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On a two parameter class of quadratic Diophantine equations

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Abstract. We characterize the solution set of a two parameter class of quadratic Diophantine equations. Our proof relies on the solvability of the positive Pell equation.

Keywords: Diophantine equation, quadratic equation, Pell equation

MSC: Primary 11D09; Secondary 40-01.

In this note we will be interested in exploring the solutions of quadratic Diophantine equations of the form

$$ax^2 + by^2 + cx + dy + f = 0,$$

where $a, b, c, d, f \in \mathbb{Z}$. Naturally, there is no hope in solving such a problem for arbitrary coefficients, and simple examples show that there are such equations which have zero, finitely many or infinitely many solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$. Equivalently, we can see this to be the case by inspecting the lattice points sitting on the algebraic curve defined by the quadratic equation above, but this geometric intuition is not the one we will be using here. Our motivation for this note stems from the following two problems proposed at the 7–8 grade level:

[3]. *Let $a, b \in \mathbb{N}$ be such that $4a^2 + 9a + 1 = 3b^2 + 7b$. Show that $3a + 3b + 7$ and $4a + 4b + 9$ are perfect squares.*

[4]. *Let $x, y \in \mathbb{N}$ such that $x > y$ and $x + 4x^2 = y + 5y^2$. Show that $x - y$ is a perfect square.*

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We wish to show that the intuition behind these problems lends itself nicely to completely solving a two parameter class of quadratic Diophantine equations. Specifically, let $\alpha, \beta \in \mathbb{N} \cup \{0\}$ and consider the following equation in variables $(x, y) \in \mathbb{Z} \times \mathbb{Z}$:

$$\alpha x^2 - (\alpha + 1)y^2 + (2\alpha\beta + 1)x - (2\alpha\beta + 2\beta + 1)y - \beta^2 = 0. \quad (1)$$

Clearly, for $\alpha = 3$ and $\beta = 1$ we obtain the equation in [3], while for $\alpha = 4$, $\beta = 0$ we get the equation in [4].

Let us remark right away that when $\alpha = 0$, the equation (1) reduces to $x = y^2 + (2\beta + 1)y + \beta^2$, which already produces the parametrization for all the infinitely many integer solutions of the equation. In fact, slightly re-writing, we have in this case $x = (y + \beta)^2 + y$, $y \in \mathbb{Z}$, which suggests that understanding the solutions of the two parameter Diophantine equation (1) could potentially be reduced to the solutions of just a one parameter Diophantine equation by appropriately substituting for x and y . We show first that this is indeed the case.

Reduction to a one parameter Diophantine equation. Let us make the change of variables

$$X = x + \beta \text{ and } Y = y + \beta.$$

Thus, replacing $x = X - \beta$ and $y = Y - \beta$ in (1) we obtain

$$\alpha(X - \beta)^2 - (\alpha + 1)(Y - \beta)^2 + (2\alpha\beta + 1)(X - \beta) - (2\alpha\beta + 2\beta + 1)(Y - \beta) = \beta^2,$$

which after some straightforward algebra simplifies to

$$\alpha X^2 - (\alpha + 1)Y^2 + X - Y = 0.$$

This simply means that in the two parameter Diophantine equation (1) we can assume without loss of generality $\beta = 0$. Moreover, if a solution of (1) exists for $\beta = 0$, the general solution for $\beta \neq 0$ is obtained from that one by translating by $-\beta$ in both unknowns.

Note also that if we let $\alpha = 0$ and $\beta = 0$ in (1), we already have the parametrization of all the integer solutions: $x = y^2 - y$, with $y \in \mathbb{Z}$. With these considerations, we will assume in the remainder of this note that $\alpha \in \mathbb{N}$ and $\beta = 0$ in (1), that is we will investigate the integer solutions of

$$\alpha x^2 - (\alpha + 1)y^2 + x - y = 0. \quad (2)$$

Our main result is the following.

Theorem 1. *Let $\alpha \in \mathbb{N}$. The non-trivial solutions $(x_n, y_n)_{n \geq 1}$ of the quadratic Diophantine equation (2) are given by*

$$x_n = (\alpha + 1)v_n^2 \pm u_n v_n, \quad y_n = \alpha v_n^2 \pm u_n v_n,$$

where

$$u_n = \frac{(\sqrt{\alpha + 1} + \sqrt{\alpha})^{2n} + (\sqrt{\alpha + 1} - \sqrt{\alpha})^{2n}}{2},$$

$$v_n = \frac{(\sqrt{\alpha+1} + \sqrt{\alpha})^{2n} - (\sqrt{\alpha+1} - \sqrt{\alpha})^{2n}}{2\sqrt{\alpha(\alpha+1)}}.$$

Moreover, the solution set of (1) is given by

$$\{(x_n - \beta, y_n - \beta) : n \in \mathbb{N}\} \cup \{(-\beta, -\beta)\},$$

with x_n, y_n as above.

The word “trivial” above refers to the pair $(0, 0)$ which obviously satisfies the given equation. We point out also that the \pm signs in the expressions of x_n and y_n , respectively, coincide. We have the following immediate consequence of our main result.

Corollary 2. *Let $(x, y) \in \mathbb{Z}$ be a solution of (2). Then $x - y$ is a perfect square and the values of $x - y$ belong precisely to the set*

$$\{0\} \cup \{v_n^2 : n \in \mathbb{N}\},$$

with v_n defined in Theorem 1.

Note that, in particular, we recover the statement in [4]. In fact, the idea of proving Theorem 1 is guided by the intuition contained in the elementary problems [3] and [4], combined with the well-understood theory of positive Pell equations. As we shall see, the statement in [3] is also a by-product of our arguments showing the main result. Before proceeding with the proof of Theorem 1, we take a brief excursion into the theory of Pell equations.

The general positive Pell equation. Let D be a positive integer that is not a perfect square. It is a known fact that the *positive Pell equation*

$$u^2 - Dv^2 = 1 \tag{3}$$

has infinitely many solutions in $\mathbb{Z} \times \mathbb{Z}$. Clearly, if $(u, v) \in \mathbb{N} \times \mathbb{N}$ is a solution to the Pell equation, then $(u, -v)$, $(-u, v)$ and $(-u, -v)$ are also solutions. Thus, without loss of generality, it suffices to consider only its solutions in the natural lattice. Any equation of the form (3) has the *trivial* solution $(1, 0)$. Besides the trivial solution, the other solutions in the natural lattice can be obtained from the *fundamental solution* (u_0, v_0) , which is the least positive integer solution to (3) different from $(1, 0)$ — the so-called *fundamental solution* of (3) for which the expression $u + v\sqrt{D}$ is minimal, via the following formulas:

$$u_n = \frac{(u_0 + v_0\sqrt{D})^n + (u_0 - v_0\sqrt{D})^n}{2}$$

$$v_n = \frac{(u_0 + v_0\sqrt{D})^n - (u_0 - v_0\sqrt{D})^n}{2\sqrt{D}}, n \in \mathbb{N}.$$

For a brief introduction to Pell equations, the interested reader can consult, for example, [1] and the references therein.

With these prerequisites we are ready to proceed with the proof of our main result.

Proof of Theorem 1. In what follow, we assume that both unknowns, x and y , are non-zero. In particular, this means that $x \neq y$.

Our first claim, following the statement in [4], is that if x, y satisfy (2), then $x - y$ must be a perfect square. Let us start by re-writing the equation (2) in two ways:

$$\begin{aligned}(x - y)[1 + \alpha(x + y)] &= y^2, \\ (x - y)[1 + (\alpha + 1)(x + y)] &= x^2.\end{aligned}$$

Denoting $A = 1 + \alpha(x + y)$ and $B = 1 + (\alpha + 1)(x + y)$ and multiplying these two equations we obtain $(x - y)^2 AB = (xy)^2$, thus showing that AB must be a perfect square. In particular, we find that A and B must be either both positive or both negative.

Let us assume for the moment that *both A and B are positive*. Note now that $(\alpha + 1)A - \alpha B = 1$. Therefore, A and B are relatively prime natural numbers. Combining this with the fact that AB is a perfect square shows that A and B have to be perfect squares as well; this is a simple consequence of the Fundamental Theorem of Arithmetic. Incidentally, the statement that A and B must be perfect squares is precisely the content of the problem [3] if we take into account also the natural change of variables reducing (1) to (2). Now, since A is a perfect square and $(x - y)A$ is a perfect square, we conclude that $x - y > 0$ must also be a perfect square. This proves our first claim.

With this information in hand, let us write then $x - y = v^2$ for some $v \in \mathbb{N}$. Our next claim is that v must be of the form v_n , $n \geq 1$, with v_n as in the statement of our Theorem 1. Substituting $x = y + v^2$ in (2) we find that

$$\alpha(y + v^2)^2 - (\alpha + 1)y^2 + v^2 = 0 \Leftrightarrow y^2 - 2\alpha v^2 y - (\alpha v^4 + v^2) = 0.$$

We are interested in the integer solutions of the quadratic equation in y , which are given by

$$y = \alpha v^2 \pm v \sqrt{\alpha(\alpha + 1)v^2 + 1}. \quad (4)$$

Clearly, for y to be an integer, we need now

$$\alpha(\alpha + 1)v^2 + 1 = u^2 \Leftrightarrow u^2 - \alpha(\alpha + 1)v^2 = 1, \quad (5)$$

for some $u \in \mathbb{Z}$. But this is exactly a positive Pell equation of the form (3), with $D = \alpha(\alpha + 1)$ obviously not being a perfect square. It is equally easy to see that the fundamental solution of (5) is given by

$$u_0 = 2\alpha + 1, \quad v_0 = 2.$$

Thus, the general solution of (5) in $\mathbb{N} \times \mathbb{N}$ can be expressed as

$$u_n = \frac{(2\alpha + 1 + 2\sqrt{\alpha(\alpha + 1)})^n + (2\alpha + 1 - 2\sqrt{\alpha(\alpha + 1)})^n}{2},$$

$$v_n = \frac{(2\alpha + 1 + 2\sqrt{\alpha(\alpha + 1)})^n - (2\alpha + 1 - 2\sqrt{\alpha(\alpha + 1)})^n}{2\sqrt{\alpha(\alpha + 1)}}.$$

It is straightforward to see that these expressions are precisely the ones stated in Theorem 1, since

$$2\alpha + 1 \pm 2\sqrt{\alpha(\alpha + 1)} = (\sqrt{\alpha + 1} \pm \sqrt{\alpha})^2.$$

Returning now to the formula (4), we thus find that the integer solutions are precisely those of the form $y_n = \alpha v_n^2 \pm v_n u_n$, and consequently

$$x_n = y_n^2 + v_n^2 = (\alpha + 1)v_n^2 \pm v_n u_n.$$

In the argument above, we assumed that both A and B are strictly positive. The remainder of our proof will show that the scenario in which A and B are both negative cannot happen. Note that if A, B would be negative, then $-A$ and $-B$ would be positive and since A and B are relatively prime, so are $-A$ and $-B$, thus implying that $-A$ and $-B$ are perfect squares. Hence, since $(y - x)(-A) = y^2$, we conclude now that $y - x > 0$ must be a perfect square.

It is worth noting that if $y > x$ and $x > 0$, the Diophantine equation (2) has no solutions. Indeed, in this case

$$\alpha x^2 - (\alpha + 1)y^2 + x - y < \alpha y^2 - (\alpha + 1)y^2 = -y^2 < 0.$$

Thus, the only possible solutions would have to have $x < 0$. Substituting $y = x + w^2$ in (2) we arrive to

$$\alpha x^2 - (\alpha + 1)(x + w^2)^2 - w^2 = 0.$$

This is now equivalent to

$$x^2 + 2(\alpha + 1)xw^2 + (\alpha + 1)w^4 + w^2 = 0,$$

and solving for x gives

$$\begin{aligned} x &= -(\alpha + 1)w^2 \pm \sqrt{(\alpha + 1)^2 w^4 - (\alpha + 1)w^4 - w^2} = \\ &= -(\alpha + 1)w^2 \pm w\sqrt{\alpha(\alpha + 1)w^2 - 1}. \end{aligned}$$

Therefore, clearly, if $x \in \mathbb{Z}$, it must be negative. We stumble here across a similar issue as in the first part of our argument, except that now we must require

$$\alpha(\alpha + 1)w^2 - 1 = z^2 \Leftrightarrow z^2 - \alpha(\alpha + 1)w^2 = -1$$

for some $z \in \mathbb{Z}$. This is an example of a *negative Pell equation*

$$z^2 - Dw^2 = -1, \tag{6}$$

with $D = \alpha(\alpha + 1)$ not a perfect square. Now, it is well known [2, Theorem 3.3.4] that the negative Pell equation (6) is solvable if and only if the period of the continued fraction expansion of \sqrt{D} is odd. For the fact that, for D

not a perfect square, the continued fraction expansion of \sqrt{D} is periodic, see [2, Theorem 2.1.21]. Notice next that we have the following identity

$$\sqrt{\alpha(\alpha+1)} - \alpha = \frac{\alpha}{\alpha + \sqrt{\alpha(\alpha+1)}}.$$

Equivalently, the continued fraction expansion of $\sqrt{\alpha(\alpha+1)}$ is given by

$$\sqrt{\alpha(\alpha+1)} = \alpha + \frac{\alpha}{2\alpha + \frac{\alpha}{2\alpha + \frac{\alpha}{\ddots}}}.$$

We have thus obtained that the period of the continued fraction expansion of $\sqrt{D} = \sqrt{\alpha(\alpha+1)}$ is even, effectively showing that (6) above has no integer solutions. We conclude that we cannot have solutions (x, y) of (2) with $x < y$, thus finishing the proof of our main theorem.

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Limits of integrals of functions over various domains

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Abstract. We prove that, under natural hypotheses, the following equality holds

$$\begin{aligned} \lim_{n \rightarrow \infty} n^k \int_0^1 v_n(x) \varphi(x) \prod_{i=1}^k \left(\sqrt[n]{f_i(x)} - 1 \right) dx \\ = \lim_{n \rightarrow \infty} \int_0^1 v_n(x) \varphi(x) \prod_{i=1}^k \ln f_i(x) dx. \end{aligned}$$

We also show that the main ideas of the proof of this result can be adapted to obtain a similar result on the unit square. Various applications are given.

Keywords: Riemann integral, multiple Riemann integral, the limit of sequences of multiple integrals

MSC: Primary 26B15; Secondary 28A35.

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1. INTRODUCTION AND A PRELIMINARY RESULT

The starting point for this paper was the following question: how can we obtain some analogous results to the following limit

$$\lim_{n \rightarrow \infty} n^2 \int_0^1 (\sqrt[n]{1+x^n} - 1) dx = \frac{\pi^2}{12} \quad (1)$$

given at Vojtěch Jarník International Mathematical Competition 2002, and proposed by Sofia University St. Kliment Ohridski; see [4]. The result of this investigation is the present paper. Let us mention that the notation and concepts used and not defined are standard, see [1]. We need the following well-known result whose proof is left to the reader.

Proposition 1. *If $M > 0$, then*

$$\begin{aligned} 0 &\leq e^a - 1 \leq \frac{e^M - 1}{M} \cdot a, \quad \forall 0 \leq a \leq M; \\ 0 &\leq e^a - 1 - a \leq \frac{e^M - 1 - M}{M^2} \cdot a^2, \quad \forall 0 \leq a \leq M. \end{aligned}$$

2. AN ILLUSTRATIVE PARTICULAR CASE

We begin by proving the first result which the author has obtained motivated by the limit (1). Since, shortly after proving this particular result we have observed that this can be extended to very general situations we think that it is instructive for the readers to present first this particular case.

