GAZETA MATEMATICĂ

## SERIA A

**ANUL XXXIII (CXII)** **Nr. 1 – 2/ 2015**

## ARTICOLE

### Two surprising series with harmonic numbers and the tail of $\zeta(2)$

Ovidiu Furdui[1)]

**Abstract**. The paper is about evaluating two special series

$$\sum_{n=1}^{\infty} \frac{H_n}{n} \left( \zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right) = \sum_{n=1}^{\infty} \frac{H_n}{n} \psi'(n+1),$$

and

$$\sum_{n=1}^{\infty} H_n \left( \zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right)^2 = \sum_{n=1}^{\infty} H_n (\psi'(n+1))^2,$$

where $H_n$ denotes the $n$th harmonic number and $\psi$ is the Digamma function.

**Keywords:** Abel's summation formula, harmonic numbers, polygamma function, Riemann zeta function.

**MSC :** 40G10, 40A05, 33B15

The celebrated Riemann zeta function $\zeta$ is a function of a complex variable (see [9, p. 265]) defined by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots + \frac{1}{n^z} + \cdots , \quad \Re(z) > 1.$$

When $z = 2$, one has that the Riemann zeta function value $\zeta(2)$ is defined by the series formula

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots .$$

[1)]Department of Mathematics, Technical University of Cluj-Napoca, 400114, Romania, `Ovidiu.Furdui@math.utcluj.ro`

The polygamma function $\psi'$ is defined (see [7, p. 22]) by

$$\psi'(z) = \frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\Gamma(z) = \frac{\mathrm{d}}{\mathrm{d}z}\psi(z), \quad z \notin \{0, -1, -2, \ldots\},$$

or, in terms of the generalized (or Hurwitz) zeta function $\zeta(\cdot,\cdot)$

$$\psi'(z) = \sum_{i=0}^{\infty} \frac{1}{(i+z)^2} = \zeta(2, z), \quad z \notin \{0, -1, -2, \ldots\}.$$

The $n$th harmonic number $H_n$ is defined, for $n \geq 1$, by

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

A famous sum, due to Euler, in which the $n$th harmonic number is involved is given below ([3], [7, p. 103], [8, p. 228]):

$$2\sum_{k=1}^{\infty} \frac{H_k}{k^n} = (n+2)\zeta(n+1) - \sum_{k=1}^{n-2} \zeta(n-k)\zeta(k+1), \quad n \in \mathbb{N} \setminus \{1\}, \tag{1}$$

where an empty sum is understood to be nil.

For a proof of (1) the reader is referred to [7, pp. 103–105].

Series involving closed form evaluation of Riemann zeta function are collected in [7] and, more recently, in [8]. Other series, involving the Riemann zeta function and harmonic numbers, that evaluate to special constants can be found in [4].

In this paper we evaluate two special series involving the product of the $n$th harmonic number $H_n$ and the tail of $\zeta(2)$.

More precisely, we calculate the series

$$\sum_{n=1}^{\infty} \frac{H_n}{n}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) = \sum_{n=1}^{\infty} \frac{H_n}{n}\psi'(n+1),$$

and the quadratic series

$$\sum_{n=1}^{\infty} H_n\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)^2 = \sum_{n=1}^{\infty} H_n(\psi'(n+1))^2.$$

The main result of this paper is the following theorem.

**Theorem 1.** ***Two harmonic series with the tail of*** $\zeta(2)$***.***

*The following identities hold*:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{H_n}{n}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) = \frac{7}{4}\zeta(4);$

(b) $\displaystyle\sum_{n=1}^{\infty} H_n\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)^2 = \frac{19}{4}\zeta(4) - 3\zeta(3).$

We need in our analysis Abel's summation formula ([1, p. 55], [4, p. 258]) which states that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two sequences of real numbers and $A_n = \sum_{k=1}^{n} a_k$, then

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1}). \tag{2}$$

We will also be using in our calculations the infinite version of the preceding formula

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} (A_n b_{n+1}) + \sum_{k=1}^{\infty} A_k (b_k - b_{k+1}). \tag{3}$$

Before we prove the main result of this paper we need the following lemmas.

**Lemma 2.** *Let $n \geq 1$ be an integer. Then,*

(i) $\sum_{k=1}^{n} H_k = (n+1)H_{n+1} - (n+1)$;

(ii) $2\sum_{k=1}^{n} \frac{H_k}{k} = H_n^2 + 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$.

*Proof.* Both parts of this lemma can be proved by induction or by an application of formula (2). For the sake of completeness we prove only part (ii) and leave the proof of part (i) to the interested reader.

We use formula (2), with $a_k = \frac{1}{k}$ and $b_k = H_k$, and we get that

$$\begin{aligned}
\sum_{k=1}^{n} \frac{H_k}{k} &= \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) H_{n+1} - \sum_{k=1}^{n} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) \frac{1}{k+1} \\
&= H_n H_{n+1} - \sum_{k=1}^{n} \frac{1}{k+1} \left(H_{k+1} - \frac{1}{k+1}\right) \\
&= H_n H_{n+1} - \sum_{k=1}^{n} \frac{H_{k+1}}{k+1} + \sum_{k=1}^{n} \frac{1}{(k+1)^2} \\
&= H_n H_{n+1} - \sum_{m=2}^{n+1} \frac{H_m}{m} + \sum_{m=2}^{n+1} \frac{1}{m^2} \\
&= H_n H_{n+1} - \sum_{m=1}^{n} \frac{H_m}{m} - \frac{H_{n+1}}{n+1} + \sum_{m=1}^{n} \frac{1}{m^2} + \frac{1}{(n+1)^2} \\
&= H_n^2 + 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} - \sum_{m=1}^{n} \frac{H_m}{m},
\end{aligned}$$

and Lemma 2 is proved. □

**Lemma 3.** *The following equalities hold:*

(i) $$\sum_{n=1}^{\infty} \frac{1}{n^2}\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)=\frac{7}{4}\zeta(4);$$

(ii) *If $k \geq 2$ is an integer, then*

$$s_k=\sum_{n=1}^{\infty} \frac{1}{n}\left(\zeta(k)-\frac{1}{1^k}-\frac{1}{2^k}-\cdots-\frac{1}{n^k}\right)=\frac{1}{2}k\zeta(k+1)-\frac{1}{2}\sum_{i=1}^{k-2}\zeta(k-i)\zeta(i+1),$$

*where the second sum is missing when $k=2$.*

*Proof.* (i) We apply formula (3), with

$$a_n=\frac{1}{n^2} \text{ and } b_n=1+\frac{1}{2^2}+\cdots+\frac{1}{n^2},$$

and we have

$$\begin{aligned}
S &= \sum_{n=1}^{\infty} \frac{1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}}{n^2} \\
&= \lim_{n\to\infty}\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)\left(1+\frac{1}{2^2}+\cdots+\frac{1}{(n+1)^2}\right) \\
&\quad -\sum_{n=1}^{\infty} \frac{\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}}{(n+1)^2} \\
&= \zeta^2(2)-\sum_{n=1}^{\infty} \frac{1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(n+1)^2}}{(n+1)^2}+\sum_{n=1}^{\infty}\frac{1}{(n+1)^4} \\
&= \zeta^2(2)-\sum_{m=2}^{\infty} \frac{1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{m^2}}{m^2}+\sum_{m=2}^{\infty}\frac{1}{m^4} \\
&= \zeta^2(2)-S+\zeta(4) \\
&= \frac{7}{2}\zeta(4)-S.
\end{aligned}$$

We used that $\zeta^2(2)=\frac{5}{2}\zeta(4)$ since $\zeta(2)=\frac{\pi^2}{6}$ and $\zeta(4)=\frac{\pi^4}{90}$ [6, p. 605].

(ii) We need in the proof of Theorem 1 only the evaluation of $s_2$, however we calculate $s_k$ for the sake of completeness. Exactly as in part (i) we use formula (3), with $a_n = \frac{1}{n}$ and $b_n = \zeta(k) - \frac{1}{1^k} - \frac{1}{2^k} - \cdots - \frac{1}{n^k}$, and we have

$$\begin{aligned}
s_k &= \sum_{n=1}^{\infty} \frac{1}{n}\left(\zeta(k) - \frac{1}{1^k} - \frac{1}{2^k} - \cdots - \frac{1}{n^k}\right)\\
&= \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)\left(\zeta(k) - \frac{1}{1^k} - \cdots - \frac{1}{(n+1)^k}\right) + \sum_{n=1}^{\infty}\frac{H_n}{(n+1)^k}\\
&= \sum_{n=1}^{\infty}\frac{H_{n+1}}{(n+1)^k} - \sum_{n=1}^{\infty}\frac{1}{(n+1)^{k+1}} \quad (n+1=m)\\
&= \sum_{m=1}^{\infty}\frac{H_m}{m^k} - \zeta(k+1),
\end{aligned}$$

and the result follows based on formula (1).

In particular $s_2 = \displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) = \zeta(3)$. □

Now we are ready to prove Theorem 1.

*Proof.* (a) We apply formula (3), with

$$a_n = \frac{H_n}{n} \text{ and } b_n = \zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2},$$

combined to part (ii) of Lemma 2 and we have

$$\begin{aligned}
A &= \sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)\\
&= \lim_{n\to\infty}\left(H_1 + \frac{H_2}{2} + \cdots + \frac{H_n}{n}\right)\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2}\right)\\
&\quad + \sum_{n=1}^{\infty}\frac{H_1 + \frac{H_2}{2} + \cdots + \frac{H_n}{n}}{(n+1)^2}\\
&= \frac{1}{2}\lim_{n\to\infty}\left(H_n^2 + 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right)\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2}\right)\\
&\quad + \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2 + 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}}{(n+1)^2}\\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2 + 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}}{(n+1)^2},
\end{aligned}$$

since

$$\lim_{n\to\infty}\left(H_n^2+1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{(n+1)^2}\right)$$
$$=\lim_{n\to\infty}\frac{H_n^2+1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}}{n+1}\lim_{n\to\infty}(n+1)\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{(n+1)^2}\right)$$
$$=0.$$

On the other hand,

$$H_n^2+1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}=H_{n+1}^2-2\frac{H_{n+1}}{n+1}+1+\frac{1}{2^2}+\cdots+\frac{1}{(n+1)^2}$$

and this implies that

$$\begin{aligned}A&=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{H_{n+1}^2}{(n+1)^2}-2\frac{H_{n+1}}{(n+1)^3}+\frac{1+\frac{1}{2^2}+\cdots+\frac{1}{(n+1)^2}}{(n+1)^2}\right)\\&=\frac{1}{2}\sum_{m=1}^{\infty}\left(\frac{H_m^2}{m^2}-2\frac{H_m}{m^3}+\frac{1+\frac{1}{2^2}+\cdots+\frac{1}{m^2}}{m^2}\right)\\&=\frac{1}{2}\left(\frac{17}{4}\zeta(4)-2\cdot\frac{5}{4}\zeta(4)+\frac{7}{4}\zeta(4)\right)\\&=\frac{7}{4}\zeta(4),\end{aligned}$$

since $\sum_{m=1}^{\infty}\frac{H_m}{m^3}=\frac{5}{4}\zeta(4)$ (see formula (1) with $n=3$) and $\sum_{m=1}^{\infty}\frac{H_m^2}{m^2}=\frac{17}{4}\zeta(4)$.

We mention that the identity $\sum_{m=1}^{\infty}\frac{H_m^2}{m^2}=\frac{17}{4}\zeta(4)$ was discovered numerically by Enrico Au-Yeung and proved rigorously by David Borwein and Jonathan Borwein in [2] who used Fourier series techniques combined to Parseval's formula for proving it, and a recent proof involving integrals of polylogarithm functions was given in [5].

(b) We apply formula (3), with $b_n=\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2$ and $a_n=H_n$, combined with part (i) of Lemma 2 and part (a) of the theorem and we get, since

$$b_n-b_{n+1}=\frac{1}{(n+1)^2}\left[2\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{(n+1)}^2\right)+\frac{1}{(n+1)^2}\right],$$

that

$$\begin{aligned}
B &= \sum_{n=1}^{\infty} H_n \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)^2 \\
&= \lim_{n\to\infty} (H_1 + H_2 + \cdots + H_n) \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2}\right)^2 \\
&+ \sum_{n=1}^{\infty} \frac{H_1 + H_2 + \cdots + H_n}{(n+1)^2} \left[2\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2}\right) + \frac{1}{(n+1)^2}\right] \\
&= \lim_{n\to\infty} ((n+1)H_{n+1} - (n+1)) \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2}\right)^2 \\
&+ \sum_{n=1}^{\infty} \frac{(n+1)H_{n+1} - (n+1)}{(n+1)^2} \left[2\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2}\right) + \frac{1}{(n+1)^2}\right] \\
&= \sum_{n=1}^{\infty} \frac{H_{n+1} - 1}{n+1} \left[2\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2}\right) + \frac{1}{(n+1)^2}\right] \\
&= \sum_{m=2}^{\infty} \frac{H_m - 1}{m} \left[2\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{m^2}\right) + \frac{1}{m^2}\right] \\
&= \sum_{m=1}^{\infty} \frac{H_m - 1}{m} \left[2\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{m^2}\right) + \frac{1}{m^2}\right],
\end{aligned}$$

since

$$\lim_{n\to\infty} ((n+1)H_{n+1} - (n+1)) \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2}\right)^2 = 0.$$

This can be proved by noting that

$$\lim_{n\to\infty} (n+1) \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2}\right) = 1,$$

which implies

$$\begin{aligned}
&\lim_{n\to\infty} ((n+1)H_{n+1} - (n+1)) \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2}\right)^2 \\
&= \lim_{n\to\infty} \frac{H_{n+1} - 1}{n+1} \cdot \lim_{n\to\infty} (n+1)^2 \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n+1)^2}\right)^2 = 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
B &= 2\sum_{m=1}^{\infty}\frac{H_m}{m}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{m^2}\right)+\sum_{m=1}^{\infty}\frac{H_m}{m^3}\\
&\quad -2\sum_{m=1}^{\infty}\frac{1}{m}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{m^2}\right)-\sum_{m=1}^{\infty}\frac{1}{m^3}\\
&= 2A+\frac{5}{4}\zeta(4)-2s_2-\zeta(3)\\
&= \frac{7}{2}\zeta(4)+\frac{5}{4}\zeta(4)-2\zeta(3)-\zeta(3)\\
&= \frac{19}{4}\zeta(4)-3\zeta(3),
\end{aligned}$$

and the theorem is proved. □

## References


[1] D. D. Bonar, M. J. Koury, *Real infinite series*, MAA Washington DC, 2006.

[2] D. Borwein, J. M. Borwein, On an intriguing integral and some series related to $\zeta(4)$, *Proc. Amer. Math. Soc.* **123** (1995), 1191–1198.

[3] J. Choi, H. M. Srivastava, Explicit evaluations of Euler and related sums, *Ramanujan J.* **10** (2005), 51–70.

[4] O. Furdui, *Limits, Series and fractional part integrals. Problems in mathematical analysis*, Springer New York, 2013.

[5] O. Furdui, Series involving products of two harmonic numbers, *Math. Mag.* **84** (2011), 371–377

[6] F. W. J. Olver (ed.), D. W. Lozier (ed.), R. F. Boisvert (ed.), C. W. Clark (ed.), *NIST Handbook of mathematical functions*, Cambridge University Press, Cambridge, 2010.

[7] H. M. Srivastava, J. Choi, *Series associated with the zeta and related functions*, Kluwer Academic Publishers, 2001.

[8] H. M. Srivastava, J. Choi, *Zeta and q-zeta functions and associated series and integrals*, Elsevier, 2012.

[9] E. T. Whittaker, G.N. Watson, *A course of modern analysis*, Cambridge University Press, 1927.

# Limits of multiple integrals over hyperbola domains

Dumitru Popa[1)]

**Abstract**. In this paper we study the asymptotic behavior of some multiple integrals on the hyperbola domains $H_a^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_1 \geq 1,\dots,x_n \geq 1,\ x_1 \cdots x_n \leq a\}$, $a > 1$.

**Keywords:** Riemann integral, multiple Riemann integral, limit of sequences of multiple integrals.

**MSC :** 26B15, 28A35.

## 1. Introduction

For every positive integer $n$ and every real $a > 1$ we call the set $H_a^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_1 \geq 1,\dots,x_n \geq 1,\ x_1 \cdots x_n \leq a\}$ a *domain limited by equilateral hyperbolas* or a *hyperbola domain*. Let us note that in the case $n = 2$ we get $H_a^2 = \left\{(x,y) \in \mathbb{R}^2 \mid x \geq 1,\ y \geq 1,\ xy \leq a\right\}$ which is a domain limited by $x = 1$, $y = 1$ and the equilateral hyperbola $xy = a$. By $\lambda_n$ we denote the Lebesgue measure on $\mathbb{R}^n$. The main purpose of this paper is to study the asymptotic behavior of some multiple integrals on $H_a^n$ and the asymptotic behavior of the $n$-dimensional Lebesgue measure of the set $H_a^n$. The notation and concepts used in this paper are standard. For details regarding the multiple Riemann integral we recommend the book [2].

## 2. Preliminary results

We begin by proving the following result. For results of different type we invite the reader to consult [4, Problems 4.1–4.3, page 98].

**Proposition 1.** *Let $a > 1$ and $h : [1,a] \to \mathbb{R}$ be a continuous function. Then:*

a) $\displaystyle\int_1^{a/x} h(xt)\,\mathrm{d}t = \frac{1}{x}\int_x^a h(u)\,\mathrm{d}u$ *for all $x \geq 1$.*

b) *For all positive integers $n$ the following equality holds*

$$\int_{H_a^n} h(x_1 \cdots x_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_n = \frac{1}{(n-1)!}\int_1^a h(x)\ln^{n-1} x\,\mathrm{d}x.$$

*Proof.* a) With the change of variables $xt = u$, $\mathrm{d}t = \frac{1}{x}\mathrm{d}u$ we get the stated equality.

b) We prove the result by induction on $n$. The case $n = 1$ is clear. Let us suppose that the equality holds for $n \geq 1$ and prove that it holds for $n+1$.

[1)]Department of Mathematics, Ovidius University of Constanta, Bd. Mamaia 124, 900527 Constanţa, Romania, `dpopa@univ-ovidius.ro`

Since

$$\begin{aligned} H_a^{n+1} &= \left\{(x_1,\ldots,x_{n+1})\in\mathbb{R}^{n+1} \mid x_1\geq 1,\ldots,x_{n+1}\geq 1,\, x_1\cdots x_{n+1}\leq a\right\} \\ &= \left\{(x_1,\ldots,x_{n+1})\in\mathbb{R}^{n+1} \mid (x_1,\ldots,x_n)\in H_a^n,\, 1\leq x_{n+1}\leq \tfrac{a}{x_1\cdots x_n}\right\} \end{aligned}$$

by a well-known result, see [2, Exercise 10, page 247 ], or [3, Theorem 6.45, page 438], we have

$$\begin{aligned} &\int_{H_a^{n+1}} h(x_1\cdots x_{n+1})\mathrm{d}x_1\cdots \mathrm{d}x_{n+1} = \\ &= \int_{H_a^n}\left(\int_1^{a/(x_1\cdots x_n)} h(x_1\cdots x_{n+1})\mathrm{d}x_{n+1}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n. \end{aligned}$$

Now by part a) we have

$$\int_1^{a/(x_1\cdots x_n)} h(x_1\cdots x_n x_{n+1})\mathrm{d}x_{n+1} = \frac{1}{x_1\cdots x_n}\int_{x_1\cdots x_n}^a h\left(t\right)\mathrm{d}t = g(x_1\cdots x_n),$$

where $g:[1,a]\to\mathbb{R}$, $g(t)=t^{-1}\int_t^a h(x)\mathrm{d}x$. We may write $g(t)=t^{-1}v(t)$, where $v:[1,a]\to\mathbb{R}$, $v(t)=\int_t^a h(x)\mathrm{d}x$, and $v'(t)=-h(t)$ for all $t\in[1,a]$, $h$ being continuous. Now we have

$$\int_{H_a^{n+1}} h(x_1\cdots x_{n+1})\mathrm{d}x_1\cdots\mathrm{d}x_{n+1} = \int_{H_a^n} g(x_1\cdots x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n.$$

By the inductive hypothesis, we have

$$\begin{aligned} \int_{H_a^n} g(x_1\cdots x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n &= \frac{1}{(n-1)!}\int_1^a g\left(x\right)\ln^{n-1}x\mathrm{d}x \\ &= \frac{1}{(n-1)!}\int_1^a \frac{1}{x}\left(\ln^{n-1}x\right)v\left(x\right)\mathrm{d}x \end{aligned}$$

and this implies

$$\int_{H_a^{n+1}} h(x_1\cdots x_{n+1})\mathrm{d}x_1\cdots\mathrm{d}x_{n+1} = \frac{1}{(n-1)!}\int_1^a \frac{1}{x}\left(\ln^{n-1}x\right)v\left(x\right)\mathrm{d}x.$$

Now integrating by parts

$$\begin{aligned} \int_1^a \frac{1}{x}\left(\ln^{n-1}x\right)v\left(x\right)\mathrm{d}x &= \frac{1}{n}\left(\ln^n x\right)v\left(x\right)\Big|_1^a - \frac{1}{n}\int_1^a\left(\ln^n x\right)v'\left(x\right)\mathrm{d}x \\ &= \frac{1}{n}\int_1^a\left(\ln^n x\right)h\left(x\right)\mathrm{d}x \end{aligned}$$

we get

$$\int_{H_a^{n+1}} h(x_1\cdots x_{n+1})\mathrm{d}x_1\cdots\mathrm{d}x_{n+1} = \frac{1}{n!}\int_1^a h(x)\ln^n x\mathrm{d}x.$$

□

**Corollary 2.** *For all positive integers $n$ and all $a > 1$ it holds*

$$\lambda_n\left(H_a^n\right) = \frac{1}{(n-1)!}\int_1^a \ln^{n-1} x\mathrm{d}x.$$

*Proof.* From Proposition 1 we have

$$\lambda_n\left(H_a^n\right) = \int_{H_a^n} 1\,\mathrm{d}x_1\ldots\mathrm{d}x_n = \frac{1}{(n-1)!}\int_1^a \ln^{n-1} x\mathrm{d}x.$$

The integral $I_{n-1} = \int_1^a \ln^{n-1} x\mathrm{d}x$ can be calculated based on the recurrence formula $I_{n-1} = a\ln^{n-1} a - (n-1)I_{n-2}$ starting with the initial value $I_0 = a - 1$. □

## 3. The main results

We need later the following result which is of independent interest.

