# GAZETA MATEMATICĂ

## SERIA A

**ANUL XXXII (CXI)** **Nr. 1 – 2/ 2014**

## ARTICOLE

### A generalisation of Hillam's theorem

Dan Ştefan Marinescu[1)], Mihai Monea[2)]

**Abstract**. The paper deals with convergence of some iterative processes. We prove a generalisation of Hillam's theorem and derive thereby convergence for a class of 'convex' iterative processes.

**Keywords:** iterative process, Hillam's theorem, Barone's lemma

**MSC:** 26A18

### 1. **Introduction**

In this paper we prove some new results on a class of iterative processes. Barone's lemma below is one of the first results on the subject [1].

**Lemma 1.** *Let $f$ be a continuous function of an interval $I \subseteq \mathbb{R}$ into itself, let $x_0$ be a point of $I$, and define the sequence $(x_n)_{n\in\mathbb{N}}$ by $x_{n+1} = f(x_n)$, $n \in \mathbb{N}$. If $\lim\limits_{n\to\infty}(x_{n+1} - x_n) = 0$, then the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent on the extended line $\overline{\mathbb{R}}$.*

Under the additional assumption that the interval $I$ in Barone's lemma is compact, we get Hillam's theorem [4]:

**Theorem 2.** *Let $f$ be a continuous function of the closed unit interval $[0,1]$ into itself, let $x_0$ be a point of $[0,1]$, and define the sequence $(x_n)_{n\in\mathbb{N}}$ by $x_{n+1} = f(x_n)$, $n \in \mathbb{N}$. If $\lim\limits_{n\to\infty}(x_{n+1} - x_n) = 0$, then the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to a fixed point of $f$.*

[1)] Colegiul Naţional „Iancu de Hunedoara“, Hunedoara, Romania, marinescuds@gmail.com

[2)] Colegiul Naţional „Decebal“, Deva, Romania, mihaimonea@yahoo.com

We refer to [4, 6] for a proof of this theorem. Our purpose here is to derive Hillam's theorem above from the two propositions in the next section. The first generalises Barone's lemma, and the second Hillam's theorem. In the final section, our results are applied to derive convergence for a class of 'convex' iterative processes.

## 2. Main results

We begin with an extension of Barone's lemma.

**Proposition 3.** *Let $f$ be a continuous real-valued function on an interval $I \subseteq \mathbb{R}$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $I$. If*

**(a)** $\lim\limits_{n\to\infty}(x_{n+1}-x_n)=0$, *and*

**(b)** $(x_{n+1}-x_n)\,(f\,(x_n)-x_n)>0$ *for all indices $n$,*

*then the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent on the extended line $\overline{\mathbb{R}}$.*

*Proof.* Let $a=\liminf\limits_{n\to\infty} x_n$ and $b=\limsup\limits_{n\to\infty} x_n$. Under the assumption that $a<b$, since $I$ is an interval, and $(x_n)_{n\in\mathbb{N}}$ is a sequence in $I$ that is frequently in every neighbourhood of $a$, respectively $b$, the open interval $(a,b)$ is contained in $I$, so $f$ is defined on $(a,b)$, and the idea is to show that $f$ fixes each point in $(a,b)$. Condition **(b)** then forces the $x_n$ outside $(a,b)$. To reach a contradiction, we may now either recall that the limit points of a sequence satisfying condition **(a)** form a compact interval of the extended line — in this case, the closed interval $[a,b]\subseteq\overline{\mathbb{R}}$ — or simply notice that $(x_n)_{n\in\mathbb{N}}$ must be frequently in every neighbourhood of $a$, respectively $b$, to contradict **(a)** and conclude that $a=b$.

To prove that $f$ fixes each point in the open interval $(a,b)$, suppose, if possible, that $f(c)\neq c$, say $f(c)>c$, for some $c$ in $(a,b)$; the case $f(c)<c$ is dealt with similarly. By continuity, $f(x)>x$ for all $x$ in some neighbourhood $(c-\epsilon,c+\epsilon)\subset(a,b)$, where $\epsilon>0$. By **(a)** and the definition of $b$, $|x_{n+1}-x_n|<\epsilon$ and $x_n>c$ for infinitely many indices $n$. Fix such an index $m$. If $x_m\geq c+\epsilon$, then $x_{m+1}>x_m-\epsilon\geq c$; and if $c<x_m<c+\epsilon$, then $f(x_m)>x_m$, so $x_{m+1}>x_m>c$ by **(b)**. It follows that $x_n>c$ for all $n\geq m$, so the sequence $(x_n)_{n\in\mathbb{N}}$ is eventually away from $a$ — a contradiction. □

The following generalisation of Hillam's theorem is a corollary to the previous proposition.