Proposition 2. *The following equality holds*

$$\lim_{n \rightarrow \infty} n^2 \int_0^1 x^n (\sqrt[n]{1+x} - 1) dx = \ln 2.$$

Proof. The idea is to observe that $\sqrt[n]{1+x} = e^{\frac{1}{n} \ln(1+x)}$ and to use Proposition 1, which was expanded to the following reasoning. Let $n \in \mathbb{N}$ and $x \in [0, 1]$. Then $0 \leq a = \frac{1}{n} \ln(1+x) \leq \frac{1}{n} \ln 2 \leq \frac{1}{n} \leq 1$ and by Proposition 1 we deduce

$$\begin{aligned} 0 &\leq \sqrt[n]{1+x} - 1 - \frac{1}{n} \ln(1+x) = e^{\frac{1}{n} \ln(1+x)} - 1 - \frac{1}{n} \ln(1+x) \\ &\leq \frac{e-2}{n^2} \ln^2(1+x) \leq \frac{(e-2) \ln^2 2}{n^2}. \end{aligned}$$

By multiplication with $n^2 x^n$ we obtain

$$0 \leq n^2 x^n \left(\sqrt[n]{1+x} - 1 - \frac{1}{n} \ln(1+x) \right) \leq (e-2) x^n \ln^2 2$$

and

$$0 \leq n^2 \int_0^1 x^n \left(\sqrt[n]{1+x} - 1 - \frac{1}{n} \ln(1+x) \right) dx \leq (e-2) \ln^2 2 \int_0^1 x^n dx \rightarrow 0.$$

From the equality

$$n^2 x^n (\sqrt[n]{1+x} - 1) = n^2 x^n \left(\sqrt[n]{1+x} - 1 - \frac{1}{n} \ln(1+x) \right) + n x^n \ln(1+x)$$

we deduce

$$\begin{aligned} n^2 \int_0^1 x^n (\sqrt[n]{1+x} - 1) dx \\ = n^2 \int_0^1 x^n \left(\sqrt[n]{1+x} - 1 - \frac{1}{n} \ln(1+x) \right) dx + n \int_0^1 x^n \ln(1+x) dx. \end{aligned}$$

Since, as it is well known, $n \int_0^1 x^n \ln(1+x) dx \rightarrow \ln 2$, the proof is finished.

Second proof. The following proof has been shown to us by the reviewer.

Integrating by parts we have

$$\begin{aligned} \int_0^1 x^n (\sqrt[n]{1+x} - 1) dx &= \frac{x^{n+1} (\sqrt[n]{1+x} - 1)}{n+1} \Big|_0^1 - \\ &\quad - \frac{1}{n(n+1)} \int_0^1 (1+x)^{\frac{1}{n}-1} x^{n+1} dx \\ &= \frac{\sqrt[n]{2} - 1}{n+1} - \frac{1}{n(n+1)} \int_0^1 (1+x)^{\frac{1}{n}-1} x^{n+1} dx. \end{aligned}$$

It follows that

$$n^2 \int_0^1 x^n (\sqrt[n]{1+x} - 1) dx = \frac{n^2 (\sqrt[n]{2} - 1)}{n+1} - \frac{n}{n+1} \int_0^1 (1+x)^{\frac{1}{n}-1} x^{n+1} dx.$$

The limit follows since $n (\sqrt[n]{2} - 1) \rightarrow \ln 2$ and

$$0 < \int_0^1 (1+x)^{\frac{1}{n}-1} x^{n+1} dx \leq \frac{\sqrt[n]{2}}{n+2}.$$

□

3. THE CASE OF RIEMANN INTEGRABLE FUNCTIONS ON THE UNIT INTERVAL: THE FIRST STEP

The main feature of the proof of Proposition 2 is that it can be extended to more general context.

Theorem 3. *Let $v_n : [0, 1] \rightarrow \mathbb{R}$ be a sequence of Riemann integrable functions such that the sequence $\left(\int_0^1 |v_n(x)| dx \right)_{n \in \mathbb{N}}$ is bounded. Then for each Riemann integrable function $\varphi : [0, 1] \rightarrow \mathbb{R}$ and each continuous function $f : [0, 1] \rightarrow [1, \infty)$ the following equality holds*

$$\lim_{n \rightarrow \infty} \left[n \int_0^1 v_n(x) \left(\sqrt[n]{f(x)} - 1 \right) \varphi(x) dx - \int_0^1 v_n(x) \varphi(x) \ln f(x) dx \right] = 0.$$

In particular, if $\lim_{n \rightarrow \infty} \int_0^1 v_n(x) \varphi(x) \ln f(x) dx \in \mathbb{R}$, then

$$\lim_{n \rightarrow \infty} n \int_0^1 v_n(x) \left(\sqrt[n]{f(x)} - 1 \right) \varphi(x) dx = \lim_{n \rightarrow \infty} \int_0^1 v_n(x) \varphi(x) \ln f(x) dx.$$

Proof. Since $[0, 1]$ is compact and f is continuous, by the Weierstrass theorem, there exists $x_0 \in [0, 1]$ such that $M = \sup_{x \in [0, 1]} f(x) = f(x_0) \in [1, \infty)$. Let

$n \in \mathbb{N}$, $n \geq \ln M$. For each $x \in [0, 1]$ we have $0 \leq a = \frac{1}{n} \ln f(x) \leq \frac{1}{n} \ln M \leq 1$ and by Proposition 1 we deduce

$$\begin{aligned} 0 &\leq \sqrt[n]{f(x)} - 1 - \frac{1}{n} \ln f(x) = e^{\frac{1}{n} \ln f(x)} - 1 - \frac{1}{n} \ln f(x) \\ &\leq \frac{1}{n^2} (e - 2) \ln^2 f(x) \leq \frac{1}{n^2} (e - 2) \ln^2 M \end{aligned}$$

and thus

$$0 \leq |v_n(x)| |\varphi(x)| \left(\sqrt[n]{f(x)} - 1 - \frac{1}{n} \ln f(x) \right) \leq \frac{1}{n^2} (e - 2) (\ln^2 M) A |v_n(x)|$$

where $A = \sup_{x \in [0, 1]} |\varphi(x)|$. We have

$$\begin{aligned} 0 &\leq \left| \int_0^1 n v_n(x) \left(\sqrt[n]{f(x)} - 1 - \frac{1}{n} \ln f(x) \right) \varphi(x) dx \right| \\ &\leq n \int_0^1 |v_n(x)| |\varphi(x)| \left(\sqrt[n]{f(x)} - 1 - \frac{1}{n} \ln f(x) \right) dx \\ &\leq \frac{1}{n} (e - 2) (\ln^2 M) A \int_0^1 |v_n(x)| dx \rightarrow 0 \end{aligned}$$

since the sequence $\left(\int_0^1 |v_n(x)| dx \right)_{n \in \mathbb{N}}$ is bounded. From

$$\begin{aligned} n v_n(x) \left(\sqrt[n]{f(x)} - 1 \right) \varphi(x) - v_n(x) \varphi(x) \ln f(x) \\ = n v_n(x) \varphi(x) \left(\sqrt[n]{f(x)} - 1 - \frac{1}{n} \ln f(x) \right) \end{aligned}$$

we deduce

$$\begin{aligned} n \int_0^1 v_n(x) \left(\sqrt[n]{f(x)} - 1 \right) \varphi(x) dx - \int_0^1 v_n(x) \varphi(x) \ln f(x) dx \\ = n \int_0^1 v_n(x) \left(\sqrt[n]{f(x)} - 1 - \frac{1}{n} \ln f(x) \right) \varphi(x) dx \rightarrow 0, \end{aligned}$$

which proves the first part of the theorem. The second assertion follows simply from the first one. \square

Corollary 4. (i) *Let $f : [0, 1] \rightarrow [1, \infty)$ be a continuous function. Then*

$$\lim_{n \rightarrow \infty} n^2 \int_0^1 x^n \left(\sqrt[n]{f(x)} - 1 \right) dx = \ln f(1).$$

(ii) Let $g : [0, 1] \rightarrow \mathbb{R}$ and $f : [0, 1] \rightarrow [1, \infty)$ be continuous functions. Then

$$\lim_{n \rightarrow \infty} n^2 \int_0^1 x^n g(x^n) \left(\sqrt[n]{f(x)} - 1 \right) dx = \left(\int_0^1 g(x) dx \right) \ln f(1).$$

(iii) Let $f : [0, 1] \rightarrow [1, \infty)$ be a continuous function. Then

$$\lim_{n \rightarrow \infty} n^3 \int_0^1 \frac{x^n}{1+x+x^2+\dots+x^n} \left(\sqrt[n]{f(x)} - 1 \right) dx = \frac{\pi^2}{6} \ln f(1).$$

Proof. (i) Take in Theorem 3, $\varphi(x) = 1$, $v_n : [0, 1] \rightarrow \mathbb{R}$, $v_n(x) = nx^n$ and use that $\lim_{n \rightarrow \infty} n \int_0^1 x^n \ln f(x) dx = \ln f(1)$.

(ii) Take in Theorem 3, $\varphi(x) = 1$, $v_n : [0, 1] \rightarrow \mathbb{R}$, $v_n(x) = x^n g(x^n)$ and use that $\lim_{n \rightarrow \infty} n \int_0^1 x^n g(x^n) \ln f(x) dx = \left(\int_0^1 g(x) dx \right) \ln f(1)$, see [2].

(iii) Take in Theorem 3, $\varphi(x) = 1$, $v_n : [0, 1] \rightarrow \mathbb{R}$, $v_n(x) = \frac{n^2 x^n}{1+x+x^2+\dots+x^n}$ and use that $\lim_{n \rightarrow \infty} n^2 \int_0^1 \frac{x^n}{1+x+x^2+\dots+x^n} \ln f(x) dx = \frac{\pi^2}{6} \ln f(1)$, see [2].

Let us mention here the following proof for the boundedness of the sequence $\int_0^1 v_n(x) dx$ indicated to us by the reviewer. By applying AM-GM inequality in the denominator we have $1+x+x^2+\dots+x^n \geq (n+1)x^{\frac{n}{2}}$, which gives us $\int_0^1 v_n(x) dx \leq \frac{n^2}{n+1} \int_0^1 x^{\frac{n}{2}} dx = \frac{2n^2}{(n+1)(n+2)} < 2$. \square

4. THE CASE OF RIEMANN INTEGRABLE FUNCTIONS ON THE UNIT INTERVAL: THE SECOND STEP

The next idea which we have taken into account was: what happen if we iterate the process in Theorem 3? In the sequel we present some possible results in this direction.

Theorem 5. Let $v_n : [0, 1] \rightarrow \mathbb{R}$ be a sequence of Riemann integrable functions such that the sequence $\left(\int_0^1 |v_n(x)| dx \right)_{n \in \mathbb{N}}$ is bounded. Let also k be a positive integer, $\varphi : [0, 1] \rightarrow \mathbb{R}$ be a Riemann integrable function and $f_1, \dots, f_k : [0, 1] \rightarrow [1, \infty)$ be continuous functions. If there exists

$$\lim_{n \rightarrow \infty} \int_0^1 v_n(x) \varphi(x) \prod_{i=1}^k \ln f_i(x) dx \in \mathbb{R}$$

then the following equality holds

$$\begin{aligned} \lim_{n \rightarrow \infty} n^k \int_0^1 v_n(x) \varphi(x) \prod_{i=1}^k \left(\sqrt[n]{f_i(x)} - 1 \right) dx &= \\ &= \lim_{n \rightarrow \infty} \int_0^1 v_n(x) \varphi(x) \prod_{i=1}^k \ln f_i(x) dx. \end{aligned}$$

Proof. We argue by induction on k . The base case $k = 1$ was shown in Theorem 3. Let us suppose the assertion is true for k , i.e., suppose that for each Riemann integrable function $g : [0, 1] \rightarrow \mathbb{R}$ and continuous functions $f_1, \dots, f_k : [0, 1] \rightarrow [1, \infty)$ with the property that there exists

$$\lim_{n \rightarrow \infty} \int_0^1 v_n(x) g(x) \prod_{i=1}^k \ln f_i(x) dx \in \mathbb{R},$$

the following equality holds

$$\begin{aligned} \lim_{n \rightarrow \infty} n^k \int_0^1 v_n(x) g(x) \prod_{i=1}^k \left(\sqrt[n]{f_i(x)} - 1 \right) dx &= \\ &= \lim_{n \rightarrow \infty} \int_0^1 v_n(x) g(x) \prod_{i=1}^k \ln f_i(x) dx. \end{aligned}$$

Now, let $\varphi : [0, 1] \rightarrow \mathbb{R}$ be Riemann integrable, $f_1, \dots, f_{k+1} : [0, 1] \rightarrow [1, \infty)$ continuous functions with the property that there exists

$$L = \lim_{n \rightarrow \infty} \int_0^1 v_n(x) \varphi(x) \prod_{i=1}^{k+1} \ln f_i(x) dx \in \mathbb{R}.$$

Let us define $g : [0, 1] \rightarrow \mathbb{R}$ by putting $g(x) = \varphi(x) \ln f_{k+1}(x)$ and note that g is Riemann integrable. Then

$$L = \lim_{n \rightarrow \infty} \int_0^1 v_n(x) g(x) \prod_{i=1}^k \ln f_i(x) dx \in \mathbb{R}$$

and by the inductive hypothesis we get

$$\begin{aligned} \lim_{n \rightarrow \infty} n^k \int_0^1 v_n(x) g(x) \prod_{i=1}^k \left(\sqrt[n]{f_i(x)} - 1 \right) dx &= \\ &= \lim_{n \rightarrow \infty} \int_0^1 v_n(x) g(x) \prod_{i=1}^k \ln f_i(x) dx, \end{aligned}$$

that is, by replacing the expression of $g(x)$ in the left member,

$$\lim_{n \rightarrow \infty} n^k \int_0^1 v_n(x) \varphi(x) \ln f_{k+1}(x) \prod_{i=1}^k \left(\sqrt[n]{f_i(x)} - 1 \right) dx = L.$$

Let us define $h_n : [0, 1] \rightarrow \mathbb{R}$ by $h_n(x) = n^k v_n(x) \varphi(x) \prod_{i=1}^k \left(\sqrt[n]{f_i(x)} - 1 \right)$.