**Proposition 3.** (i) *Let $a > 1$ and $h : [1, a] \to \mathbb{R}$ be a Riemann integrable function. Then for all positive integers $n$ we have*

$$\lim_{a\to\infty} \frac{\int_1^a h\left(\frac{x}{a}\right)\ln^{n-1} x\mathrm{d}x}{a\left(\ln a\right)^{n-1}} = \int_0^1 h\left(t\right)\mathrm{d}t.$$

(ii) *Let $a > 1$ and $h : [1, a] \to \mathbb{R}$ be a continuous function. Then for all positive integers $n$ we have*

$$\lim_{a\to 1,\ a>1} \frac{\int_1^a h\left(\frac{x}{a}\right)\ln^{n-1} x\mathrm{d}x}{\left(a-1\right)^n} = \frac{h\left(1\right)}{n}.$$

(iii) *Let $a > 1$ and $h : [1, a] \to \mathbb{R}$ be a continuous function. Then*

$$\lim_{n\to\infty} \frac{n\int_1^a h\left(\frac{x}{a}\right)\ln^{n-1} x\mathrm{d}x}{\left(\ln a\right)^n} = ah\left(1\right).$$

*Proof.* (i) With the change of variables $\frac{x}{a} = t$, $\mathrm{d}x = a\mathrm{d}t$, we deduce

$$\begin{aligned}\int_1^a h\left(\frac{x}{a}\right)\ln^{n-1} x\mathrm{d}x &= a\int_{\frac{1}{a}}^1 h\left(t\right)\left[\ln\left(ta\right)\right]^{n-1}\mathrm{d}t\\ &= a\left(\ln a\right)^{n-1}\int_{\frac{1}{a}}^1 h\left(t\right)\left(1+\frac{\ln t}{\ln a}\right)^{n-1}\mathrm{d}t\end{aligned}$$

and it follows that

$$\frac{\int_1^a h\left(\frac{x}{a}\right)\ln^{n-1} x\mathrm{d}x}{a\left(\ln a\right)^{n-1}} = \int_{\frac{1}{a}}^1 h\left(t\right)\left(1+\frac{\ln t}{\ln a}\right)^{n-1}\mathrm{d}t. \tag{3.1}$$

For $a > 1$ let $f_a : [0, 1] \to \mathbb{R}$, $f_a(t) = \begin{cases} h\left(t\right)\left(1+\frac{\ln t}{\ln a}\right)^{n-1} & \text{if} \quad a^{-1} \le t \le 1,\\ 0 & \text{if} \quad 0 \le t < a^{-1}.\end{cases}$
We prove that for all $t \in (0, 1]$, $\lim\limits_{a\to\infty} f_a\left(t\right) = h(t)$. Indeed, let $0 < t \le 1$. Since

$\lim\limits_{a\to\infty}\frac{1}{a}=0$, there exists $a_t\geq 1$ such that $\frac{1}{a}\leq t$, $\forall a\geq a_t$. Thus for all $a\geq a_t$ we have $f_a(t)=h(t)\left(1+\frac{\ln t}{\ln a}\right)^{n-1}$and passing to the limit for $a\to\infty$ we deduce $\lim\limits_{a\to\infty}f_a(t)=h(t)$. Let $a>1$. For all $\frac{1}{a}\leq t\leq 1$ we have $-1\leq\frac{\ln t}{\ln a}\leq 0$, $0\leq\left(1+\frac{\ln t}{\ln a}\right)^{n-1}\leq 1$, and thus $|f_a(t)|\leq|h(t)|$, $\forall t\in[0,1]$. By Lebesgue Dominated Convergence Theorem it follows that $\lim\limits_{a\to\infty}\int_0^1 f_a(t)\mathrm{d}t=\int_0^1 h(t)\mathrm{d}t$, i.e.,

$$\lim_{a\to\infty}\int_{\frac{1}{a}}^1 h(t)\left(1+\frac{\ln t}{\ln a}\right)^{n-1}\mathrm{d}t=\int_0^1 h(t)\,\mathrm{d}t. \tag{3.2}$$

From relations (3.1) and (3.2) we get the statement.

(ii) Let us define $F_n:(1,\infty)\to\mathbb{R}$, $F_n(a)=\int_1^a h\left(\frac{x}{a}\right)\ln^{n-1}x\mathrm{d}x$. We must prove that

$$\lim_{a\to 1,\ a>1}\frac{F_n(a)}{(a-1)^n}=\frac{h(1)}{n}.$$

By the change of variables $\frac{x}{a}=u$, $\mathrm{d}x=a\mathrm{d}u$ and the Newton binomial formula we deduce

$$\begin{aligned}F_n(a)&=\int_1^a h\left(\frac{x}{a}\right)\ln^{n-1}x\mathrm{d}x=a\int_{\frac{1}{a}}^1 h(u)\ln^{n-1}(au)\,\mathrm{d}u\\&=a\int_{\frac{1}{a}}^1 h(u)(\ln u+\ln a)^{n-1}\mathrm{d}u\\&=a\sum_{k=0}^{n-1}C_{n-1}^k(\ln a)^k\int_{\frac{1}{a}}^1 h(u)(\ln u)^{n-1-k}\,\mathrm{d}u.\end{aligned}$$

Then

$$\begin{aligned}\lim_{a\to 1,\ a>1}\frac{F_n(a)}{(a-1)^n}&=\sum_{k=0}^{n-1}C_{n-1}^k\lim_{a\to 1,\ a>1}\frac{a(\ln a)^k}{(a-1)^k}\cdot\frac{\int_{\frac{1}{a}}^1 h(u)(\ln u)^{n-1-k}\,\mathrm{d}u}{(a-1)^{n-k}}\\&=\sum_{k=0}^{n-1}\frac{C_{n-1}^k}{n-k}(-1)^{n-k-1}=\frac{1}{n}.\end{aligned}$$

Above we have used that if $m\in\mathbb{N}\cup\{0\}$ and $h:[0,1]\to\mathbb{R}$ is continuous then, by L'Hôpital rule,

$$\lim_{a\to 1,\ a>1}\frac{\int_{\frac{1}{a}}^1 h(u)(\ln u)^m\,\mathrm{d}u}{(a-1)^{m+1}}=\lim_{a\to 1,\ a>1}\frac{\frac{1}{a^2}h\left(\frac{1}{a}\right)\left(\ln\frac{1}{a}\right)^m}{(m+1)(a-1)^m}=\frac{(-1)^m h(1)}{m+1}.$$

The equality $\sum\limits_{k=0}^{n-1} \frac{C_{n-1}^k}{n-k} (-1)^{n-k-1} = \frac{1}{n}$ follows from the Newton binomial formula $(1-x)^{n-1} = \sum\limits_{k=0}^{n-1} C_{n-1}^k x^{n-k-1} (-1)^{n-k-1}$, $\forall x \in \mathbb{R}$, and by integration

$$\sum_{k=0}^{n-1} C_{n-1}^k (-1)^{n-k-1} \int_0^1 x^{n-k-1} \mathrm{d}x = \int_0^1 (1-x)^{n-1} \mathrm{d}x.$$

(iii) With the change of variables $\frac{\ln x}{\ln a} = t$, $x = a^t$, $\mathrm{d}x = a^t \ln a \, \mathrm{d}t$ we deduce

$$\int_1^a h\left(\frac{x}{a}\right) \ln^{n-1} x \mathrm{d}x = (\ln a)^n \int_0^1 t^{n-1} h\left(a^{t-1}\right) a^t \mathrm{d}t$$

and this implies

$$\frac{n \int_1^a h\left(\frac{x}{a}\right) \ln^{n-1} x \mathrm{d}x}{(\ln a)^n} = n \int_0^1 t^{n-1} h\left(a^{t-1}\right) a^t \mathrm{d}t.$$

Since by a well-known result, see, e.g., [6, Exercise 3.4, page 51],

$$\lim_{n\to\infty} n \int_0^1 t^{n-1} h\left(a^{t-1}\right) a^t \mathrm{d}t = a h(1),$$

we get the statement. □

In the next proposition, which is the main result of this paper, we study the asymptotic behavior of some multiple integrals on $H_a^n$.

**Proposition 4.** a) *Let $n$ be a positive integer. For each Riemann integrable function $h : [0,1] \to \mathbb{R}$ the following equality holds*

$$\lim_{a\to\infty} \frac{\int_{H_a^n} h\left(\frac{x_1\cdots x_n}{a}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n}{a (\ln a)^{n-1}} = \frac{1}{(n-1)!} \int_0^1 h(x) \mathrm{d}x.$$

b) *Let $n$ be a positive integer. For each continuous function $h : [0,1] \to \mathbb{R}$ the following equality holds*

$$\lim_{a\to 1,\ a>1} \frac{\int_{H_a^n} h\left(\frac{x_1\cdots x_n}{a}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n}{(a-1)^n} = \frac{1}{n!} h(1).$$

c) *Let $a > 1$. For each continuous function $h : [0,1] \to \mathbb{R}$ the following equality holds*

$$\lim_{n\to\infty} \frac{n^n \sqrt{n}}{e^n (\ln a)^n} \int_{H_a^n} h\left(\frac{x_1 \cdots x_n}{a}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n = \frac{a}{\sqrt{2\pi}} h(1)$$

*where $e$ is the base of the natural logarithm.*

*Proof.* Let $h:[0,1]\to\mathbb{R}$ be a continuous function. From Proposition 1 applied to $f:[1,a]\to\mathbb{R}$, $f(t)=h\left(\frac{t}{a}\right)$, we have

$$\int_{H_a^n} h\left(\frac{x_1\cdots x_n}{a}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n=\frac{1}{(n-1)!}\int_1^a h\left(\frac{x}{a}\right)\ln^{n-1}x\mathrm{d}x. \tag{3.3}$$

a) **The case of continuous function**. From the relation (3.2) and Proposition 3(i) we deduce

$$\lim_{a\to\infty}\frac{\int_{H_a^n} h\left(\frac{x_1\cdots x_n}{a}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n}{a(\ln a)^{n-1}}=$$

$$=\frac{1}{(n-1)!}\lim_{a\to\infty}\frac{1}{a(\ln a)^{n-1}}\int_1^a h\left(\frac{x}{a}\right)\ln^{n-1}x\mathrm{d}x=\frac{1}{(n-1)!}\int_0^1 h(x)\,\mathrm{d}x.$$

**The case of Riemann integrable function**. Let $h:[0,1]\to\mathbb{R}$ be a Riemann integrable function. Let $\varepsilon>0$. From a well-known result, there exist two continuous functions $\varphi,\psi:[0,1]\to\mathbb{R}$ such that $\varphi(x)\le h(x)\le\psi(x)$, $\forall x\in[0,1]$, and $\int_0^1(\psi(x)-\varphi(x))\,\mathrm{d}x\le\varepsilon$; for a proof of this result see [1, Lemma 1] and for an $n$-dimensional version see [3, Theorem 6.8.2, page 456]. Then

$$\begin{aligned}\frac{\int_{H_a^n}\varphi\left(\frac{x_1\cdots x_n}{a}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n}{a(\ln a)^{n-1}}&\le\frac{\int_{H_a^n}h\left(\frac{x_1\cdots x_n}{a}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n}{a(\ln a)^{n-1}}\\&\le\frac{\int_{H_a^n}\psi\left(\frac{x_1\cdots x_n}{a}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n}{a(\ln a)^{n-1}}.\end{aligned}$$

Passing to the limit as $a\to\infty$ and using the preceding case ($\varphi$ and $\psi$ are continuous) we obtain

$$\begin{aligned}\frac{1}{(n-1)!}\int_0^1\varphi(x)\mathrm{d}x&\le\liminf_{a\to\infty}\frac{\int_{H_a^n}h\left(\frac{x_1\cdots x_n}{a}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n}{a(\ln a)^{n-1}}\\&\le\limsup_{a\to\infty}\frac{\int_{H_a^n}h\left(\frac{x_1\cdots x_n}{a}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n}{a(\ln a)^{n-1}}\\&\le\frac{1}{(n-1)!}\int_0^1\psi(x)\mathrm{d}x.\end{aligned}$$

Now since

$$\int_0^1\psi(x)\mathrm{d}x\le\int_0^1\big(\psi(x)-\varphi(x)\big)\mathrm{d}x+\int_0^1\varphi(x)\mathrm{d}x\le\varepsilon+\int_0^1 h(x)\mathrm{d}x$$

and similarly

$$\int_0^1\varphi(x)\mathrm{d}x\ge\int_0^1\psi(x)\mathrm{d}x-\int_0^1\big(\psi(x)-\varphi(x)\big)\mathrm{d}x\ge\int_0^1 h(x)\mathrm{d}x-\varepsilon,$$

we obtain

$$\begin{aligned}\frac{1}{(n-1)!}\left(\int_0^1 h(x)\mathrm{d}x-\varepsilon\right) &\le \liminf_{a\to\infty}\frac{\int_{H_a^n} h\left(\frac{x_1\cdots x_n}{a}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n}{a\left(\ln a\right)^{n-1}}\\ &\le \limsup_{a\to\infty}\frac{\int_{H_a^n} h\left(\frac{x_1\cdots x_n}{a}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n}{a\left(\ln a\right)^{n-1}}\\ &\le \frac{1}{(n-1)!}\left(\int_0^1 h(x)\mathrm{d}x+\varepsilon\right).\end{aligned}$$

Since $\varepsilon>0$ is arbitrarily taken, passing to the limit as $\varepsilon\to 0$, $\varepsilon>0$, we obtain

$$\begin{aligned}\liminf_{a\to\infty}\frac{\int_{H_a^n} h\left(\frac{x_1\cdots x_n}{a}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n}{a\left(\ln a\right)^{n-1}} &= \limsup_{a\to\infty}\frac{\int_{H_a^n} h\left(\frac{x_1\cdots x_n}{a}\right)\mathrm{d}x_1\cdots\mathrm{d}x_n}{a\left(\ln a\right)^{n-1}}\\ &= \frac{1}{(n-1)!}\int_0^1 h\left(x\right)\mathrm{d}x.\end{aligned}$$

which ends the proof of a).

b) It follows from relation (3.3) and part (ii) of Proposition 3.

c) From relation (3.3) we deduce

$$\frac{n!\int_{H_a^n} h\left(\frac{x_1\cdots x_n}{a}\right)\mathrm{d}x_1\ldots\mathrm{d}x_n}{\left(\ln a\right)^n}=\frac{n\int_1^a h\left(\frac{x}{a}\right)\ln^{n-1}x\mathrm{d}x}{\left(\ln a\right)^n}$$

for all positive integers $n$. The limit from the statement follows from Stirling's formula $n!\sim\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n$ and part (iii) of Proposition 3. □

Taking $h\left(x\right)=1$ in Proposition 4 we get the asymptotic behavior of the $n$-dimensional Lebesgue measure of the set $H_a^n$. For the results of different type we refer the reader to [4, problems 6.17, page 106 and 6.23, 6.25, page 107] or [5, Example 4, page 229].

**Corollary 5.** a) *Let $n$ be a positive integer. The following evaluations hold:*

$$\lim_{a\to\infty}\frac{\lambda_n(H_a^n)}{a\left(\ln a\right)^{n-1}}=\frac{1}{(n-1)!},$$
$$\lim_{a\to 1,\ a>1}\frac{\lambda_n(H_a^n)}{(a-1)^n}=\frac{1}{n!}.$$

b) *Let $a>1$. The following evaluation holds*

$$\lim_{n\to\infty}\frac{n^n\sqrt{n}}{e^n\left(\ln a\right)^n}\lambda_n\left(H_a^n\right)=\frac{a}{\sqrt{2\pi}}.$$

## References

[1] M. Bănescu, D. Popa, New extensions of some classical theorems in number theory, *J. Number Theory* **133** (2013), 3771–3795.
[2] N. Boboc, *Analiză matematică*, partea a II-a, Bucureşti, 1993.
[3] J. J. Duistermaat, J. A. C. Kolk, *Multidimensional real analysis I: Differentiation*, Cambridge Studies in Advanced Mathematics 86, Cambridge University Press, 2004.
[4] B. M. Makarov, M. G. Goluzina, A.A. Lodkin, A. N. Podkoryotov, *Selected problems in real analysis*, Transl. Math. Monographs, 107, A.M.S., 1992.
[5] B. Makarov, A. Podkoryotov, *Real analysis: measures, integrals and applications*, Universitext. London, Springer, 2013.
[6] D. Popa, *Exerciţii de analiză matematică*, Biblioteca S. S. M. R., Editura Mira, Bucureşti, 2007.

# Inegalităţi de tip Chebyshev-Grüss pentru operatorii Bernstein-Euler-Jacobi

Heiner Gonska[1)], Maria-Daniela Rusu[2)],
Elena-Dorina Stănilă[3)]

**Abstract**. The classical form of Grüss' inequality was first published by G. Grüss and gives an estimate of the difference between the integral of the product and the product of the integrals of two functions. In the subsequent years, many variants of this inequality appeared in the literature. The aim of this paper is to consider some Chebyshev-Grüss-type inequalities and apply them to the Bernstein-Euler-Jacobi (BEJ) operators of first and second kind. The first and second moments of the operators will be of great interest.

**Keywords:** Chebyshev functional, Chebyshev-type inequality, Grüss-type inequality, Chebyshev-Grüss-type inequalities, Bernstein-Euler-Jacobi (BEJ) operators of first and second kind, first and second moments.
**MSC :** 41A17, 41A36, 26D15.

## 1. Introducere

În cele ce urmează vom prezenta rezultate clasice din literatură.

Inegalităţile de tip Chebyshev-Grüss au fost intens studiate de-a lungul anilor, mai ales datorită numeroaselor aplicaţii. Ele reprezintă legătura între funcţionala lui Chebyshev şi inegalitatea lui G. Grüss.

Funcţionala descrisă de

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \cdot \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x,$$

1) University of Duisburg-Essen, Faculty of Mathematics, heiner.gonska@uni-due.de
2) University of Duisburg-Essen, Faculty of Mathematics, maria.rusu@uni-due.de
3) University of Duisburg-Essen, Faculty of Mathematics,
elena.stanila@stud.uni-due.de

unde $f, g : [a, b] \to \mathbb{R}$ sunt funcţii integrabile, este binecunoscută în literatură drept funcţionala clasică a lui Chebyshev. Pentru detalii, articolul [3] poate fi de ajutor.

Un prim rezultat pe care îl readucem în atenţia cititorului este dat de următoarea teoremă.

**Teorema 1.** (vezi [16]) *Fie $f, g : [a, b] \to \mathbb{R}$ două funcţii mărginite şi integrabile, ambele crescătoare sau descrescătoare. Mai mult, fie $p : [a, b] \to \mathbb{R}_0^+$ o funcţie mărginită şi integrabilă. Atunci avem*

$$\int_a^b p(x)\mathrm{d}x \int_a^b p(x)f(x)g(x)\mathrm{d}x \geq \int_a^b p(x)f(x)\mathrm{d}x \int_a^b p(x)g(x)\mathrm{d}x. \tag{1.1}$$

*Dacă una din funcţiile $f$ sau $g$ este crescătoare şi cealaltă este descrescătoare, atunci inegalitatea (1.1) se inversează.*

**Remarca 2.** Relaţia (1.1) a fost introdusă pentru prima oară de către P. L. Chebyshev în 1882 (vezi [2]). Din acest motiv, este cunoscută sub numele de inegalitatea lui Chebyshev.

În continuare amintim unul din rezultatele esenţiale pe care se bazează cercetarea noastră, şi anume inegalitatea lui Grüss pentru funcţionala lui Chebyshev.

**Teorema 3.** (vezi [11]) *Fie $f, g$ funcţii integrabile, definite pe intervalul $[a, b]$ cu valori în $\mathbb{R}$, astfel încât $m \leq f(x) \leq M$, $p \leq g(x) \leq P$, pentru orice $x \in [a, b]$, unde $m, M, p, P \in \mathbb{R}$. Atunci are loc inegalitatea*

$$|T(f, g)| \leq \frac{1}{4}(M - m)(P - p). \tag{1.2}$$

Următoarele inegalităţi de tip Chebyshev-Grüss vor fi folosite în continuare, pentru a introduce rezultatele noastre.

**Teorema 4.** (vezi [21]) *Dacă $f, g \in C[a, b]$ şi $x \in [a, b]$ este fixat, atunci are loc inegalitatea*

$$|T(f, g; x)| \leq \frac{1}{4}\widetilde{\omega}\left(f; 2\sqrt{H((e_1 - x)^2; x)}\right)\widetilde{\omega}\left(g; 2\sqrt{H((e_1 - x)^2; x)}\right). \tag{1.3}$$

**Remarca 5.** Rezultatul de mai sus implică folosirea celui mai mic majorant concav $\widetilde{\omega}$ al primului modul de continuitate. O definiţie şi detalii cu privire la acesta se găsesc, de exemplu, în [8].

**Remarca 6.** Scopul nostru este să aplicăm inegalitatea de mai sus în cazul operatorilor BEJ de tipul $I$ şi $II$, pentru diferite cazuri, luând în considerare diferitele momente de ordinul doi.

În cazul operatorilor liniari şi pozitivi care reproduc funcţiile constante, dar nu le reproduc pe cele liniare, avem următorul rezultat.

**Teorema 7.** (vezi Corolarul 5.1 din [8]) *Dacă* $H : C[a,b] \to C[a,b]$ *este un operator liniar şi pozitiv care reproduce funcţii constante, atunci pentru* $f, g \in C[a,b]$ *şi* $x \in [a,b]$ *fixat, au loc următoarele inegalităţi:*

$$|T(f,g;x)| \leq \frac{1}{4}\widetilde{\omega}\left(f; 2\sqrt{H(e_2;x) - H(e_1;x)^2}\right)\widetilde{\omega}\left(g; 2\sqrt{H(e_2;x) - H(e_1;x)^2}\right)$$
$$\leq \frac{1}{4}\widetilde{\omega}\left(f; 2\sqrt{H((e_1 - x)^2;x)}\right)\widetilde{\omega}\left(g; 2\sqrt{H((e_1 - x)^2;x)}\right). \qquad (1.4)$$

**Remarca 8.** Pentru a aplica inegalitatea de mai sus unora din cazurile operatorilor BEJ de tipul $I$ sau $II$, avem în primul rând nevoie de momentele de ordinul întâi. Apoi trebuie să calculăm diferenţe de tipul $T(e_1, e_1; x) := H(e_2;x) - H(e_1;x)^2$. Dacă operatorul $H$ nu reproduce funcţiile liniare, se obţine o îmbunătăţire a inegalităţii (1.3).

Operatorii Bernstein-Euler-Jacobi (BEJ) sunt împărţiţi în două clase: BEJ de tip $I$ şi BEJ de tip $II$. Scopul acestui articol este de a aplica inegalităţile de tip Chebyshev-Grüss de mai sus acestor tipuri de operatori. În rezultatele următoare momentele de ordinul unu şi cele de ordinul doi vor fi de mare interes.

Operatorii BEJ de tipul $I$, ca şi clasă de operatori pozitivi şi liniari, pot fi definiţi după cum urmează. Pentru mai multe detalii, vezi [23].

**Definiţia 9.** *Definim* $R_{m,n}^{(r,a,b)} : C[0,1] \to C[0,1]$ *ca fiind*

$$R_{m,n}^{(r,a,b)} = B_m \circ \mathcal{B}_r^{a,b} \circ B_n,$$

*pentru* $r > 0$, $a, b \geq -1$, $n, m > 1$. *În ecuaţia de mai sus,* $\mathcal{B}_r^{a,b}$ *este operatorul Euler-Jacobi-Beta definit în* [9] *şi* $B_n, B_m$ *sunt operatorii Bernstein de ordin* $n$ *şi* $m$.

**Remarca 10.** Operatorii BEJ de tip $I$ reproduc constantele. Pentru anumite valori ale lui $a$, respectiv $b$, şi anume pentru $a = b = -1$, sunt reproduse şi funcţiile liniare.

A doua clasă de operatori liniari şi pozitivi pe care îi considerăm sunt BEJ de tipul $II$. Definiţia este dată în continuare.

**Definiţia 11.** *Pentru* $r, s > 0$, $a, b, c, d \geq -1$ *şi* $n > 1$, *definim* $R_n^{s,c,d;r,a,b} : C[0,1] \to C[0,1]$ *ca fiind*

$$R_n^{s,c,d;r,a,b} = \mathcal{B}_s^{c,d} \circ B_n \circ \mathcal{B}_r^{a,b}.$$

$\mathcal{B}_r^{a,b}$ *şi* $\mathcal{B}_s^{c,d}$ *sunt operatori de tip Euler-Jacobi Beta şi* $B_n$ *este operatorul Bernstein de ordin* $n$.

**Remarca 12.** Operatorii BEJ de tip $II$ reproduc constantele. Pentru anumite valori ale lui $a, b$ şi $c, d$, şi anume pentru $a = b = c = d = -1$, sunt reproduse şi funcţiile liniare.

## 2. Momentele de ordinul unu şi doi

**Lema 13.** *Pentru clasa de operatori BEJ de tipul I momentele de ordinul 1 şi 2 sunt date de:*

$$R_{m,n}^{(r,a,b)}((e_1-xe_0)^1;x)=\frac{a+1-x(a+b+2)}{r+a+b+2}. \tag{2.1}$$

$$\begin{aligned}&R_{m,n}^{(r,a,b)}((e_1-xe_0)^2;x)=\\&=\frac{x^2[mn(a^2+b^2+5a+5b+2ab+6-r)+r^2(1-m-n)]}{mn(r+a+b+2)(r+a+b+3)}-\\&-\frac{x[mn(2a^2+2ab+8a+2b+6-r)+r^2(1-m-n)+mr(a-b)]}{mn(r+a+b+2)(r+a+b+3)}+\\&+\frac{[mn(a+1)(a+2)+m(r(a+1)+ab+a+b+1)]}{mn(r+a+b+2)(r+a+b+3)}.\end{aligned} \tag{2.2}$$

Pentru detalii în ceea ce priveşte demonstraţia, vezi [23].

