-Carnot line.

Definition 2.2. Given the triangle ABC, let B' and C' two points on its circumcircle and let A' be the point on the same circle such that

$$\mathrm{m}(\widehat{\Delta_{A'}, B'C'}) = \pi - \varphi,$$

where $\Delta_{A'}$ is the φ-Carnot line of the point A' with respect to the triangle ABC. We say that the triangle $A'B'C'$ is a *φ-S-triangle with respect to triangle ABC*.

The point A' is obtained by the following construction: From A we draw the straight line AA'' such that $\mathrm{m}(\widehat{AA'', B'C'}) = \pi - \varphi$, where A'' lies on the circumcircle of ABC, and from A'' we draw the straight line $A''A'$ such that $\mathrm{m}(\widehat{A''A', BC}) = \varphi$, where A' lies on the same circle. This point A' is the searched point.

Theorem 2.3. *The pair of triangles ABC and $A'B'C'$ has the following properties*:

1) $\mathrm{m}(\overset{\frown}{AA'}) + \mathrm{m}(\overset{\frown}{BB'}) + \mathrm{m}(\overset{\frown}{CC'}) \equiv 4\varphi \pmod{2\pi}$. (2.1)

2) *The measure of the angle between the φ-Carnot line of each vertex of the triangle $A'B'C'$ with the opposite side of this triangle is $\pi - \varphi$.*

3) *The φ-Carnot lines of A', B' and C' with respect to the triangle ABC concur in a point I_φ.*

4) *The triangle ABC is a $(\pi - \varphi)$-S-triangle with respect to the triangle $A'B'C'$.*

5) *The $(\pi - \varphi)$-Carnot lines of A, B and C with respect to the triangle $A'B'C'$ concur in the same point I_φ ([3]).*

Remarks. 1) The point I_φ is the φ-isopole of each side of the triangle $A'B'C'$ with respect to the triangle ABC and the $(\pi - \varphi)$-isopole of each side of the triangle ABC with respect to the triangle $A'B'C'$.

2) From $\mathrm{m}(\overset{\frown}{AA'}) + \mathrm{m}(\overset{\frown}{BB'}) + \mathrm{m}(\overset{\frown}{CC'}) \equiv 4\varphi \pmod{2\pi}$ we may say that every pair of triangles inscribed in the same circle are in relationship $(\varphi - S)$ and $(\pi - \varphi) - S$ each with respect to the other. Actually, there are two values φ_1 and φ_2 of $\varphi \in (0, \pi)$, with $|\varphi_1 - \varphi_2| = \dfrac{\pi}{2}$, such that the relation (2.1) holds.

A connection between the left and right isoscelizers and S-triangles is given by the following theorem.

Theorem 2.4. ([2]) *Given the acute triangle ABC such that* $\mathrm{m}(\widehat{B}) \neq \frac{1}{2}\mathrm{m}(\widehat{A}) \neq \mathrm{m}(\widehat{C})$, $\mathrm{m}(\widehat{C}) \neq \frac{1}{2}\mathrm{m}(\widehat{B}) \neq \mathrm{m}(\widehat{A})$ *and* $m(\widehat{A}) \neq \frac{1}{2}m(\widehat{C}) \neq m(\widehat{B})$. Let $A_\ell, A_r \in BC$, $B_\ell, B_r \in CA$ and $C_\ell, C_r \in AB$ *such that $B_\ell C_\ell$, $C_\ell A_\ell$ and $A_\ell B_\ell$ are A-, B- and C-left isoscelizers and $B_r C_r$, $C_r A_r$ and $A_r B_r$ are*

A-, B- and C-right isoscelizers. Then the triangles $A_\ell B_\ell C_\ell$ and $A_r B_r C_r$ are inscribed in the same circle and they are S-triangles (see Fig. 3 and Fig. 4).

Proof. The construction of the triangles $A_\ell B_\ell C_\ell$ and $A_r B_r C_r$ follows the procedure to inscribe a triangle having parallel sides with three given directions, into an other triangle.