**Proposition 4.** *Let $f$ be a continuous real-valued function on a compact interval $I\subset\mathbb{R}$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $I$. If $\lim\limits_{n\to\infty}(x_{n+1}-x_n)=0$ and $(x_{n+1}-x_n)\,(f\,(x_n)-x_n)>0$ for all indices $n$, then the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to a point in $I$.*

*Proof.* Compactness of $I$ and Proposition 3 imply the conclusion. □

Proposition 4 yields the following straightforward proof of Hillam's theorem: If $f$ fixes some term of the sequence $(x_n)_{n\in\mathbb{N}}$, then the latter is eventually constant, hence convergent to a fixed point of $f$. Otherwise, $(x_{n+1} - x_n)\,(f\,(x_n) - x_n) = (f\,(x_n) - x_n)^2 > 0$ for all indices $n$, and Proposition 4 applies to show $(x_n)_{n\in\mathbb{N}}$ converges to a point in $I$. Since $x_{n+1} = f(x_n)$, the conclusion follows by continuity.

## 3. Consequences

In this section, our results are applied to 'convex' iterative processes. We begin with an extension of a result in [2].

**Corollary 5.** *Let $f$ be a continuous function of an interval $I \subseteq \mathbb{R}$ into itself, and let $(t_n)_{n\in\mathbb{N}}$ be a sequence in the closed unit interval $[0,1]$ such that $\lim\limits_{n\to\infty} t_n = 0$. Given a point $x_0$ in $I$, define the sequence $(x_n)_{n\in\mathbb{N}}$ of points of $I$ by*

$$x_{n+1} = (1 - t_n)\,x_n + t_n f(x_n);$$

*the definition makes sense for $f$ takes on values in $I$, and the latter is convex.*

**(a)** *If the sequence $(f(x_n) - x_n)_{n\in\mathbb{N}}$ is bounded (e.g., if $I$ is bounded), then the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent on the extended line $\overline{\mathbb{R}}$ (if $I$ is bounded, then $(x_n)_{n\in\mathbb{N}}$ is convergent in the compact $\bar{I}$, the closure of $I$ in $\mathbb{R}$).*

**(b)** *If the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded in $I$ (e.g., if $I$ is compact [2]), then it is convergent to a point in $I$.*

*And if, in addition, the series $\sum\limits_{n\in\mathbb{N}} t_n$ is divergent (e.g., if $t_n = 1/(n+a)$, $a > 0$), then the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to a fixed point of $f$.*

*Proof.* **(a)** If $f$ fixes some $x_n$ or $t_n = 0$ for all but finitely many indices $n$, then $(x_n)_{n\in\mathbb{N}}$ is eventually constant, hence convergent in $I \subset \overline{\mathbb{R}}$.

Next, write $x_{n+1} - x_n = t_n(f(x_n) - x_n)$, refer to the boundedness of the $f(x_n) - x_n$, and recall that $\lim\limits_{n\to\infty} t_n = 0$, to deduce that $\lim\limits_{n\to\infty}(x_{n+1} - x_n) = 0$.

Assume henceforth that $f$ fixes no $x_n$, and $t_n \neq 0$ for infinitely many indices $n$. Label the elements of the set $\{n\colon t_n \neq 0\}$ increasingly, $n_0 < n_1 < \cdots < n_k < \cdots$, set $y_k = x_{n_k+1}$ and $u_k = t_{n_{k+1}}$, $k \in \mathbb{N}$, and notice that $y_{k+1} - y_k = x_{n_{k+1}+1} - x_{n_{k+1}}$, since $x_n = y_k$, $n_k + 1 \leq n \leq n_{k+1}$, so $\lim\limits_{k\to\infty}(y_{k+1} - y_k) = 0$, by the preceding paragraph. Moreover, one has $y_{k+1} = (1-u_k)y_k + u_k f(y_k)$, so $(y_{k+1} - y_k)(f(y_k) - y_k) = u_k^2\,(f(y_k) - y_k)^2 > 0$ for all indices $k$.

Hence, $(y_k)_{k\in\mathbb{N}}$ satisfies both conditions in Proposition 3, so it coverges on the extended line $\overline{\mathbb{R}}$. Finally, since $x_n = y_k$, $n_k + 1 \le n \le n_{k+1}$, $k \in \mathbb{N}$, the conclusion follows.

**(b)** With minor changes, the proof of the first part goes along the same lines. This time, the idea is to apply Proposition 4 on a suitable compact interval $K \subseteq I$.