**Lema 14.** *Pentru clasa de operatori BEJ de tipul II momentele de ordinul 1 şi 2 sunt date de:*

$$\begin{aligned}&R_n^{(s,c,d;r,a,b)}((e_1-xe_0)^1;x)=\\&=\frac{-x\,[r(c+d+2)+(a+b+2)(s+c+d+2)]}{(r+a+b+2)(s+c+d+2)}+\\&+\frac{r(c+1)+(s+c+d+2)(a+1)}{(r+a+b+2)(s+c+d+2)}.\end{aligned} \tag{2.3}$$

$$\begin{aligned}&R_n^{(s,c,d;r,a,b)}((e_1-xe_0)^2;x)=\\&=\frac{(n-1)((a+b)^2+2ab+5a+5b+6-r)(sx+c+1)(sx+c+2)}{n(r+a+b+2)(r+a+b+3)(s+c+d+2)(s+c+d+3)}+\\&+\frac{((a+b)^2+5a+5b+6-r-n(2ab+2a^2+8a+2b+6-r))(sx+c+1)}{n(r+a+b+2)(r+a+b+3)(s+c+d+2)}\\&+\frac{a^2+3a+2}{(r+a+b+2)(r+a+b+3)}+\frac{(sx+c+1)(sx+c+2)}{n(s+c+d+2)(s+c+d+3)}-\\&-\frac{sx+c+1}{n(s+c+d+2)}-\frac{c^2+d^2+2cd+5c+5d+6-s}{(s+c+d+2)(s+c+d+3)}x^2+\\&+\frac{2cd+2c^2+8c+2d+6-s}{(s+c+d+2)(s+c+d+3)}x-\frac{c^2+3c+2}{(s+c+d+2)(s+c+d+3)}+\end{aligned}$$

$$
\begin{aligned}
&+\frac{2r(sx+c+1)}{n(r+a+b+2)(s+c+d+2)}-\\
&-\frac{2r(sx+c+1)(sx+c+2)}{n(r+a+b+2)(s+c+d+2)(s+c+d+3)}+\\
&+\frac{2r(sx+c+1)(sx+c+2)}{(r+a+b+2)(s+c+d+2)(s+c+d+3)}+\\
&+\frac{2(a+1-x(2r+a+b+2))(sx+c+1)}{(r+a+b+2)(s+c+d+2)}-\frac{2(a+1)x}{r+a+b+2}+2x^2.
\end{aligned}
\tag{2.4}
$$

Pentru detalii în ceea ce priveşte demonstraţia, vezi [23].

Mulţi operatori uzuali pot fi regăsiţi dacă particularizăm valorile indicilor. Astfel în primele cinci tabele prezentate in Anexă regăsim momentele de ordinul doi, pentru cazurile particulare pe care am reuşit să le localizăm în literatură, determinate pe baza ecuaţiilor (2.2) şi (2.4), folosind convenţia $B_\infty = \mathcal{B}_\infty^{a,b} = \mathrm{Id}$.

## 3. Inegalităţi Chebyshev-Grüss pentru operatorii BEJ de tip I şi II

În continuare aplicăm inegalitatea de tip Chebyshev-Grüss, dată de (1.3), operatorilor BEJ de tipul I şi II şi obţinem următoarele teoreme.

**Teorema 15.** *Dacă $f,g\in C[a,b]$ şi $x\in[a,b]$ este fixat, atunci inegalitatea*

$$|T(f,g;x)|\le \tfrac{1}{4}\widetilde{\omega}\left(f;2\sqrt{R_{m,n}^{(r,a,b)}((e_1-x)^2;x)}\right)\widetilde{\omega}\left(g;2\sqrt{R_{m,n}^{(r,a,b)}((e_1-x)^2;x)}\right)$$

*are loc, pentru $r>0$, $a,b\ge -1$, $n,m>1$.*

**Teorema 16.** *Dacă $f,g\in C[a,b]$ şi $x\in[a,b]$ este fixat, atunci inegalitatea*

$$
\begin{aligned}
|T(f,g;x)|\le \frac{1}{4}\widetilde{\omega}\left(f;2\sqrt{R_n^{(s,c,d;r,a,b)}((e_1-x)^2;x)}\right)\cdot\\
\cdot\,\widetilde{\omega}\left(g;2\sqrt{R_n^{(s,c,d;r,a,b)}((e_1-x)^2;x)}\right)
\end{aligned}
$$

*are loc, pentru $r,s>0$, $a,b,c,d\ge -1$, $n>1$.*

**Remarca 17.** Particularizând valorile indicilor în ecuaţiile (2.2) şi (2.4), obţinem inegalităţile Chebyshev-Grüss corespunzătoare operatorilor binecunoscuţi în literatură. Momentele de ordinul doi în cazurile particulare sunt prezentate în Tabelele $1-5$ din Anexă.

Dacă aplicăm inegalitatea (1.4) operatorilor BEJ de tipul I şi II care nu reproduc funcţiile liniare, obţinem următorul rezultat.

**Teorema 18.** *Pentru $f, g \in C[a,b]$ şi $x \in [a,b]$ fixat, următoarea inegalitate are loc:*

$$|T(f,g;x)| \leq \frac{1}{4} \cdot \widetilde{\omega}\left(f; 2 \cdot \sqrt{T(e_1,e_1;x)}\right) \cdot \widetilde{\omega}\left(g; 2 \cdot \sqrt{T(e_1,e_1;x)}\right).$$

*Diferenţele de mai sus sunt date de ecuaţiile:*

$$T(e_1,e_1;x) := R_{m,n}^{(r,a,b)}((e_1 - xe_0)^2;x) - [R_{m,n}^{(r,a,b)}((e_1 - xe_0)^1;x)]^2,$$

*respectiv*

$$T(e_1,e_1;x) := R_n^{(s,c,d;r,a,b)}((e_1 - xe_0)^2;x) - [R_n^{(s,c,d;r,a,b)}((e_1 - xe_0)^1;x)]^2.$$

**Remarca 19.** Momentele de ordinul întâi şi diferenţele de tip $T(e_1,e_1;x)$ pentru cazurile particulare se regăsesc în Tabelele $6-7$ din Anexă.

## REFERENCES


[1] S.N. Bernstein, Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités, *Comm. Soc. Math. Kharkow* **13** (1912/13), 1–2. [Also appears in Russian translation in Bernstein's Collected Works].

[2] P.L. Chebyshev, O priblizhennyh vyrazhenijah odnih integralov cherez drugie, *Soobschenija i Protokoly Zasedanij Mathematicheskogo Obschestva pri Imperatorskom Khar'kovskom Universitete* **2** (1882), 93–98; *Polnoe Sobranie Sochinenii P. L. Chebysheva. Moskva-Leningrad*, **3**(1978), 128–131.

[3] P.L. Chebyshev, Sur les expressions approximatives des intégrales définies par les autres prises entre les mêmes limites, *Proc. Math. Soc. Kharkov* **2** (1882), 93–98 (Russian), translated in Oeuvres, **2** (1907), 716–719.

[4] W. Chen, On the modified Bernstein-Durrmeyer operator. In *Report of the Fifth Chinese Conference on Approximation Theory*, Zhen Zhou, China, 1987.

[5] J.L. Durrmeyer, *Une formule d'inversion de la transformée de Laplace: Application à la théorie des moments*, Thèse de 3e cycle, Faculté des Scieces de l'Université de Paris, 1967.

[6] Z. Finta, Quantitative estimates for some linear and positive operators, *Studia Univ. Babeş-Bolyai*, Mathematica **47** (2002), 71–84.

[7] H. Gonska, P. Piţul, I. Raşa, On differences of positive linear operators, *Carpathian J. Math.* **22** (2006), 65–78.

[8] H. Gonska, I. Raşa, M.-D. Rusu, Čebyšev-Grüss-type inequalities revisited, *Math. Slovaca* **63**, no. 5, (2013), 1007–1024.

[9] H. Gonska, I. Raşa, E.D. Stănilă, Beta operators with Jacobi weights, arXiv: 1402.3485 (2014).

[10] T.N.T. Goodman, A. Sharma, A Bernstein-type operator on the Simplex, *Math. Balkanica* **5** (1991), 129–145.

[11] G. Grüss, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \cdot \int_a^b g(x)dx$, *Math. Z.* **39** (1935), 215–226.

[12] A. Lupaş, *Die Folge der Betaoperatoren*, Ph.D. Thesis, Universität Stuttgart, 1972.

[13] A. Lupaş, The approximation by means of some linear positive operators. In *Approximation Theory* (M.W. Müller et al., eds.), 201–227, Berlin: Akademie-Verlag 1995.

[14] A. Lupaş, L. Lupaş, Polynomials of binomial type and approximation operators, *Studia Univ. Babeş-Bolyai, Mathematica* XXXII **4**, 1987.

[15] D.H. Mache, D.X. Zhou, Characterization theorems for the approximation by a family of operators, *J. Approx. Theory* **84** (1996), 145–161.

[16] Mitrinović, D. S., Pečarić, J. E., Fink, A. M., Classical and New Inequalities in Analysis, *Dordrecht et al.: Kluwer Academic Publishers*, (1993).

[17] G. Mühlbach, Rekursionsformeln für die zentralen Momente der Pólya und der Beta-Verteilung, *Metrika* **19** (1972), 171–177.

[18] R. Păltănea, Sur un opérateur polynomial défini sur l'ensemble des fonctions intégrables. In: *Babeş-Bolyai Univ., Faculty of Math., Research Seminar*, 1983, Vol. 2, pp. 101–106.

[19] R. Păltănea, A class of Durrmeyer type operators preserving linear functions, *Ann. Tiberiu Popoviciu Sem. Funct. Equat. Approxim. Convex.* (Cluj-Napoca) **5** (2007), 109–117.

[20] P. Piţul, *Evaluation of the approximation order by positive linear operators*, Ph.D. Thesis, Babeş-Bolyai University, Cluj-Napoca, 2007.

[21] M. D. Rusu, On Grüss-type inequalities for positive linear operators, *Stud. Univ. Babeş-Bolyai Math.* **56** (2011), no.2, 551–565.

[22] D.D. Stancu, Approximation of functions by a new class of linear polynomial operators, *Rev. Roumaine Math. Pures Appl.* XIII **8** (1968), 1173–1194.

[23] E.D. Stănilă, Remarks on Bernstein-Euler-Jacobi (BEJ) type operators, *General Mathematics*, Vol. 20, **5** (2012), Special Issue, 123–132.

## 4. Anexe

| Notaţie | Denumire | Momentul de ordinul doi | Referinţe |
|---|---|---|---|
| $U_n = R_{n,\infty}^{(n,-1,-1)} = R_n^{\infty,-,-;n,-1,-1}$ | operatorul "original" Bernstein-Durrmeyer | $U_n((e_1-xe_0)^2;x) := \frac{2X}{n+1}$ | [4] [10] |
| $M_n = R_{n,\infty}^{(n,0,0)} = R_n^{\infty,-,-;n,0,0}$ | operatorul lui Durrmeyer | $M_n((e_1-xe_0)^2;x) := \frac{2X(n-3)+2}{(n+2)(n+3)}$ | [5] |
| $D^{<\alpha>} = R_{n,\infty}^{(n,\alpha,\alpha)} = R_n^{\infty,-,-;n,\alpha,\alpha}$ | operatorul Bernstein-Durrmeyer cu ponderi simetrice | $D^{<\alpha>}((e_1-xe_0)^2;x) := \frac{X(2n-4a^2-10a-6)+(a+1)(a+2)}{(n+2a+2)(n+2a+3)}$ | [13] |
| $M_n^{ab} = R_{n,\infty}^{(n,a,b)} = R_n^{\infty,-,-;n,a,b}$ | operatorul lui Durrmeyer cu ponderi Jacobi | $M_n^{ab}((e_1-xe_0)^2;x) := \frac{a^2+b^2+2ab+5a+5b+6-2n}{(n+a+b+2)(n+a+b+3)}\cdot x^2 - \frac{2a^2+2ab+8a+2b+6-2n}{(n+a+b+2)(n+a+b+3)}\cdot x + \frac{(a+1)(a+2)}{(n+a+b+2)(n+a+b+3)}$ | [18] |
| $U_n^{\varrho} = R_{n,\infty}^{(n\varrho,-1,-1)} = R_n^{\infty,-,-;n\varrho,-1,-1}$ | $\varrho$-operatorul lui Păltănea | $U_n^{\varrho}((e_1-xe_0)^2;x) := \frac{X(1+\varrho)}{n\varrho+1}$ | [19] |
| $P_n = R_{n,\infty}^{(nc,a,b)} = R_n^{\infty,-,-;nc,a,b}$ | operatorul lui Mache-Zhou | $P_n((e_1-xe_0)^2;x) := \frac{a^2+b^2+2ab+5a+5b+6-nc-nc^2}{(nc+a+b+2)(nc+a+b+3)}\cdot x^2 - \frac{(2a^2+2ab+8a+2b+6-nc-nc^2)\cdot x+(a+1)(a+2)}{(nc+a+b+2)(nc+a+b+3)}$ | [15] |
| $L_n^{\nabla} = R_{\infty,n}^{(n,-1,-1)} = R_n^{n,-1,-1;\infty,-,-}$ | operator de tip Stancu | $L_n^{\nabla}((e_1-xe_0)^2;x) := \frac{2X}{2+1}$ | [14] |
| $V_n^{0,0} = R_{\infty,n}^{(n,0,0)} = R_n^{n,0,0;\infty,-,-}$ | operatorul lui Lupaş | $V_n^{0,0}((e_1-xe_0)^2;x) := \frac{X(2n^2-6n)+3n+1}{n(n+2)(n+3)}$ | [13] |

**Tabelul 1**: Momentele de ordinul doi - partea I

| Notaţie | Denumire | Momentul de ordinul doi | Referinţe |
|---|---|---|---|
| $V_n^{\alpha,\beta} = R_{\infty,n}^{(n,\alpha,\beta)} = R_n^{n,\alpha,\beta;\infty,-,-}$ | operatorul lui Lupaş cu ponderi Jacobi | $V_n^{\alpha,\beta}((e_1 - xe_0)^2; x) := \frac{\alpha^2+\beta^2+2\alpha\beta+5\alpha+5\beta+6-2n}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)} \cdot x^2 - \frac{2\alpha^2+2\alpha\beta+9\alpha+\beta+6-2n}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)} \cdot x + \frac{n\alpha^2+4n\alpha+\alpha\beta+3n+\alpha+\beta+1}{n(n+\alpha+\beta+2)(n+\alpha+\beta+3)}$ | I. Raşa handwritten notes, 19 August 2008. |
| $S_n^{\alpha} = R_{\infty,n}^{(\frac{1}{\alpha},-1,-1)} = R_n^{\frac{1}{\alpha},-1,-1;\infty,-,-}$ | operatorul lui Stancu | $S_n^{\alpha}((e_1 - xe_0)^2; x) := \frac{X(n^2+1)}{n(n+1)}$ | [22] |
| $Q_n^{\varrho,c,d} = R_{\infty,n\varrho}^{(n,c,d)} = R_n^{n\varrho,c,d;\infty,-,-}$ | operator de tip Stancu cu parametri $c$ şi $d$ | $Q_n^{\varrho,c,d}((e_1 - xe_0)^2; x) := \frac{c^2+d^2+2cd+5c+5d+6-n\varrho-n\varrho^2}{(n\varrho+c+d+2)(n\varrho+c+d+3)} \cdot x^2 - \frac{2c^2+2cd+8c+2d+6-n\varrho-n\varrho^2}{(n\varrho+c+d+2)(n\varrho+c+d+3)} \cdot x + \frac{nc^2+3nc+nc\varrho+n\varrho+2n+c+d+cd+1}{n(n\varrho+c+d+2)(n\varrho+c+d+3)}$ | H. Gonska handwritten notes, 18 March 2009. |
| $\overline{\mathbb{B}}_n$, $\mathcal{B}_n^{-1,-1} = R_{\infty,\infty}^{(n,-1,-1)} = R_{\infty}^{\infty,-,-;n,-1,-1} = R_{\infty}^{n,-1,-1;\infty,-,-}$ | operatorul Beta de a doua speţa al lui Lupaş | $\overline{\mathbb{B}}_n((e_1 - xe_0)^2; x) := \frac{X}{n+1}$ | [12] |
| $\widetilde{\mathbb{B}}_\lambda$, $T_\lambda = R_{\infty,\infty}^{(\frac{1}{\lambda},-1,-1)} = R_{\infty}^{\infty,-,-;\frac{1}{\lambda},-1,-1} = R_{\infty}^{\frac{1}{\lambda},-1,-1;\infty,-,-}$ | operatorul Beta al lui Mühlbach | $\widetilde{\mathbb{B}}_\lambda((e_1 - xe_0)^2; x) := \frac{\lambda X}{1+\lambda}$ | [17] |
| $\mathbb{B}_n$, $\mathcal{B}_n^{0,0} = R_{\infty,\infty}^{(n,0,0)} = R_{\infty}^{\infty,-,-;n,0,0} = R_{\infty}^{n,0,0;\infty,-,-}$ | operatorul Beta de prima speţa al lui Lupaş | $\mathbb{B}_n((e_1 - xe_0)^2; x) := \frac{X(n-6)+2}{(n+2)(n+3)}$ | [12] |

**Tabelul 2**: Momentele de ordinul doi - partea a II-a

| Notaţie | Denumire | Momentul de ordinul doi | Referinţe |
|---|---|---|---|
| $\mathcal{B}_n^{-1,\beta} = R_{\infty,\infty}^{(n,-1,\beta)} = R_{\infty}^{\infty,-,-;n,-1,\beta} = R_{\infty}^{n,-1,\beta;\infty,-,-}$ | | $\mathcal{B}_n^{-1,\beta}((e_1 - xe_0)^2; x) := \frac{nX + (\beta+1)(\beta+2)x^2}{(n+\beta+1)(n+\beta+2)}$ | [9] |
| $\mathcal{B}_n^{\alpha,-1} = R_{\infty,\infty}^{(n,\alpha,-1)} = R_{\infty}^{\infty,-,-;n,\alpha,-1} = R_{\infty}^{n,\alpha,-1;\infty,-,-}$ | | $\mathcal{B}_n^{\alpha,-1}((e_1 - xe_0)^2; x) := \frac{nX + (\alpha+1)(\alpha+2)(x-1)^2}{(n+\alpha+1)(n+\alpha+2)}$ | [9] |
| $\mathcal{B}_n^{\alpha,\beta} = R_{\infty,\infty}^{(n,\alpha,\beta)} = R_{\infty}^{\infty,-,-;n,\alpha,\beta} = R_{\infty}^{n,\alpha,\beta;\infty,-,-}$ | operatorul Beta cu ponderi Jacobi | $\mathcal{B}_n^{\alpha,\beta}((e_1 - xe_0)^2; x) := \frac{\alpha^2+\beta^2+2\alpha\beta+5\alpha+5\beta+6-n}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)} \cdot x^2 - \frac{2\alpha^2+2\alpha\beta+8\alpha+2\beta+6-n}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)} \cdot x + \frac{(\alpha+1)(\alpha+2)}{(n+\alpha+\beta+2)(n+\alpha+\beta+3)}$ | [9] |
| $B_n = R_{n,\infty}^{(\infty,-,-)} = R_{\infty,n}^{(\infty,-,-)} = R_n^{\infty,-,-;\infty,-,-}$ | operatorul lui Bernstein | $B_n((e_1 - xe_0)^2; x) := \frac{X}{n}$ | [1] |

**Tabelul 3**: Momentele de ordinul doi - partea a III-a

| Notaţie | Denumire | Momentul de ordinul doi | Referinţe |
|---|---|---|---|
| $D_n = R_{n,n+1}^{(\infty,-,-)}$ | | $D_n((e_1 - xe_0)^2; x) := \frac{2X}{n+1}$ | [7] |
| $R_{m,n}^{\infty} = R_{m,n}^{(\infty,-,-)}$ | | $R_{m,n}^{\infty}((e_1 - xe_0)^2; x) := \frac{X(n+m-1)}{nm}$ | [23] |
| $R_{m,n}^{\varrho} = R_{m,n}^{(n\varrho,-1,-1)}$ | | $R_{m,n}^{\varrho}((e_1 - xe_0)^2; x) := \frac{X(n\varrho + m\varrho - \varrho + m)}{m(n\varrho + 1)}$ | [23] |

**Tabelul 4**: Momentele de ordinul doi - pentru operatori BEJ de primul tip

| Notaţie | Denumire | Momentul de ordinul doi | Referinţe |
|---|---|---|---|
| $\mathbb{B}_{\infty}^{(\alpha,\lambda)} = R_{\infty}^{\frac{1}{\alpha},-1,-1;\frac{1}{\lambda},-1,-1}$ | operatori de tip Beta generalizaţi | $\mathbb{B}_{\infty}^{\alpha,\lambda}((e_1 - xe_0)^2; x) := \frac{X(\alpha + \lambda + \alpha\lambda)}{(1+\alpha)(1+\lambda)}$ | [20] |
| $F_n^{\alpha} = R_n^{\frac{1}{\alpha},-1,-1;n-1,-1}$ | operatorul lui Finta | $F_n^{\alpha}((e_1 - xe_0)^2; x) := \frac{X(n\alpha + \alpha + 2)}{(\alpha+1)(n+1)}$ | [6] |
| $\mathbb{B}_n^{(\alpha,\lambda)} = R_n^{\frac{1}{\alpha},-1,-1;\frac{1}{\lambda},-1,-1}$ | operatorul de tip Beta al lui Piţul | $\mathbb{B}_n^{(\alpha,\lambda)}((e_1 - xe_0)^2; x) := \frac{X(n\alpha + n\lambda + n\alpha\lambda + 1)}{n(1+\alpha)(1+\lambda)}$ | [20] |

**Tabelul 5**: Momentele de ordinul doi - pentru operatori BEJ de tipul al doilea

| Notaţie | Momentul de ordinul unu | Diferenţe de tipul $T(e_1, e_1; x)$ |
|---|---|---|
| $M_n$ | $M_n((e_1 - xe_0)^1; x) := \frac{1 - 2 \cdot x}{n+2}$ | $T(e_1, e_1; x) := \frac{(2 \cdot n \cdot X + 1) \cdot (n+1)}{(n+2)^2 \cdot (n+3)}$ |
| $D^{<\alpha>}$ | $D^{<\alpha>}((e_1 - xe_0)^1; x) := \frac{-2 \cdot x \cdot (\alpha+1) + \alpha + 1}{n + 2 \cdot \alpha + 2}$ | $T(e_1, e_1; x) := \frac{X \cdot (2 \cdot n^2 + 2 \cdot n + 2 \cdot \alpha \cdot n) + 1 + n + 2\alpha + \alpha n + \alpha^2}{(n + 2\alpha + 2)^2 (n + 2\alpha + 3)}$ |
| $M_n^{ab}$ | $M_n^{ab}((e_1 - xe_0)^1; x) := \frac{a + 1 - x \cdot (a + b + 2)}{n + a + b + 2}$ | $T(e_1, e_1; x) := \frac{2nX(n + b + 1) + n \cdot x^2 (b - a) + 1 + n + a + b + an + ab}{(n + a + b + 2)^2 (n + a + b + 3)}$ |
| $P_n$ | $P_n((e_1 - xe_0)^1; x) := \frac{a + 1 - x(a + b + 2)}{n \cdot c + a + b + 2}$ | $T(e_1, e_1; x) := \frac{X \cdot n \cdot c^2 (n \cdot c + a + b + n + 2) + x \cdot n \cdot c(a + b)}{(nc + a + b + 2)^2 (nc + a + b + 3)}$ $- \frac{10a^2 + 5anc + 15a + 3b + 2a^2 nc + 3nc + 2a^2 b + 2a^3 + 5ab + 7}{(nc + a + b + 2)^2 (nc + a + b + 3)}$ |
| $V_n^{0,0}$ | $V_n^{0,0}((e_1 - xe_0)^1; x) := \frac{1 - 2x}{n+2}$ | $T(e_1, e_1; x) := \frac{2(n+1)(n^2 \cdot X + n + 1)}{n(n+2)^2 (n+3)}$ |
| $V_n^{\alpha,\beta}$ | $V_n^{\alpha,\beta}((e_1 - xe_0)^1; x) := \frac{\alpha + 1 - x \cdot (\alpha + \beta + 2)}{n + \alpha + \beta + 2}$ | $T(e_1, e_1; x) := \frac{2n^3 + n^2\beta + n^2\alpha + 2n^2}{(n + \alpha + \beta + 2)^2 (n + \alpha + \beta + 3)n} \cdot x^2$ $+ \frac{n\alpha^2 - 2n^2 + n^2\alpha + 2n\alpha - n\beta^2 - 2\beta n - 2n^3 - 3n^2\beta}{(n + \alpha + \beta + 2)^2 (n + \alpha + \beta + 3)n} \cdot x +$ $\frac{-2 - 4n - 3\alpha - 3\beta - n\alpha^2 - 4\alpha\beta - 3\alpha\beta n - \alpha^2 - \beta^2 - 5n\alpha - \beta^2\alpha - 3\beta n - 2n^2 - \alpha^2\beta - 2n^2\alpha}{n(n + \alpha + \beta + 2)^2 (n + \alpha + \beta + 3)}$ |