Let $n \in \mathbb{N}$, $n \geq \max_{1 \leq i \leq k} \ln M_i$ ($M_i = \sup_{x \in [0, 1]} f_i(x) \in [1, \infty)$). For each $x \in [0, 1]$

we have $0 \leq a = \frac{1}{n} \ln f_i(x) \leq \frac{1}{n} \ln M_i \leq 1$. Then from Proposition 1 we deduce

$$0 \leq \sqrt[n]{f_i(x)} - 1 = e^{\frac{1}{n} \ln f_i(x)} - 1 \leq \frac{e-1}{n} \ln f_i(x) \leq \frac{e-1}{n} \ln M_i,$$

whence

$$|h_n(x)| \leq n^k |v_n(x)| |\varphi(x)| \prod_{i=1}^k \left(\sqrt[n]{f_i(x)} - 1 \right) \leq (e-1)^k |v_n(x)| M \prod_{i=1}^k \ln M_i,$$

where $M = \sup_{x \in [0,1]} |\varphi(x)|$. Since $\left(\int_0^1 |v_n(x)| dx \right)_{n \in \mathbb{N}}$ is bounded, it follows that $\left(\int_0^1 |h_n(x)| dx \right)_{n \geq \max_{1 \leq i \leq k} \ln M_i}$ is bounded, hence $\left(\int_0^1 |h_n(x)| dx \right)_{n \in \mathbb{N}}$ is bounded. Since

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^1 h_n(x) \ln f_{k+1}(x) dx \\ = \lim_{n \rightarrow \infty} n^k \int_0^1 v_n(x) \varphi(x) \prod_{i=1}^k \left(\sqrt[n]{f_i(x)} - 1 \right) \ln f_{k+1}(x) dx = L, \end{aligned}$$

from Theorem 3 we deduce

$$\lim_{n \rightarrow \infty} n \int_0^1 h_n(x) \left(\sqrt[n]{f_{k+1}(x)} - 1 \right) dx = \lim_{n \rightarrow \infty} \int_0^1 h_n(x) \ln f_{k+1}(x) dx,$$

that is, by replacing in the left member the expression of $h_n(x)$,

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{k+1} \int_0^1 v_n(x) \varphi(x) \prod_{i=1}^{k+1} \left(\sqrt[n]{f_i(x)} - 1 \right) dx \\ = \lim_{n \rightarrow \infty} n^k \int_0^1 v_n(x) \varphi(x) \prod_{i=1}^k \left(\sqrt[n]{f_i(x)} - 1 \right) \ln f_{k+1}(x) dx = L. \end{aligned}$$

This means that the statement is true for $k+1$. \square

Taking in Theorem 5, $\varphi(x) = 1$ for all $x \in [0, 1]$, we get the following corollary.

Corollary 6. *Let $v_n : [0, 1] \rightarrow \mathbb{R}$ be a sequence of Riemann integrable functions such that the sequence $\left(\int_0^1 |v_n(x)| dx \right)_{n \in \mathbb{N}}$ is bounded. Let also k be a positive integer and $f_1, \dots, f_k : [0, 1] \rightarrow [1, \infty)$ be continuous functions with the property that there exists*

$$\lim_{n \rightarrow \infty} \int_0^1 v_n(x) \prod_{i=1}^k \ln f_i(x) dx \in \mathbb{R}.$$

Then the following equality holds

$$\lim_{n \rightarrow \infty} n^k \int_0^1 v_n(x) \prod_{i=1}^k \left(\sqrt[n]{f_i(x)} - 1 \right) dx = \lim_{n \rightarrow \infty} \int_0^1 v_n(x) \prod_{i=1}^k \ln f_i(x) dx.$$

We recall a well known result, see [2, Ex. 3.13, pag. 53–54].

Lemma 7. *Let $v_n : [0, 1] \rightarrow \mathbb{R}$ be a sequence of Riemann integrable functions such that the sequence $\left(\int_0^1 |v_n(x)| dx \right)_{n \in \mathbb{N}}$ is bounded, and for all $0 < u < 1$ we have $\lim_{n \rightarrow \infty} \int_0^u |v_n(x)| dx = 0$ and $\lim_{n \rightarrow \infty} \int_0^1 v_n(x) dx \in \mathbb{R}$. Then for each continuous function $f : [0, 1] \rightarrow \mathbb{R}$ the following equality holds*

$$\lim_{n \rightarrow \infty} \int_0^1 v_n(x) f(x) dx = f(1) \lim_{n \rightarrow \infty} \int_0^1 v_n(x) dx.$$

From Corollary 6 and Lemma 7 we get the following result which is an extension of Proposition 2 and Corollary 4.

Corollary 8. *Let $v_n : [0, 1] \rightarrow \mathbb{R}$ be a sequence of Riemann integrable functions such that the sequence $\left(\int_0^1 |v_n(x)| dx \right)_{n \in \mathbb{N}}$ is bounded, and for all $0 < u < 1$ we have $\lim_{n \rightarrow \infty} \int_0^u |v_n(x)| dx = 0$ and $\lim_{n \rightarrow \infty} \int_0^1 v_n(x) dx \in \mathbb{R}$. Let also k be a positive integer and $f_1, \dots, f_k : [0, 1] \rightarrow [1, \infty)$ be continuous functions. Then the following equality holds*

$$\lim_{n \rightarrow \infty} n^k \int_0^1 v_n(x) \prod_{i=1}^k \left(\sqrt[n]{f_i(x)} - 1 \right) dx = \left[\prod_{i=1}^k \ln f_i(1) \right] \lim_{n \rightarrow \infty} \int_0^1 v_n(x) dx.$$

Taking in Theorem 5, $v_n(x) = 1$ for all $x \in [0, 1]$ and $n \in \mathbb{N}$, we get the following corollary.

Corollary 9. *Let k be a positive integer, $\varphi : [0, 1] \rightarrow \mathbb{R}$ a Riemann integrable function and $f_1, \dots, f_k : [0, 1] \rightarrow [1, \infty)$ continuous functions. Then the following equality holds*

$$\lim_{n \rightarrow \infty} n^k \int_0^1 \varphi(x) \prod_{i=1}^k \left(\sqrt[n]{f_i(x)} - 1 \right) dx = \int_0^1 \varphi(x) \prod_{i=1}^k \ln f_i(x) dx.$$

Corollary 10. *For each positive integer k the following equality holds*

$$\lim_{n \rightarrow \infty} n^k \int_0^1 \left(\sqrt[n]{1+x} - 1 \right)^k dx = I_k,$$

where $I_1 = 2 \ln 2 - 1$, $I_k = 2 \ln^k 2 - k I_{k-1}$, for $k \geq 2$, and I_k are as above.

Proof. From Corollary 9 we have

$$\lim_{n \rightarrow \infty} n^k \int_0^1 \left(\sqrt[n]{1+x} - 1 \right)^k dx = \int_0^1 [\ln(1+x)]^k dx = I_k.$$

It is a standard exercise to prove that I_k has the stated properties. \square

5. THE CASE OF RIEMANN INTEGRABLE FUNCTIONS ON THE UNIT SQUARE

In the sequel we show that, following similar ideas, we can prove that Theorems 3 and 5 can be extended to Riemann integrable functions on the unit square.

Theorem 11. *Let $h_n : [0, 1]^2 \rightarrow \mathbb{R}$ be a sequence of Riemann integrable functions such that the sequence $\left(\iint_{[0,1]^2} |h_n(x, y)| \, dx dy \right)_{n \in \mathbb{N}}$ is bounded. Then*

for each continuous function $f : [0, 1]^2 \rightarrow [1, \infty)$ the following equality holds

$$\lim_{n \rightarrow \infty} \left(n \iint_{[0,1]^2} h_n(x, y) (\sqrt[n]{f(x, y)} - 1) \, dx dy - \iint_{[0,1]^2} h_n(x, y) \ln f(x, y) \, dx dy \right) = 0.$$

In particular, if there exists $\lim_{n \rightarrow \infty} \iint_{[0,1]^2} h_n(x, y) \ln f(x, y) \, dx dy \in \mathbb{R}$, then

$$\lim_{n \rightarrow \infty} n \iint_{[0,1]^2} h_n(x, y) (\sqrt[n]{f(x, y)} - 1) \, dx dy = \lim_{n \rightarrow \infty} \iint_{[0,1]^2} h_n(x, y) \ln f(x, y) \, dx dy.$$

Proof. The proof of Theorem 3 uses the Weierstrass theorem: continuous functions on compact sets are bounded and attain their bounds (the set $[0, 1]$ is compact); here $[0, 1]^2$ is compact and again we use the Weierstrass theorem. Now the reader will have no difficulty to mimic the proof of Theorem 3 in this new context. \square

Theorem 12. *Let $h_n : [0, 1]^2 \rightarrow \mathbb{R}$ be a sequence of Riemann integrable functions such that the sequence $\left(\iint_{[0,1]^2} |h_n(x, y)| \, dx dy \right)_{n \in \mathbb{N}}$ is bounded. Let*

also k be a positive integer, $\varphi : [0, 1]^2 \rightarrow \mathbb{R}$ be a Riemann integrable function and $f_1, \dots, f_k : [0, 1]^2 \rightarrow [1, \infty)$ be continuous functions. If there exists

$$\lim_{n \rightarrow \infty} \iint_{[0,1]^2} h_n(x, y) \varphi(x, y) \prod_{i=1}^k \ln f_i(x, y) \, dx dy \in \mathbb{R}$$

then the following equality holds

$$\begin{aligned} & \lim_{n \rightarrow \infty} n^k \iint_{[0,1]^2} h_n(x, y) \varphi(x, y) \prod_{i=1}^k \left(\sqrt[n]{f_i(x, y)} - 1 \right) \, dx dy \\ &= \lim_{n \rightarrow \infty} \iint_{[0,1]^2} h_n(x, y) \varphi(x, y) \prod_{i=1}^k \ln f_i(x, y) \, dx dy. \end{aligned}$$

Proof. The idea of the proof of Theorem 5 was to use the induction and Theorem 3. In this new context we apply again the induction and now we use Theorem 11 instead of Theorem 3. We leave the details to the reader. \square

Taking in Theorem 12, $h_n(x, y) = 1$ for all $x, y \in [0, 1]$ and $n \in \mathbb{N}$, we get the following corollary.

Corollary 13. *For each positive integer k , each Riemann integrable function $\varphi : [0, 1]^2 \rightarrow \mathbb{R}$ and arbitrary continuous functions $f_1, \dots, f_k : [0, 1]^2 \rightarrow [1, \infty)$ the following equality holds*

$$\lim_{n \rightarrow \infty} n^k \iint_{[0,1]^2} \varphi(x, y) \prod_{i=1}^k (\sqrt[n]{f_i(x, y)} - 1) dx dy = \iint_{[0,1]^2} \varphi(x, y) \prod_{i=1}^k \ln f_i(x, y) dx dy.$$

Corollary 14. *For each positive integer k the following equality holds*

$$\lim_{n \rightarrow \infty} n^k \iint_{[0,1]^2} \left(\sqrt[n]{(1+x)(1+y)} - 1 \right)^k dx dy = A_k,$$

where $A_1 = 4 \ln 2 - 2$, $A_k = 2I_k + \sum_{i=1}^{k-1} \binom{k}{i} I_{k-i} I_i$, for $k \geq 2$, and I_k are as in Corollary 10.

Proof. From Corollary 13 we get $\lim_{n \rightarrow \infty} n^k \iint_{[0,1]^2} \left(\sqrt[n]{(1+x)(1+y)} - 1 \right)^k dx dy = A_k$, where $A_k = \iint_{[0,1]^2} [\ln(1+x)(1+y)]^k dx dy$. Further, by Fubini's theorem,

$$A_1 = \iint_{[0,1]^2} [\ln(1+x) + \ln(1+y)] dx dy = 2I_1 = 4 \ln 2 - 2,$$

where I_k ($k \geq 1$) are defined in Corollary 10. For $k \geq 2$, by the Newton binomial formula and Fubini's Theorem we have

$$\begin{aligned} A_k &= \iint_{[0,1]^2} [\ln(1+x) + \ln(1+y)]^k dx dy \\ &= 2 \int_0^1 [\ln(1+x)]^k dx + \sum_{i=1}^{k-1} \binom{k}{i} \iint_{[0,1]^2} [\ln(1+x)]^{k-i} [\ln(1+y)]^i dx dy \\ &= 2I_k + \sum_{i=1}^{k-1} \binom{k}{i} I_{k-i} I_i. \end{aligned}$$

\square

Now we recall the following result, see [3, Theorem 5].

Theorem 15. Let $h_n : [0, 1]^2 \rightarrow \mathbb{R}$ be a sequence of continuous functions such that the sequence $\left(\iint_{[0,1]^2} |h_n(x, y)| \, dx dy \right)_{n \in \mathbb{N}}$ is bounded and the following symmetric conditions are satisfied:

$$\forall 0 < u < 1, \lim_{n \rightarrow \infty} \iint_{[0,u] \times [0,1]} |h_n(x, y)| \, dx dy = 0;$$

$$\forall 0 < v < 1, \lim_{n \rightarrow \infty} \iint_{[0,1] \times [0,v]} |h_n(x, y)| \, dx dy = 0.$$

If $\lim_{n \rightarrow \infty} \iint_{[0,1]^2} h_n(x, y) \, dx dy \in \mathbb{R}$, then for each continuous function $f : [0, 1]^2 \rightarrow \mathbb{R}$ the following equality holds

$$\lim_{n \rightarrow \infty} \iint_{[0,1]^2} h_n(x, y) f(x, y) \, dx dy = f(1, 1) \lim_{n \rightarrow \infty} \iint_{[0,1]^2} h_n(x, y) \, dx dy.$$

The result shown in Theorem 12 can be made more precise, as we prove next.

Theorem 16. Let $h_n : [0, 1]^2 \rightarrow \mathbb{R}$ be a sequence of continuous functions such that the sequence $\left(\iint_{[0,1]^2} |h_n(x, y)| \, dx dy \right)_{n \in \mathbb{N}}$ is bounded, the following symmetric conditions are satisfied:

$$\forall 0 < u < 1, \lim_{n \rightarrow \infty} \iint_{[0,u] \times [0,1]} |h_n(x, y)| \, dx dy = 0,$$

$$\forall 0 < v < 1, \lim_{n \rightarrow \infty} \iint_{[0,1] \times [0,v]} |h_n(x, y)| \, dx dy = 0,$$

and there exists $\lim_{n \rightarrow \infty} \iint_{[0,1]^2} h_n(x, y) \, dx dy \in \mathbb{R}$. Then for each positive integer k

and arbitrary continuous functions $f_1, \dots, f_k : [0, 1]^2 \rightarrow [1, \infty)$ the following equality holds

$$\begin{aligned} & \lim_{n \rightarrow \infty} n^k \iint_{[0,1]^2} h_n(x, y) \prod_{i=1}^k \left(\sqrt[n]{f_i(x, y)} - 1 \right) \, dx dy \\ &= \left(\lim_{n \rightarrow \infty} \iint_{[0,1]^2} h_n(x, y) \, dx dy \right) \prod_{i=1}^k \ln f_i(1, 1). \end{aligned}$$

Proof. Under our hypothesis and Theorem 15 it follows that

$$\lim_{n \rightarrow \infty} \iint_{[0,1]^2} h_n(x, y) \prod_{i=1}^k \ln f_i(x, y) dx dy = \prod_{i=1}^k \ln f_i(1, 1) \in \mathbb{R}.$$

The conclusion now follows from Theorem 12. \square

As applications of Theorem 16 we give the following corollary.

Corollary 17. (i) *For every $a, b > 0$, each positive integer k and each continuous function $f : [0, 1]^2 \rightarrow [1, \infty)$ the following equality holds*

$$\lim_{n \rightarrow \infty} n^{k+2} \iint_{[0,1]^2} \left(\frac{ax+by}{a+b} \right)^n \left(\sqrt[n]{f(x, y)} - 1 \right)^k dx dy = \frac{(a+b)^2}{ab} \cdot [\ln f(1, 1)]^k.$$

(ii) *For each continuous function $f : [0, 1]^2 \rightarrow [1, \infty)$ and each positive integer k the limit*

$$\lim_{n \rightarrow \infty} n^{k+4} \iint_{[0,1]^2} \frac{x^n y^n}{(1+x+x^2+\dots+x^n)(1+y+y^2+\dots+y^n)} \left(\sqrt[n]{f(x, y)} - 1 \right)^k dx dy$$

has the value $\frac{\pi^4}{36} \cdot [\ln f(1, 1)]^k$.

Proof. (i) and (ii).