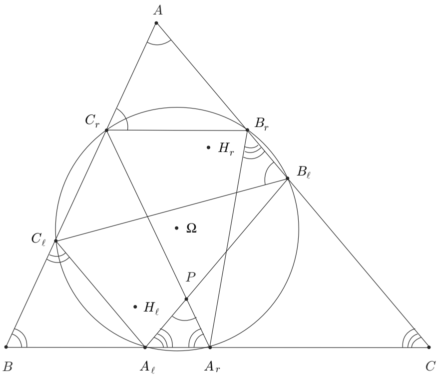

Figure 3

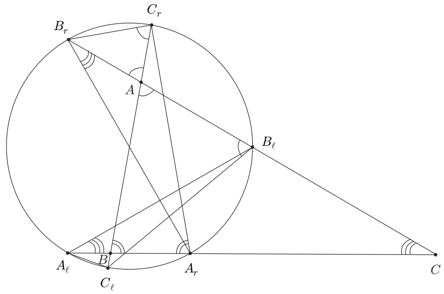

Figure 4

In Fig. 3, according to the case

$$\mathrm{m}(\widehat{A}), \mathrm{m}(\widehat{B}) > \frac{1}{2}\mathrm{m}(\widehat{C}), \quad \mathrm{m}(\widehat{B}), \mathrm{m}(\widehat{C}) > \frac{1}{2}\mathrm{m}(\widehat{A}), \quad \mathrm{m}(\widehat{C}), \mathrm{m}(\widehat{A}) > \frac{1}{2}\mathrm{m}(\widehat{B}),$$

we have

$$\mathrm{m}(\widehat{A_\ell}) = 2\mathrm{m}(\widehat{B}) - \mathrm{m}(\widehat{C}),\ \mathrm{m}(\widehat{B_\ell}) = 2\mathrm{m}(\widehat{C}) - \mathrm{m}(\widehat{A}),\ \mathrm{m}(\widehat{C_\ell}) = 2\mathrm{m}(\widehat{A}) - \mathrm{m}(\widehat{B})$$

and

$$\mathrm{m}(\widehat{A_r}) = 2\mathrm{m}(\widehat{B}) - \mathrm{m}(\widehat{C}),\ \mathrm{m}(\widehat{B_r}) = 2\mathrm{m}(\widehat{A}) - \mathrm{m}(\widehat{C}),\ \mathrm{m}(\widehat{C_r}) = 2\mathrm{m}(\widehat{B}) - \mathrm{m}(\widehat{A}).$$

In Fig. 4, $\mathrm{m}(\widehat{C}) < \frac{1}{2}\mathrm{m}(\widehat{A}), \frac{1}{2}\mathrm{m}(\widehat{B})$.

The triangles $A_\ell B_\ell C_\ell$ and $A_r B_r C_r$ are inscribed in the same circle with center having triangle center function $\alpha = \cos 3A$, which is X_{49}, and circumradius

$$R' = \frac{R}{1 + 8\cos A \cos B \cos C},$$

where R is the circumradius of triangle ABC. This circle is called the sine-triple-angle circle (see [4], [11]).

Let $\{P\} = A_l B_l \cap A_r C_r$ (see Fig. 3). Then

$$\begin{aligned}\mathrm{m}(\widehat{A}) = \mathrm{m}(\widehat{A_l P A_r}) &= \frac{1}{2}[\mathrm{m}(\overset{\frown}{A_l A_r}) + \mathrm{m}(\overset{\frown}{B_l C_r})] \\ &= \frac{1}{2}[\mathrm{m}(\overset{\frown}{A_l A_r}) + \mathrm{m}(\overset{\frown}{B_l B_r}) + \mathrm{m}(\overset{\frown}{B_r C_r})].\end{aligned}$$

But

$$\begin{aligned}\mathrm{m}(\widehat{A}) = \mathrm{m}(\widehat{C_l B_l A}) = \frac{1}{2}\mathrm{m}(\overset{\frown}{B_r C_l}) \\ = \frac{1}{2}[\mathrm{m}(\overset{\frown}{B_r C_r}) + \mathrm{m}(\overset{\frown}{C_r C_l})],\end{aligned}$$

hence

$$\mathrm{m}(\overset{\frown}{A_l A_r}) + \mathrm{m}(\overset{\frown}{B_l B_r}) = \mathrm{m}(\overset{\frown}{C_r C_l}),$$

therefore

$$\mathrm{m}(\overset{\frown}{A_l A_r}) + \mathrm{m}(\overset{\frown}{B_l B_r}) + \mathrm{m}(\overset{\frown}{C_l C_r}) \equiv 0 \pmod{2\pi}.$$

This fact means that $\mathrm{tr}A_lB_lC_l$ and $\mathrm{tr}A_rB_rC_r$ are S-triangles.