**Tabelul 6**: Diferenţe de tipul $T(e_1, e_1; x)$- Partea I

| Notaţie | Momentul de ordinul unu | Diferenţe de tipul $T(e_1, e_1; x)$ |
|---|---|---|
| $Q_n^{\rho,c,d}$ | $Q_n^{\rho,c,d}((e_1 - xe_0)^1; x) := \frac{c+1-x\cdot(c+d+2)}{n\cdot\rho+c+d+2}$ | $T(e_1, e_1; x) := \frac{2n^2\rho^2 + n^3\rho^2 + n^2\rho^2 d + n^2 c\rho^2 + n^3\rho^3}{(n\rho+c+d+2)^2(n\rho+c+d+3)n}\cdot x^2 - \frac{-n^2c\rho + n^2 d\rho + 2n^2\rho^2 + n^3\rho^2 + n^2\rho^2 d + n^2 c\rho^2 + n^3\rho^3}{(n\rho+c+d+2)^2(n\rho+c+d+3)n}\cdot x + \frac{-2-n-4nc\rho-2cdn\rho-n^2\rho-nd-3\rho n-3c-nc-4cd-c^2-d^2-c^2n\rho}{n(n\rho+c+d+2)^2(n\rho+c+d+3)} + \frac{-n^2c\rho - n^2c\rho^2 - ncd - 3d - d^2c - n^2\rho^2 - c^2d - 2dn\rho}{n(n\rho+c+d+2)^2(n\rho+c+d+3)}$ |
| $\mathbb{B}_n$, $\mathcal{B}_n^{0,0}$ | $\mathbb{B}_n((e_1 - xe_0)^1; x) := \frac{1-2x}{n+2}$ | $T(e_1, e_1; x) := \frac{X(n^2+12n+24)+n+1}{(n+2)^2(n+3)}$ |
| $\mathcal{B}_n^{-1,\beta}$ | $\mathcal{B}_n^{-1,\beta}((e_1 - xe_0)^1; x) := \frac{-x(\beta+1)}{n+\beta+1}$ | $T(e_1, e_1; x) := \frac{n^2\cdot X + n\beta + n}{(n+\beta+1)^2(n+\beta+2)}$ |
| $\mathcal{B}_n^{\alpha,-1}$ | $\mathcal{B}_n^{\alpha,-1}((e_1 - xe_0)^1; x) := \frac{(\alpha+1)(1-x)}{n+\alpha+1}$ | $T(e_1, e_1; x) := \frac{n^2\cdot X - n\cdot x(1+\alpha) + n + n\cdot\alpha}{(n+\alpha+1)^2(n+\alpha+2)}$ |
| $\mathcal{B}_n^{\alpha,\beta}$ | $\mathcal{B}_n^{\alpha,\beta}((e_1 - xe_0)^1; x) := \frac{\alpha+1-x(\alpha+\beta+2)}{n+\alpha+\beta+2}$ | $T(e_1, e_1; x) := \frac{X\cdot n^2 + nx(\beta-\alpha) + \alpha + \beta + \alpha\beta + \alpha n + n + 1}{(n+\alpha+\beta+2)^2(n+\alpha+\beta+3)}$ |

**Tabelul 7**: Diferenţe de tipul $T(e_1, e_1; x)$- Partea a II-a

# Olimpiada de Matematică a studenţilor din sud-estul Europei, SEEMOUS 2015

Cornel Băeţica[1)], Gabriel Mincu[2)], Radu Strugariu[3)]

**Abstract**. This note deals with the problems of the 9th South Eastern European Mathematical Olympiad for University Students, SEEMOUS 2015, organized by the Union of Macedonian Mathematicians in Ohrid, Macedonia, between March 3 and March 8, 2015.

**Keywords:** Determinants, block matrices, Riemann integral, sequences and series.

**MSC :** 11C20, 26A42, 40A05

Cea de-a noua ediţie a Olimpiadei de Matematică a studenţilor din sud-estul Europei, SEEMOUS 2015, a fost organizată de Uniunea Matematicienilor din Macedonia şi de Societatea de Matematică din Sud-Estul Europei în localitatea Ohrid din Macedonia, în perioada 3–8 martie 2015. Au participat 91 de studenţi de la 20 de universităţi din Bulgaria, Grecia, Iran, Macedonia, România, Slovenia şi Turkmenistan.

Concursul a avut o singură probă, care a constat în patru probleme. Prezentăm mai jos aceste probleme însoţite de soluţii. Unele idei sau soluţii au fost preluate din lucrările concurenţilor. Soluţiile oficiale pot fi consultate pe situl `http://www.seemous2015.smm.com.mk`.

**Problema 1.** Să se arate că pentru orice $x \in (0,1)$ are loc inegalitatea

$$\int_0^1 \sqrt{1+\cos^2 y}\,\mathrm{d}y > \sqrt{x^2+\sin^2 x}.$$

**Pirmyrat Gurbanov**, Turkmenistan

*Aceasta a fost considerată de juriu drept o problemă uşoară. Majoritatea studenţilor care au rezolvat problema au preferat o abordare calculatorie în spiritul celei de-a doua soluţii pe care o prezentăm.*

**Soluţia 1.** Se observă că membrul stâng al relaţiei cerute reprezintă lungimea graficului funcţiei $f : [0,1] \to \mathbb{R}$, $f(x) = \sin x$, în timp ce membrul drept reprezintă lungimea coardei ce uneşte punctele $(0,0)$ şi $(x, \sin x)$ de pe graficul respectiv. În această interpretare, inegalitatea cerută este evidentă. □

1) Department of Mathematics, University of Bucharest, 14, Academiei str., Bucharest, Romania, `cornel.baetica@fmi.unibuc.ro`

2) Department of Mathematics, University of Bucharest, 14, Academiei str.,Bucharest, Romania, `gamin@fmi.unibuc.ro`

3) Universitatea Tehnică „Gheorghe Asachi” din Iaşi, Departamentul de Matematică şi Informatică, `rstrugariu@tuiasi.ro`

**Soluţia 2.** Considerăm funcţia $F : [0,1] \to \mathbb{R}$,

$$F(x) = \int_0^x \sqrt{1+\cos^2 y}\,\mathrm{d}y - \sqrt{x^2+\sin^2 x}.$$

Constatăm că $F'(x) = \begin{cases} \sqrt{1+\cos^2 x} - \frac{x+\sin x\cos x}{\sqrt{x^2+\sin^2 x}} & \text{, dacă } x \neq 0, \\ 0 & \text{, dacă } x = 0. \end{cases}$.
Ţinând cont de inegalitatea $(1+\cos^2 x)(x^2+\sin^2 x) > (x+\sin x\cos x)^2$, valabilă pentru orice $x \in (0,1]$, căci pe acest interval avem $\operatorname{tg} x \neq x$, obţinem $F'(x) > 0$ pentru orice $x \in (0,1]$, deci $F$ este strict crescătoare. Prin urmare, pentru orice $x \in (0,1]$ avem $F(x) > F(0) = 0$, deci $\int_0^x \sqrt{1+\cos^2 y}\,\mathrm{d}y > \sqrt{x^2+\sin^2 x}$. Cum însă integrandul este pozitiv,

$$\int_0^1 \sqrt{1+\cos^2 y}\,\mathrm{d}y \geq \int_0^x \sqrt{1+\cos^2 y}\,\mathrm{d}y$$

şi obţinem inegalitatea cerută. □

*Observaţie.* Soluţia 2 nu face altceva decât să concretizeze de o manieră calculatorie intuiţia inegalităţii dintre lungimea curbei şi lungimea coardei pe care am folosit-o la soluţia 1.

**Problema 2.** Considerăm funcţiile $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}^*$, definite prin $f_1(x) = 3x - 4x^3$ şi $f_{n+1}(x) = f_1(f_n(x))$. Rezolvaţi ecuaţia $f_n(x) = 0$.

*Problema este una standard. Soluţiile date de concurenţi au făcut prin urmare uz de idei de acelaşi tip. Prezentăm mai jos două soluţii, prima în spiritul celei propuse de autorul problemei, iar a doua, dată în concurs de Andrei Bud, care ilustrează anumite variaţii de abordare.*

**Soluţia 1.** Începem prin a observa că pentru $|x| > 1$ avem $|f_1(x)| = |x||3-4x^2| > 1$. De aici se obţine inductiv faptul că dacă $|x| > 1$, atunci $|f_n(x)| > 1$ pentru orice $n \in \mathbb{N}^*$. În consecinţă, ecuaţia propusă nu poate avea soluţii decât în intervalul $[-1,1]$. Pentru $x \in [-1,1]$, $f_1(x) = f_1(\sin(\arcsin x))$ $= \sin(3\arcsin x)$. Inductiv, $f_n(x) = \sin(3^n \arcsin x)$. În consecinţă, soluţiile ecuaţiei $f_n(x) = 0$ sunt $\sin \frac{k\pi}{3^n}$, $k \in \left\{\frac{1-3^n}{2}, \ldots, \frac{3^n-1}{2}\right\}$. □

**Soluţia 2.** Observăm că $f_1(x) = -T_3(x)$, $T_3$ fiind al treilea polinom Cebâşev. Prin urmare, pentru $x \in [-1,1]$ are loc $f_1(\cos(\arccos x)) =$ $= -\cos(3\arccos x)$; inductiv, $f_n(x) = (-1)^n \cos(3^n \arccos x)$, deci ecuaţia din enunţ admite drept soluţii numerele

$$\cos\frac{(2k+1)\pi}{2\cdot 3^n}, \quad k \in \{0,1,\ldots,3^n-1\}. \tag{1}$$

Cum funcţia cos este strict descrescătoare pe $[0,\pi]$, aceste soluţii sunt distincte două câte două, deci ele sunt în număr de $3^n$. Pe de altă parte, se constată inductiv că $f_n$ este o funcţie polinomială de grad $3^n$, deci ea are cel mult $3^n$ rădăcini reale. În consecinţă, soluţiile ecuaţiei propuse sunt exact cele date în (1). □

**Problema 3.** Fie $n \geq 2$ un număr întreg şi $A, B, C, D \in M_n(\mathbb{R})$ matrice care satisfac următoarele relaţii:

$$AC - BD = I_n,$$

$$AD + BC = 0_n,$$

unde $I_n$ este matricea unitate iar $0_n$ este matricea nulă din $M_n(\mathbb{R})$. Arătaţi că:

(a) $CA - DB = I_n$ şi $DA + CB = 0_n$.

(b) $\det(AC) \geq 0$ şi $(-1)^n \det(BD) \geq 0$.

**Vasile Pop**, România

*Prima parte a problemei este standard, bazându-se pe faptul că dacă două matrice pătratice $X$ şi $Y$ satisfac $XY = I_n$, atunci şi $YX = I_n$. Cea de-a doua parte însă a pus probleme serioase concurenţilor, doar doi dintre aceştia, Andrei Bud şi Anca Băltărigă, ambii de la Facultatea de Matematică şi Informatică a Universităţii din Bucureşti, reuşind să dea soluţii complete. Prezentăm mai jos trei soluţii, prima folosind matrice bloc iar celelalte două date în concurs de către cei doi studenţi.*

**Soluţie.** (a) Din $AC - BD = I_n$ şi $AD + BC = 0_n$ obţinem $(AC - BD) + i(AD + BC) = I_n$, ceea ce este echivalent cu $(A + iB)(C + iD) = I_n$. Aceasta ne arată că matricele $A + iB$ şi $C + iD$ sunt inversabile, fiind una inversa celeilalte. Astfel $(C + iD)(A + iB) = I_n$ şi de aici rezultă $(CA - DB) + i(DA + CB) = I_n$. Cum $CA - DB, DA + CB \in M_n(\mathbb{R})$ obţinem $CA - DB = I_n$ şi $DA + CB = 0_n$.

(b) **Soluţia 1.** Începem prin a observa că dacă $A, B \in M_n(\mathbb{R})$ atunci

$$\det \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \geq 0.$$

Aceasta se demonstrează astfel:

$$\det \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \det \begin{pmatrix} A - iB & B + iA \\ -B & A \end{pmatrix} = \det \begin{pmatrix} 0_n & B + iA \\ -B + iA & A \end{pmatrix} =$$

$$= \det(B + iA) \det(B - iA) = \det(B + iA) \overline{\det(B + iA)} = |\det(B + iA)|^2 \geq 0.$$

Acum trecem la determinanţi următoarea relaţie:

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} C & 0_n \\ -D & C \end{pmatrix} = \begin{pmatrix} I_n & BC \\ 0_n & AC \end{pmatrix}$$

şi obţinem $\det(AC) = |\det(B + iA)|^2 \det(C)^2 \geq 0$.

Pentru a arăta că $(-1)^n \det(BD) \geq 0$ folosim o relaţie asemănătoare celei anterioare:

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\begin{pmatrix} C & D \\ -D & 0_n \end{pmatrix} = \begin{pmatrix} I_n & AD \\ 0_n & -BD \end{pmatrix},$$

din care deducem $(-1)^n \det(BD) = |\det(B+iA)|^2 \det(D)^2 \geq 0$.

**Soluţia 2.** Să demonstrăm mai întâi că $\det(AC) \geq 0$. Dacă $\det A = 0$ sau $\det C = 0$ nu este nimic de demonstrat. Să presupunem atunci că $\det(AC) \neq 0$. Din $AD = -BC$ rezultă $D = -A^{-1}BC$. Înlocuind în $AC - BD = I_n$ obţinem $AC + BA^{-1}BC = I_n \Rightarrow C + A^{-1}BA^{-1}BC = A^{-1} \Rightarrow I_n + (A^{-1}B)^2 = A^{-1}C^{-1} \Rightarrow [I_n + (A^{-1}B)^2]CA = I_n$. Trecem la determinanţi şi avem că $\det[I_n + (A^{-1}B)^2]\det(AC) = 1$. Cum $\det[I_n + (A^{-1}B)^2] \geq 0$ conchidem că $\det(AC) > 0$.

Pentru a arăta că $(-1)^n \det(BD) \geq 0$ procedăm asemănător: presupunem $\det(BD) \neq 0$ şi din $AD = -BC$ rezultă $A = -BCD^{-1}$. Înlocuind în $AC - BD = I_n$ obţinem $BCD^{-1}C + BD = -I_n \Rightarrow CD^{-1}C + D = -B^{-1} \Rightarrow I_n + (D^{-1}C)^2 = -D^{-1}B^{-1} \Rightarrow [I_n + (D^{-1}C)^2]BD = -I_n$. Trecem la determinanţi şi avem că $\det[I_n + (D^{-1}C)^2]\det(BD) = (-1)^n$. Cum $\det[I_n + (D^{-1}C)^2] \geq 0$ rezultă că $(-1)^n \det(BD) > 0$.

**Soluţia 3.** Din (a) ştim că $CA - DB = I_n$. De aici rezultă $A(CA - DB)C = AC \Rightarrow (AC)^2 - (AD)(BC) = AC \Rightarrow (AC)^2 + (AD)^2 = AC$, deci $\det(AC) = \det[(AC)^2 + (AD)^2]$. Pe de altă parte $(AD)(AC) = A(DA)C = A(-CB)C = -(AC)(BC) = (AC)(AD)$. Fie $X = AC$ şi $Y = AD$. Din $XY = YX$ deducem că $X^2 + Y^2 = (X + iY)(X - iY)$, deci $\det(X^2 + Y^2) = |\det(X + iY)|^2 \geq 0$, ceea ce era de demonstrat.

Procedăm analog pentru cea de-a doua inegalitate. Din $CA - DB = I_n$ rezultă $B(CA - DB)D = BD \Rightarrow (BC)(AD) - (BD)^2 = BD \Rightarrow (AD)^2 + (BD)^2 = -BD \Rightarrow \det[(AD)^2 + (BD)^2] = (-1)^n \det(BD)$. Dar $(BD)(AD) = B(DA)D = B(-CB)D = -(BC)(BD) = (AD)(BD)$, deci putem conchide că $\det[(AD)^2 + (BD)^2] \geq 0$. □

**Problema 4.** Fie $I \subset \mathbb{R}$ un interval deschis, $0 \in I$ iar $f : I \to \mathbb{R}$ o funcţie de clasă $C^{2016}(I)$ astfel încât

$$f(0) = 0,\ f'(0) = 1,\ f''(0) = f'''(0) = \cdots = f^{(2015)}(0) = 0,\ f^{(2016)}(0) < 0.$$

(i) Arătaţi că există $\delta > 0$ astfel încât

$$0 < f(x) < x, \quad \forall x \in (0, \delta). \tag{2}$$

(ii) Cu $\delta$ dat de (i), definim şirul $(a_n)$ prin

$$a_1 := \frac{\delta}{2},\ a_{n+1} := f(a_n),\ \forall n \geq 1. \tag{3}$$

Studiaţi convergenţa seriei

$$\sum_{n=1}^{\infty} a_n^r,$$

unde $r \in \mathbb{R}$ este un parametru arbitrar.

**Radu Strugariu**, România

*Problema a fost considerată de către juriu drept o problemă dificilă, iar rezultatele au confirmat acest lucru, punctajul maxim obţinut fiind de* 7 *puncte din* 10 *posibile. Prezentăm pentru început soluţia autorului.*

**Soluţie.** (i) Vom arăta că există $\alpha > 0$ astfel încât $f(x) > 0$ pentru orice $x \in (0, \alpha)$. Pentru aceasta, observăm că, deoarece $f$ este de clasă $C^1$ şi $f'(0) = 1 > 0$, există $\alpha > 0$ astfel încât $f'(x) > 0$ pe $(0, \alpha)$. Cum $f(0) = 0$, iar $f$ este strict crescătoare pe $(0, \alpha)$, rezultă concluzia dorită.

Demonstrăm că există $\beta > 0$ astfel încât $f(x) < x$ pentru orice $x$ din $(0, \beta)$. Cum $f^{(2016)}(0) < 0$, iar $f$ este de clasă $C^{2016}$, există $\beta > 0$ astfel încât $f^{(2016)}(t) < 0$ pentru orice $t \in (0, \beta)$. Folosind formula lui Taylor, pentru orice $x \in (0, \beta)$ există $\theta \in [0, 1]$ astfel încât

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \cdots + \frac{f^{(2015)}(0)}{2015!}x^{2015} + \frac{f^{(2016)}(\theta x)}{2016!}x^{2016}, \quad (4)$$

deci

$$f(x) - x = \frac{f^{(2016)}(\theta x)}{2016!}x^{2016} < 0, \quad \forall x \in (0, \beta).$$

Luând $\delta := \min\{\alpha, \beta\} > 0$, punctul (i) este complet demonstrat.

*Observaţie.* Diverse variante alternative de demonstraţie pentru inegalitatea $f(x) < x$ de la punctul (i) au fost găsite de mulţi dintre studenţii participanţi. O parte dintre acestea s-au bazat pe faptul că $f^{(2016)}(0) < 0$ antrenează, în baza continuităţii funcţiei $f^{(2016)}$, că $f^{(2016)}(x) < 0$ pe o vecinătate a originii. Cuplat cu $f^{(2015)}(0) = 0$, aceasta implică faptul că $f^{(2015)}(x) < 0$ pentru $x > 0$ dintr-o vecinătate a originii. Inductiv, se ajunge că $f''(x) < 0$ pentru $x > 0$ dintr-o vecinătate a originii. Cum ecuaţia tangentei în $(0, 0) = (0, f(0))$ la graficul funcţiei $f$ este $y = x$, iar $f$ este strict concavă pe o vecinătate a originii, rezultă $f(x) < x$ pe un interval $(0, \delta)$.

(ii) **Soluţia 1.** Vom arăta mai întâi că şirul $(a_n)$ definit de (3) converge la 0. Într-adevăr, datorită relaţiei (2), rezultă că

$$0 < a_{n+1} < a_n, \ \forall n \geq 1,$$

deci şirul $(a_n)$ este strict descrescător şi mărginit inferior de 0. Rezultă că $(a_n)$ este convergent la un $\ell \in \left[0, \frac{\delta}{2}\right)$. Trecând la limită în relaţia de recurenţă, obţinem că $\ell = f(\ell)$. Având în vedere (2), rezultă $\ell = 0$.

În continuare, calculăm

$$\lim_{n\to\infty} n a_n^{2015}.$$

Cum $a_n \downarrow 0$, folosind Teorema Stolz-Cesàro, avem că

$$\lim_{n\to\infty} na_n^{2015} = \lim_{n\to\infty} \frac{n}{\frac{1}{a_n^{2015}}} = \lim_{n\to\infty} \frac{(n+1)-n}{\frac{1}{a_{n+1}^{2015}} - \frac{1}{a_n^{2015}}} = \lim_{n\to\infty} \frac{1}{\frac{1}{f(a_n)^{2015}} - \frac{1}{a_n^{2015}}} =$$

$$= \lim_{x\to 0} \frac{1}{\frac{1}{f(x)^{2015}} - \frac{1}{x^{2015}}} = \lim_{x\to 0} \frac{(xf(x))^{2015}}{x^{2015} - f(x)^{2015}}.$$

Folosind (4), obţinem că pentru orice $x$ suficient de mic, există $\theta \in [0,1]$ astfel încât

$$(xf(x))^{2015} = x^{4030} + O(x^{6045})$$

şi

$$x^{2015} - f(x)^{2015} = -2015 \cdot \frac{f^{(2016)}(\theta x)}{2016!} x^{4030} + O(x^{6045}),$$

ceea ce arată, folosind şi faptul că $f$ este de clasă $C^{2016}$, că $\lim\limits_{x\to 0} f^{(2016)}(\theta x) = f^{(2016)}(0)$ şi

$$\lim_{x\to 0} \frac{(xf(x))^{2015}}{x^{2015} - f(x)^{2015}} = -\frac{2016!}{2015 \cdot f^{(2016)}(0)} > 0.$$

Aceasta implică, în baza criteriului de comparaţie cu limită pentru serii cu termeni pozitivi, că seriile $\sum\limits_{n=1}^{\infty} a_n^r$ şi $\sum\limits_{n=1}^{\infty} \frac{1}{n^{\frac{r}{2015}}}$ au aceeaşi natură, deci seria $\sum\limits_{n=1}^{\infty} a_n^r$ este convergentă pentru $r > 2015$ şi divergentă pentru $r \leq 2015$. □

*Observaţie.* Ideea de considerare a limitei $\lim\limits_{n\to\infty} na_n^{2015}$ apare natural atunci când se încearcă aplicarea criteriului de convergenţă de tip Gauss: dacă seria $\sum\limits_{n=1}^{\infty} b_n$ cu termeni pozitivi are proprietatea că $\frac{b_{n+1}}{b_n}$ se poate scrie sub forma

$$\frac{b_{n+1}}{b_n} = 1 - \frac{\lambda}{n} + \frac{B(n)}{n^p}, \tag{5}$$

unde $B$ este o funcţie mărginită, iar $p > 1$, atunci seria $\sum\limits_{n=1}^{\infty} b_n$ este convergentă dacă şi numai dacă $\lambda > 1$. În cazul nostru, notând $\beta_n := \frac{f^{(2016)}(\theta_n a_n)}{2016!}$, avem folosind (4) pentru $x := a_n$ că

$$\left(\frac{a_{n+1}}{a_n}\right)^r = \left(1 + \beta_n a_n^{2015}\right)^r = 1 + r\beta_n a_n^{2015} + \frac{r(r-1)}{2}\beta_n^2 a_n^{4030} + \cdots,$$

folosind expresia seriei binomiale şi faptul că $\beta_n \to \frac{f^{(2016)}(0)}{2016!}$, $a_n \to 0$ implică $\beta_n a_n^{2015} \in (-1,1)$ pentru $n$ suficient de mare. Este clar deci că se impune calculul limitei $\lim\limits_{n\to\infty} na_n^{2015}$ pentru a putea scrie $\left(\frac{a_{n+1}}{a_n}\right)^r$ sub forma (5). Făcând calculele şi demonstrând mărginirea funcţiei $B(n)$, obţinem aceeaşi concluzie ca mai sus, deoarece ne rezultă $\lambda = \frac{r}{2015}$.