The sequence of continuous functions $h_n : [0, 1]^2 \rightarrow \mathbb{R}$, $h_n(x, y) = n^2 \left(\frac{ax+by}{a+b} \right)^n$ (respectively $h_n(x, y) = \frac{n^4 x^n y^n}{(1+x+x^2+\dots+x^n)(1+y+y^2+\dots+y^n)}$) satisfies the hypotheses in Theorem 16 and $\lim_{n \rightarrow \infty} \iint_{[0,1]^2} h_n(x, y) dx dy = \frac{(a+b)^2}{ab}$ (respectively $\lim_{n \rightarrow \infty} \iint_{[0,1]^2} h_n(x, y) dx dy = \frac{\pi^4}{36}$), see [3, Prop. 6 and Cor. 7]. \square

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Asupra funcțiilor convexe cu graficele tangente – cazul unei mulțimi oarecare de puncte

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Abstract. Examples of two real, strictly convex, indefinitely derivable functions whose difference is non-negative and vanishes on a set M and whose graphics are tangent on points of abscissa in M are thought for. It is shown that under the hypothesis of being increasing, such functions exist if and only if M is closed. A similar result is proved if instead of monotonicity it is required that both functions are unbounded.

Keywords: strict convex function, indefinitely derivable function, closed set

MSC: Primary 26A51; Secondary 26A48.

În [1], autorul a construit două exemple de funcții cu graficele tangente într-un număr finit de puncte. Din păcate, tehnica respectivă nu se poate aplica pentru un număr infinit de puncte. În cele ce urmează, ne propunem să determinăm **toate mulțimile de puncte** M pentru care sunt posibile construcții care să respecte cerințele din [1], mai exact să rezolvăm următoarele probleme:

Problema 1. Să se studieze existența a două funcții strict convexe $f, g : \mathbb{R} \rightarrow \mathbb{R}$ cu următoarele proprietăți:

- $f(x) \geq g(x)$, $\forall x \in \mathbb{R}$, cu egalitate dacă și numai dacă $x \in M$;
- f și g sunt indefinit derivabile;
- graficele funcțiilor f și g sunt tangente în punctele de abscisă $x \in M$.

Problema 2. Notăm $\mathbb{R} \setminus M = \Delta$. Să se studieze existența a două funcții strict convexe $f, g : \mathbb{R} \rightarrow \mathbb{R}$ și a unei partiții $\Delta = \Delta_1 \cup \Delta_2$, în care Δ_1 și Δ_2 să fie cardinal echivalente, cu următoarele proprietăți:

- $f(x) = g(x)$, $\forall x \in M$;
- $f - g|_{\Delta_1} > 0$ și $f - g|_{\Delta_2} < 0$
- f și g sunt indefinit derivabile;
- graficele funcțiilor f și g sunt tangente în punctele de abscisă $x \in M$.

În plus, să se realizeze construcțiile în fiecare din **ipotezele suplimentare**:

- $\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} g(x) = +\infty$;
- f și g strict crescătoare.

Reamintim în prealabil următoarele noțiuni de topologie

1) O mulțime $D \subseteq \mathbb{R}$ se numește *deschisă* dacă pentru orice $x \in D$ există $I_x \subset \mathbb{R}$, interval deschis, astfel încât $x \in I_x \subset D$

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2) O mulțime $F \subseteq \mathbb{R}$ se numește *închisă* dacă $\mathbb{R} \setminus F$ este deschisă.

3) (*Teorema de structură a mulțimilor deschise din \mathbb{R}*) Orice submulțime deschisă a lui \mathbb{R} se scrie în mod unic ca o submulțime cel mult numărabilă de intervale deschise disjuncte.

4) (*Teorema de caracterizare cu șiruri a mulțimilor închise din \mathbb{R}*) O mulțime $M \subset \mathbb{R}$ este închisă dacă și numai dacă pentru orice șir convergent $(x_n)_{n \in \mathbb{N}}$ de elemente din M avem $\lim_{n \rightarrow \infty} x_n \in M$.

Prezentăm în continuare rezultatul principal al lucrării.

Teorema 1. *Problema 1 cu ipoteza i) admite soluții dacă și numai dacă mulțimea M este închisă.*

Demonstrație. „ \implies ” Fie $M \subseteq \mathbb{R}$ o mulțime și $f, g : \mathbb{R} \rightarrow \mathbb{R}$ două funcții cu proprietățile căutate. Din convexitate, rezultă că f și g sunt continue. Fie $(x_n)_{n \in \mathbb{N}} \subset M$ un șir convergent arbitrar și $l = \lim_{n \rightarrow \infty} x_n \in \mathbb{R}$. Cum $f(x_n) = g(x_n), \forall n \in \mathbb{N}$, prin trecere la limită obținem $f(l) = g(l)$.

Așadar, $l \in M$, ceea ce arată că M este o mulțime închisă.

„ \impliedby ” Fie $M \subset \mathbb{R}$ o mulțime închisă. Cum $\mathbb{R} \setminus M$ este mulțime deschisă, conform teoremei de structură a mulțimilor deschise există două șiruri $(a_n)_{n \in I}, (b_n)_{n \in I} \subset \mathbb{R}$ cu proprietățile:

- 1) $a_n < b_n, \forall n \in I$, unde $I \subseteq \mathbb{N}$ este o mulțime nevidă de numere consecutive;
- 2) $(a_m, b_m) \cap (a_n, b_n) = \emptyset, \quad \forall m \neq n$;
- 3) $\mathbb{R} \setminus M = \bigcup_{n \in I} (a_n, b_n)$.

Observăm că $\{a_n, b_n \mid n \in I\} \cap \mathbb{R} \subset M$.

Vom utiliza funcțiile $\varphi_m : \mathbb{R} \rightarrow \mathbb{R}, m \in \mathbb{Z}$,

$$\varphi_m(x) = \begin{cases} x^m e^{-x^{-2}}, & \text{dacă } x \neq 0, \\ 0, & \text{dacă } x = 0, \end{cases}$$

care au proprietatea că sunt indefinit derivabile și $\varphi_m^{(k)}(0) = 0, \forall k \in \mathbb{N}$. În plus, pentru $m < 0$, aceste funcții sunt mărginite, deoarece $\lim_{x \rightarrow \pm\infty} \varphi_m(x) = 0$. Totodată, derivatele lor de orice ordin au aceleași proprietăți, fiind combinații liniare de funcții de același tip.

În continuare vom considera $m = -2$ și vom nota $\varphi(x) = \varphi_{-2}(x)$.

I) Alegem funcția diferență $h = f - g$ astfel:

$$h(x) = \begin{cases} \varphi\left(\frac{(x-a_n)(x-b_n)}{b_n-a_n+1}\right), & x \in (a_n, b_n) \quad (\text{dacă } a_n, b_n \in \mathbb{R}), \\ \varphi(x - a_k), & x > a_k \quad (\text{dacă } b_k = +\infty), \\ \varphi(x - b_l), & x < b_l \quad (\text{dacă } a_l = -\infty), \\ 0, & x \in M. \end{cases}$$

Funcția h este indefinit derivabilă pe $\mathbb{R} \setminus M$. Să observăm, de asemenea, că avem $h(x) \geq 0, \forall x \in \mathbb{R}$, cu egalitate dacă și numai dacă $x \in M$. Dacă $\mathbb{R} \setminus M$

este o reuniune finită de intervale deschise disjuncte, atunci derivabilitatea lui h rezultă din cea a lui φ .

Analizăm în continuare cazul în care h are un număr infinit de ramuri.

Studiem derivabilitatea lui h într-un punct $\alpha \in M$. Mai precis, arătăm că $h'(\alpha) = 0, \forall \alpha \in M$.

Fie $\alpha \in M$, arbitrar, fixat. Dacă α este un punct izolat al mulțimii M , atunci $\alpha \in \{a_n, b_n \mid n \in I\} \cap \mathbb{R}$, deci h este derivabilă în α și $h'(\alpha) = 0$.

Presupunem în continuare că α este un punct de acumulare al lui M . Vom arăta că $h'_s(\alpha) = 0$.

Dacă α nu este punct de acumulare la stânga al mulțimii $\mathbb{R} \setminus M$, atunci $h(x) = 0$ pe o vecinătate la stânga a lui α , deci $h'_s(\alpha) = 0$.

În caz contrar, presupunem în cele ce urmează și că α este punct de acumulare la stânga al mulțimii $\mathbb{R} \setminus M$.

Fie $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R} \setminus M$ un șir crescător astfel încât $x_n \rightarrow \alpha$. În acest caz, pentru orice $n \in \mathbb{N}$, există și este unic $k_n \in \mathbb{N}$ (șir crescător și nestaționar) astfel încât $x_n \in (a_{k_n}, b_{k_n})$. Vom avea nevoie de următoarea observație.

Lema 2. $a_{k_n}, b_{k_n} \rightarrow \alpha$.

Demonstrație. Șirurile (a_{k_n}) și (b_{k_n}) sunt crescătoare și mărginite superior de α , deci sunt convergente, având limitele $l_1 \leq l_2 \leq \alpha$. Deoarece $x_n < b_{k_n}$, prin trecere la limită obținem $\alpha \leq l_2$, deci $\alpha = l_2$.

Presupunem prin absurd că $l_1 < \alpha$. Cum $b_{k_n} \rightarrow \alpha$, există $m \in \mathbb{N}$ astfel încât $b_{k_n} > l_1, \forall n \geq m$. De aici rezultă că $l_1 \in (a_{k_n}, b_{k_n}), \forall n \geq m$, ceea ce contrazice faptul că șirul $(k_n)_{n \in \mathbb{N}}$ este nestaționar (întrucât intervalele (a_n, b_n) sunt disjuncte două câte două).

Așadar, $l_1 = \alpha$, ceea ce încheie demonstrația lemei. \square

Revenim la demonstrația Teoremei 1. Avem

$$\frac{h(x_n) - h(\alpha)}{x_n - \alpha} = \frac{\varphi\left(\frac{(x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right)}{x_n - \alpha} = \frac{e^{-\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right]^{-2}}}{\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right]^2 (x_n - \alpha)}.$$

Dar $|x_n - a_{k_n}| \leq 1$, pentru n suficient de mare, deci

$$\begin{aligned} & \frac{e^{-\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right]^{-2}}}{\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right]^2 (x_n - \alpha)} \leq \frac{e^{-\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right]^{-2}}}{\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right]^2 |x_n - b_{k_n}|} \leq \\ & \leq \frac{e^{-\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right]^{-2}}}{\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right]^2 |x_n - b_{k_n}| |x_n - a_{k_n}|} = \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{b_{k_n} - a_{k_n} + 1} \cdot \frac{e^{-\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right]^{-2}}}{\left[\frac{(x_n - a_{k_n})(x_n - b_{k_n})}{b_{k_n} - a_{k_n} + 1}\right]^2} = \\
&= \frac{1}{b_{k_n} - a_{k_n} + 1} \cdot \varphi_{-3} \left(\frac{|x_n - a_{k_n}| |x_n - b_{k_n}|}{b_{k_n} - a_{k_n} + 1} \right)
\end{aligned}$$

și ultima expresie tinde la 0 deoarece

$$\frac{|x_n - a_{k_n}| |x_n - b_{k_n}|}{b_{k_n} - a_{k_n} + 1} \leq |x_n - a_{k_n}| \cdot \frac{b_{k_n} - a_{k_n}}{b_{k_n} - a_{k_n} + 1} \leq |x_n - a_{k_n}| \rightarrow 0.$$

Prin urmare, din criteriul majorării rezultă că

$$\lim_{n \rightarrow \infty} \frac{h(x_n) - h(\alpha)}{x_n - \alpha} = 0$$

pentru orice șir crescător $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R} \setminus M$ convergent la α .

De fapt, constatăm că afirmația este adevărată pentru orice șir $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R} \setminus M$, $x_n \rightarrow \alpha$, $x_n < \alpha$ (putem permuta termenii unui șir convergent, obținând un șir monoton, având aceeași limită ca șirul inițial).

Considerăm acum un șir oarecare $(y_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ convergent la α pe care îl partiționăm, eventual, în două subșiruri $(y_{k_n})_{n \in \mathbb{N}} \subset \mathbb{R} \setminus M$ și $(y_l)_{n \in \mathbb{N}} \subset M$, $y_{k_n}, y_l \rightarrow \alpha$. Cum $y_l \in M$, rezultă că $h(y_l) = 0$ și, împreună cu lema, obținem

$$\lim_{n \rightarrow \infty} \frac{h(y_n) - h(\alpha)}{y_n - \alpha} = 0,$$

ceea ce arată că $h'_s(\alpha) = 0$.

Analog se arată că $h'_d(\alpha) = 0$.

Mai mult, obținem că h este indefinit derivabilă, întrucât derivatele sale de orice ordin sunt sume de funcții de tipul φ_m , $m < 0$.

II) Determinăm acum funcțiile f și g .

Mai întâi, pentru o funcție oarecare $f : \mathbb{R} \rightarrow \mathbb{R}$, notăm $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$.

Constatăm că

$$h'(x) = \begin{cases} \varphi' \left(\frac{(x - a_n)(x - b_n)}{b_n - a_n + 1} \right) \cdot \frac{(2x - (a_n + b_n))}{b_n - a_n + 1}, & x \in (a_n, b_n) \quad (\text{dacă } a_n, b_n \in \mathbb{R}), \\ \varphi'(x - a_k), & x > a_k \quad (\text{dacă } b_k = +\infty), \\ \varphi'(x - b_l), & x < b_l \quad (\text{dacă } a_l = -\infty), \\ 0, & x \in M, \end{cases}$$

de unde rezultă

$$\|h'\| \leq \max \left\{ \|\varphi'\|, \|\varphi'\| \cdot \sup_n \frac{b_n - a_n}{b_n - a_n + 1} \right\} \leq \|\varphi'\| < \beta \in \mathbb{R}_+^*. \quad (1)$$

De asemenea

$$h''(x) = \begin{cases} \varphi'' \left(\frac{(x-a_n)(x-b_n)}{b_n-a_n+1} \right) \left[\frac{(2x-a_n-b_n)}{b_n-a_n+1} \right]^2 \\ + \varphi' \left(\frac{(x-a_n)(x-b_n)}{b_n-a_n+1} \right) \frac{2}{b_n-a_n+1}, & x \in (a_n, b_n) \quad (\text{dacă } a_n, b_n \in \mathbb{R}), \\ \varphi''(x - a_k), & x > a_k \quad (\text{dacă } b_k = +\infty), \\ \varphi''(x - b_l), & x < b_l \quad (\text{dacă } a_l = -\infty), \\ 0, & x \in M, \end{cases}$$

deci

$$\|h''\| \leq \|\varphi''\| + 2\|\varphi'\| < \beta, \quad (2)$$

unde β este ales convenabil pentru a verifica relațiile (1) și (2).