Since $(x_n)_{n\in\mathbb{N}}$ is bounded in $I$, it is a sequence in some compact interval $K \subseteq I$, the images $f(x_n)$ form a sequence in the compact interval $f(K) \subseteq I$, so $(f(x_n))_{n\in\mathbb{N}}$ is also bounded in $I$, and the $f(x_n)-x_n$ are therefore bounded. Proceed now as in the proof of **(a)** to complete the proof of the first part.

To prove the second part, let $x = \lim\limits_{n\to\infty} x_n$, $x \in I$. Suppose, if possible, that $f(x) \ne x$ and separate the two by $\epsilon$-neighbourhoods contained in $I$, $0 < \epsilon < \frac{1}{2}|f(x) - x|$, to get $|f(x_n) - x_n| > |f(x) - x| - 2\epsilon$ for all but finitely many indices $n$, say all $n \ge m$, where $m$ is a fixed positive integer. Write again $x_{n+1}-x_n = t_n(f(x_n)-x_n)$, to deduce that $|x_n-x_m| > (|f(x)-x|-2\epsilon)\sum\limits_{k=m}^{n-1} t_k$ for all $n > m$, and reach thereby a contradiction by recalling divergence of $\sum\limits_{n\in\mathbb{N}} t_n$ and boundedness of $(x_n)_{n\in\mathbb{N}}$. □

**Remarks.** Since a sequence in a closed interval is bounded if and only if it is bounded in that interval, if $I$ in Corollary 5 is closed, and $(x_n)_{n\in\mathbb{N}}$ is bounded, then the latter is convergent in $I$, by Corollary 5 **(b)**. In this case, $(f(x_n)-x_n)_{n\in\mathbb{N}}$ is also bounded. However, boundedness of the latter does not necessarily imply that of the former. For instance, if $I = [0,\infty)$, $f\colon I \to I$, $f(x) = x+1$, and $t_n = 1/(n+1)$, $n \in \mathbb{N}$, then $x_n = x_0 + 1 + 1/2 + \cdots + 1/n$, $n \in \mathbb{N}^*$, is unbounded, whereas $f(x_n) - x_n = 1$, $n \in \mathbb{N}$.

It is worth noticing that divergence of the series $\sum\limits_{n\in\mathbb{N}} t_n$ is essential for the conclusion in the second half of Corollary 5 **(b)** to hold in general, as the following example shows. Let $I$ be the closed unit interval $[0,1]$, let $f\colon I \to I$, $f(x) = x^2$, let $t_n = 2^{-n}$, $n \in \mathbb{N}$, and start at a point $x_0$ in the open interval $(1/2,1)$, to obtain a strictly decreasing sequence $(x_n)_{n\in\mathbb{N}}$ in the closed interval $[x_0 - 1/2, x_0] \subset (0,1)$:

$$\begin{aligned} 1 > x_0 \ge x_n = x_0 + \sum_{k=0}^{n-1}(x_{k+1} - x_k) &= x_0 + \sum_{k=0}^{n-1} 2^{-k} x_k(x_k - 1) \\ &\ge x_0 - \sum_{k=0}^{n-1} 2^{-k-2} > x_0 - 1/2 > 0, \quad n \in \mathbb{N}. \end{aligned}$$

Since $f$ fixes only 0 and 1, the limit of $(x_n)_{n\in\mathbb{N}}$ is not a fixed point of $f$.

Corollary 5 applies to the 'weighted' iterative processes below — see [3, 5] for special cases.

**Corollary 6.** *Let $f$ be a continuous function of an interval $I \subseteq \mathbb{R}$ into itself, and let $(w_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that*

$$\lim_{n\to\infty} \frac{w_n}{w_0 + \cdots + w_n} = 0;$$

*e.g., all $w_n = 1$. Given two points $x_0$ and $y_0$ in $I$, define the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of points of $I$ by*

$$x_{n+1} = \frac{w_0 f(x_0) + \cdots + w_n f(x_n)}{w_0 + \cdots + w_n} \quad \text{and} \quad y_{n+1} = f\left(\frac{w_0 y_0 + \cdots + w_n y_n}{w_0 + \cdots + w_n}\right).$$

**(a)** *If the sequence $(f(x_n) - x_n)_{n\in\mathbb{N}}$ is bounded (e.g., if $I$ is bounded), then the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent on the extended line $\overline{\mathbb{R}}$ (if $I$ is bounded, then $(x_n)_{n\in\mathbb{N}}$ is convergent in the compact $\bar{I}$, the closure of $I$ in $\mathbb{R}$).*