Majoritatea studenţilor care au dat rezolvări parţiale legate de convergenţa seriei au încercat compararea acesteia cu diverse serii telescopice. Este şi cazul studentului Andrei Bud de la Facultatea de Matematică şi Informatică a Universităţii din Bucureşti, care a rezolvat complet cazul $r \geq 2016$, precum şi parţial cazul $r \leq 2015$ prin acest procedeu. Prezentăm în continuare o soluţie a problemei ce porneşte de la această idee.

**Soluţia 2.** În primul rând să observăm că formula lui Taylor asigură că pentru orice $n \geq 1$,

$$a_{n+1} = a_n + \frac{f^{(2016)}(\theta_n a_n)}{2016!} \cdot a_n^{2016}, \tag{6}$$

unde $\theta_n \in [0,1]$. De asemenea, deoarece există $\omega$ astfel încât $f^{(2016)}(0) < \omega < 0$ şi $f^{(2016)}$ este continuă, avem că există $T > 0$ astfel încât $f^{(2016)}(x) < \omega$ pentru orice $x \in [0,T]$. Mai mult, din Teorema lui Weierstrass,

$$A := \max_{x\in[0,T]} \frac{f^{(2016)}(x)}{2016!} < 0,$$
$$B := \min_{x\in[0,T]} \frac{f^{(2016)}(x)}{2016!} < 0.$$

Notând $\beta_n := \frac{f^{(2016)}(\theta_n a_n)}{2016!}$, observăm că $\beta_n \leq A < 0$, pentru orice $n$ suficient de mare. Presupunem, fără a restrânge generalitatea, că relaţia are loc pentru orice $n \geq 1$. Atunci

$$-Aa_n^{2016} \leq -\beta_n a_n^{2016} = a_n - a_{n+1},$$

de unde prin sumare

$$-A\sum_{n=1}^{m} a_n^{2016} \leq a_1 - a_{m+1}.$$

Rezultă convergenţa seriei $\sum\limits_{n=1}^{\infty} a_n^{2016}$. Prin intermediul unui criteriu de comparaţie, concluzionăm că seria $\sum\limits_{n=1}^{\infty} a_n^r$ converge pentru orice $r \geq 2016$.

Să observăm acum că

$$\frac{a_n - a_{n+1}}{a_n} = -\beta_n a_n^{2015} \leq -Ba_n^{2015}, \quad \forall n.$$

Prin sumare, rezultă

$$\sum_{n=m}^{\infty} \frac{a_n - a_{n+1}}{a_n} \leq -B\sum_{n=m}^{\infty} a_n^{2015}. \tag{7}$$

Cum $(a_n)$ este descrescător, avem că

$$\frac{a_n - a_{n+1}}{a_n} \geq \frac{a_n - a_{n+1}}{a_m}, \quad \forall n \geq m,$$

de unde

$$\sum_{n=m}^{m+p} \frac{a_n - a_{n+1}}{a_n} \geq \frac{a_m - a_{m+p+1}}{a_m}, \quad \forall p \in \mathbb{N}.$$

Cum şirul din dreapta converge la 1 pentru $p \to \infty$, rezultă

$$-B \sum_{n=m}^{\infty} a_n^{2015} \geq \sum_{n=m}^{\infty} \frac{a_n - a_{n+1}}{a_n} \geq 1,$$

adică restul de ordin $m$ al seriei $\sum\limits_{n=1}^{\infty} a_n^{2015}$ nu converge la 0. Rezultă că $\sum\limits_{n=1}^{\infty} a_n^{2015}$ este divergentă. Prin aplicarea unui criteriu de comparaţie, avem că $\sum\limits_{n=1}^{\infty} a_n^{r}$ este divergentă pentru orice $r \leq 2015$.

Rămâne de studiat cazul $r \in (2015, 2016)$. Observăm că relaţia (6) asigură faptul că

$$a_{n+1}^k = a_n^k \left(1 + \beta_n a_n^{2015}\right)^k, \quad \forall k \in \mathbb{R},$$

de unde, folosind expresia seriei binomiale, avem că, pentru orice $n$ suficient de mare,

$$a_{n+1}^k - a_n^k = k\beta_n a_n^{2015+k} + \frac{k(k-1)}{2}\beta_n^2 a_n^{2\cdot 2015+k} + \frac{k(k-1)(k-2)}{6}\beta_n^3 a_n^{3\cdot 2015+k} + \ldots$$

Pentru $k \in (0,1)$, cum $\beta_n < 0$, rezultă că toţi termenii sumei din dreapta sunt negativi, deci

$$a_{n+1}^k - a_n^k \leq k\beta_n a_n^{2015+k}.$$

Rezultă

$$-Aka_n^{2015+k} \leq -k\beta_n a_n^{2015+k} \leq a_n^k - a_{n+1}^k.$$

Sumând, obţinem că

$$-Ak \sum_{n=1}^{m} a_n^{2015+k} \leq a_1^k - a_{m+1}^k.$$

Cum şirul din dreapta este convergent, rezultă convergenţa seriei $\sum\limits_{n=1}^{\infty} a_n^{2015+k}$ pentru orice $k \in (0,1)$. Această afirmaţie încheie rezolvarea problemei. □

**Notă.** Contribuţia celui de-al treilea autor este susţinută de proiectul „Sistem integrat de îmbunătăţire a calităţii cercetării doctorale şi postdoctorale din România şi de promovare a rolului ştiinţei în societate", POSDRU/159/1.5/S/133652, finanţat prin Fondul Social European, Programul Operaţional Sectorial Dezvoltarea Resurselor Umane 2007–2013.

# NOTE MATEMATICE

## A multiple sum and the Riemann zeta function

Nicolae Anghel[1)]

**Abstract**. In this short note we express a certain multiple sum as a linear combination with rational coefficients of values of the Riemann zeta function.

**Keywords:** Multiple sum, series, Riemann zeta function.
**MSC :** 14G10, 40B05, 11M06

The Riemann zeta function is one of the most important functions in all of mathematics. On its tail rests the last historically great unsolved problem, the Riemann hypothesis. With only minimal knowledge about convergent series the beginning undergraduate student or the high school mathematics teacher can get a glimpse of it, and this short note is certainly written with this idea in mind.

The Riemann zeta function, $\zeta(z)$, is defined for $z \in \mathbf{C}$, $\mathrm{Re}(z) > 1$, as the sum of the convergent series

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}. \tag{1}$$

The series (1) converges because its associated absolute series, $\sum\limits_{n=1}^{\infty} \frac{1}{n^x}$, where $z = x+iy$, converges for $x > 1$ to a sum not exceeding $1+\int_1^{\infty} \frac{\mathrm{d}t}{t^x} = \frac{x}{x-1}$, by the integral test for convergence of series with positive terms. $\zeta(z)$ extends then by analytic continuation to the whole complex plane, with only a simple pole at $z = 1$ [2]. The Riemann hypothesis asserts that the so-called non-trivial zeros of $\zeta(z)$ all belong to the critical line $\mathrm{Re}(z) = \frac{1}{2}$.

The purpose of this note is to express, for any positive integer $p$, the multiple sum

$$S_p(z) := \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_p=1}^{\infty} \frac{1}{(n_1+n_2+\cdots+n_p)^z}, \quad \mathrm{Re}(z) > p, \tag{2}$$

in terms of values of the Riemann zeta function. For a somewhat similar multiple sum problem, see [1]. Notice first that the multiple sum (2) converges when $\mathrm{Re}(z) > p$. Indeed, by the arithmetic-geometric mean inequality

[1)]Department of Mathematics, PO Box 311430, University of North Texas, Denton, TX 76203, USA, anghel@unt.edu

$\frac{n_1+n_2+\cdots+n_p}{p} \geq (n_1 n_2 \cdots n_p)^{1/p}$, for the absolute multiple sum we have

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_p=1}^{\infty}\frac{1}{(n_1+n_2+\cdots+n_p)^x} \leq \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_p=1}^{\infty}\frac{1}{p^x(n_1 n_2\cdots n_p)^{x/p}} = \frac{1}{p^x}\zeta^p\left(\frac{x}{p}\right).$$

Being absolutely convergent, it is then legitimate to replace the multiple summation in $S_p(z)$ by a single sum, via the substitution $n_1 + n_2 + \cdots + n_p = m$, $m \geq p$. Notice that there are exactly $\binom{m-1}{p-1}$ possible $p$-tuples of positive integers $(n_1, n_2, \ldots, n_p)$ summing up to a fixed $m \geq p$. Indeed, since $1 \leq n_1 < n_1 + n_2 < \cdots < n_1 + n_2 + \cdots + n_{p-1} \leq m - 1$, choosing a $p$-tuple $(n_1, n_2, \ldots, n_p)$, $n_1 + n_2 + \cdots + n_p = m$, is equivalent to choosing $p - 1$ members out of the set with $m - 1$ elements $\{1, 2, \ldots, m - 1\}$, which may then be ordered to yield $(n_1, n_1 + n_2, \ldots, n_1 + n_2 + \cdots + n_{p-1})$, and so $(n_1, n_2, \ldots, n_p)$. As a result,

$$\begin{aligned} S_p(z) &= \sum_{m=p}^{\infty}\binom{m-1}{p-1}\frac{1}{m^z} = \sum_{m=p}^{\infty}\frac{(m-1)(m-2)\cdots(m-p+1)}{(p-1)!}\frac{1}{m^z} \\ &= \sum_{m=1}^{\infty}\frac{(m-1)(m-2)\cdots(m-p+1)}{(p-1)!}\frac{1}{m^z} \\ &= \frac{1}{(p-1)!}\sum_{m=1}^{\infty}\frac{m(m-1)\cdots(m-p+1)}{m^{z+1}} \\ &= \frac{1}{(p-1)!}\sum_{m=1}^{\infty}\frac{\sum_{k=1}^{p} s(p,k)m^k}{m^{z+1}} = \frac{1}{(p-1)!}\sum_{k=1}^{p} s(p,k)\sum_{m=1}^{\infty}\frac{1}{m^{z+1-k}} \\ &= \frac{1}{(p-1)!}\sum_{k=1}^{p} s(p,k)\zeta(z+1-k), \end{aligned}$$

where $s(p, k)$ are the Stirling numbers of the first kind. Recall [3] that they are defined for a positive integer $p$ by the generating function

$$t(t-1)(t-2)\cdots(t-p+1) = \sum_{k=1}^{p} s(p,k)t^k.$$

Alternatively, up to parity signs they are given by the elementary symmetric functions of the first $p - 1$ positive integers.

In conclusion, for $\mathrm{Re}(z) > p$,

$$S_p(z) = \sum_{k=1}^{p}\frac{s(p,k)}{(p-1)!}\zeta(z+1-k), \qquad (3)$$

and (3) is the linear combination of values of the Riemann zeta function for our multiple sum we were looking for. In particular, $S_1(z) = \zeta(z)$, $S_2(z) = -\zeta(z) + \zeta(z-1)$, and $S_3(z) = \zeta(z) - \frac{3}{2}\zeta(z-1) + \frac{1}{2}\zeta(z-2)$.

*Acknowledgment.* The paper benefited from the insightful remarks of the referee.

## References

[1] N. Anghel, Proposed Problem No. 10522, *Amer. Math. Monthly* **103** (1996), 426. Solution to Problem No. 10522, *Amer. Math. Monthly* **105** (1998), 676.
[2] J. Conway, *Functions of one complex variable I*, Springer, New York, 1978.
[3] H. M. Srivastava, J. Choi, *Series associated with the zeta and related functions*, Kluwer Academic Publishers, Dordrecht, 2001.

# Finitely generated modules and a theorem of Orzech

Cornel Băeţica[1)]

*Dedicated to Professor Ion D. Ion*
*on the occasion of his 80th birthday*

**Abstract**. The aim of this note is to fill a gap in the proof of Orzech's theorem (see M. Orzech, Onto Endomorphisms are Isomorphisms, *Amer. Math. Monthly* **78** (1971), 357–362), and give some sufficient conditions on finitely generated modules to be free.

**Keywords:** Finitely generated modules, free modules.
**MSC :** 13C10

In [2] M. Orzech proved the following

**Theorem 1.** *Let $M$ be a finitely generated module over the commutative ring $R$. Let $N$ be any $R$-submodule of $M$. Let $f : N \to M$ be an $R$-module epimorphism. Then $f$ is an isomorphism.*

The proof goes by reduction to the noetherian case. Unfortunately this part of the proof has a gap in the original paper and the main purpose of this note is to fix it. For the sake of completeness we give a full proof by recording the noetherian case as well.

*Proof.* **Step 1. Reduction to the noetherian case.**
Let $0 \neq x'_0 \in N$. It suffices to prove $f(x'_0) \neq 0$.
Set $f(x'_0) = x_0$. Let $x_1, \ldots, x_n$ be a system of generators for $M$. Then $x'_0 = \sum_{i=1}^n a'_i x_i$ and $x_0 = \sum_{i=1}^n a_i x_i$ with $a_i, a'_i \in R$. Since $f$ is surjective there is $x'_i \in N$ such that $f(x'_i) = x_i$ and write $x'_i = \sum_{j=1}^n a_{ij} x_j$ with $a_{ij} \in R$ for all $i = 1, \ldots, n$.
Let $R' = \mathbb{Z}[a_{ij}, a_i, a'_i : i, j = 1, \ldots, n]$. Then $R'$ is a noetherian subring of

1) Department of Mathematics, University of Bucharest, 14, Academiei str., Bucharest, Romania, cornel.baetica@fmi.unibuc.ro

$R$. Now let $N' = R'x'_0 + R'x'_1 + \cdots + R'x'_n$, $M' = R'x_1 + \cdots + R'x_n$, and $f' : N' \to M'$ the restriction of $f$ to $N'$. It's easily seen that the image of $f'$ is contained in $M'$ and $f'$ is surjective since each generator of $M'$ is in $f(N')$. From the noetherian case we get that $f'$ is injective, and therefore $f(x'_0) \neq 0$.

**Step 2. The noetherian case.**

We assume that $M$ is a noetherian $R$-module. Set $K_0 = \ker f$ and define $K_n = f^{-1}(K_{n-1})$ for $n \geq 1$. By induction we show that $K_{n-1} \subseteq K_n$ for all $n \geq 1$. Since $N$ is also noetherian the ascending chain $K_0 \subseteq K_1 \subseteq \cdots$ of submodules of $N$ is stationary, that is, there exists an index $m$ such that $K_m = K_{m+1} = \cdots$. Now suppose that $f(x_0) = 0$ for some $x_0 \in N$. By using the surjectivity of $f$ we find $x_n \in K_n$ such that $f(x_n) = x_{n-1}$ for all $n \geq 1$. Since $x_{m+1} \in K_{m+1}$ and $K_{m+1} = K_m$ we get $x_{m+1} \in K_m$ and this implies $x_m \in K_{m-1}$. Successively we get $x_1 \in K_0$, that is, $f(x_1) = 0$. But $f(x_1) = x_0$ hence $x_0 = 0$. $\square$

Orzech's theorem has some interesting applications to finitely generated free modules.

**Corollary 2.** *Let $R$ be a commutative ring and $M$ a finitely generated $R$-module.*
*(i) If $M$ has $n$ generators then it does not contain a linearly independent set with $n+1$ elements.*
*(ii) If $M$ has $n$ generators and $n$ linearly independent elements, then it is free of rank $n$.*

*Proof.* (i) Suppose the contrary and let $F$ be a submodule of $M$ generated by $n+1$ linearly independent elements. We have an isomorphism $F \to R^{n+1}$. Since $M$ is generated by $n$ elements there exists an $R$-submodule $K$ of $R^n$ and an isomorphism $R^n/K \to M$. Now let $p : R^{n+1} \to R^n$ be the canonical projection which sends $(a_1, \ldots, a_{n+1})$ to $(a_1, \ldots, a_n)$. The composition of these maps gives rise to a surjective homomorphism $F \to M$. By Theorem 1 this must be an isomorphism. In particular, $p$ is injective, a contradiction.
(ii) Let $\sigma : R^n \to M$ be an injective homomorphism, and $\pi : R^n \to M$ a surjective homomorphism. Set $N = \sigma(R^n)$ and notice that $\sigma : R^n \to N$ is an isomorphism. Now consider $\pi\sigma^{-1} : N \to M$. This is a surjective homomorphism, hence by Theorem 1 an isomorphism. $\square$

**Remark 3.** (i) Here is an alternative proof of Corollary 2(i).
Let $x_1, \ldots, x_m \in M$ be linearly independent over $R$, and $y_1, \ldots, y_n \in M$ a system of generators. Now define an injective homomorphism $\sigma : R^m \to M$ by $\sigma(e_i) = x_i$ (here $(e_i)_{i=1,\ldots,m}$ denotes the canonical basis of $R^m$), and a surjective homomorphism $\pi : R^n \to M$ by $\pi(f_j) = y_j$ (where $(f_j)_{j=1,\ldots,n}$ is the canonical basis of $R^n$). Write $x_i = \sum_{j=1}^n a_{ij} y_j$ for $i = 1, \ldots, m$, and define a homomorphism $\varphi : R^m \to R^n$ by $\varphi(e_i) = \sum_{j=1}^n a_{ij} f_j$ for $i = 1, \ldots, m$. It

is easily checked that $\pi\varphi = \sigma$, so $\varphi$ is injective. Now use the Exercise 2.11 from [1] to conclude that $m \le n$.

(ii) Corollary 2(ii) can be restated as follows: *Let $R$ be a commutative ring and $M$ an $R$-module. If there is an injective $R$-module homomorphism $R^n \to M$, and a surjective $R$-module homomorphism $R^n \to M$ then $M$ is free of rank $n$.*

**Corollary 4.** *Let $M$ be a finitely generated module over a local integral domain $R$. Let $k$ be the residue field of $R$ and $K$ its field of fractions. Then $\dim_K(M \otimes_R K) \le \dim_k(M \otimes_R k)$, and the equality holds if and only if $M$ is free.*

*Proof.* Let $s = \dim_k(M \otimes_R k)$ and $t = \dim_K(M \otimes_R K)$. Then $s$ is the minimum number of a system of generators of $M$, and $t$ is the maximum number of linearly independent elements of $M$. From Corollary 2(i) we have $s \le t$.

If $s = t$ then from Corollary 2(ii) we get that $M$ is free. □

## References


[1] M.F. Atiyah, I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[2] M. Orzech, Onto Endomorphisms are Isomorphisms, *Amer. Math. Monthly* **78** (1971), 357–362.


# A note on problem 7 of day 2 of IMC 2015

Ovidiu Furdui[1)]

**Abstract**. In this note we give a new proof of Problem 7 of Day 2 of IMC 2015, Blagoevgrad, Bulgaria, and we generalize this problem by considering under the integral sign a function which verifies a certain condition.

**Keywords:** Integrals, limits, l'Hôspital's rule.

**MSC :** 26A06, 26A42

## 1. Introduction and the main result

Problem 7 of Day 2 of IMC 2015, Blagoevgrad, Bulgaria, is about calculating the interesting limit

$$\lim_{A\to\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x.$$

Two official solutions of this problem can be found online on the site of the competition at `http://www.imc-math.org.uk`. The first solution is based

[1)] Department of Mathematics, Technical University of Cluj-Napoca, 400114, Romania, `Ovidiu.Furdui@math.utcluj.ro`

on splitting the interval $[1, A]$ into three intervals and then by estimating the function under the integral sign on each of these intervals and the second solution uses a technique based on l'Hôspital's rule.

In the first part of this note we give a new solution of this problem, which is different than the second official solution, and in the second part of the paper we generalize the problem by considering under the integral sign a function which verifies a certain condition.

First we give a new solution of this problem by rewriting the integral into an equivalent form and then by applying l'Hôspital's rule twice.

Using the substitution $\frac{\ln A}{x} = t$ the integral becomes

$$\frac{1}{A}\int_1^A A^{\frac{1}{x}}\mathrm{d}x = \frac{1}{A}\int_1^A \mathrm{e}^{\frac{1}{x}\ln A}\mathrm{d}x = \frac{\ln A}{A}\int_{\frac{\ln A}{A}}^{\ln A}\frac{\mathrm{e}^t}{t^2}\mathrm{d}t.$$

We calculate the limit by applying l'Hôspital's rule twice and we have that

$$\begin{aligned}
\lim_{A\to\infty}\frac{1}{A}\int_1^A A^{\frac{1}{x}}\mathrm{d}x &= \lim_{A\to\infty}\frac{\ln A\int_{\frac{\ln A}{A}}^{\ln A}\frac{\mathrm{e}^t}{t^2}\mathrm{d}t}{A}\\
&= \lim_{A\to\infty}\left(\frac{1}{A}\int_{\frac{\ln A}{A}}^{\ln A}\frac{\mathrm{e}^t}{t^2}\mathrm{d}t + \frac{1}{\ln A} - \mathrm{e}^{\frac{\ln A}{A}}\cdot\frac{1-\ln A}{\ln A}\right)\\
&= 1 + \lim_{A\to\infty}\frac{\int_{\frac{\ln A}{A}}^{\ln A}\frac{\mathrm{e}^t}{t^2}\mathrm{d}t}{A}\\
&= 1 + \lim_{A\to\infty}\left(\frac{1}{\ln^2 A} - \mathrm{e}^{\frac{\ln A}{A}}\cdot\frac{1-\ln A}{\ln^2 A}\right)\\
&= 1,
\end{aligned}$$

and the problem is solved.

Now we solve the following generalization.

**A generalization of Problem 7.** *If $f : [1,\infty) \to \mathbb{R}$ is a continuous function such that $\lim_{x\to\infty} f(x) = L$, then*

$$\lim_{A\to\infty}\frac{1}{A}\int_1^A A^{\frac{1}{x}}f(x)\mathrm{d}x = L.$$

We will be using in our calculations the limit

$$\lim_{A\to\infty}\frac{1}{A}\int_1^A A^{\frac{1}{x}}\mathrm{d}x = 1,$$

whose proof was given above.

First recall that, since $f$ has a finite limit at $\infty$ one has that $f$ is bounded, i.e. there exists a positive real number $M$ such that $|f(x)| \leq M$, for all $x \geq 1$.

We have

$$\frac{1}{A}\int_1^A A^{\frac{1}{x}} f(x)\mathrm{d}x = \frac{1}{A}\int_1^A A^{\frac{1}{x}}(f(x)-L)\mathrm{d}x + \frac{L}{A}\int_1^A A^{\frac{1}{x}}\mathrm{d}x. \tag{1}$$

Let $\epsilon > 0$. Since $\lim\limits_{x\to\infty} f(x) = L$ there exists $\delta = \delta(\epsilon) > 0$ such that $|f(x) - L| < \epsilon$ for all $x > \delta$. We distinguish here two cases according to whether $\delta \leq 1$ or $\delta > 1$.