Considerăm

$$f(x) = h(x) + \frac{\beta}{2}x^2 \quad \text{și} \quad g(x) = \frac{\beta}{2}x^2.$$

Se arată ușor că funcțiile f și g sunt strict convexe. În plus, doar în punctele $\alpha \in M$ avem $h(\alpha) = h'(\alpha)$, de unde rezultă că $f(\alpha) = g(\alpha)$ și $f'(\alpha) = g'(\alpha)$, deci graficele celor două funcții f și g sunt tangente în aceste puncte. \square

Corolar 3. *Problema 2 cu ipoteza i) admite soluții dacă și numai dacă mulțimea M este închisă, iar $\mathbb{R} \setminus M$ nu este interval.*

Demonstrație. Cu argumentul din demonstrația teoremei 1 se arată că mulțimea M trebuie să fie închisă. Pentru a avea sens construcția, trebuie ca $\mathbb{R} \setminus M$ să se scrie ca o reuniune de cel puțin două intervale deschise disjuncte. Procedăm ca la demonstrația teoremei 1, singura deosebire fiind aceea că alegem $h = \pm\varphi$, de exemplu alternativ, pe intervalele din definiție. Considerăm $\Delta_1 = \{x \in \mathbb{R} \mid h(x) > 0\}$ și $\Delta_2 = \{x \in \mathbb{R} \mid h(x) < 0\}$. Egalitatea cardinalelor mulțimilor Δ_1 și Δ_2 rezultă din faptul că ambele sunt de puterea continuului. \square

Teorema 4. *Problema 1 cu ipoteza ii) admite soluții dacă și numai dacă mulțimea M este închisă.*

Demonstrație. Considerăm o nouă funcție diferență $q = f - g$, anume

$$q(x) = e^x h(x),$$

unde h este funcția din demonstrația teoremei 1.

Constatăm următoarele:

- 1) $q \geq 0$,
- 2) $h, h', h'' > -\gamma$ (unde $\gamma \geq \beta > 0$ este convenabil ales, iar β este constanta determinată în cadrul demonstrației Teoremei 1),
- 3) $q'(x) = e^x(h(x) + h'(x))$,
- 4) $q''(x) = e^x(h(x) + 2h'(x) + h''(x))$.

Alegem

$$f(x) = q(x) + 4\gamma e^x \text{ și } g(x) = 4\gamma e^x.$$

Funcțiile f și g sunt strict crescătoare, strict convexe, cu $\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} g(x) = +\infty$ și $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} g(x) = 0$. \square

Corolar 5. Problema 2 cu ipoteza ii) admite soluții dacă și numai dacă mulțimea M este închisă, iar $\mathbb{R} \setminus M$ nu este interval.

Demonstrație. Procedăm în maniera anterioară, alegând funcția diferență $\pm q$ pe intervale. \square

Observația 6. Tehnica utilizată în această lucrare generează o nouă soluție a problemei inițiale din [1].

Observația 7. În acest articol, din cauza dificultății construcției, nu ne-am propus să tratăm toate cazurile posibile, ca în [2], mărginindu-ne doar la ipotezele i) și ii), pe care le considerăm relevante.

Observația 8. Se poate înlocui funcția φ cu orice altă funcție care are proprietățile:

- 1) φ este de două ori derivabilă;
- 2) $\varphi(0) = \varphi'(0) = 0$ și $\varphi(x) > 0, \forall x \neq 0$ (în locul lui 0 putem alege orice punct $a \in \mathbb{R}$);
- 3) $\varphi, \varphi', \varphi''$ sunt mărginite.

Un astfel de exemplu, altul decât cel prezentat, este funcția $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, $\varphi(x) = \frac{x^2}{1+x^2}$. Totuși, funcțiile φ_m au proprietatea că derivatele lor de orice ordin se anulează în 0.

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Olimpiada de Matematică a studenților din sud-estul Europei, SEEMOUS 2016

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Abstract. We discuss the problems of the 10th South Eastern European Mathematical Olympiad for University Students, SEEMOUS 2016, organized by The Mathematical Society of South Eastern Europe and Cyprus Mathematical Society, that took place in Protaras, Cyprus, between March 1 and March 6, 2016.

Keywords: Integrals, series, matrices, Jordan form, rank

MSC: Primary 15A03; Secondary 15A21, 26D15.

În perioada 1–6 martie 2016 s-a desfășurat cea de a zecea ediție a Olimpiadei de Matematică pentru studenții din sud-estul Europei (South Eastern European Mathematical Olympiad for University Students), SEEMOUS 2016. Ediția din acest an a avut loc în Protaras, Cipru, și a fost organizată de Societatea de Matematică din Sud-Estul Europei (MASSEE) și de Societatea de Matematică din Cipru (CYMS), sub auspiciile Ministerului Educației și Culturii din Cipru. Au participat 23 de echipe din Bulgaria, Cipru, Grecia, Iran, Fosta Republică Yugoslavă a Macedoniei, Marea Britanie, România, Turkmenistan, Uzbekistan.

Concursul a constat dintr-o singură probă desfășurată pe durata a cinci ore, timp în care studenții au avut de rezolvat patru probleme selectate de către juriu în funcție de nivelul de dificultate: prima problemă a fost considerată de juriu ca fiind una ușoară, următoarele două probleme au fost considerate de dificultate medie, iar ultima problemă s-a încadrat la nivel de dificultate ridicat.

Fiecare problemă s-a punctat cu un număr întreg de la 0 la 10. La finalul concursului s-au acordat 10 medalii de aur, 20 medalii de argint și 21 medalii de bronz, la un total de 90 participanți. Patru concurenți au obținut punctajul maxim, dintre care doi din România, și anume Vlad Mihai Mihaly de la Universitatea Tehnică din Cluj-Napoca și Emanuel Necula de la Universitatea Politehnică din București.

Mai multe detalii despre concurs se pot consulta pe pagina web oficială: <http://www.massee-org.eu/index.php/news/item/49-seemous-2016>.

Prezentăm, în continuare, problemele date la concurs, însoțite de soluții și comentarii pe marginea acestora.

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Problema 1. *Să se arate că pentru orice funcție continuă și descrescătoare $f : \left[0, \frac{\pi}{2}\right] \rightarrow \mathbb{R}$ au loc inegalitățile:*

$$\int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} f(x) dx \leq \int_0^{\frac{\pi}{2}} f(x) \cos x dx \leq \int_0^1 f(x) dx.$$

Să se precizeze când are loc fiecare dintre egalități.

Pirmyrat Gurbanov, Turkmenistan

Comentarii și observații. Această problemă a primit diferite soluții din partea participanților la concurs (23 de concurenți au obținut punctaj maxim) precum și din partea membrilor juriului.

Notăm faptul că pentru fiecare inegalitate în parte, cazul de egalitate are loc doar pentru funcțiile constante, fapt ce rezultă imediat pentru fiecare dintre abordările pe care le vom prezenta în continuare. Astfel, vom omite să mai discutăm acest aspect.

De asemenea, au existat câteva abordări ale problemei (pe care nu le vom mai prezenta aici) în care s-a presupus că funcția f este pozitivă. De notat că această presupunere nu reduce din generalitate problemei (este suficient să se adune la f o constantă suficient de mare pentru a o face pozitivă, deoarece pentru funcții constante are loc egalitate iar funcția f este mărginită).

Soluția 1 (a autorului). Pentru a demonstra prima inegalitate, se minorează diferența dintre a doua și prima integrală, ținând cont de monotonia funcției f . Astfel:

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} f(x) \cos x dx - \int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} f(x) dx = \\ &= \int_0^{\frac{\pi}{2}-1} f(x) \cos x dx - \int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} f(x) \cdot (1 - \cos x) dx \\ &\geq \int_0^{\frac{\pi}{2}-1} f(x) \cos x dx - f\left(\frac{\pi}{2} - 1\right) \int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} (1 - \cos x) dx \\ &= \int_0^{\frac{\pi}{2}-1} f(x) \cos x dx - f\left(\frac{\pi}{2} - 1\right) \left(1 - \int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} \cos x dx\right) \\ &= \int_0^{\frac{\pi}{2}-1} f(x) \cos x dx - f\left(\frac{\pi}{2} - 1\right) \left(\int_0^{\frac{\pi}{2}} \cos x dx - \int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} \cos x dx\right) \\ &= \int_0^{\frac{\pi}{2}-1} f(x) \cos x dx - f\left(\frac{\pi}{2} - 1\right) \int_0^{\frac{\pi}{2}-1} \cos x dx \end{aligned}$$

$$= \int_0^{\frac{\pi}{2}-1} \left(f(x) - f\left(\frac{\pi}{2} - 1\right) \right) \cos x \, dx \geq 0.$$

Pentru a doua inegalitate se procedează similar, astfel:

$$\begin{aligned} \int_0^1 f(x) \, dx - \int_0^{\frac{\pi}{2}} f(x) \cos x \, dx &= \\ &= \int_0^1 f(x) \cdot (1 - \cos x) \, dx - \int_1^{\frac{\pi}{2}} f(x) \cos x \, dx \\ &\geq f(1) \int_0^1 (1 - \cos x) \, dx - \int_1^{\frac{\pi}{2}} f(x) \cos x \, dx \\ &= f(1) \left(1 - \int_0^1 \cos x \, dx \right) - \int_1^{\frac{\pi}{2}} f(x) \cos x \, dx \\ &= f(1) \left(\int_0^{\frac{\pi}{2}} \cos x \, dx - \int_0^1 \cos x \, dx \right) - \int_1^{\frac{\pi}{2}} f(x) \cos x \, dx \\ &= f(1) \int_1^{\frac{\pi}{2}} \cos x \, dx - \int_1^{\frac{\pi}{2}} f(x) \cos x \, dx \\ &= \int_1^{\frac{\pi}{2}} (f(x) - f(1)) \cos x \, dx \geq 0. \end{aligned}$$

Soluția 2. Făcând schimbarea de variabilă $t = x - \left(\frac{\pi}{2} - 1\right)$ în prima integrală, respectiv $t = \sin x$ în a doua integrală, rezultă că șirul de inegalități ce trebuie demonstrat se rescrie echivalent

$$\int_0^1 f\left(t + \frac{\pi}{2} - 1\right) dt \leq \int_0^1 f(\arcsin t) dt \leq \int_0^1 f(t) dt.$$

Se arată ușor că

$$t + \frac{\pi}{2} - 1 \geq \arcsin t \geq t \quad \text{pentru orice } t \in [0, 1],$$

astfel că șirul de inegalități de mai sus este o consecință imediată a monotoniei funcției f și a integralei.

Soluția 3. Pentru a demonstra prima inegalitate, se folosește inegalitatea lui Cebâșev sub formă integrală pentru funcțiile f și \cos . Astfel,

$$\int_0^{\frac{\pi}{2}} f(x) \cos x \, dx \geq \frac{2}{\pi} \left(\int_0^{\frac{\pi}{2}} f(x) \, dx \right) \left(\int_0^{\frac{\pi}{2}} \cos x \, dx \right) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} f(x) \, dx.$$

Mai departe, făcând schimbarea de variabilă $x = \frac{\pi}{2} \left(t - \frac{\pi}{2} + 1\right)$, rezultă că

$$\frac{2}{\pi} \int_0^{\frac{\pi}{2}} f(x) \, dx = \int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} f\left(\frac{\pi}{2} \left(t - \frac{\pi}{2} + 1\right)\right) dt \geq \int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} f(t) \, dt,$$

unde ultima inegalitatea rezultă din monotonia funcției f și din faptul că

$$\frac{\pi}{2} \left(t - \frac{\pi}{2} + 1 \right) \leq t \quad \text{pentru orice } t \leq \frac{\pi}{2}.$$

Combinând cele două inegalități obținute, rezultă prima inegalitate din enunț.

Pentru a obține a doua inegalitate, se consideră substituția $x = \sin t$ în integrala din membrul drept, astfel că

$$\int_0^1 f(x) dx = \int_0^{\frac{\pi}{2}} f(\sin t) \cos t dt \geq \int_0^{\frac{\pi}{2}} f(t) \cos t dt,$$

unde s-a ținut cont de monotonia funcției f și de faptul că $\sin t \leq t$ pentru orice $t \in \left[0, \frac{\pi}{2}\right]$.

Problema 2.

- Să se demonstreze că pentru orice matrice $X \in \mathcal{M}_2(\mathbb{C})$ există o matrice $Y \in \mathcal{M}_2(\mathbb{C})$ astfel ca $Y^3 = X^2$.
- Să se demonstreze că există o matrice $X \in \mathcal{M}_3(\mathbb{C})$ astfel încât pentru orice matrice $Y \in \mathcal{M}_3(\mathbb{C})$ să aibă loc $Y^3 \neq X^2$.

Vasile Pop, Universitatea Tehnică din Cluj-Napoca, România

Comentarii și observații. Dacă pentru $n, k \geq 2$ notăm

$$\mathcal{P}(n, k) = \left\{ X^k : X \in \mathcal{M}_n(\mathbb{C}) \right\},$$

atunci punctul a) al problemei cere incluziunea $\mathcal{P}(2, 2) \subseteq \mathcal{P}(2, 3)$. Se poate arăta, urmând soluția prezentată mai jos, că și incluziunea inversă are loc, deci $\mathcal{P}(2, 3) \subseteq \mathcal{P}(2, 2)$. De asemenea, la punctul b), se arată că $\mathcal{P}(3, 2) \not\subseteq \mathcal{P}(3, 3)$. Reciproc, se poate arăta, însă, că $\mathcal{P}(3, 3) \subseteq \mathcal{P}(3, 2)$.

Pentru această problemă, 15 dintre concurenți au primit punctajul maxim.

Soluția autorului.

a) Fie J_X forma canonică Jordan a matricei X și fie P matricea de pasaj, astfel că

$$X = P \cdot J_X \cdot P^{-1}.$$

Căutăm o matrice $Y \in \mathcal{M}_2(\mathbb{C})$ de forma

$$Y = P \cdot Y_1 \cdot P^{-1}$$

astfel că egalitatea $Y^3 = X^2$ devine $Y_1^3 = J_X^2$.

Distingem două cazuri, după felul formei canonice Jordan.

Astfel, dacă $J_X = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, atunci $J_X^2 = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$ și putem considera

$Y_1 = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$, unde $\mu_1, \mu_2 \in \mathbb{C}$ sunt astfel încât $\mu_1^3 = \lambda_1^2$ și $\mu_2^3 = \lambda_2^2$.

Dacă $J_X = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, atunci $J_X^2 = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix}$ și căutăm Y_1 de forma $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$. Atunci $Y_1^3 = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$, de unde rezultă egalitățile $a^3 = \lambda^2$ și $3a^2b = 2\lambda$. Dacă $\lambda = 0$, atunci se alege $a = 0$ și $b \in \mathbb{C}$ oarecare, astfel că $Y_1 = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$. Dacă $\lambda \neq 0$, atunci se alege $a \in \mathbb{C}^*$ astfel încât $a^3 = \lambda^2$, de unde $b = \frac{2\lambda}{3a^2}$.

b) Este suficient să alegem $X \in \mathcal{M}_3(\mathbb{C})$ astfel încât $X^2 \neq O_3$ și $X^3 = O_3$, spre exemplu, $X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Astfel, dacă există $Y \in \mathcal{M}_3(\mathbb{C})$ astfel încât $Y^3 = X^2$, rezultă că $Y^6 = X^4 = O_3$, deci Y este o matrice nilpotentă, de unde rezultă că $Y^3 = O_3$, ceea ce reprezintă o contradicție cu $Y^3 = X^2 \neq O_3$.

Problema 3. Fie $n, k \in \mathbb{N}^*$ și matricele idempotente $A_1, \dots, A_k \in \mathcal{M}_n(\mathbb{R})$. Să se arate că

$$\sum_{i=1}^k (n - \text{rang}(A_i)) \geq \text{rang} \left(I_n - \prod_{i=1}^k A_i \right).$$

* * *

Comentarii și observații. Problema a fost propusă de delegația din Turmenistan, însă s-a dovedit ulterior concursului ca fiind o problemă relativ cunoscută. Mai exact, concluzia problemei se obține prin simpla combinare a două proprietăți legate de rangul matricelor, proprietăți ce se regăsesc ca exerciții în lucrarea Fuzhen Zhang, *Matrix Theory. Basic Results and Techniques*. Universitext. Springer-Verlag, New York, 2011, pg. 55.