Let $B_lC_l = u_l$, $C_lA_l = v_l$ and $A_lB_l = w_l$. Then we have

$$\begin{cases} u_l + 2v_l \cos B = c, \\ v_l + 2w_l \cos C = a, \\ w_l + 2u_l \cos A = b. \end{cases}$$

Therefore, the sides of the triangle $A_lB_lC_l$ are:

$$u_l = \frac{c + 4b\cos B\cos C - 2a\cos B}{1 + 8\cos A\cos B\cos C},$$

$$v_l = \frac{a + 4c\cos C\cos A - 2b\cos C}{1 + 8\cos A\cos B\cos C},$$

$$w_l = \frac{b + 4a\cos A\cos B - 2c\cos A}{1 + 8\cos A\cos B\cos C}.$$

The actual trilinears of A_l, B_l, C_l are

$$\begin{aligned} A_l = 0 : \frac{b + 4a\cos A\cos B - 2c\cos A}{1 + 8\cos A\cos B\cos C} \cdot \sin 2C \\ : \frac{a + 4c\cos C\cos A - 2b\cos C}{1 + 8\cos A\cos B\cos C} \cdot \sin B, \\ B_l = \frac{b + 4a\cos A\cos B - 2c\cos A}{1 + 8\cos A\cos B\cos C} \cdot \sin C : 0 \\ : \frac{c + 4b\cos B\cos C - 2a\cos B}{1 + 8\cos A\cos B\cos C} \cdot \sin 2A, \\ C_l = \frac{a + 4c\cos C\cos A - 2b\cos C}{1 + 8\cos A\cos B\cos C} \cdot \sin 2B \\ : \frac{c + 4b\cos B\cos C - 2a\cos B}{1 + 8\cos A\cos B\cos C} \cdot \sin A : 0. \end{aligned}$$

Let H_l be the orthocenter of the triangle $A_lB_lC_l$. In Fig. 3, we have

$d(H_l, BC) = A_lH_l\cos(2A + B - \pi) = 2R\cos(2B - C)\cos(A - C),$

where R is the radius of the cyclic hexagon $A_lA_rB_lB_rC_rC_l$. Hence, the trilinears of H_l are

$$H_l = \cos(2B-C)\cos(A-C) : \cos(2C-A)\cos(B-A) \\ : \cos(2A-B)\cos(C-B).$$

□

Remark. H_l is denoted $P(61)$ in [7] and is named 1st Văcăreţu Point.

Similarly, we have for the triangle $A_rB_rC_r$

$$\begin{cases} u_r + 2w_r\cos C = b, \\ v_r + 2u_r\cos A = c, \\ w_r + 2v_r\cos B = a, \end{cases}$$

hence

$$u_r = \frac{b+4c\cos B\cos C - 2a\cos C}{1+8\cos A\cos B\cos C},$$
$$v_r = \frac{c+4a\cos C\cos A - 2b\cos A}{1+8\cos A\cos B\cos C},$$
$$w_r = \frac{a+4b\cos A\cos B - 2c\cos B}{1+8\cos A\cos B\cos C}.$$

The actual trilinears of A_r, B_r, C_r are

$$A_r = 0 : \frac{a+4b\cos A\cos B - 2c\cos B}{1+8\cos A\cos B\cos C}\cdot\sin C \\ : \frac{c+4a\cos C\cos A - 2b\cos A}{1+8\cos A\cos B\cos C}\cdot\sin 2B,$$
$$B_r = \frac{a+4b\cos A\cos B - 2c\cos B}{1+8\cos A\cos B\cos C}\cdot\sin 2C : 0 \\ : \frac{b+4c\cos B\cos C - 2a\cos C}{1+8\cos A\cos B\cos C}\cdot\sin A,$$
$$C_r = \frac{c+4a\cos C\cos A - 2b\cos A}{1+8\cos A\cos B\cos C}\cdot\sin B \\ : \frac{b+4c\cos B\cos C - 2a\cos C}{1+8\cos A\cos B\cos C}\cdot\sin 2A : 0.$$

Let H_r be the orthocenter of the triangle $A_rB_rC_r$. The trilinears of H_r are

$$H_r = \cos(2C-B)\cos(A-B) : \cos(2A-C)\cos(B-C) \\ : \cos(2B-A)\cos(C-A).$$

The pair (H_l, H_r) is a pair of bicentric points (see [4], [5], [6]).