**The case $\delta \leq 1$.** We have

$$\left| \frac{1}{A}\int_1^A A^{\frac{1}{x}}(f(x)-L)\mathrm{d}x \right| \leq \frac{1}{A}\int_1^A A^{\frac{1}{x}}|f(x)-L|\mathrm{d}x < \frac{\epsilon}{A}\int_1^A A^{\frac{1}{x}}\mathrm{d}x$$

and it follows that

$$\lim_{A\to\infty} \left| \frac{1}{A}\int_1^A A^{\frac{1}{x}}(f(x)-L)\mathrm{d}x \right| \leq \epsilon,$$

from which we get, since $\epsilon$ is arbitrary taken, that

$$\lim_{A\to\infty} \frac{1}{A}\int_1^A A^{\frac{1}{x}}(f(x)-L)\mathrm{d}x = 0.$$

This implies, based on (1), that

$$\lim_{A\to\infty} \frac{1}{A}\int_1^A A^{\frac{1}{x}} f(x)\mathrm{d}x = L.$$

**The case $\delta > 1$.** We have

$$\frac{1}{A}\int_1^A A^{\frac{1}{x}}(f(x)-L)\mathrm{d}x = \frac{1}{A}\int_1^\delta A^{\frac{1}{x}}(f(x)-L)\mathrm{d}x + \frac{1}{A}\int_\delta^A A^{\frac{1}{x}}(f(x)-L)\mathrm{d}x.$$

On one hand

$$\left| \frac{1}{A}\int_\delta^A A^{\frac{1}{x}}(f(x)-L)\mathrm{d}x \right| \leq \frac{\epsilon}{A}\int_\delta^A A^{\frac{1}{x}}\mathrm{d}x \leq \frac{\epsilon}{A}\int_1^A A^{\frac{1}{x}}\mathrm{d}x,$$

and it follows that

$$\lim_{A\to\infty} \left| \frac{1}{A}\int_\delta^A A^{\frac{1}{x}}(f(x)-L)\mathrm{d}x \right| \leq \epsilon$$

and since $\epsilon$ was arbitrary taken one has that

$$\lim_{A\to\infty} \frac{1}{A}\int_\delta^A A^{\frac{1}{x}}(f(x)-L)\mathrm{d}x = 0. \tag{2}$$

On the other hand, the substitution $\frac{\ln A}{x} = t$ implies that

$$\left| \frac{1}{A}\int_1^\delta A^{\frac{1}{x}}(f(x)-L)\mathrm{d}x \right| \leq \frac{M+|L|}{A}\int_1^\delta A^{\frac{1}{x}}\mathrm{d}x = (M+|L|)\cdot\frac{\ln A}{A}\int_{\frac{\ln A}{\delta}}^{\ln A} \frac{\mathrm{e}^t}{t^2}\mathrm{d}t.$$

We have, by applying l'Hôspital's rule twice, that

$$\begin{aligned}\lim_{A\to\infty}\frac{1}{A}\int_1^{\delta}A^{\frac{1}{x}}\mathrm{d}x &= \lim_{A\to\infty}\frac{\ln A\int_{\frac{\ln A}{\delta}}^{\ln A}\frac{\mathrm{e}^t}{t^2}\mathrm{d}t}{A}\\ &= \lim_{A\to\infty}\left(\frac{1}{A}\int_{\frac{\ln A}{\delta}}^{\ln A}\frac{\mathrm{e}^t}{t^2}\mathrm{d}t+\frac{1}{\ln A}-\delta\frac{A^{\frac{1}{\delta}}}{A\ln A}\right)\\ &= \lim_{A\to\infty}\frac{\int_{\frac{\ln A}{\delta}}^{\ln A}\frac{\mathrm{e}^t}{t^2}\mathrm{d}t}{A}\\ &= \lim_{A\to\infty}\left(\frac{1}{\ln^2 A}-\frac{\delta A^{\frac{1}{\delta}}}{A\ln^2 A}\right)\\ &= 0,\end{aligned}$$

which implies that

$$\lim_{A\to\infty}\frac{1}{A}\int_1^{\delta}A^{\frac{1}{x}}(f(x)-L)\mathrm{d}x=0. \tag{3}$$

It follows, based on (1), (2) and (3), that

$$\lim_{A\to\infty}\frac{1}{A}\int_1^{A}A^{\frac{1}{x}}f(x)\mathrm{d}x=L,$$

and the problem is solved.

## Note added to the paper "Functions for which mixed partial derivatives are distinct", GMA, 3–4 (2014), 1–10, by D. Popa

DUMITRU POPA[1)]

In the paper mentioned in the title we referred to the function $f:\mathbb{R}^2\to\mathbb{R}$ defined by

$$f(x,y)=\begin{cases}\frac{xy\left(x^2-y^2\right)}{x^2+y^2} \text{ if } (x,y)\neq(0,0)\\ 0 \text{ if } (x,y)=(0,0)\end{cases}$$

and which has the property that $\frac{\partial^2 f}{\partial x\partial y}(0,0)=1$ and $\frac{\partial^2 f}{\partial y\partial x}(0,0)=-1$, as *Dieudonné example*. In fact, as we recently found out, this example appears in the book of A. Genocchi, G. Peano, *Calcolo differenziale e principii di calcolo integrale*, Roma Torino Firenze, Fratelli Boca, 1884, at page 174, so the correct name for the above function should be *Genocchi-Peano example*.

The book of A. Genocchi and G. Peano is freely available at `https://ia601406.us.archive.org/17/items/calcolodifferen00peangoog`.

[1)]Department of Mathematics, Ovidius University of Constanta, Bd. Mamaia 124, 900527 Constanţa, Romania, dpopa@univ-ovidius.ro

# PROBLEMS

Authors should submit proposed problems to gmaproblems@rms.unibuc.ro. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of May 2016**.

## PROPOSED PROBLEMS

**429.** Let $f$ be a $C^1$-class real valued function on $[0,1]$, infinitely differentiable at $x=0$. If $f^{(n)}(0)=0$ for every $n\in\mathbb{N}$ and, for some $C>0$ one has $|xf'(x)|\leq C|f(x)|$ for every $x\in[0,1]$, then $f(x)=0$ for every $x\in[0,1]$.

George Stoica, Department of Mathematical Sciences, University of New Brunswick, Canada.

**430.** Suppose that $r\in\mathbb{Q}$ and $m\in\mathbb{Z}$, $m>0$, such that $\cos^m r\pi\in\mathbb{Q}$. Determine all possible values of $\cos r\pi$.

Proposed by Marius Cavachi, Ovidius University, Constanţa, Romania.

**431.** Let $f:[0,1]^2\to\mathbb{R}$ be such that $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}:[0,1]^2\to\mathbb{R}$ are continuous. Find the value of the limit

$$\lim_{n\to\infty} n\left(n^2\iint\limits_{[0,1]^2} x^n y^n f(x,y)\,\mathrm{d}x\mathrm{d}y - f(1,1)\right).$$

Proposed by Dumitru Popa, Ovidius University Constanţa, Romania.

**432.** Let $k\geq 1$ be an integer. Find all $\alpha\in\mathbb{R}$ with the property that there is a sequence $(a_n)_{n\geq 1}$ such that $a_1+\cdots+a_{n+k}<\alpha a_n$ $\forall n\geq 1$.

Proposed by Constantin-Nicolae Beli, IMAR, Bucureşti, Romania.

**433.** Prove that

$$\sum_{k=1}^{\infty} k\left(k+1-\zeta(2)-\zeta(3)-\cdots-\zeta(k+1)\right)=\zeta(3),$$

where $\zeta$ denotes the Riemann zeta function.

Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

**434.** If $-1 \le \alpha < 0$, show that the function

$$f(x) = (x+4)^\alpha - 3(x+6)^\alpha + x(x+3)^\alpha - x(x+5)^\alpha$$

is strictly increasing for $x \ge 0$.

Proposed by Ioan Tomescu, Faculty of Mathematics and Informatics, University of Bucharest, Romania.

**435.** Let $f(x) = \frac{e^x}{(a+be^x)^2}$, where $a$, $b$ are two integers with $a+b=1$.

(i) Prove that $f^{(k)}(0) \in \mathbb{Z}$ for any $k \ge 0$. (Here $f^{(0)} = f$ and for $k \ge 1$ $f^{(k)}$ is the $k$th derivative of $f$.)

(ii) Prove that if $p$ is a prime and $\alpha \ge 1$ then for any integers $k, l \ge 0$ with $k \equiv l \pmod{p^{\alpha-1}(p-1)}$ we have $f^{(k)}(0) \equiv f^{(l)}(0) \pmod{p^\alpha}$.

Proposed by Constantin-Nicolae Beli, IMAR, Bucureşti, Romania.

**436.** Calculate

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln\sqrt{n(n+1)} - \gamma\right),$$

where $\gamma$ denotes the Euler–Mascheroni constant.

Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, România.

**437.** Let $f : [0,1] \to \mathbb{R}$ with $f(0) \in \mathbb{Q}$. Suppose that for every $x \in [0,1]$ there exists $\varepsilon_x > 0$ such that $f(x) - f(y) \in \mathbb{Q}$ for $y \in [0,1]$ and $|x-y| < \varepsilon_x$. Prove that $f(x) \in \mathbb{Q}$ for every $x \in [0,1]$.

Proposed by George Stoica, Department of Mathematical Sciences, University of New Brunswick, Canada.

**438.** Let $f_1, \ldots, f_n$ be polynomials with real coefficients with positive leading coefficients. Suppose that there is $M \in \mathbb{R}$ and $f \in \mathbb{R}[X]$ such that $f_i(x) > 0$ for $1 \le i \le n$ and $\sqrt{f_1(x)} + \cdots + \sqrt{f_n(x)} = f(x)$ $\forall x > M$. Prove that for any $1 \le i \le n$ we have $f_i = g_i^2$ for some $g_i \in \mathbb{R}[X]$.

Proposed by Marius Cavachi, Ovidius University, Constanţa, Romania.

## SOLUTIONS

**405.** Let $f \in C^2([0,1])$ such that

$$f(0) + 2f\left(\frac{1}{p}\right) + 2f\left(\frac{2}{p}\right) + \cdots + 2f\left(\frac{p-1}{p}\right) + f(1) = 0,$$

where $p \geq 2$ is an integer. Prove that

$$\left(\int_0^1 f(x)\mathrm{d}x\right)^2 \leq \frac{1}{120p^4}\int_0^1 \left(f''(x)\right)^2 \mathrm{d}x.$$

Proposed by Cristian Chiser, Craiova, Romania.

*Solution by the author.* For $1 \leq k \leq p$ we have:

$$\int_{\frac{k-1}{p}}^{\frac{k}{p}} \left(x - \frac{k-1}{p}\right)\left(x - \frac{k}{p}\right) f''(x)\mathrm{d}x$$
$$= \left(x - \frac{k-1}{p}\right)\left(x - \frac{k}{p}\right) f'(x)\Big|_{\frac{k-1}{p}}^{\frac{k}{p}} - \int_{\frac{k-1}{p}}^{\frac{k}{p}} \left(2x - \frac{2k-1}{p}\right) f'(x)\mathrm{d}x$$
$$= -\left(2x - \frac{2k-1}{p}\right) f(x)\Big|_{\frac{k-1}{p}}^{\frac{k}{p}} + 2\int_{\frac{k-1}{p}}^{\frac{k}{p}} f(x)\mathrm{d}x$$
$$= -\frac{1}{p}\left(f\left(\frac{k-1}{p}\right) + f\left(\frac{k}{p}\right)\right) + 2\int_{\frac{k-1}{p}}^{\frac{k}{p}} f(x)\mathrm{d}x.$$

By adding these relations we get

$$\sum_{k=1}^{p}\int_{\frac{k-1}{p}}^{\frac{k}{p}} \left(x - \frac{k-1}{p}\right)\left(x - \frac{k}{p}\right) f''(x)\mathrm{d}x$$
$$= -\frac{1}{p}\left(f(0) + 2f\left(\frac{1}{p}\right) + \cdots + 2f\left(\frac{p-1}{p}\right) + f(1)\right) + 2\int_0^1 f(x)\mathrm{d}x$$
$$= 2\int_0^1 f(x)\mathrm{d}x.$$

Let $\phi : [0,1] \to \mathbb{R}$ be given by $\phi(x) = (x - \frac{k-1}{p})(x - \frac{k}{p})$ when $x \in [\frac{k-1}{p}, \frac{k}{p}]$ for $k = 1, \ldots, p$. Note that $\phi$ is continuous and it vanishes precisely when $x = \frac{k}{p}$ with $k = 0, \ldots, p$. By the integral Cauchy inequality we get

$$4\left(\int_0^1 f(x)\mathrm{d}x\right)^2 = \left(\int_0^1 \phi(x)f''(x)\mathrm{d}x\right)^2 \leq \int_0^1 (\phi(x))^2\,\mathrm{d}x \int_0^1 \left(f''(x)\right)^2 \mathrm{d}x.$$

Note that for $x \in [\frac{k-1}{p}, \frac{k}{p}]$ we have $\phi(x + \frac{2k-1}{2p}) = x^2 - \frac{1}{4p^2}$. So, after making the substitution $u = x + \frac{2k-1}{2}$ we get

$$\int_{\frac{k-1}{p}}^{\frac{k}{p}} \phi(x)^2 \mathrm{d}x = \int_{-\frac{1}{2p}}^{\frac{1}{2p}} \left(x^2 - \frac{1}{4p^2}\right)^2 \mathrm{d}x = \left(\frac{1}{5}x^5 - \frac{1}{6p^2}x^3 + \frac{1}{16p^4}x\right)\Big|_{-\frac{1}{2p}}^{\frac{1}{2p}}$$
$$= \frac{1}{80p^5} - \frac{1}{24p^5} + \frac{1}{16p^5} = \frac{1}{30p^5}.$$

It follows that

$$4\left(\int_0^1 f(x)\mathrm{d}x\right)^2 \le p\frac{1}{30p^5}\int_0^1 f''(x)^2 \mathrm{d}x = \frac{1}{30p^4}\int_0^1 \left(f''(x)\right)^2 \mathrm{d}x,$$

which implies the desired inequality.

Since $\phi$ is continuous and vanishes only on a finite set the inequality becomes an equality precisely when $f'' = C\phi$ for some $C \in \mathbb{R}$ and $f$ satisfies the hypothesis $f(0) + 2f(\frac{1}{p}) + \cdots + 2f(\frac{p-1}{p}) + f(1) = 0$. If we consider $g_0 : [0,1] \to \mathbb{R}$ an arbitrary primitive of a primitive of $\phi$ then $g_0'' = \phi$.

If we take $f_0 = g_0 - \frac{1}{2p}(g_0(0) + 2g_0(\frac{1}{p}) + \cdots + 2g_0(\frac{p-1}{p}) + g_0(1))$ then, besides $f_0'' = \phi$, we also have $f_0(0) + 2f_0(\frac{1}{p}) + \cdots + 2f_0(\frac{p-1}{p}) + f_0(1) = 0$. If $f$ is arbitrary the condition $f'' = C\phi$ writes as $f'' = Cf_0''$, so $f = Cf_0 + ax + b$ for some $a, b \in \mathbb{R}$. Since $f_0(0) + 2f_0(\frac{1}{p}) + \cdots + 2f_0(\frac{p-1}{p}) + f_0(1) = 0$, the condition $f(0) + 2f(\frac{1}{p}) + \cdots + 2f(\frac{p-1}{p}) + f(1) = 0$ writes as

$$0 = b + 2\left(\frac{1}{p}a + b + \cdots + \frac{p-1}{p}a + b\right) + a + b = pa + 2pb.$$

Hence we have $f = Cf_0 + Dp(2x-1)$ for some $D \in \mathbb{R}$. With some work one can find an explicit function $g_0$ with $g_0'' = \phi$ and the corresponding $f_0$. Both $g_0$ and $f_0$ are given by polynomial formulas on each of the intervals $[\frac{k-1}{p}, \frac{k}{p}]$. □

**406.** Let $d$ be a positive integer. Define a $2d \times 2d$ matrix $\mathbf{M}(d)$ with entries in $\{-1, 0, 1\}$ as follows: For $1 \le a \le 2d$ and $1 \le b \le d$,

$$M_{a,2b-1} = \begin{cases} 1 & \text{if } a = 2b, \\ -1 & \text{if } a = 2b+2, \\ 0 & \text{otherwise} \end{cases} \quad M_{a,2b} = \begin{cases} 1 & \text{if } a = 2b-1, \\ -1 & \text{if } a = b-1, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that for every positive integer $d$, $\det\ \mathbf{M}(d) = (-1)^d$.

`Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.`

*Solution by the author.* For $d = 1$ one has $\mathbf{M}(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\det\ \mathbf{M}(1) = -1$. For $d > 1$, we proceed by calculating a $2 \times 2$ block matrix. As it is well known, the determinant of an arbitrary square matrix can be expressed in terms of the determinants of $2 \times 2$ matrices, via minor expansion [1, 2]. The solution of the problem can be obtained by considering the following equation

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{D}^{-1}\mathbf{C} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}, \tag{1}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are $k \times k$, $k \times (N-k)$, $(N-k) \times k$, and $(N-k) \times (N-k)$ matrices, respectively, and $\mathbf{I}$ and $\mathbf{0}$ are the identity and zero matrix, respectively (taken to be of the appropriate dimension), and it is assumed that $\mathbf{D}$ is invertible.

In our problem let us define

$$\mathbf{M}(d) = \begin{bmatrix} \mathbf{M}(d-1) & \mathbf{B}(d) \\ \mathbf{C}(d) & \mathbf{D} \end{bmatrix},$$

so we have $\mathbf{A} = \mathbf{M}(d-1)$ in Eq. (1), and $\mathbf{D} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{B}(d)$ and $\mathbf{C}(d)$ are the corresponding $(2d-2) \times 2$ and $2 \times (2d-2)$ matrices. Since $\mathbf{D}^{-1} = \mathbf{D}$, and $\det\ \mathbf{D} = -1$, taking the determinant of Eq. (1) we have that $\det\ (\mathbf{M}(d)) = (-1)\det\ (\mathbf{M}(d-1) - \mathbf{B}(d)\mathbf{D}\mathbf{C}(d))$.

The conclusion of the problem follows by induction if the product $\mathbf{B}(d)\mathbf{D}\mathbf{C}(d))$ is the zero matrix.

Note that the only non null entry of matrix $\mathbf{B}(d)$ is $B_{d-1,2} = -1$. Also the product $\mathbf{B}(d)\mathbf{D}$ is equivalent to change the columns of matrix $\mathbf{B(d)}$. Hence the only non null entry of matrix $\mathbf{B}(d)\mathbf{D}$ is $BD_{d-1,1} = -1$. On the other hand, the only non null entry of matrix $\mathbf{C}(d)$ is $C_{2,2d-1} = -1$. Therefore $\mathbf{B}(d)\mathbf{D}\mathbf{C}(d)) = \mathbf{0}$, and the problem is done. □

## References


[1] Philip D. Powell, Calculating Determinants of Block Matrices, arXiv: 1112.4379v1, `http://arxiv.org/abs/1112.4379`.

[2] John R. Silvester, Determinants of block matrices, *The Mathematical Gazette* **84** (501) (2000), 460–467.


**407.** (i) Let $i, n$ be two integers with $2 \leq i \leq n-2$ and let $x_k, y_k \in \mathbb{R}$ for $k = 1, \ldots, n$ such that $x_1 \geq \ldots \geq x_n$, $y_1 \geq \ldots \geq y_n$. Prove that

$$n \left( \sum_{j=1}^{i-1} x_j y_j + x_i y_{i+1} + x_{i+1} y_i + \sum_{j=i+2}^{n} x_j y_j \right) \geq \sum_{j=1}^{n} x_j \sum_{j=1}^{n} y_j.$$

(ii) As an application, prove that if $a_k, b_k \in \mathbb{R}$ for $k = 1, 2, 3, 4$ such that $a_1 \geq a_2 \geq a_3 \geq a_4 \geq 0$ and $b_1 \geq b_2 \geq b_3 \geq b_4 \geq 0$ then

$$\frac{a_1a_2b_1b_2 + a_2a_3b_2b_3 + a_3a_4b_3b_4 + a_4a_1b_4b_1}{4} \geq \frac{a_1a_2 + a_2a_3 + a_3a_4 + a_4a_1}{4} \times \frac{b_1b_2 + b_2b_3 + b_3b_4 + b_4b_1}{4}.$$

Proposed by Ovidiu Pop, Satu Mare, Romania.

*Solution by the author.* (i) Writing the known identity

$$n\sum_{k=1}^{n} x_ky_k - \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right) = \sum_{1\leq k<l\leq n} (x_k - x_l)(y_k - y_l)$$

for $y_1, \ldots, y_{i-1}, y_{i+1}, y_i, y_{i+2}, \ldots, y_n$ instead of $y_1, \ldots, y_n$, we see that the inequality (i) becomes

$$\sum_{\substack{1\leq k<l\leq n\\ k,l\notin\{i,i+1\}}} (x_k - x_l)(y_k - y_l) + \sum_{\substack{1\leq k\leq n\\ k\notin\{i,i+1\}}} \big((x_k - x_i)(y_k - y_{i+1}) + (x_k - x_{i+1})(y_k - y_i)\big) \geq (x_i - x_{i+1})(y_i - y_{i+1}).$$

We have $x_1 \geq x_i \geq x_{i+1} \geq x_n$ and $y_1 \geq y_i \geq y_{i+1} \geq y_n$, so all the products are non-negative. It is enough to find a term in the left hand side of the inequality which is $\geq (x_i-x_{i+1})(y_i-y_{i+1})$. One such term is $(x_1-x_n)(y_1-y_n)$. This term appears in the sum because $2 \leq i \leq n - 2$, so $1, n \notin \{i, i + 1\}$. Since $x_1 - x_n \geq x_i - x_{i+1} \geq 0$ and $y_1 - y_n \geq y_i - y_{i+1} \geq 0$, we also have $(x_1 - x_n)(y_1 - y_n) \geq (x_i - x_{i+1})(y_i - y_{i+1})$. □

**Remark 1.** Inequality (i) is a Chebyshev type inequality in the sense that $x_iy_{i+1}+x_{i+1}y_i$ replaces $x_iy_i+x_{i+1}y_{i+1}$. Unfortunately it only works for consecutive indices $i, i + 1$ with $i > 1$, $i + 1 < n$.

**Remark 2.** Our result (i) is stronger than Chebyshev's inequality since

$$\sum_{k=1}^{n} x_ky_k \geq x_1y_1 + \cdots + x_{i-1}y_{i-1} + x_iy_{i+1} + x_{i+1}y_i + x_{i+2}y_{i+2} + \cdots + x_ny_n.$$

Indeed, this inequality is equivalent to $(x_i - x_{i+1})(y_i - y_{i+1}) \geq 0$.

(ii) Let $x_1, x_2, x_3, x_4$ be the sequence $a_1a_2, a_2a_3, a_3a_4, a_4a_1$ written in decreasing order. Since $a_1 \geq a_2 \geq a_3 \geq a_4 \geq 0$ we have that $a_1a_2$ is the largest of all and $a_3a_4$ is the smallest. Hence $x_1 = a_1a_2$, $x_4 = a_3a_4$ and $(x_2, x_3) = (a_2a_3, a_4a_1)$ or $(a_4a_1, a_2a_3)$. Similarly, if $y_1, y_2, y_3, y_4$ are $b_1b_2, b_2b_3, b_3b_4, b_4b_1$ written in decreasing order then $y_1 = b_1b_2$, $y_4 = b_3b_4$ and $(y_2, y_3) = (b_2b_3, b_4b_1)$ or $(b_4b_1, b_2b_3)$.

In all cases, $a_1a_2b_1b_2 + a_2a_3b_2b_3 + a_3a_4b_3b_4 + a_4a_1b_4b_1$ writes either as $x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$ or as $x_1y_1 + x_2y_3 + x_3y_2 + x_4y_4$. In both cases it holds $4\sum_{cyc} a_ia_{i+1}b_ib_{i+1} \geq \sum_{cyc} a_ia_{i+1} \sum_{cyc} b_ib_{i+1}$, either by Chebyshev inequality or by (i). So we get the desired inequality. □

**Note from the editor.** Note that (ii) can be false without the hypothesis $a_4 \geq 0$, $b_4 \geq 0$, as the example of the sequences $1, 0, 0, -1$ and $2, 1, 1, 1$ shows.

**408.** Let $p$ be a prime and let $n \geq 2$ be an integer with $p \nmid n$. Prove that

$$\mathbb{Z}_p[X^n] \cap \mathbb{Z}_p[X^n + X^p] = \mathbb{Z}_p.$$

Proposed by Victor Alexandru, University of Bucharest, Romania.

*Solution by the author.* Assume that $\mathbb{Z}_p[X^n] \cap \mathbb{Z}_p[X^n + X^p] \neq \mathbb{Z}_p$.

Then there are nonconstant polynomials $P, Q \in \mathbb{Z}_p[X]$ such that $P(X^n) = Q(X^n + X^p)$. We assume also that $P, Q$ with this property are chosen such that $\deg P$ is minimum.

By taking derivatives we get $nX^{n-1}P'(X^n) = nX^{n-1}Q'(X^n + X^p)$, so $P'(X^n) = Q'(X^n + X^p)$. We have $\deg P' < \deg P$ so, by the minimality of $\deg P$, $P'$ and $Q'$ are constant, i.e., $P' = Q' = a$, with $a \in \mathbb{Z}_p$. It follows that $P = aX + f(X^p)$ and $Q = aX + g(X^p)$ for some $f, g \in \mathbb{Z}_p[X]$.