Un număr de 18 concurenți au primit punctajul maxim pentru această problemă.

Soluție. Rezultatul se obține prin combinarea următoarelor două rezultate auxiliare:

(1) Pentru orice matrice $A \in \mathcal{M}_n(\mathbb{R})$ idempotentă are loc relația

$$\text{rang } A + \text{rang}(I_n - A) = n.$$

(2) Pentru orice matrice $A, B \in \mathcal{M}_n(\mathbb{R})$ are loc relația

$$\text{rang}(I_n - AB) \leq \text{rang}(I_n - A) + \text{rang}(I_n - B).$$

Rezultatul (1) este o consecință a proprietății de subaditivitate a rangului, respectiv a inegalității lui Sylvester. Astfel,

$$\text{rang}(A + B) \leq \text{rang}(A) + \text{rang}(B) \leq n - \text{rang}(AB), \quad A, B \in \mathcal{M}_n(\mathbb{R}),$$

de unde, pentru $B = I_n - A$, se obține concluzia.

Rezultatul (2) se obține folosind tot proprietatea de subaditivitate a rangului, astfel că

$$\begin{aligned} \text{rang}(I_n - AB) &= \text{rang}((I_n - A) + A(I_n - B)) \\ &\leq \text{rang}(I_n - A) + \text{rang}(A(I_n - B)) \\ &\leq \text{rang}(I_n - A) + \text{rang}(I_n - B), \end{aligned}$$

unde ultima inegalitate rezultă din aplicarea proprietății

$$\text{rang}(AB) \leq \min \{ \text{rang}(A), \text{rang}(B) \}$$

pentru orice $A, B \in \mathcal{M}_n(\mathbb{R})$.

Rezultatul (2) se extinde natural, prin inducție, la oricare k matrice $A_1, A_2, \dots, A_k \in \mathcal{M}_n(\mathbb{R})$, astfel că are loc inegalitatea

$$\text{rang} \left(I_n - \prod_{i=1}^k A_i \right) \leq \sum_{i=1}^k \text{rang}(I_n - A_i),$$

de unde, aplicând apoi rezultatul (1) se obține direct concluzia problemei.

Problema 4. Pentru orice $n \in \mathbb{N}^*$, fie

$$I_n = \int_0^\infty \frac{\arctg x}{(1+x^2)^n} dx.$$

Să se demonstreze că:

- a) $\sum_{n=1}^{\infty} \frac{I_n}{n} = \frac{\pi^2}{6}$;
- b) $\int_0^\infty \arctg x \cdot \ln \left(1 + \frac{1}{x^2} \right) dx = \frac{\pi^2}{6}$.

Ovidiu Furdui, Universitatea Tehnică din Cluj-Napoca, România

Comentarii și observații. Pe lângă soluția autorului, prezentăm încă două soluții: prima propusă de conf. univ. dr. Tiberiu Trif de la Universitatea „Babeș-Bolyai” din Cluj-Napoca, a doua de prof. univ. dr. Mircea Ivan de la Universitatea Tehnică din Cluj-Napoca.

Numărul concurenților ce au primit punctajul maxim pentru această problemă a fost 7.

Soluția 1 (a autorului). a) Calculând I_n prin părți, cu alegerile

$$\begin{aligned} f(x) &= \frac{\arctg x}{(1+x^2)^n}, & f'(x) &= \frac{1-2nx \arctg x}{(1+x^2)^{n+1}}, \\ g'(x) &= 1, & g(x) &= x, \end{aligned}$$

se obține că

$$\begin{aligned}
 I_n &= \frac{x \operatorname{arctg} x}{(1+x^2)^{n+1}} \Big|_0^\infty - \int_0^\infty \frac{x - 2nx^2 \operatorname{arctg} x}{(1+x^2)^{n+1}} dx = \\
 &= - \int_0^\infty \frac{x}{(1+x^2)^{n+1}} dx + 2n \int_0^\infty \frac{x^2}{(1+x^2)^{n+1}} \operatorname{arctg} x dx = \\
 &= \frac{1}{2n(1+x^2)^n} \Big|_0^\infty + 2n \int_0^\infty \left(\frac{\operatorname{arctg} x}{(1+x^2)^n} - \frac{\operatorname{arctg} x}{(1+x^2)^{n+1}} \right) dx = \\
 &= -\frac{1}{2n} + 2n(I_n - I_{n+1}).
 \end{aligned}$$

Concluzionând,

$$\frac{I_n}{n} = -\frac{1}{2n^2} + 2(I_n - I_{n+1}) \quad \text{pentru orice } n \geq 1$$

de unde, mai departe,

$$\sum_{n=1}^{\infty} \frac{I_n}{n} = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} + 2 \sum_{n=1}^{\infty} (I_n - I_{n+1}) = -\frac{1}{2} \cdot \frac{\pi^2}{6} + 2I_1 = \frac{\pi^2}{6},$$

deoarece

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6}, \\
 \lim_{n \rightarrow \infty} I_n &= 0
 \end{aligned}$$

și

$$I_1 = \int_0^\infty \frac{\operatorname{arctg} x}{1+x^2} dx = \frac{\operatorname{arctg}^2 x}{2} \Big|_0^\infty = \frac{\pi^2}{8}.$$

b) Are loc șirul de egalități:

$$\begin{aligned}
 \int_0^\infty \operatorname{arctg} x \cdot \ln \left(1 + \frac{1}{x^2} \right) dx &= - \int_0^\infty \operatorname{arctg} x \cdot \ln \left(1 - \frac{1}{1+x^2} \right) dx = \\
 &= \int_0^\infty \operatorname{arctg} x \cdot \left(\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{1+x^2} \right)^n \right) dx = \\
 &= \sum_{n=1}^{\infty} \frac{I_n}{n} = \frac{\pi^2}{6},
 \end{aligned}$$

unde ultima egalitate rezultă de la punctul a). Permutarea integralei cu suma se poate justifica, spre exemplu, folosind teorema lui Tonelli, deoarece toți termenii sunt pozitivi.

Soluția 2 (Tiberiu Trif). Egalitatea dintre suma $\sum_{n=1}^{\infty} \frac{I_n}{n}$ de la punctul a)

și integrala $\int_0^\infty \operatorname{arctg} x \cdot \ln \left(1 + \frac{1}{x^2} \right) dx$ de la punctul b) se obține la fel ca

în Soluția 1 pentru punctul b), astfel că este suficient să se dea o deducere directă a identității de la punctul b).

Astfel, prin schimbarea de variabilă $t = \operatorname{arctg} x$ și apoi folosind metoda de integrare prin părți se obține că

$$\begin{aligned} I &:= \int_0^\infty \operatorname{arctg} x \cdot \ln \left(1 + \frac{1}{x^2} \right) dx = -2 \int_0^{\frac{\pi}{2}} \frac{t}{\cos^2 t} \ln(\sin t) dt = \\ &= -2 \int_0^{\frac{\pi}{2}} (t \cdot \operatorname{tg} t + \ln(\cos t))' \cdot \ln(\sin t) dt = \\ &= -2 (t \cdot \operatorname{tg} t + \ln(\cos t)) \cdot \ln(\sin t) \Big|_0^{\frac{\pi}{2}} + 2 \int_0^{\frac{\pi}{2}} (t \cdot \operatorname{tg} t + \ln(\cos t)) \frac{\cos t}{\sin t} dt = \\ &= 2 \int_0^{\frac{\pi}{2}} t dt + 2 \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t} \ln(\cos t) dt = \frac{\pi^2}{4} + 2J, \end{aligned}$$

unde

$$J := \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t} \ln(\cos t) dt.$$

De notat că s-a ținut cont în calculele intermediare de relațiile:

$$\begin{aligned} \int \frac{t}{\cos^2 t} dt &= t \cdot \operatorname{tg} t + \ln(\cos t) + C, \\ (t \cdot \operatorname{tg} t + \ln(\cos t)) \cdot \ln(\sin t) &\Big|_0^{\frac{\pi}{2}} = 0. \end{aligned}$$

Mai departe,

$$J = \frac{1}{4} \int_0^{\frac{\pi}{2}} \frac{2 \sin t \cos t}{\sin^2 t} \ln(1 - \sin^2 t) dt,$$

astfel că prin schimbarea de variabilă $x = \sin^2 t$, se obține

$$J = \frac{1}{4} \int_0^1 \frac{\ln(1-x)}{x} dx.$$

În final,

$$\frac{\ln(1-x)}{x} = - \sum_{n=0}^{\infty} \frac{x^n}{n+1},$$

de unde

$$J = -\frac{1}{4} \sum_{n=0}^{\infty} \int_0^1 \frac{x^n}{n+1} dx = -\frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{\pi^2}{24}.$$

Revenind, rezultă

$$I = \frac{\pi^2}{4} + 2J = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6}.$$

Soluția 3 (Mircea Ivan). Diferența față de soluția anterioară este dată de modul prin care este calculată integrala de la punctul b). Integrând prin părți, se obține că

$$\begin{aligned} I &:= \int_0^\infty \operatorname{arctg} x \cdot \ln \left(1 + \frac{1}{x^2} \right) dx = x \cdot \operatorname{arctg} x \cdot \ln \left(1 + \frac{1}{x^2} \right) \Big|_0^\infty - \\ &- \int_0^\infty x \left(\frac{1}{1+x^2} \ln \left(1 + \frac{1}{x^2} \right) + \operatorname{arctg} x \cdot \left(-\frac{2}{x(x^2+1)} \right) \right) dx = \\ &= 0 - \int_0^\infty \frac{x}{1+x^2} \ln \left(1 + \frac{1}{x^2} \right) dx + 2 \int_0^\infty \frac{\operatorname{arctg} x}{1+x^2} dx = \\ &= (\operatorname{arctg} x)^2 \Big|_0^\infty - \frac{1}{2} \int_0^\infty (1+x^2) \ln \left(1 - \frac{1}{1+x^2} \right) \cdot \left(\frac{1}{1+x^2} \right)' dx, \end{aligned}$$

iar cu schimbarea de variabilă $u = \frac{1}{1+x^2}$ se obține

$$I = \frac{\pi^2}{4} + \frac{1}{2} \int_0^1 \frac{\ln(1-u)}{u} du.$$

Folosind formula dedusă la soluția anterioară

$$\int_0^1 \frac{\ln(1-u)}{u} du = -\frac{\pi^2}{6},$$

se obține în final că

$$I = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6}.$$

Left and Right Isoscelizers and S-Triangles

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Abstract. In *Gazeta Matematică, Seria A*, **22(101)** (2004), 222–231 we defined the left and right isoscelizers and showed some related bicentric pairs of points similarly with the Yff’s points. The aim of this paper is to link the left and right isoscelizers with the S-triangles (orthopolar triangles). The definition and properties of S-triangles have been introduced by Traian Lalescu in *Gazeta Matematică* volume XX, in February 1915, p. 213.

Keywords: Left and right isoscelizers, S-triangles, sine-triple-angle-circle
MSC: Primary 51M04.

1. LEFT AND RIGHT ISOSCELIZERS

Definition. Given a triangle ABC with acute angles, we consider the points $A_1, A_2 \in BC$, $B_1, B_2 \in CA$ and $C_1, C_2 \in AB$, such that $\text{tr}C_2B_1A$, $\text{tr}A_2C_1B$ and $\text{tr}B_2A_1C$ are isosceles triangles: $\widehat{C_2B_1A} = \widehat{A}$, $\widehat{A_2C_1B} = \widehat{B}$ and $\widehat{B_2A_1C} = \widehat{C}$.

We call the line segments C_2B_1 , A_2C_1 , B_2A_1 , A -left isoscelizer, B -left isoscelizer and C -left isoscelizer (see Fig. 1.ℓ).

If the points $A_1, A_2 \in BC$, $B_1, B_2 \in CA$ and $C_1, C_2 \in AB$, are such that $\widehat{B_1C_2A} = \widehat{A}$, $\widehat{C_1A_2B} = \widehat{B}$ and $\widehat{A_1B_2C} = \widehat{C}$, then the line segments B_1C_2 , C_1A_2 and A_1B_2 are A -right isoscelizer, B -right isoscelizer and C -right isoscelizer (see Fig. 1.r).

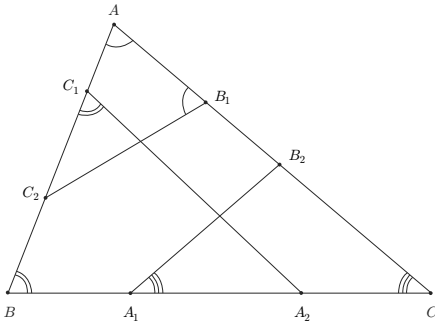


Figure 1.ℓ

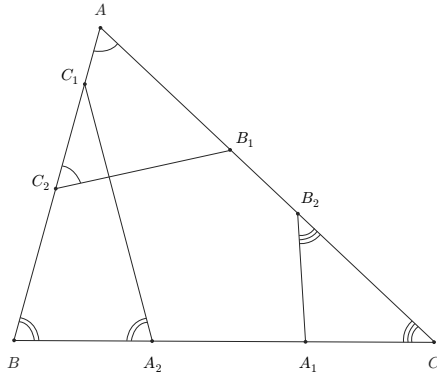


Figure 1.r

Denote the left-isoscelizer lengths as follows:

$$u = C_2B_1 = C_2A, \quad v = A_2C_1 = A_2B, \quad w = B_2A_1 = B_2C.$$

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Theorem 1.ℓ. *The three lines C_2B_1 , A_2C_1 and B_2A_1 concur (see Fig. 2.ℓ) if and only if*

$$\begin{aligned} & 8 \cos A \cos B \cos C (ab \cdot u + bc \cdot v + ca \cdot w) \\ & - 4(a^2 \cos A \cos B \cdot u + b^2 \cos B \cos C \cdot v + c^2 \cos C \cos A \cdot w) \\ & + 2(ca \cos A \cdot u + ab \cos B \cdot v + bc \cos C \cdot w) \\ & = abc(8 \cos A \cos B \cos C + 1). \end{aligned} \quad (1.\ell)$$

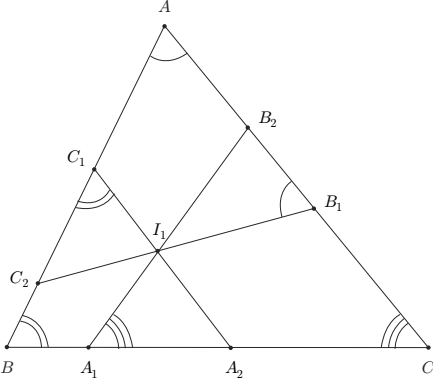


Figure 2.ℓ

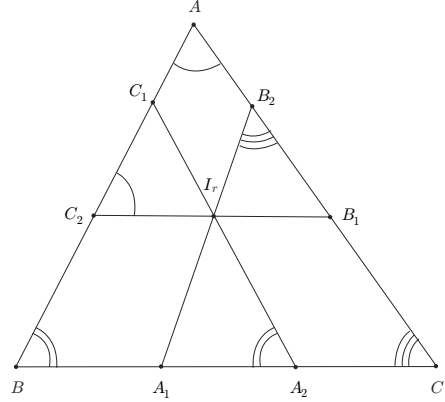


Figure 2.r

Proof. The barycentric coordinates of A_1 and B_2 are:

$$\begin{aligned} A_1 &= 0 : 2w \cos C : a - 2w \cos C, \\ B_2 &= w : 0 : b - w. \end{aligned}$$

The line A_1B_2 is then given by

$$A_1B_2 : \begin{vmatrix} \alpha & \beta & \gamma \\ 0 & 2w \cos C & a - 2w \cos C \\ w & 0 & b - w \end{vmatrix} = 0,$$

that is

$$A_1B_2 : 2(b - w) \cos C \cdot \alpha + (a - 2w \cos C) \cdot \beta - 2w \cos C \cdot \gamma = 0,$$

and similarly for B_1C_2 and C_1A_2 :

$$B_1C_2 : \mathbb{Q} - 2u \cos A \cdot \alpha + 2(c - u) \cos A \cdot \beta + (b - 2u \cos A) \cdot \gamma = 0,$$

$$C_1A_2 : \mathbb{Q}(c - 2v \cos B) \cdot \alpha - 2v \cos B \cdot \beta + 2(a - v) \cos B \cdot \gamma = 0.$$

The three lines concur if and only if

$$\begin{vmatrix} 2(b - w) \cos C & a - 2w \cos C & -2w \cos C \\ -2u \cos A & 2(c - u) \cos A & b - 2u \cos A \\ c - 2v \cos B & -2v \cos B & 2(a - v) \cos B \end{vmatrix} = 0,$$

which is equivalent with (1.ℓ). \square

Let I_ℓ be the point of intersection of left-isoscelizers C_2B_1 , A_2C_1 and B_2A_1 . The trilinears of I_ℓ are:

$$\alpha_\ell = 4 \cos A \cos B \sin 2C \cdot w - 2 \cos B \sin 2A \cdot u + \sin 2B \cdot v,$$

$$\beta_\ell = 4 \cos B \cos C \sin 2A \cdot u - 2 \cos C \sin 2B \cdot v + \sin 2C \cdot w,$$

$$\gamma_\ell = 4 \cos C \cos A \sin 2B \cdot v - 2 \cos A \sin 2C \cdot w + \sin 2A \cdot u.$$

Similarly, for the right-isoscelizers

$$B_1C_2 = B_1A = u, \quad C_1A_2 = C_1B = v, \quad A_1B_2 = A_1C = w,$$

the following theorem holds.