The intersection point of the six Simson lines of A_l, B_l, C_l with respect to the triangle $A_rB_rC_r$ and of A_r, B_r, C_r with respect to the triangle $A_lB_lC_l$

(= the common orthopole = the midpoint of the segment line $H_lH_r = X_{1594}$, Rigby-Lalescu orthopole) is $H_l \oplus H_r$ and its trilinears are

$$\alpha = \cos(2B - C)\cos(A - C) + \cos(2C - B)\cos(A - B),$$
$$\beta = \cos(2C - A)\cos(B - A) + \cos(2A - C)\cos(B - C),$$
$$\gamma = \cos(2A - B)\cos(C - B) + \cos(2B - A)\cos(C - A).$$

From the sine law, we may compute the radius R of the circumcircle of the triangles $A_lB_lC_l$, $A_rB_rC_r$:

$$\begin{aligned}\frac{u_l}{\sin(2B-C)} = \frac{v_l}{\sin(2C-A)} &= \frac{w_l}{\sin(2A-B)} = \frac{u_r}{\sin(2C-B)}\\ &= \frac{v_r}{\sin(2A-C)} = \frac{w_r}{\sin(2B-A)} = 2R.\end{aligned}$$

Finally, let Ω be the center of the circumcircle of the triangles $A_lB_lC_l$ and $A_rB_rC_r$. We have

$$\begin{aligned}\mathrm{m}(\widehat{\Omega A_r C_r}) &= \frac{1}{2}\{\pi - 2[2\mathrm{m}(\widehat{A}) - \mathrm{m}(\widehat{C})]\}\\ &= \frac{\pi}{2} - 2\mathrm{m}(\widehat{A}) + \mathrm{m}(\widehat{C}).\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{m}(\widehat{\Omega A_r C_r}) &= \mathrm{m}(\widehat{\Omega A_r C_r}) + \mathrm{m}(\widehat{C_r A_r B})\\ &= \frac{\pi}{2} - 2\mathrm{m}(\widehat{A}) + \mathrm{m}(\widehat{C}) + \mathrm{m}(\widehat{B})\\ &= \frac{3\pi}{2} - 3\mathrm{m}(\widehat{A}).\end{aligned}$$

Hence

$$d(\Omega, BC) = R\sin\left[\frac{3\pi}{2} - 3\mathrm{m}(\widehat{A})\right] = -R\cos(3A).$$

Therefore the trilinears of Ω are

$$\Omega = \cos(3A) : \cos(3B) : \cos(3C),$$

hence $\Omega = X_{49}$ = center of sine-triple-angle-circle ([4], [5]).

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NOTE MATEMATICE

On a new family of semi-perfect cuboids

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Abstract. In this paper, we give a new parametric solution of one of the three variations of one classical open problem relating to the perfect cuboid. With this solution, one can find infinitely many semi-perfect cuboids each of which has integer sides such that two face diagonals and the body diagonal are of integral lengths.

Keywords: Rational cuboids, Diophantine equations, perfect cuboids, semi-perfect cuboids.

MSC: Primary 11D41; Secondary 11D72.

Introduction

The question whether there can be a *perfect rational cuboid*, which is a rectangular parallelepiped whose three edges, three face diagonals and body diagonal all have integer lengths, is yet to be answered. If a, b, c represent the lengths of the edges; p, q, r, the face diagonals; and d, the body diagonal, what is sought is a solution to the following system of equations:

$$\begin{aligned} a^2 + b^2 &= p^2; \\ a^2 + c^2 &= q^2; \\ b^2 + c^2 &= r^2; \\ a^2 + b^2 + c^2 &= d^2, \end{aligned} \tag{1}$$

where a, b, c, p, q, r and d are positive integers. Here, we have to find seven related integer quantities: the lengths of three edges, three face diagonals, and the body diagonal. Till May, 2016, we do not have the solution for the system of Diophantine equations (1). But, if we ask for only six of these to be integers, then there are three variations of this problem as follows:

I. The body diagonal does not have integer length.
II. One of the face diagonals does not have integer length.
III. One of the edges does not have integer length.

The rational cuboids falling into any one of these three variations are called *semi-perfect cuboids*. Infinitely many solutions do exist for all the three variations, though none of them is complete (see [2] and [9]). Many papers dealing with different aspects of this problem are published (see [1], [3], [4], [7], [8], [10]–[13]). The historical details of this problem can be found in [5], and for other related research one may refer [6].