If $a = 0$ then $P = f(X^p) = f(X)^p$ and $Q = g(X^p) = g(X)^p$, so $P(X^n) = Q(X^n + X^p)$ writes as $f(X^n)^p = g(X^n + X^p)^p$, which implies $f(X^n) = g(X^n + X^p)$. But this contradicts the minimality of $\deg P$. (We have $\deg P = p \deg f$, so $0 < \deg f < \deg P$.) We conclude that $a \neq 0$, in which case $P(X^n) = Q(X^n + X^p)$ writes as

$$aX^n + f(X^{np}) = a(X^n + X^p) + g(X^{np} + X^{p^2}),$$

so that one has $aX^p = g(X^{np} + X^{p^2}) - f(X^{np})$. But this is impossible since, except the constant term, $g(X^{np} + X^{p^2}) - f(X^{np})$ has no monomials of degree smaller than $\min\{np, p^2\}$. □

**409.** Let $n, k$ be two integers with $n \geq 3$ and $0 < k \leq n$. Given a convex polygon with $n$ sides determine how many sets of $k$ sides have the property that no two sides are adjacent.

Proposed by Ionel Popescu, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

*Solution by the author.* We first solve the similar problem when the sides of a polygon are replaced by the segments of a broken line. We don't require here that $n \geq 3$. Let $A_1, \ldots, A_n$ be the consecutive vertices of a broken line. Let $A_{i_1}, A_{i_1+1}, \ldots, A_{i_k}, A_{i_k+1}$, with $1 \leq i_1 < \ldots < i_k \leq n-1$, be $k$ segments of the broken line. The condition that these pairs are mutually not adjacent is equivalent to $i_{j+1} - i_j \geq 2$ for $1 \leq j \leq k-1$. So we want to determine $|A|$, where $A = \{(i_1, \ldots, i_k) \mid 1 \leq i \leq n-1,\ i_{j+1} - i_j \geq 2\}$.

If $f(i_1, i_2, \ldots, i_k) = (i_1, i_2 - 1, \ldots, i_k - k + 1)$ then $f$ is a bijection between $A$ and $B = \{(l_1, \ldots, l_k) \mid 1 \leq l_1 < \ldots < l_k \leq n - 2k\}$.

Indeed, if $(l_1, \ldots, l_k) = (i_1, i_2 - 1, \ldots, i_k - k + 1)$ then $l_1 = i_1$ and $l_k = i_k - k + 1$, so $1 \leq i_1$ iff $1 \leq l_1$ and $i_k \leq n - 1$ iff $l_k \leq n - k$. Also

$l_{j+1}-l_j=(i_{j+1}-j)-(i_j-j+1)=i_{j+1}-i_j-1$, so $i_{j+1}-i_j\geq 2$ iff $l_{j+1}-l_j\geq 1$, i.e., iff $l_j<l_{j+1}$. Hence $(i_1,i_2,\ldots,i_k)\in A$ iff $(l_1,\ldots,l_k)\in B$.

Hence $|A|=|B|=\binom{n-k}{k}$.

Suppose now that $A_1,\ldots,A_n$ are the vertices of a polygon $P$ and let $1\leq k\leq n/2$. We want to estimate $|S|$, where $S$ is the set of all sets of $k$ mutually not adjacent sides of $P$.

The set $S$ writes as a disjoint union $S=T\cup U$, where

$$T=\{M\in S\mid A_nA_1\notin M\},\quad U=\{M\in S\mid A_nA_1\in M\}.$$

If $M\in T$ then $M\subseteq\{A_1A_2,\ldots,A_{n-1}A_n\}$. So we are in the case of a broken line with $n$ vertices. Thus $|T|=\binom{n-k}{k}$. If $M\in U$ then $M=\{A_nA_1\}\cup M'$, where $M'$ is a set with $k-1$ mutually non adjacent sides of $P$ which are also not adjacent to $A_nA_1$, so $M'\subseteq\{A_2A_3,\ldots,A_{n-2}A_{n-1}\}$. So we are in the case of sets of $k-1$ mutually disjoint segments on a broken line with $n-2$ vertices. It follows that $|U|=\binom{n-k-1}{k-1}$. The formula holds even when $k=1$, that is, $k-1=0$. In this case $U=\{A_nA_1\}$ and $\binom{n-k-1}{k-1}=\binom{n-2}{0}=1=|U|$.

In conclusion, $|S|=\binom{n-k}{k}+\binom{n-k-1}{k-1}=\frac{n}{k}\binom{n-k-1}{k-1}$. □

**410.** Let $a,b\in(0,1)$ and let $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ be two sequences such that $a_0=a$, $b_0=b$ and for any $n\geq 0$ we have either $a_{n+1}=a_n^{\alpha_n}$, $b_{n+1}=b_n^{\alpha_n}$ for some $\alpha_n\geq 1$ or $a_{n+1}=\lambda_na_n+(1-\lambda_n)$, $b_{n+1}=\lambda_nb_n+(1-\lambda_n)$ for some $\lambda_n\in[0,1]$.

Prove the following.

(i) $(a_n)$ is convergent iff $(b_n)$ is convergent.

(ii) We have $\lim_{n\to\infty}a_n=0$ or 1 iff $\lim_{n\to\infty}b_n=0$ or 1, respectively.

(iii) If $(a_n)$, $(b_n)$ are divergent then $\lim_{n\to\infty}(a_n-b_n)=0$.

Proposed by Liviu Păunescu and Constantin-Nicolae Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

*Solution by the authors.* In the case $a_{n+1}=\lambda_na_n+(1-\lambda_n)$, $b_{n+1}=\lambda_nb_n+(1-\lambda_n)$ we may assume that $\lambda_n\neq 0$ since otherwise $a_{n+1}=b_{n+1}=1$, which, by an immediate induction, implies $a_m=b_m=1$, $\forall m>n$ and our statements become trivial. Also if $a=b$ then $a_n=b_n$ $\forall n$, so our statements are trivial. So we may assume that $a>b$. Since both mappings $x\mapsto x^\alpha$, where $\alpha\geq 1$, and $x\mapsto\lambda x+(1-\lambda)$, where $0<\lambda\leq 1$, are strictly increasing, we have by induction $a_n>b_n$ $\forall n\geq 0$. Also by induction we get $a_n,b_n\in(0,1)$ $\forall n\geq 0$.

Let $c_n=\frac{1-a_n}{1-b_n}$, $d_n=\frac{\log a_n}{\log b_n}$. As $0<b_n<a_n<1$, we have $0<c_n,d_n<1$. We claim that both sequences $(c_n)$ and $(d_n)$ are increasing.

If $a_{n+1}=\lambda_na_n+(1-\lambda_n)$, $b_{n+1}=\lambda_nb_n+(1-\lambda_n)$ then $c_{n+1}=\frac{1-a_{n+1}}{1-b_{n+1}}=\frac{1-a_n}{1-b_n}=c_n$. If $a_{n+1}=a_n^{\alpha_n}$, $b_{n+1}=b_n^{\alpha_n}$ then $c_{n+1}=\frac{1-a_n^{\alpha_n}}{1-b_n^{\alpha_n}}$. To prove that

$c_{n+1} \geq c_n = \frac{1-a_n}{1-b_n}$ we need the inequality $\frac{1-a_n^{\alpha_n}}{1-a_n} \geq \frac{1-b_n^{\alpha_n}}{1-b_n}$. But this follows from the fact that the mapping $x \mapsto \frac{1-x^\alpha}{1-x}$ is increasing on $(0,1)$ when $\alpha > 1$. We have $\left(\frac{1-x^\alpha}{1-x}\right)' = \frac{1-\alpha x^{\alpha-1}+(\alpha-1)x^\alpha}{(1-x)^2}$ and the inequality $1 - \alpha x^{\alpha-1} + (\alpha - 1)x^\alpha > 0$ follows from the fact that the function $\alpha \mapsto x^\alpha$ is strictly convex when $x < 1$. (We have $x^{\frac{1}{\alpha}0+(1-\frac{1}{\alpha})\alpha} < \frac{1}{\alpha}x^0 + (1-\frac{1}{\alpha})x^\alpha$.)

If $a_{n+1} = a_n^{\alpha_n}$, $b_{n+1} = b_n^{\alpha_n}$ then $d_{n+1} = \frac{\log a_{n+1}}{\log b_{n+1}} = \frac{\log a_n}{\log b_n} = d_n$. If $a_{n+1} = \lambda_n a_n + (1-\lambda_n)$, $b_{n+1} = \lambda_n b_n + (1-\lambda_n)$ then $d_{n+1} \geq d_n$ is equivalent to $\frac{\log a_{n+1}}{\log a_n} \geq \frac{\log b_{n+1}}{\log b_n}$, i.e., $\frac{\log(\lambda_n a_n+(1-\lambda_n))}{\log a_n} \geq \frac{\log(\lambda_n b_n+(1-\lambda_n))}{\log b_n}$, which follows from the fact that the mapping $x \mapsto \frac{\log(\lambda x+(1-\lambda))}{\log x}$ is increasing on $(0,1)$ when $0 \leq \lambda < 1$. We have

$$\left(\frac{\log(\lambda x + (1-\lambda))}{\log x}\right)' = \frac{\log(\lambda x + (1-\lambda))}{\log x} \times \left(\frac{\lambda}{(\lambda x + (1-\lambda))\log(\lambda x + (1-\lambda))} - \frac{1}{x\log x}\right).$$

To prove that this derivative is positive we need the inequality $(\lambda x + (1-\lambda))\log(\lambda x+(1-\lambda)) < \lambda x \log x$. But this follows from the fact that $(x\log x)'' = \frac{1}{x} > 0$, which shows that the mapping $x \mapsto x\log x$ is strictly convex. (We have $(\lambda x + (1-\lambda)1)\log(\lambda x + (1-\lambda)1) < \lambda x \log x + (1-\lambda)1\log 1$.)

Since $(c_n)$, $(d_n)$ are both increasing we have $c_0 \leq c_n < 1$ and $d_0 \leq d_n < 1$, i.e., $0 < c_0 \leq \frac{1-a_n}{1-b_n} < 1$ and $0 < d_0 \leq \frac{\log a_n}{\log b_n} < 1$. The first implies that $\lim_{n\to\infty}(1-a_n) = 0$ iff $\lim_{n\to\infty}(1-b_n) = 0$, i.e., $\lim_{n\to\infty} a_n = 1$ iff $\lim_{n\to\infty} b_n = 1$. From the second we get $\lim_{n\to\infty} \log a_n = -\infty$ iff $\lim_{n\to\infty} \log b_n = -\infty$, i.e., $\lim_{n\to\infty} a_n = 0$ iff $\lim_{n\to\infty} b_n = 0$. Hence we have (ii).

The monotony of $(c_n)$ and $(d_n)$ also implies that $\lim_{n\to\infty} c_n = c$ and $\lim_{n\to\infty} d_n = d$ for some $c \in [c_0, 1] \subset (0,1]$ and $d \in [d_0, 1] \subset (0,1]$. We also have $c_n \leq c$, $d_n \leq d$ $\forall n$.

To complete our proof we will show that

(1) If $c = 1$ then $\lim_{n\to\infty}(a_n - b_n) = 0$.

(2) If $c < 1$ then both $(a_n)$ and $(b_n)$ are convergent.

Obviously (1) implies both (i) and (iii) in the case when $c = 1$ and (2) implies (i) and (iii) in the case when $c < 1$.

If $c = 1$ then $\frac{1-a_n}{1-b_n} = c_n$ implies $a_n - b_n = (1-c_n)(1-b_n)$, so $0 < a_n - b_n \leq (1-c_n)$. Since $\lim_{n\to\infty}(1-c_n) = (1-c) = 0$, we have $\lim_{n\to\infty}(a_n - b_n) = 0$.

Hence we have (1). For (2) we need the following lemma.

**Lemma.** If $A := \{(x,y) \in \mathbb{R}^2 \mid 0 < y < x < 1\}$ then we have a diffeomorphism $\phi : A \to A$ given by $(x,y) \mapsto (\frac{1-x}{1-y}, \frac{\log x}{\log y})$.

*Proof.* We denote $\phi = (z, t)$, where $z = \frac{1-x}{1-y}$, $t = \frac{\log x}{\log y}$. From $0 < y < x < 1$ we get $z, t \in (0, 1)$. The condition $t = \frac{\log x}{\log y}$ writes as $y = x^{1/t}$, so $z = \phi_t(x)$, where $\phi_t : (0, 1) \to \mathbb{R}$ is given by $x \mapsto \frac{1-x}{1-x^{1/t}}$. Since $\frac{1}{t} > 1$, the mapping $x \mapsto \frac{1-x^{1/t}}{1-x}$ is strictly increasing on $(0, 1)$ (see above), so $\phi_t$ is strictly decreasing.

Now $\lim\limits_{x\searrow 0} \phi_t(x) = 1$ and by l'Hospital's rule $\lim\limits_{x\nearrow 1} \phi_t(x) = \lim\limits_{x\nearrow 1} \frac{-1}{-\frac{1}{t}x^{\frac{1}{t}-1}} = t$. Hence $\phi_t$ is a decreasing bijection between $(0, 1)$ and $(t, 1)$.

In conclusion, $\phi$ is a bijection between $A$ and the set of all $(z, t) \in (0, 1)^2$ with $z \in (t, 1)$, i.e., $\{(z, t) \in \mathbb{R}^2 \mid 0 < t < z < 1\} = A$. The inverse of $\phi$ is given by $(z, t) \mapsto (x, x^{1/t})$, where $x = \phi_t^{-1}(z)$.

If $(x, y) \in A$ then one calculates

$$\begin{vmatrix} \dfrac{\mathrm{d}z}{\mathrm{d}x} & \dfrac{\mathrm{d}z}{\mathrm{d}y} \\ \dfrac{\mathrm{d}t}{\mathrm{d}x} & \dfrac{\mathrm{d}t}{\mathrm{d}y} \end{vmatrix} = zt\left(\frac{-1}{(x-1)y\log y} + \frac{1}{(y-1)x\log x}\right)$$

$$= \frac{zt}{xy\log x\log y}\left(\frac{x\log x}{1-x} - \frac{y\log y}{1-y}\right).$$

We prove that the Jacobian above is negative by showing that $\frac{x\log x}{1-x} > \frac{y\log y}{1-y}$, which will follow from the fact that $u \mapsto \frac{u\log u}{1-u}$ is decreasing on $(0, 1)$, i.e., that $0 > \left(\frac{u\log u}{1-u}\right)' = \frac{u\log u}{1-u}\left(\frac{1}{u} + \frac{1}{u\log u} - \frac{1}{u-1}\right) = \frac{1}{(1-u)^2}(\log u + 1 - u)\ \forall u \in (0, 1)$. But if $\varepsilon = 1 - u \in (0, 1)$ then $\log u + 1 - u = \log(1-\varepsilon) + \varepsilon = -\sum\limits_{n\geq 2} \frac{1}{n}\varepsilon^n < 0$, so we are done.

Since $\phi : A \to A$ is a differentiable bijection and its Jacobian vanishes nowhere, it is a diffeomorphism. $\square$

Let $\psi : A \to A$ be the inverse of $\phi$. Then $\psi$ is differentiable, so continuous.

We now prove (2). We have $(c_n, d_n) = (\frac{1-a_n}{1-b_n}, \frac{\log a_n}{\log b_n}) = \phi(a_n, b_n) \in A$, so $(a_n, b_n) = \psi(c_n, b_n)$. We have $d_0 \leq d_n < c_n < 1$, so, by considering limits, $0 < d_0 \leq d \leq c \leq 1$. By hypothesis $c < 1$.

Assume first that $d < c$. Then $(c, d) \in A$. Since $\lim\limits_{n\to\infty}(c_n, d_n) = (c, d)$ and $\psi$ is continuous on $A$, we have $\psi(c, d) = \lim\limits_{n\to\infty} \psi(c_n, d_n) = \lim\limits_{n\to\infty}(a_n, b_n)$, which implies that $(a_n)$ and $(b_n)$ are both convergent.
In the remaining case $c = d \in (0, 1)$, we prove that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 1$. We have $d_n = \frac{\log a_n}{\log b_n}$, so $a_n = b_n^{d_n}$. Hence $c_n = \frac{1-b_n^{d_n}}{1-b_n}$. If $\varepsilon = 1 - b_n \in (0, 1)$

we get

$$c_n = \frac{1-(1-\varepsilon)^{d_n}}{\varepsilon} = \frac{1}{\varepsilon}\left(1 - \sum_{k\geq 0}(-1)^k\binom{d_n}{k}\varepsilon^k\right)$$
$$= \sum_{k\geq 1}(-1)^{k-1}\binom{d_n}{k}\varepsilon^{k-1}.$$

But for any $k \geq 1$ we have

$$(-1)^{k-1}\binom{d_n}{k} = (-1)^{k-1}\frac{1}{k!}d_n(d_n-1)\cdots(d_n-k+1)$$
$$= \frac{1}{k!}d_n(1-d_n)\cdots(k-1-d_n) > 0,$$

so all the terms in the sum above are positive. It follows that

$$c_n > \binom{d_n}{1} - \binom{d_n}{2}\varepsilon = d_n + \frac{d_n(1-d_n)}{2}\varepsilon = d_n + \frac{d_n(1-d_n)}{2}(1-b_n),$$

so $0 < 1-b_n < \frac{2(c_n-d_n)}{d_n(1-d_n)}$. But $\lim\limits_{n\to\infty}\frac{2(c_n-d_n)}{d_n(1-d_n)} = \frac{2(c-d)}{d(1-d)} = 0$, whence $\lim\limits_{n\to\infty} b_n = 1$. Since $b_n < a_n < 1$, we also have $\lim\limits_{n\to\infty} a_n = 1$. □

**Remark.** We have $0 < d \leq c \leq 1$. The only case when $(a_n)$ and $(b_n)$ can be divergent is when $d = c = 1$. In all other cases $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ exist and they are uniquely determined by $c$ and $d$. Indeed, if $d < c < 1$ then $(c,d) \in A$, so $(\lim\limits_{n\to\infty} a_n, \lim\limits_{n\to\infty} b_n) = \psi(c,d)$. If $d = c < 1$ then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 1$. We claim that in the remaining case $d < c = 1$ one has $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$.

We have $\frac{\log a_n}{\log b_n} = d_n$, so $a_n = b_n^{d_n} \geq b_n^d$. Then $c_n = \frac{1-a_n}{1-b_n} \leq \frac{1-b_n^d}{1-b_n} < 1$. (Recall that $b_n < 1$ and $d_n \leq d < 1$.)

The function $f : [0,1) \to \mathbb{R}$, $f(x) = \frac{1-x^d}{1-x}$ is decreasing. We have $f'(x) = \frac{1-dx^{d-1}-(1-d)x^d}{(1-x)^2}$ and the inequality $1 - dx^{d-1} - (1-d)x^d < 0$ follows from the strict convexity of the function $d \mapsto x^d$ when $x < 1$. (We have $1 = x^{d(d-1)+(1-d)d} < dx^{d-1} + (1-d)x^d$.)

Since $c_n \leq f(b_n) < 1$ and $\lim\limits_{n\to\infty} c_n = c = 1$, we have $\lim\limits_{n\to\infty} f(b_n) = 1$. Since $f(0) = 1$ and $f$ is strictly decreasing and continuous, this implies that $\lim\limits_{n\to\infty} b_n = 0$. We also have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n^{d_n} = 0^d = 0$. □

**411.** Consider the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \int_0^x \cos\frac{1}{t}\cos\frac{3}{t}\cos\frac{5}{t}\cos\frac{7}{t}\mathrm{d}t.$$

Prove that $f$ is well defined, differentiable and $f'(0) = \frac{1}{8}$.

Proposed by Eugen J. Ionaşcu, Department of Mathematics, Columbus State University, Columbus, Georgia, U.S.A.

*Solution by the author.* From the formulas

$$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big), \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

it follows that

$$\begin{aligned}\cos\frac{1}{t}\cos\frac{3}{t}\cos\frac{5}{t}\cos\frac{7}{t} &= \frac{1}{2}\left(\cos\frac{2}{t} + \cos\frac{4}{t}\right)\frac{1}{2}\left(\cos\frac{2}{t} + \cos\frac{12}{t}\right)\\ &= \frac{1}{4}\left(\cos^2\frac{2}{t} + \cos\frac{2}{t}\cos\frac{4}{t} + \cos\frac{2}{t}\cos\frac{12}{t} + \cos\frac{4}{t}\cos\frac{12}{t}\right)\\ &= \frac{1}{8}\left(1 + \cos\frac{2}{t} + \cos\frac{4}{t} + \cos\frac{6}{t} + \cos\frac{8}{t} + \cos\frac{10}{t} + \cos\frac{14}{t} + \cos\frac{16}{t}\right).\end{aligned}$$

Hence, in order to prove our statement is suffices to show that $f_0(x) = \int_0^x 1\mathrm{d}t$ is well defined and differentiable with $f_0'(0) = 1$, and that if $a > 0$ then $f_a(x) = \int_0^x \cos\frac{a}{t}\mathrm{d}t$ is well defined and differentiable with $f_a'(0) = 0$.

The first statement is trivial. For the second statement we note that if $0 < y < z$ then, by a substitution and an integration by parts we have $\int_y^z \cos\frac{a}{t}\mathrm{d}t = a\int_{a/z}^{a/y}\frac{1}{u^2}\cos u\,\mathrm{d}u = a\frac{1}{u^2}\sin u\Big|_{a/z}^{a/y} + a\int_{a/z}^{a/y}\frac{2}{u^3}\sin u\,\mathrm{d}u$.

But from the inequalities

$$\Big|a\frac{1}{u^2}\sin u\Big|_{a/z}^{a/y}\Big| = \Big|\frac{y^2}{a}\sin a/y - \frac{z^2}{a}\sin a/z\Big| \le \frac{y^2}{a} + \frac{z^2}{a}$$

and

$$\Big|a\int_{a/z}^{a/y}\frac{2}{u^3}\sin u\,\mathrm{d}u\Big| \le a\int_{a/z}^{a/y}\frac{2}{u^3}\mathrm{d}u = \frac{z^2}{a} - \frac{y^2}{a}$$

it follows that $\Big|\int_y^z \cos\frac{a}{t}\mathrm{d}t\Big| \le \frac{2z^2}{a}$.

Let now $x > 0$ then for $0 < y < z < x$ we have

$$\Big|\int_y^x \cos\frac{a}{t}\mathrm{d}t - \int_z^x \cos\frac{a}{t}\mathrm{d}t\Big| = \Big|\int_y^z \cos\frac{a}{t}\mathrm{d}t\Big| \le \frac{2z^2}{a}.$$

Since $\frac{z^2}{a} \to 0$ as $z \to 0$, the map $y \mapsto \int_y^x \cos\frac{a}{t}\mathrm{d}t$ has the Cauchy property at 0, so $\int_0^x \cos\frac{a}{t}\mathrm{d}t = \lim\limits_{y\to 0}\int_y^x \cos\frac{a}{t}\mathrm{d}t$ exists, i.e., $f_a(x)$ is defined.

If $x > 0$ then for any $0 < y < x$ one has $\Big|\int_y^x \cos\frac{a}{t}\mathrm{d}t\Big| \le \frac{2x^2}{a}$. By taking limits as $y \to 0$ we get $|f_a(x)| \le \frac{2x^2}{a}$. It follows that we have $|\frac{f_a(x)}{x}| \le \frac{2x}{a}$, so that $f_a'(x) = \lim\limits_{x\to 0}\frac{f_a(x) - f_a(0)}{x - 0} = \lim\limits_{x\to 0}\frac{f_a(x)}{x} = 0$.

Note that so far we only proved our statement on the interval $[0, \infty)$. (And by $f_a'(0)$ we mean the derivative to the right of $f_a$ at 0.) For the interval

$(-\infty, 0]$ we use the fact that $t \mapsto \cos\frac{a}{t}$ is an even function. Then $f_a$ will be defined everywhere and odd. □

*A similar solution was given by Moubinool Omarjee, Lycée Henri IV, Paris, France.*

**412.** Let $[A, B, C, D]$ be an equifacial tetrahedron with the lengths of the sides $a, b, c$ and the lenght of the heights $h$. Prove that $h > \frac{1}{\sqrt{2}}\max\{a, b, c\}$ and

$$\frac{a^2b^2c^2}{(2h^2-a^2)(2h^2-b^2)(2h^2-c^2)} \geq 27.$$

Proposed by Marius Olteanu, S.C. Hidroconstrucţia S.A., sucursala Muntenia, Râmnicu Vâlcea, România.