Theorem 1.r. *The three lines C_2B_1 , A_2C_1 , B_2A_1 concur in a point I_r (see Fig. 2.r) if and only if*

$$\begin{aligned} & 8 \cos A \cos B \cos C (ca \cdot u + ab \cdot v + bc \cdot w) \\ & - 4(a^2 \cos C \cos A \cdot u + b^2 \cos A \cos B \cdot v + c^2 \cos B \cos C \cdot w) \\ & + 2(ab \cos A \cdot u + bc \cos B \cdot v + ca \cos C \cdot w) \\ & = abc(8 \cos A \cos B \cos C + 1) \end{aligned} \tag{1.r}$$

The trilinear coordinates of I_r are

$$I_r = \alpha_r : \beta_r : \gamma_r,$$

where

$$\alpha_r = 4 \cos C \cos A \sin 2B \cdot v - 2 \cos C \sin 2A \cdot u + \sin 2C \cdot w,$$

$$\beta_r = 4 \cos A \cos B \sin 2C \cdot w - 2 \cos A \sin 2B \cdot v + \sin 2A \cdot u,$$

$$\gamma_r = 4 \cos B \cos C \sin 2A \cdot u - 2 \cos B \sin 2C \cdot w + \sin 2B \cdot v.$$

Choosing in Theorems 1. ℓ and 1.r appropriate values for u , v , w such that triangles C_2B_1A , A_2C_1B , B_2A_1C have certain properties, we obtain several pairs of bicentric points, (I_ℓ, I_r) , similar to the centers X_{173} (= congruent isoscelizers point), X_{362} (= congruent circumcircles isoscelizer point), X_{363} (= equal perimeters isoscelizer point), X_{364} (= Wabash center = equal areas isoscelizer point), X_{164} (= congruent altitudes), X_{258} (= congruent incircles) discovered by Peter Yff (see [4], [5]).

2. S-TRIANGLES (ORTHOPOLAR TRIANGLES)

In a note [8] published in *Gazeta Matematică* in 1915, Traian Lalescu wrote:

“Given a triangle ABC and two points B' and C' on its circumcircle. Let A' be the point on the same circle for which the Simson line is perpendicular on $B'C'$. It is known that this point is obtained as follows:

From A we draw a straight line perpendicular on $B'C'$ which intersect the circle in A'' and from A'' we draw the straight line perpendicular on BC which intersect the circle in the searched point A' . We say that the triangle $A'B'C'$ is a S-triangle with respect to the triangle ABC .”

Next, Traian Lalescu proved the following properties:

1) The algebraic sum of arcs $\widehat{AA'}$, $\widehat{BB'}$ and $\widehat{CC'}$ considered with the same orientation is zero:

$$m(\widehat{AA'}) + m(\widehat{BB'}) + m(\widehat{CC'}) \equiv 0 \pmod{2\pi}.$$

2) The Simson line of each vertex of the triangle $A'B'C'$ is perpendicular on the opposite side of this triangle.

3) The Simson lines of A', B', C' with respect to triangle ABC are concurrent.

4) The triangle ABC is an S-triangle with respect to the triangle $A'B'C'$.

5) The intersection point of the six Simson lines is the midpoint of the line segment joining the orthocenters of triangles ABC and $A'B'C'$ (see [8], [9], [10]).

Other authors showed that the point of intersection of the six Simson lines is the common orthopole of each side of the triangle $A'B'C'$ with respect to the triangle ABC and of each side of the triangle ABC with respect to the triangle $A'B'C'$, and for this reason the S-triangles are also named *orthopolar triangles*.

A classical example of S-triangles is the pair formed by the orthic triangle and the medial triangle, inscribed in the nine-point circle. The common orthopole is the midpoint of the line segment joining X_3 and X_{52} , which is X_{389} , the center of the Taylor's circle = the Spiecker point (incenter of the medial triangle) of the orthic triangle (notations from [5]).

A natural generalization of the S-triangles is obtained by replacing the Wallace-Simson lines with the Carnot lines.

Theorem 2.1. (Carnot) *Given a triangle ABC and M a point on its circumcircle. Let $P \in BC$, $Q \in CA$ and $R \in AB$ such that*

$$m(\widehat{MP, BC}) = m(\widehat{MQ, CA}) = m(\widehat{MR, AB}) = \varphi.$$

Then the points P , Q and R are collinear, and the straight line PQR is called the φ -Carnot line.

Definition 2.2. Given the triangle ABC , let B' and C' two points on its circumcircle and let A' be the point on the same circle such that

$$m(\widehat{\Delta_{A'}, B'C'}) = \pi - \varphi,$$

where $\Delta_{A'}$ is the φ -Carnot line of the point A' with respect to the triangle ABC . We say that the triangle $A'B'C'$ is a φ -S-triangle with respect to triangle ABC .

The point A' is obtained by the following construction: From A we draw the straight line AA'' such that $m(\widehat{AA'', B'C'}) = \pi - \varphi$, where A'' lies on the circumcircle of ABC , and from A'' we draw the straight line $A''A'$ such that $m(\widehat{A''A', BC}) = \varphi$, where A' lies on the same circle. This point A' is the searched point.

Theorem 2.3. *The pair of triangles ABC and $A'B'C'$ has the following properties:*

- 1) $m(\widehat{AA'}) + m(\widehat{BB'}) + m(\widehat{CC'}) \equiv 4\varphi \pmod{2\pi}$. (2.1)
- 2) *The measure of the angle between the φ -Carnot line of each vertex of the triangle $A'B'C'$ with the opposite side of this triangle is $\pi - \varphi$.*
- 3) *The φ -Carnot lines of A' , B' and C' with respect to the triangle ABC concur in a point I_φ .*
- 4) *The triangle ABC is a $(\pi - \varphi)$ -S-triangle with respect to the triangle $A'B'C'$.*
- 5) *The $(\pi - \varphi)$ -Carnot lines of A , B and C with respect to the triangle $A'B'C'$ concur in the same point I_φ ([3]).*

Remarks. 1) The point I_φ is the φ -isopole of each side of the triangle $A'B'C'$ with respect to the triangle ABC and the $(\pi - \varphi)$ -isopole of each side of the triangle ABC with respect to the triangle $A'B'C'$.

2) From $m(\widehat{AA'}) + m(\widehat{BB'}) + m(\widehat{CC'}) \equiv 4\varphi \pmod{2\pi}$ we may say that every pair of triangles inscribed in the same circle are in relationship $(\varphi - S)$ and $(\pi - \varphi) - S$ each with respect to the other. Actually, there are two values φ_1 and φ_2 of $\varphi \in (0, \pi)$, with $|\varphi_1 - \varphi_2| = \frac{\pi}{2}$, such that the relation (2.1) holds.

A connection between the left and right isoscelizers and S-triangles is given by the following theorem.

Theorem 2.4. ([2]) *Given the acute triangle ABC such that $m(\widehat{B}) \neq \frac{1}{2}m(\widehat{A}) \neq m(\widehat{C})$, $m(\widehat{C}) \neq \frac{1}{2}m(\widehat{B}) \neq m(\widehat{A})$ and $m(\widehat{A}) \neq \frac{1}{2}m(\widehat{C}) \neq m(\widehat{B})$. Let $A_\ell, A_r \in BC$, $B_\ell, B_r \in CA$ and $C_\ell, C_r \in AB$ such that $B_\ell C_\ell$, $C_\ell A_\ell$ and $A_\ell B_\ell$ are A-, B- and C-left isoscelizers and $B_r C_r$, $C_r A_r$ and $A_r B_r$ are*

A-, *B*- and *C*-right isoscelizers. Then the triangles $A_\ell B_\ell C_\ell$ and $A_r B_r C_r$ are inscribed in the same circle and they are *S*-triangles (see Fig. 3 and Fig. 4).

Proof. The construction of the triangles $A_\ell B_\ell C_\ell$ and $A_r B_r C_r$ follows the procedure to inscribe a triangle having parallel sides with three given directions, into an other triangle.

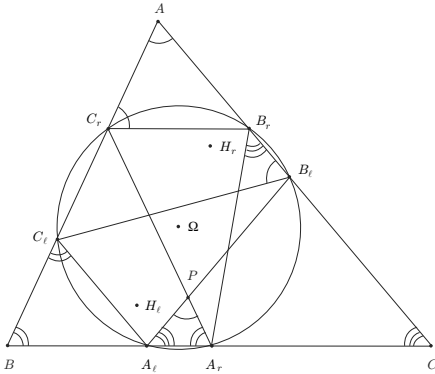


Figure 3

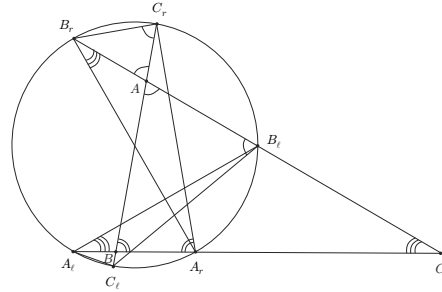


Figure 4

In Fig. 3, according to the case

$$m(\widehat{A}), m(\widehat{B}) > \frac{1}{2}m(\widehat{C}), \quad m(\widehat{B}), m(\widehat{C}) > \frac{1}{2}m(\widehat{A}), \quad m(\widehat{C}), m(\widehat{A}) > \frac{1}{2}m(\widehat{B}),$$

we have

$$m(\widehat{A}_\ell) = 2m(\widehat{B}) - m(\widehat{C}), \quad m(\widehat{B}_\ell) = 2m(\widehat{C}) - m(\widehat{A}), \quad m(\widehat{C}_\ell) = 2m(\widehat{A}) - m(\widehat{B})$$

and

$$m(\widehat{A}_r) = 2m(\widehat{B}) - m(\widehat{C}), \quad m(\widehat{B}_r) = 2m(\widehat{A}) - m(\widehat{C}), \quad m(\widehat{C}_r) = 2m(\widehat{B}) - m(\widehat{A}).$$

In Fig. 4, $m(\widehat{C}) < \frac{1}{2}m(\widehat{A}), \frac{1}{2}m(\widehat{B})$.

The triangles $A_\ell B_\ell C_\ell$ and $A_r B_r C_r$ are inscribed in the same circle with center having triangle center function $\alpha = \cos 3A$, which is X_{49} , and circumradius

$$R' = \frac{R}{1 + 8 \cos A \cos B \cos C},$$

where R is the circumradius of triangle ABC . This circle is called the sine-triple-angle circle (see [4], [11]).

Let $\{P\} = A_l B_l \cap A_r C_r$ (see Fig. 3). Then

$$\begin{aligned} m(\widehat{A}) &= m(\widehat{A_l P A_r}) = \frac{1}{2}[m(\widehat{A_l A_r}) + m(\widehat{B_l C_r})] \\ &= \frac{1}{2}[m(\widehat{A_l A_r}) + m(\widehat{B_l B_r}) + m(\widehat{B_r C_r})]. \end{aligned}$$

But

$$\begin{aligned} m(\widehat{A}) &= m(\widehat{C_l B_l A}) = \frac{1}{2} m(\widehat{B_r C_l}) \\ &= \frac{1}{2} [m(\widehat{B_r C_r}) + m(\widehat{C_r C_l})], \end{aligned}$$

hence

$$m(\widehat{A_l A_r}) + m(\widehat{B_l B_r}) = m(\widehat{C_r C_l}),$$

therefore

$$m(\widehat{A_l A_r}) + m(\widehat{B_l B_r}) + m(\widehat{C_l C_r}) \equiv 0 \pmod{2\pi}.$$

This fact means that $\text{tr} A_l B_l C_l$ and $\text{tr} A_r B_r C_r$ are S-triangles.

Let $B_l C_l = u_l$, $C_l A_l = v_l$ and $A_l B_l = w_l$. Then we have

$$\begin{cases} u_l + 2v_l \cos B = c, \\ v_l + 2w_l \cos C = a, \\ w_l + 2u_l \cos A = b. \end{cases}$$

Therefore, the sides of the triangle $A_l B_l C_l$ are:

$$\begin{aligned} u_l &= \frac{c + 4b \cos B \cos C - 2a \cos B}{1 + 8 \cos A \cos B \cos C}, \\ v_l &= \frac{a + 4c \cos C \cos A - 2b \cos C}{1 + 8 \cos A \cos B \cos C}, \\ w_l &= \frac{b + 4a \cos A \cos B - 2c \cos A}{1 + 8 \cos A \cos B \cos C}. \end{aligned}$$

The actual trilinears of A_l, B_l, C_l are

$$\begin{aligned} A_l &= 0 : \frac{b + 4a \cos A \cos B - 2c \cos A}{1 + 8 \cos A \cos B \cos C} \cdot \sin 2C \\ &\quad : \frac{a + 4c \cos C \cos A - 2b \cos C}{1 + 8 \cos A \cos B \cos C} \cdot \sin B, \\ B_l &= \frac{b + 4a \cos A \cos B - 2c \cos A}{1 + 8 \cos A \cos B \cos C} \cdot \sin C : 0 \\ &\quad : \frac{c + 4b \cos B \cos C - 2a \cos B}{1 + 8 \cos A \cos B \cos C} \cdot \sin 2A, \\ C_l &= \frac{a + 4c \cos C \cos A - 2b \cos C}{1 + 8 \cos A \cos B \cos C} \cdot \sin 2B \\ &\quad : \frac{c + 4b \cos B \cos C - 2a \cos B}{1 + 8 \cos A \cos B \cos C} \cdot \sin A : 0. \end{aligned}$$

Let H_l be the orthocenter of the triangle $A_l B_l C_l$. In Fig. 3, we have

$$d(H_l, BC) = A_l H_l \cos(2A + B - \pi) = 2R \cos(2B - C) \cos(A - C),$$

where R is the radius of the cyclic hexagon $A_l A_r B_l B_r C_r C_l$. Hence, the trilinears of H_l are

$$H_l = \cos(2B - C) \cos(A - C) : \cos(2C - A) \cos(B - A) \\ : \cos(2A - B) \cos(C - B).$$

□

Remark. H_l is denoted $P(61)$ in [7] and is named 1st Văcărețu Point.