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In this paper, we give a new parametric solution for the *semi-perfect cuboids* of *Variation* III followed by a table containing some numerical results. Here, our approach is purely elementary.

1. The Problem and the solution

For given positive integers a, b, c, d, p, q, we want to find the primitive and non-trivial solutions of the following system of Diophantine equations:

$$\begin{gathered} a^2 + b^2 = p^2; \\ a^2 + c^2 = q^2; \\ \text{and} \\ a^2 + b^2 + c^2 = d^2. \end{gathered} \tag{2}$$

In (2), take

$$a = 2\sqrt{xyzt}\,, b = xy - zt\,, c = xz - yt. \tag{3}$$

So, we get

$$\begin{aligned} p^2 &= a^2 + b^2 = (2\sqrt{xyzt})^2 + (xy - zt)^2 = (xy + zt)^2; \\ q^2 &= a^2 + c^2 = (2\sqrt{xyzt})^2 + (xz - yt)^2 = (xz + yt)^2; \\ d^2 &= a^2 + b^2 + c^2 = (2\sqrt{xyzt})^2 + (xy - zt)^2 + (xz - yt)^2 \\ &= x^2y^2 + z^2t^2 + x^2z^2 + y^2t^2 = (y^2 + z^2)(x^2 + t^2). \end{aligned} \tag{4}$$

Thus, from (4) we have

$$\begin{aligned} p &= (xy + zt); \\ q &= (xz + yt); \\ d &= \sqrt{(y^2 + z^2)(x^2 + t^2)}. \end{aligned} \tag{5}$$

Taking $x = \alpha^2 - \beta^2$, $y = m^2 - n^2$, $z = 2mn$, and $t = 2\alpha\beta$ in (3) and (5), and rearranging, we get

$$\begin{aligned} a &= 4\sqrt{mn\alpha\beta(m^2 - n^2)(\alpha^2 - \beta^2)}; \\ b &= (m^2 - n^2)(\alpha^2 - \beta^2) - 4mn\alpha\beta; \\ c &= 2mn(\alpha^2 - \beta^2) - 2\alpha\beta(m^2 - n^2); \\ p &= (m^2 - n^2)(\alpha^2 - \beta^2) + 4mn\alpha\beta; \\ q &= 2mn(\alpha^2 - \beta^2) + 2\alpha\beta(m^2 - n^2); \\ d &= \sqrt{(m^2 + n^2)^2(\alpha^2 + \beta^2)^2} \\ &= (m^2 + n^2)(\alpha^2 + \beta^2). \end{aligned} \tag{6}$$

We state a lemma which will be used to modify (6) after appropriate substitutions to suit our requirement.

Lemma 1. *For any non-zero integers m, n, α, β, if $\alpha = (m^2-n^2+2mn)^2$ and $\beta = (m^2-n^2-2mn)^2$, then, $\alpha\beta = (m^4-6m^2n^2+n^4)^2$, $\alpha^2-\beta^2 = 16mn(m^2-n^2)(m^2+n^2)^2$ and $\alpha^2+\beta^2 = 2(m^8+20m^6n^2-26m^4n^4+20m^2n^6+n^8)$.*

Proof. The proof is too elementary, and can be easily verified. □

In (6), substituting the values of α, β, $\alpha\beta$, $(\alpha^2-\beta^2)$ and $(\alpha^2+\beta^2)$ from Lemma 1, and then, scaling down the calculated expressions for a, b, c, p, q, d by their common factor 2, we get

$$\begin{aligned}
a &= 8mn(m^8-6m^6n^2+6m^2n^6-n^8);\\
b &= 2mn(3m^8+12m^6n^2-46m^4n^4+12m^2n^6+3n^8);\\
c &= (m^2-n^2)(-m^8+28m^6n^2-6m^4n^4+28m^2n^6-n^8);\\
p &= 2mn(5m^8-12m^6n^2+30m^4n^4-12m^2n^6+5n^8);\\
q &= (m^2-n^2)(m^8+4m^6n^2+70m^4n^4+4m^2n^6+n^8);\\
d &= (m^2+n^2)(m^8+20m^6n^2-26m^4n^4+20m^2n^6+n^8).
\end{aligned} \tag{7}$$