*Solution by the author.* By the geometric mean–harmonic mean inequality we have

$$\sqrt[3]{\frac{a^2b^2c^2}{(2h^2-a^2)(2h^2-b^2)(2h^2-c^2)}} \geq \frac{3}{\frac{2h^2-a^2}{a^2}+\frac{2h^2-b^2}{b^2}+\frac{2h^2-c^2}{c^2}} = \frac{3}{2h^2(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2})-3},$$

that is

$$\frac{a^2b^2c^2}{(2h^2-a^2)(2h^2-b^2)(2h^2-c^2)} \geq \frac{27}{(2h^2(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2})-3)^3}.$$

To complete the proof we show that

$$\frac{27}{(2h^2(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2})-3)^3} \geq 27,$$

which is equivalent to $2h^2(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2})-3 \leq 1$, i.e., to $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \leq \frac{2}{h^2}$.

Since $[A, B, C, D]$ is an equifacial tetrahedron, we have $h = 4r$, where $r$ is the radius of the sphere inscribed in the tetrahedron. The inequality to prove becomes $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \leq \frac{1}{8r^2}$. But by point g) of Consecinţe from [1], page 102, we know that in any tetrahedron [ABCD] we have

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{l^2}+\frac{1}{m^2}+\frac{1}{n^2} \leq \frac{1}{4r^2},$$

where $a, b, c, l, m, n$ are the edges $BC$, $AC$, $AB$, $AD$, $BD$, $CD$ of $[ABCD]$. But $[ABCD]$ is equifacial, so $a = l$, $b = m$, $c = n$. Hence the last inequality writes as $(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}) \leq \frac{1}{4r^2}$, which yields the desired result. □

## References

[1] Marius Olteanu, Noi rafinări ale inegalităţii lui Durrande în tetraedru, GMA, **26** (2008), 98–108.

**413.** Prove that $\displaystyle\int_0^\infty \frac{\cos x - \cos(tx)}{x}\mathrm{d}x = \ln|t|$ for any $t \neq 0$.

`Proposed by George Stoica, Department of Mathematical Sciences, University of New Brunswick, Canada.`

*Solution by Victor Makanin, Sankt Petersburg, Russia.* First we prove the following

**Proposition.** Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function such that $\int_k^\infty \frac{f(x)}{x}\mathrm{d}x$ is convergent for some $k > 0$. Then, for every positive $a$ and $b$, we have

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\mathrm{d}x = f(0)\ln\frac{b}{a}.$$

*Proof.* We have, for $\alpha, A > 0$,

$$\int_\alpha^A \frac{f(ax) - f(bx)}{x}\mathrm{d}x = \int_{\alpha a}^{Aa} \frac{f(x)}{x}\mathrm{d}x - \int_{\alpha b}^{Ab} \frac{f(x)}{x}\mathrm{d}x$$

$$= \int_{\alpha a}^{\alpha b} \frac{f(x)}{x}\mathrm{d}x - \int_{Aa}^{Ab} \frac{f(x)}{x}\mathrm{d}x = f(c)\int_{\alpha a}^{\alpha b} \frac{1}{x}\mathrm{d}x - \int_{Aa}^{Ab} \frac{f(x)}{x}\mathrm{d}x$$

$$= f(c)\ln\frac{b}{a} - \int_{Aa}^{Ab} \frac{f(x)}{x}\mathrm{d}x;$$

we used obvious changes of variables, then the additivity of the integral and the mean value theorem. Of course, $c = c(\alpha, a, b)$ is in the interval $[\alpha a, \alpha b]$ (we can consider $a < b$ without loss of generality), so it goes to 0 when $\alpha$ becomes small; consequently, the first term above has limit $f(0)\ln(b/a)$, by the continuity of $f$ at the origin. Yet, because $\int_k^\infty \frac{f(x)}{x}\mathrm{d}x$ converges, the absolute value of the second term is as small as we want if $A$ is big enough. Thus

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\mathrm{d}x = \lim_{\alpha\to 0^+,\, A\to\infty} \int_\alpha^A \frac{f(ax) - f(bx)}{x}\mathrm{d}x = f(0)\ln\frac{b}{a},$$

as claimed. The formula that we got is known as Frullani's formula (actually a variant of it). □

Now for the problem it is enough to consider $f(x) = \cos x$, $a = 1$, and $b = t$ to get

$$\int_0^\infty \frac{\cos x - \cos tx}{x}\mathrm{d}x = \ln t$$

for $t > 0$. (If $k > 0$ then by integration by parts we get $\int_k^\infty \frac{\cos x}{x}\mathrm{d}x = \frac{\sin x}{x}\Big|_k^\infty + \int_k^\infty \frac{\sin x}{x^2}\mathrm{d}x = -\frac{\sin k}{k} + \int_k^\infty \frac{\sin x}{x^2}\mathrm{d}x$. The integral $\int_k^\infty \frac{\sin x}{x^2}\mathrm{d}x$ is convergent because $\int_k^\infty \left|\frac{\sin x}{x^2}\right| \mathrm{d}x \le \int_k^\infty \frac{1}{x^2}\mathrm{d}x = \frac{1}{k} < \infty$. It follows that $\int_k^\infty \frac{\cos x}{x}\mathrm{d}x$ is convergent as well.)

Because the cosine is an even function, for $t < 0$ the integral is the same as for $-t$, thus it evaluates to $\ln(-t) = \ln|t|$ in this case, too. □

*Solution by Moubinool Omarjee, Lycée Henri IV, Paris, France.* The integral $A(t)$ converges at 0 because $\frac{\cos x - \cos tx}{x} \to 0$ as $t \to 0$. The convergence at $\infty$ follows from the convergence at $\infty$ of $\int \frac{\cos ax}{x}\mathrm{d}x$ for $a = 1, t$. (Integration by parts.) Hence $A(t)$ is defined.

Suppose that $t > 0$. We have $A(t) = \lim\limits_{\epsilon \to 0^+} I_\epsilon$, where

$$I_\epsilon = \int_\epsilon^\infty \frac{\cos x - \cos tx}{x}\mathrm{d}x.$$

We have

$$\begin{aligned} I_\epsilon &= \int_\epsilon^\infty \frac{\cos x}{x}\mathrm{d}x - \int_\epsilon^\infty \frac{\cos tx}{x}\mathrm{d}x = \int_\epsilon^\infty \frac{\cos x}{x}\mathrm{d}x - \int_{t\epsilon}^\infty \frac{\cos x}{x}\mathrm{d}x \\ &= \int_\epsilon^{t\epsilon} \frac{\cos x}{x}\mathrm{d}x = \int_\epsilon^{t\epsilon} \frac{1}{x}\mathrm{d}x + \int_\epsilon^{t\epsilon} \frac{\cos x - 1}{x}\mathrm{d}x \\ &= \ln t + \int_\epsilon^{t\epsilon} \frac{\cos x - 1}{x}\mathrm{d}x. \end{aligned}$$

Since $\frac{\cos x - 1}{x} \to 0$ as $x \to 0$, we have $\lim\limits_{\epsilon \to 0^+} \int_\epsilon^{t\epsilon} \frac{\cos x - 1}{x}\mathrm{d}x = 0$, so $A(t) = \lim\limits_{\epsilon \to 0^+} I_\epsilon = \ln t$.

For $t < 0$ we use the parity of cos, which implies that $A(t) = A(-t) = \ln(-t) = \ln|t|$. □

**414.** Does there exist a sequence $(a_n)_{n\ge1}$ of terms greater than or equal to 1 such that the following conditions are satisfied:

(i) $\prod\limits_{k=1}^{n} a_k < n^n, n \ge 1$,

(ii) the sequence $(x_n)_{n\ge1}$ given by

$$x_n = \sum_{i=1}^{n} \frac{1}{1 + a_i}$$

is bounded for all $n \ge 1$?

`Proposed by Cezar Lupu, University of Pittsburgh, USA.`

*Solution by Victor Makanin, Sankt Petersburg, Russia.* First we observe that $\prod_{k=1}^{n} a_k < n^n$ for $n = 1$ and $a_1 \geq 1$ are in contradiction – but that is not a major issue if we accept possibility of equality in (i) – at least for $n = 1$.

Anyway, such a sequence does not exist. Indeed, assuming the contrary, we would have that $\sum_{n=1}^{\infty} \frac{1}{1+a_n}$ is convergent, and, by Carleman's inequality,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{(1+a_1)\cdots(1+a_n)}} \leq e \sum_{n=1}^{\infty} \frac{1}{1+a_n}.$$

The inequality implies, of course, that the series from the left is also convergent. But this is not the case, because

$$\left(1+\frac{1}{a_1}\right)\cdots\left(1+\frac{1}{a_n}\right) \leq 2^n$$

(according to $a_k \geq 1$ for all $k$), therefore (using the above and (i))

$$\frac{1}{\sqrt[n]{(1+a_1)\cdots(1+a_n)}} \geq \frac{1}{\sqrt[n]{2^n a_1 \cdots a_n}} \geq \frac{1}{2n}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{(1+a_1)\cdots(1+a_n)}} \geq \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

□

**Remark.** Actually one may assume only that $a_n > 0$ and still there exists no sequence $(a_n)_{n\geq 1}$ satisfying (i) and (ii). Indeed, if there are infinitely many $a_n$ less than 1, then for any such term one has $\frac{1}{1+a_n} > \frac{1}{2}$, hence the series $\sum_{n=1}^{\infty} \frac{1}{1+a_n}$ is not convergent. Otherwise we have $a_n \geq 1$ for all $n > N$, hence

$$\left(1+\frac{1}{a_1}\right)\cdots\left(1+\frac{1}{a_n}\right) \leq A \cdot 2^{n-N}$$

for all $n > N$, with $A = (1 + 1/a_1)\cdots(1 + 1/a_N)$, thus we obtain

$$\frac{1}{\sqrt[n]{(1+a_1)\cdots(1+a_n)}} \geq \frac{1}{\sqrt[n]{2^n B a_1 \cdots a_n}} \geq \frac{1}{2n\sqrt[n]{B}}$$

for $B = A/2^N$. The series $\sum_{n=1}^{\infty} \frac{1}{2n\sqrt[n]{B}}$ being divergent (by comparison test we see that it behaves like the harmonic series), the divergence of $\sum_{n=1}^{\infty} \frac{1}{1+a_n}$ follows as well (via Carleman's inequality), finishing our proof.

*Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.* Suppose that such sequence exists. Then $\frac{1}{1+a_i} \to 0$, so

$a_i \to \infty$ when $i \to \infty$. Then, since $\sum\limits_{i=1}^{\infty} \frac{1}{1+a_i}$ converges, $\sum\limits_{i=1}^{\infty} \frac{1}{a_i}$ converges as well. (We have $\lim\limits_{i\to\infty} \frac{\frac{1}{1+a_i}}{\frac{1}{a_i}} = 1$.)

By Carleman's inequality, we would have

$$\infty = \sum_{n=1}^{\infty} \frac{1}{n} \le \sum_{n=1}^{\infty} \frac{1}{\left(\prod\limits_{k=1}^{n} a_k\right)^{1/n}} \le e \sum_{n=1}^{\infty} \frac{1}{a_n} < \infty,$$

contradiction. So the answer is NO. □

*A very similar solution was given by Moubinool Omarjee, Lycée Henri IV, Paris, France.*

**415.** For any be a positive integer $n$ we define the polynomial

$$P_n = \sum_{\substack{1\le k\le n\\(k,n)=1}} X^k.$$

Prove that $P_n$ is divisible by the cyclotomic polynomial $\Phi_n$ if and only if $n$ is not squarefree.

`Proposed by Filip-Andrei Chindea, student, University of Bucharest, Romania.`

*Solution by Victor Makanin, Sankt Petersburg, Russia.* All the roots of $\Phi_n$ are simple, therefore $P_n$ is divisible by $\Phi_n$ if and only if $P_n(z) = 0$ for every $z$ such that $\Phi_n(z) = 0$. Such a $z$ is actually a primitive $n$th root of unity, thus the condition that $P_n$ is divisible by $\Phi_n$ is equivalent to the fact that any primitive $n$th root of unity is also a root of $P_n$. Now, when $k$ runs over all positive integers relatively prime to $n$ and $z$ is any primitive $n$th root of unity, we know that $z^k$ runs over all primitive roots of unity of order $n$, thus $P_n(z)$ is precisely the sum of these roots, which is well-known to be $\mu(n)$ (with $\mu$ the Möbius function). For the last sentence see, for example, G.H. Hardy and E.M. Wright, *Introduction to the Theory of Numbers*, Oxford University Press, 1975, p. 239.

Summarizing the above facts, we see that $P_n$ is divisible by $\Phi_n$ if and only if $\mu(n) = 0$, which is, on its turn, equivalent to $n$ being not square-free (by the very definition of $\mu$), as desired. □

**Note from the editor.** The result quoted by V. Makanin is not widely known. For self-containment we give a short proof. We denote by $f(n)$ the sum of all primitive $n$th roots of unity. If $F(n) = \sum\limits_{d|n} f(d)$ then $F(n)$ is the sum of all $n$th roots of unity. We have $F(1) = 1$ and if $n > 1$ and $\zeta$ is a primitive $n$th root of unity then $F(n) = 1 + \zeta + \cdots + \zeta^{n-1} = \frac{\zeta^n - 1}{\zeta - 1} = 0$. By

Möbius inversion formula, $f(n)=\sum_{d|n}F(d)\mu(\frac{n}{d})$. Since $F(1)=1$ and $F(d)=0$ if $d>1$, we have $f(n)=\mu(n)$, as claimed.

**416.** Let $M_n$ be the set consisting of $\emptyset$ and all non-equivalent expressions in $n$ variables $X_1,\dots,X_n$, which are sets that can be obtained by using only $\cup$ and $\cap$. Two expressions $E_1,E_2\in M_n$ are considered to be equivalent if $E_1(A_1,\dots,A_n)=E_2(A_1,\dots,A_n)$ for any sets $A_1,\dots,A_n$.

E.g. $M_2=\{\emptyset,X_1,X_2,X_1\cup X_2,X_1\cap X_2\}$.

(i) Describe all elements of $M_n$.

(ii) Prove that $2^{\binom{n}{\lfloor n/2\rfloor}}\le |M_n|\le 2^{2^n}$.

**Open problem** By Stirling's formula we have $\binom{n}{\lfloor n/2\rfloor}\sim\sqrt{\frac{2}{\pi}}\cdot\frac{2^n}{\sqrt{n}}$, so

$$\sqrt{\frac{2}{\pi}}\cdot\frac{2^n}{\sqrt{n}}\sim\binom{n}{\lfloor n/2\rfloor}\le\log_2|M_n|\le 2^n.$$

Try to determine how fast $\log_2|M_n|$ grows or at least find tighter bounds for it.

`Proposed by Constantin-Nicolae Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.`

*Solution by the author.* Let $I=\{1,\dots,n\}$. For any set $X$ we denote $P(X)=\{Y\mid Y\subseteq X\}$ (the parts of $X$) and $P'(X)=P(X)\setminus\{\emptyset\}$. Hence $|P(X)|=2^{|X|}$ and $|P'(X)|=2^{|X|}-1$.

We have $M_n\subseteq N_n$, where $N_n$ is the set consisting of $\emptyset$ and all expressions that can be obtained by using subtraction together with $\cup$ and $\cap$. For any $S\in P'(I)$ we consider the "elementary set" $T_S=(\bigcap_{i\in S}X_i))\setminus(\bigcup_{i\in I\setminus S}X_i)$, i.e., $T_S$ is the set of all elements in $X_1\cup\dots\cup X_n$ belonging precisely to those $X_i$ with $i\in S$. Obviously the sets $E_S$ are mutually disjoint.

Now $\emptyset=\bigcup_{S\in\emptyset}T_S$ and for any $i\in I$ we have $X_i=\bigcup_{S\in\Phi_i}T_S$, where $\Phi_i=\{S\in P'(I)\mid i\in S\}$. Also if $X=\cup_{S\in\Phi}T_S$, $Y=\bigcup_{S\in\Psi}T_S$ then $X\cup Y=\bigcup_{S\in\Phi\cup\Psi}T_S$, $X\cap Y=\bigcup_{S\in\Phi\cap\Psi}T_S$, $X\setminus Y=\bigcup_{S\in\Phi\setminus\Psi}T_S$. (Here we use the fact that the sets $T_S$ are mutually disjoint.) Then, by induction, one proves that $E(X_1,\dots,X_n)=\bigcup_{S\in E(\Phi_1,\dots,\Phi_n)}T_S$ for any $E\in N_n$. In particular, every $E\in N_n$ can be written as $\bigcup_{S\in\Phi}T_S$ for some $\Phi\subseteq P'(I)$. Let $E=\bigcup_{S\in\Phi}T_S$, $E'=\bigcup_{S\in\Phi'}T_S$ with $\Phi\ne\Phi'$. Then we may assume that $\Phi\not\subseteq\Phi'$, so let $S_0\in\Phi\setminus\Phi'$. We consider the sets $A_1,\dots,A_n$, where $A_i=\{0\}$ of $i\in S_0$ and $A_i=\emptyset$ otherwise. Then $T_{S_0}(A_1,\dots,A_n)=\{0\}$ and $T_S(A_1,\dots,A_n)=\emptyset$ for any $S\in P'(I)$, $S\ne S_0$. Since $S_0\in\Phi$, $S_0\notin\Phi'$, we have $E(A_1,\dots,A_n)=$

$\bigcup_{S\in\Phi} T_S(A_1,\dots,A_n)=\{0\}$, $E'(A_1,\dots,A_n)=\bigcup_{S\in\Phi'} T_S(A_1,\dots,A_n)=\emptyset$. Hence $E,E'$ are not equivalent. In conclusion, the map $\Phi\mapsto\bigcup_{S\in\Phi} T_S$ is a bijection between $P(P'(I))$ and $N_n$. It follows that $N_n=2^{2^n-1}$. But $M_n\subseteq N_n$, so $|M_n|\le|N_n|$ and we have the second inequality of (ii).

Now for any $S\in P'(I)$ we consider $U_S\in M_n$, $U_S=\bigcap_{i\in S} X_i$. Note that $X_i=U_{\{i\}}$. Also $U_S\cap U_{S'}=U_{S\cup S'}$. In particular, if $S\subseteq S'$ then $U_S\supseteq U_{S'}$. We claim that any $E\in M_n$ writes as $E=\bigcup_{S\in\Phi} U_S$ for some $\Phi\in P(P'(I))$. Indeed, if $E=\emptyset$ then we take $\Phi=\emptyset$, and if $E=X_i$ then we take $\Phi=\{\{i\}\}$. For the induction step we note that if $E=\bigcup_{S\in\Phi} U_S$, $E'=\bigcup_{S\in\Phi'} U_S$ then $E\cup E'=\bigcup_{S\in\Phi\cup\Phi'} U_S$ and $E\cap E'=\bigcup_{S\in\Phi}\bigcup_{S'\in\Phi'}(U_S\cap U'_S)=\bigcup_{S\in\Psi} U_S$, where $\Psi=\{S\cup S'\mid S\in\Phi,\ S'\in\Phi'\}$.

If $E=\bigcup_{S\in\Phi} U_S$ and $S_1,S_2\in\Phi$ with $S_1\subset S_2$ then $U_{S_1}\supseteq U_{S_2}$ so $U_{S_2}$ is superfluous in the formula for $E$ and so it can be removed, i.e., $S_2$ can be removed from $\Phi$. Hence whenever there are $S_1,S_2\in\Phi$ with $S_1\subset S_2$ we may remove $S_2$ from $\Phi$ without altering $E$. By repeating the procedure we may assume that $\Phi\in Q(P'(i))$, where

$$Q(P'(I))=\{\Phi\in P(P'(I))\mid S_1\not\subset S_2\,\forall S_1,S_2\in\Phi\}.$$

Let now $E=\bigcup_{S\in\Phi} U_S$, $E'=\bigcup_{S\in\Phi'} U_S$, with $\Phi,\Phi'\in Q(P'(I))$, $\Phi\neq\Phi'$. Let $S_0\in(\Phi\setminus\Phi')\cup(\Phi'\setminus\Phi)$ with $|S_0|$ minimal. We may assume that $S_0\in\Phi\setminus\Phi'$. Let now $S\in\Phi'$. If $S\in\Phi$ then $S\not\subset S_0$ since $S_0,S\in\Phi$ and $\Phi\in Q(P'(I))$. If $S\in\Phi'\setminus\Phi$ then $S\not\subset S_0$ since otherwise $|S|<|S_0|$, which contradicts the minimality of $|S_0|$. Hence $S\not\subset S_0$ $\forall S\in\Phi'$. But $S_0\notin\Phi'$, so in fact $S\not\subseteq S_0$. Let now $A_i=\{0\}$ if $i\in S_0$ and $A_i=\emptyset$ if $i\in I\setminus S_0$. Then $U_S(A_1,\dots,A_n)=S_0$ if $S\subseteq S_0$ and $U_S(A_1,\dots,A_n)=\emptyset$ otherwise. Since $S_0\in\Phi$, we have $E(A_1,\dots,A_n)=\bigcup_{S\in\Phi} U_S=\{0\}$. On the other hand, for any $S\in\Phi'$ we have $S\not\subseteq S_0$, so $U_S(A_1,\dots,A_n)=\emptyset$. Hence $E'(A_1,\dots,A_n)=\bigcup_{S\in\Phi'} U_S=\emptyset$. Thence $E$ and $E'$ are not equivalent.

In conclusion, the map $\Phi\mapsto\bigcup_{S\in\Phi} U_S$ is a bijection between $Q(P'(I))$ and $M_n$. Hence $|M_n|=|Q(P'(I))|$. This solves (i).

Unfortunately it is not easy to find assymptotic formulas for $|Q(P'(I))|$. In order to obtain the first inequality in (ii) we note that $Q(P'(I))\supseteq R(P'(I))$, so $|Q(P'(I))|\ge|R(P'(I))|$, where

$$R(P'(I))=\{\Phi\in P(P'(I))\mid |S|=|S'|\,\forall S,S'\in\Phi\}.$$

(If $\Phi \in R(P'(I))$ then for any $S, S' \in \Phi$ we have $|S| = |S'|$, so $S \not\subset S'$. Hence $\Phi \in Q(P'(I))$.) We have $P'(I) = P_1(I) \cup \cdots \cup P_n(I)$, where

$$P_k(I) = \{S \subset I \mid |S| = k\}.$$

For any $1 \leq k \leq n$ we have $P(P_k(I)) \subseteq R(P'(I))$, so

$$|R(P'(I))| \geq |P(P_k(I))| = 2^{|P_k(I)|} = 2^{\binom{n}{k}}.$$

When we take $k = \lfloor n/2 \rfloor$ we get the desired result. □

**Erratum.**

In Remark 2 of the paper *Existence of a Hamiltonian path in a plane configuration*, GMA, 3-4 (2014), 37–41, by M. Cavachi, the author has made some statements that are not always true. Namely, the first paragraph holds only for $1 < k < n-1$. In the remaining cases, $k = 1$ and $k \in \{n-1, n\}$, $k$ odd, i.e. $k = 2\left[\frac{n-1}{2}\right] + 1$, with the same notation, we have the following.

Of the four segments of circles that meet at $S$, we consider the one which is different from $SS'$ and $ST$ and is not on the same circle with $S'$. We denote it by $SA$. Similarly $TB$ is the segment in $T$ different from $TT'$ and $TS$ and not on the same circle with $T'$. Then, same as in the case $1 < k < n-1$, $SA$ and $TB$ are the only possible candidates to being $\ell_k$ and $\ell_{k-2}$. The difference in the case $k = 1$ is that there is no $F_{-1}$ so there is no $\ell_{-1}$, which joins $F_{-1}$ to $F_1$. Hence we have either $SA = \ell_1$ and $T$ is one end of the hamiltonian path or $TB = \ell_1$ and $S$ is one end of the hamiltonian path. Similarly, if $k \in \{n-1, n\}$ then $k+2 > n$ so there is no $F_{k+2}$ and so no $\ell_k$ which joins $F_k$ to $F_{k+2}$. Hence in this case either $SA = \ell_{k-2}$ and $T$ is one end of the hamiltonian path or $TB = \ell_{k-2}$ and $S$ is one end of the hamiltonian path.

As a consequence, in the last paragraph of Remark 2, the statement that the hamiltonian path is uniquely determined by the edge $ST$ of the cycle $F_k$ missing from the path holds only when $1 < k < n-1$. If $k = 1$ or $2\left[\frac{n-1}{2}\right] + 1$ then, in order to have unicity, we also need to know which of $S$ and $T$ is an end of the hamiltonian path. Otherwise there are (at most) two possible hamiltonian paths satisfying the rules and not containing $ST$.

However, this mistake does not affect the rest of the proof.