Similarly, we have for the triangle $A_r B_r C_r$

$$\begin{cases} u_r + 2w_r \cos C = b, \\ v_r + 2u_r \cos A = c, \\ w_r + 2v_r \cos B = a, \end{cases}$$

hence

$$u_r = \frac{b + 4c \cos B \cos C - 2a \cos C}{1 + 8 \cos A \cos B \cos C}, \\ v_r = \frac{c + 4a \cos C \cos A - 2b \cos A}{1 + 8 \cos A \cos B \cos C}, \\ w_r = \frac{a + 4b \cos A \cos B - 2c \cos B}{1 + 8 \cos A \cos B \cos C}.$$

The actual trilinears of A_r, B_r, C_r are

$$A_r = 0 : \frac{a + 4b \cos A \cos B - 2c \cos B}{1 + 8 \cos A \cos B \cos C} \cdot \sin C \\ : \frac{c + 4a \cos C \cos A - 2b \cos A}{1 + 8 \cos A \cos B \cos C} \cdot \sin 2B, \\ B_r = \frac{a + 4b \cos A \cos B - 2c \cos B}{1 + 8 \cos A \cos B \cos C} \cdot \sin 2C : 0 \\ : \frac{b + 4c \cos B \cos C - 2a \cos C}{1 + 8 \cos A \cos B \cos C} \cdot \sin A, \\ C_r = \frac{c + 4a \cos C \cos A - 2b \cos A}{1 + 8 \cos A \cos B \cos C} \cdot \sin B \\ : \frac{b + 4c \cos B \cos C - 2a \cos C}{1 + 8 \cos A \cos B \cos C} \cdot \sin 2A : 0.$$

Let H_r be the orthocenter of the triangle $A_r B_r C_r$. The trilinears of H_r are

$$H_r = \cos(2C - B) \cos(A - B) : \cos(2A - C) \cos(B - C) \\ : \cos(2B - A) \cos(C - A).$$

The pair (H_l, H_r) is a pair of bicentric points (see [4], [5], [6]).

The intersection point of the six Simson lines of A_l, B_l, C_l with respect to the triangle $A_r B_r C_r$ and of A_r, B_r, C_r with respect to the triangle $A_l B_l C_l$

(= the common orthopole = the midpoint of the segment line $H_l H_r = X_{1594}$, Rigby-Lalescu orthopole) is $H_l \oplus H_r$ and its trilinears are

$$\alpha = \cos(2B - C) \cos(A - C) + \cos(2C - B) \cos(A - B),$$

$$\beta = \cos(2C - A) \cos(B - A) + \cos(2A - C) \cos(B - C),$$

$$\gamma = \cos(2A - B) \cos(C - B) + \cos(2B - A) \cos(C - A).$$

From the sine law, we may compute the radius R of the circumcircle of the triangles $A_l B_l C_l$, $A_r B_r C_r$:

$$\begin{aligned} \frac{u_l}{\sin(2B - C)} &= \frac{v_l}{\sin(2C - A)} = \frac{w_l}{\sin(2A - B)} = \frac{u_r}{\sin(2C - B)} \\ &= \frac{v_r}{\sin(2A - C)} = \frac{w_r}{\sin(2B - A)} = 2R. \end{aligned}$$

Finally, let Ω be the center of the circumcircle of the triangles $A_l B_l C_l$ and $A_r B_r C_r$. We have

$$\begin{aligned} m(\widehat{\Omega A_r C_r}) &= \frac{1}{2} \{ \pi - 2[2m(\widehat{A}) - m(\widehat{C})] \} \\ &= \frac{\pi}{2} - 2m(\widehat{A}) + m(\widehat{C}). \end{aligned}$$

Therefore

$$\begin{aligned} m(\widehat{\Omega A_r C_r}) &= m(\widehat{\Omega A_r C_r}) + m(\widehat{C_r A_r B}) \\ &= \frac{\pi}{2} - 2m(\widehat{A}) + m(\widehat{C}) + m(\widehat{B}) \\ &= \frac{3\pi}{2} - 3m(\widehat{A}). \end{aligned}$$

Hence

$$d(\Omega, BC) = R \sin \left[\frac{3\pi}{2} - 3m(\widehat{A}) \right] = -R \cos(3A).$$

Therefore the trilinears of Ω are

$$\Omega = \cos(3A) : \cos(3B) : \cos(3C),$$

hence $\Omega = X_{49}$ = center of sine-triple-angle-circle ([4], [5]).

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NOTE MATEMATICE

On a new family of semi-perfect cuboids

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Abstract. In this paper, we give a new parametric solution of one of the three variations of one classical open problem relating to the perfect cuboid. With this solution, one can find infinitely many semi-perfect cuboids each of which has integer sides such that two face diagonals and the body diagonal are of integral lengths.

Keywords: Rational cuboids, Diophantine equations, perfect cuboids, semi-perfect cuboids.

MSC: Primary 11D41; Secondary 11D72.

INTRODUCTION

The question whether there can be a *perfect rational cuboid*, which is a rectangular parallelepiped whose three edges, three face diagonals and body diagonal all have integer lengths, is yet to be answered. If a, b, c represent the lengths of the edges; p, q, r , the face diagonals; and d , the body diagonal, what is sought is a solution to the following system of equations:

$$\begin{aligned} a^2 + b^2 &= p^2; \\ a^2 + c^2 &= q^2; \\ b^2 + c^2 &= r^2; \\ a^2 + b^2 + c^2 &= d^2, \end{aligned} \tag{1}$$

where a, b, c, p, q, r and d are positive integers. Here, we have to find seven related integer quantities: the lengths of three edges, three face diagonals, and the body diagonal. Till May, 2016, we do not have the solution for the system of Diophantine equations (1). But, if we ask for only six of these to be integers, then there are three variations of this problem as follows:

- I. The body diagonal does not have integer length.
- II. One of the face diagonals does not have integer length.
- III. One of the edges does not have integer length.

The rational cuboids falling into any one of these three variations are called *semi-perfect cuboids*. Infinitely many solutions do exist for all the three variations, though none of them is complete (see [2] and [9]). Many papers dealing with different aspects of this problem are published (see [1], [3], [4], [7], [8], [10]–[13]). The historical details of this problem can be found in [5], and for other related research one may refer [6].

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In this paper, we give a new parametric solution for the *semi-perfect cuboids* of *Variation III* followed by a table containing some numerical results. Here, our approach is purely elementary.

1. THE PROBLEM AND THE SOLUTION

For given positive integers a, b, c, d, p, q , we want to find the primitive and non-trivial solutions of the following system of Diophantine equations:

$$\begin{aligned} a^2 + b^2 &= p^2; \\ a^2 + c^2 &= q^2; \\ \text{and} \\ a^2 + b^2 + c^2 &= d^2. \end{aligned} \tag{2}$$

In (2), take

$$a = 2\sqrt{xyzt}, b = xy - zt, c = xz - yt. \tag{3}$$

So, we get

$$\begin{aligned} p^2 &= a^2 + b^2 = (2\sqrt{xyzt})^2 + (xy - zt)^2 = (xy + zt)^2; \\ q^2 &= a^2 + c^2 = (2\sqrt{xyzt})^2 + (xz - yt)^2 = (xz + yt)^2; \\ d^2 &= a^2 + b^2 + c^2 = (2\sqrt{xyzt})^2 + (xy - zt)^2 + (xz - yt)^2 \\ &= x^2y^2 + z^2t^2 + x^2z^2 + y^2t^2 = (y^2 + z^2)(x^2 + t^2). \end{aligned} \tag{4}$$

Thus, from (4) we have

$$\begin{aligned} p &= (xy + zt); \\ q &= (xz + yt); \\ d &= \sqrt{(y^2 + z^2)(x^2 + t^2)}. \end{aligned} \tag{5}$$

Taking $x = \alpha^2 - \beta^2$, $y = m^2 - n^2$, $z = 2mn$, and $t = 2\alpha\beta$ in (3) and (5), and rearranging, we get

$$\begin{aligned} a &= 4\sqrt{mn\alpha\beta(m^2 - n^2)(\alpha^2 - \beta^2)}; \\ b &= (m^2 - n^2)(\alpha^2 - \beta^2) - 4mn\alpha\beta; \\ c &= 2mn(\alpha^2 - \beta^2) - 2\alpha\beta(m^2 - n^2); \\ p &= (m^2 - n^2)(\alpha^2 - \beta^2) + 4mn\alpha\beta; \\ q &= 2mn(\alpha^2 - \beta^2) + 2\alpha\beta(m^2 - n^2); \\ d &= \sqrt{(m^2 + n^2)^2(\alpha^2 + \beta^2)^2} \\ &= (m^2 + n^2)(\alpha^2 + \beta^2). \end{aligned} \tag{6}$$

We state a lemma which will be used to modify (6) after appropriate substitutions to suit our requirement.

Lemma 1. For any non-zero integers m, n, α, β , if $\alpha = (m^2 - n^2 + 2mn)^2$ and $\beta = (m^2 - n^2 - 2mn)^2$, then, $\alpha\beta = (m^4 - 6m^2n^2 + n^4)^2$, $\alpha^2 - \beta^2 = 16mn(m^2 - n^2)(m^2 + n^2)^2$ and $\alpha^2 + \beta^2 = 2(m^8 + 20m^6n^2 - 26m^4n^4 + 20m^2n^6 + n^8)$.

Proof. The proof is too elementary, and can be easily verified. \square

In (6), substituting the values of $\alpha, \beta, \alpha\beta, (\alpha^2 - \beta^2)$ and $(\alpha^2 + \beta^2)$ from Lemma 1, and then, scaling down the calculated expressions for a, b, c, p, q, d by their common factor 2, we get

$$\begin{aligned} a &= 8mn(m^8 - 6m^6n^2 + 6m^2n^6 - n^8); \\ b &= 2mn(3m^8 + 12m^6n^2 - 46m^4n^4 + 12m^2n^6 + 3n^8); \\ c &= (m^2 - n^2)(-m^8 + 28m^6n^2 - 6m^4n^4 + 28m^2n^6 - n^8); \\ p &= 2mn(5m^8 - 12m^6n^2 + 30m^4n^4 - 12m^2n^6 + 5n^8); \\ q &= (m^2 - n^2)(m^8 + 4m^6n^2 + 70m^4n^4 + 4m^2n^6 + n^8); \\ d &= (m^2 + n^2)(m^8 + 20m^6n^2 - 26m^4n^4 + 20m^2n^6 + n^8). \end{aligned} \quad (7)$$

If we take $r^2 = b^2 + c^2$, then based on (7) we calculate

$$\begin{aligned} r &= (m^{20} - 22m^{18}n^2 + 1197m^{16}n^4 - 2568m^{14}n^6 - 942m^{12}n^8 \\ &\quad + 5692m^{10}n^{10} - 942m^8n^{12} - 2568m^6n^{14} + 1197m^4n^{16} \\ &\quad - 22m^2n^{18} + n^{20})^{1/2}. \end{aligned} \quad (8)$$

2. SOME NUMERICAL SOLUTIONS

In the following table we give the absolute value of numerical solutions to (2) by using (7) in the range $2 \leq m \leq 10$ and $1 \leq n \leq 10$ where $\gcd(m, n) = 1$. In (7), inter-changing $m \leftrightarrow n$ does not affect the absolute values of the functions involved. So, in the calculation table we take only one pair of values for (m, n) instead of two pairs which are obtained by inter-changing the values of m and n . When both m and n are odd, we get 32 as a common factor in the calculated values of (a, b, c, p, q, d) . In this case, we have to pass over this factor to get primitive solutions for (a, b, c, p, q, d) . In all these numerical solutions as given in the table, we calculated the values of r to be irrational in accordance with (8). Now, there are two options: either we prove that for any integer pair (m, n) , r is never an integer; or, we may like to search extensively to find an (m, n) pair for which r is a positive integer. If one is lucky enough with the second option, the search for a perfect cuboid will be over.

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Table 1.

m, n	$ a $ $ p $	$ b $ $ q $	$ c $ $ d $
2, 1	1680 3796	3404 4947	4653 6005
3, 1	32×1680 32×4947	32×4653 32×3796	32×3404 32×6005
4, 1	1313760 2288168	1873432 1498575	720945 2399057
5, 1	32×371280 32×557525	32×415915 32×372732	32×32868 32×558493
6, 1	67194960 94519932	66474468 68498675	13300595 95451157
7, 1	32×8853600 32×12024103	32×8135897 32×9605496	32×3725496 32×12588025
8, 1	973103040 1293799504	852635696 1141104447	595978047 1424467265
9, 1	32×89681760 32×117591849	32×76059351 32×114080200	32×70508680 32×137110601
10, 1	7520047920 9765976100	6230824060 10365339699	7133662899 12093942101
3, 2	371280 372732	32868 557525	415915 558493
5, 2	1997520 29704100	29636860 28292901	28222299 40973549

Table 1. continued...

m, n	$ a $ $ p $	$ b $ $ q $	$ c $ $ d $
7, 2	331495920 680203468	593958932 465709725	327102435 754766693
9, 2	4366509840 6941260404	5395802796 4536769853	1231207747 7049607685
4, 3	8853600 9605496	3725496 12024103	8135897 12588025
5, 3	32×1313760 32×1498575	32×720945 32×2288168	32×1873432 32×2399057
7, 3	32×1997520 32×28292901	32×28222299 32×29704100	32×29636860 32×40973549
8, 3	555804480 3120035568	3070130832 2729754775	2672572265 4108194793
10, 3	11143401360 24927480300	22298068980 17562832651	13574892149 28384097149
5, 4	89681760 114080200	70508680 117591849	76059351 137110601
7, 4	983542560 1265281304	795977896 1887224097	1610670303 2048217665
9, 4	1741390560 11515088232	11382654168 12741605825	12622047295 17085471457
6, 5	560258160 834264300	618148620 722544251	456268549 950882701
7, 5	32×67194960 32×68498675	32×13300595 32×94519932	32×66474468 32×95451157
8, 5	5419202880 5759680400	1950937840 8836707399	6979945401 9049505849
9, 5	32×331495920 32×465709725	32×327102435 32×680203468	32×593958932 32×754766693
7, 6	2557005360 4394448996	3573920196 3267539197	2034339203 4842490885
8, 7	9354555840 18283984432	15709754032 11886826575	7334093745 19700076593
9, 7	32×973103040 32×1141104447	32×595978047 32×1293799504	32×852635696 32×1424467265
10, 7	72338224560 72792328700	8118151580 103505744451	74031212349 103823617349
9, 8	29031468480 63703398096	56703586896 36753615953	22538902447 67573108945
10, 9	79331127120 193284792900	176254314660 100174139299	61167233501 202732438501