If we take $r^2 = b^2+c^2$, then based on (7) we calculate

$$\begin{aligned}
r = (&m^{20}-22m^{18}n^2+1197m^{16}n^4-2568m^{14}n^6-942m^{12}n^8\\
&+5692m^{10}n^{10}-942m^8n^{12}-2568m^6n^{14}+1197m^4n^{16}\\
&-22m^2n^{18}+n^{20})^{1/2}.
\end{aligned} \tag{8}$$

2. Some numerical solutions

In the following table we give the absolute value of numerical solutions to (2) by using (7) in the range $2 \le m \le 10$ and $1 \le n \le 10$ where $\gcd(m,n) = 1$. In (7), inter-changing $m \leftrightarrow n$ does not affect the absolute values of the functions involved. So, in the calculation table we take only one pair of values for (m,n) instead of two pairs which are obtained by inter-changing the values of m and n. When both m and n are odd, we get 32 as a common factor in the calculated values of (a,b,c,p,q,d). In this case, we have to pass over this factor to get primitive solutions for (a,b,c,p,q,d). In all these numerical solutions as given in the table, we calculated the values of r to be irrational in accordance with (8). Now, there are two options: either we prove that for any integer pair (m,n), r is never an integer; or, we may like to search extensively to find an (m,n) pair for which r is a positive integer. If one is lucky enough with the second option, the search for a perfect cuboid will be over.

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Table 1.

m, n	$\lvert a\rvert$ $\lvert p\rvert$	$\lvert b\rvert$ $\lvert q\rvert$	$\lvert c\rvert$ $\lvert d\rvert$
2, 1	1680	3404	4653
	3796	4947	6005
3, 1	32 × 1680	32× 4653	32× 3404
	32× 4947	32× 3796	32× 6005
4, 1	1313760	1873432	720945
	2288168	1498575	2399057
5, 1	32× 371280	32× 415915	32× 32868
	32× 557525	32× 372732	32× 558493
6, 1	67194960	66474468	13300595
	94519932	68498675	95451157
7, 1	32× 8853600	32× 8135897	32× 3725496
	32× 12024103	32× 9605496	32× 12588025
8, 1	973103040	852635696	595978047
	1293799504	1141104447	1424467265
9, 1	32× 89681760	32× 76059351	32× 70508680
	32× 117591849	32× 114080200	32× 137110601
10, 1	7520047920	6230824060	7133662899
	9765976100	10365339699	12093942101
3, 2	371280	32868	415915
	372732	557525	558493
5, 2	1997520	29636860	28222299
	29704100	28292901	40973549

Table 1. continued...

m, n	$\|a\|$ $\|p\|$	$\|b\|$ $\|q\|$	$\|c\|$ $\|d\|$
7, 2	331495920	593958932	327102435
	680203468	465709725	754766693
9, 2	4366509840	5395802796	1231207747
	6941260404	4536769853	7049607685
4, 3	8853600	3725496	8135897
	9605496	12024103	12588025
5, 3	32× 1313760	32× 720945	32× 1873432
	32× 1498575	32× 2288168	32× 2399057
7, 3	32× 1997520	32× 28222299	32× 29636860
	32× 28292901	32× 29704100	32× 40973549
8, 3	555804480	3070130832	2672572265
	3120035568	2729754775	4108194793
10, 3	11143401360	22298068980	13574892149
	24927480300	17562832651	28384097149
5, 4	89681760	70508680	76059351
	114080200	117591849	137110601
7, 4	983542560	795977896	1610670303
	1265281304	1887224097	2048217665
9, 4	1741390560	11382654168	12622047295
	11515088232	12741605825	17085471457
6, 5	560258160	618148620	456268549
	834264300	722544251	950882701
7, 5	32× 67194960	32× 13300595	32× 66474468
	32× 68498675	32× 94519932	32× 95451157
8, 5	5419202880	1950937840	6979945401
	5759680400	8836707399	9049505849
9, 5	32× 331495920	32× 327102435	32× 593958932
	32× 465709725	32× 680203468	32× 754766693
7, 6	2557005360	3573920196	2034339203
	4394448996	3267539197	4842490885
8, 7	9354555840	15709754032	7334093745
	18283984432	11886826575	19700076593
9, 7	32× 973103040	32× 595978047	32× 852635696
	32× 1141104447	32× 1293799504	32× 1424467265
10, 7	72338224560	8118151580	74031212349
	72792328700	103505744451	103823617349
9, 8	29031468480	56703586896	22538902447
	63703398096	36753615953	67573108945
10, 9	79331127120	176254314660	61167233501
	193284792900	100174139299	202732438501