*If the limit of $f$ exists at each end-point of $I$, then boundedness of the sequence $(y_{n+1} - \bar{y}_n)_{n\in\mathbb{N}}$, where $\bar{y}_n = (w_0 y_0 + \cdots + w_n y_n)/(w_0 + \cdots + w_n)$, implies convergence of $(y_n)_{n\in\mathbb{N}}$ on the extended line $\overline{\mathbb{R}}$.*

**(b)** *If the sequence $(x_n)_{n\in\mathbb{N}}$, respectively $(y_n)_{n\in\mathbb{N}}$, is bounded in $I$ (e.g., if $I$ is compact), then it is convergent to a point in $I$.*

*And if, in addition, the series $\sum_{n\in\mathbb{N}} w_n/(w_0 + \cdots + w_n)$ is divergent (e.g., if all $w_n = 1$), then the sequence $(x_n)_{n\in\mathbb{N}}$, respectively $(y_n)_{n\in\mathbb{N}}$, is convergent to a fixed point of $f$.*

*Proof.* To deal with $(x_n)_{n\in\mathbb{N}}$, notice that

$$x_{n+1} = \left(1 - \frac{w_n}{w_0 + \cdots + w_n}\right) x_n + \frac{w_n}{w_0 + \cdots + w_n} f(x_n),$$

and let $t_n = w_n/(w_0 + \cdots + w_n)$ in Corollary 5.

To deal with $(y_n)_{n\in\mathbb{N}}$, notice that $y_{n+1} = f(\bar{y}_n)$,

$$\bar{y}_{n+1} = \left(1 - \frac{w_{n+1}}{w_0 + \cdots + w_{n+1}}\right) \bar{y}_n + \frac{w_{n+1}}{w_0 + \cdots + w_{n+1}} f(\bar{y}_n),$$

and $\min\{y_k : k = 0, \dots, n\} \le \bar{y}_n \le \max\{y_k : k = 0, \dots, n\}$, and let

$$t_n = \frac{w_{n+1}}{w_0 + \cdots + w_{n+1}}$$

in Corollary 5 to get the desired conclusions for $(\bar{y}_n)_{n\in\mathbb{N}}$, whence also for the sequence $(y_{n+1})_{n\in\mathbb{N}} = (f(\bar{y}_n))_{n\in\mathbb{N}}$. □

**Remarks.** A well-kown sufficient condition for $\sum\limits_{n\in\mathbb{N}} w_n/(w_0+\cdots+w_n)$ to diverge is that $\sum\limits_{n\in\mathbb{N}} w_n$ be divergent – this follows from the inequality

$$\sum_{k=1}^{n} \frac{w_{m+k}}{w_0+\cdots+w_{m+k}} \geq 1 - \frac{w_0+\cdots+w_m}{w_0+\cdots+w_{m+n}}.$$

If $\sum\limits_{n\in\mathbb{N}} w_n$ diverges, then the conclusion in the second half of Corollary 6 **(b)** also follows from the Stolz-Cesàro theorem.

**Acknowledgements.** The authors are indebted to Călin Popescu, without whom the paper would have been published earlier.

## References

[1] H. G. Barone, Limit points of sequences and their transforms by methods of summability, *Duke Math. J.* **5** (1939), no. 4, 740–752.
[2] D. Borwein, J. Borwein, Fixed point iteration for real functions, *J. Math. Anal. Appl.* **157** (1991), 112–126.
[3] R. L. Franks, R. P. Marzec, A theorem of mean value iteration, *Proc. Amer. Math. Soc.* **30** (1971), No. 2, 324–326.
[4] B. P. Hillam, A characterizations of the convergence of succesive approximations, *Amer. Math. Monthly* **83** (1976), 273.
[5] W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.* **4** (1953), 505–510.
[6] V. Radu, *Lecţii de matematici elementare*, Ed. Augusta, Timişoara, 2000.

# An infinite family of inequalities involving cosecant sums

Mircea Merca[1)]

**Abstract**. A very special case of a Ramus's identity is used in this note to derive an infinite family of inequalities involving finite sums with cosecant function. As a corollary of this result, we obtain the Wallis's formula.

**Keywords:** inequalities, cosine power sum, cosecant sums, Wallis's formula

**MSC:** 05A10, 26D20, 33B10

## 1. Introduction

Let $n$ and $p$ be two positive integers and let $r$ be a nonnegative integer such that $r < n$. The following identity was proved by Ramus [4] in 1834:

$$\sum_{k\geqslant 0}\binom{p}{r+kn} = \frac{1}{n}\sum_{k=1}^{n}\omega^{-rk}(1+\omega^k)^p = \frac{2^p}{n}\sum_{k=1}^{n}\cos^p\left(\frac{k\pi}{n}\right)\cos\left((p-2r)\frac{k\pi}{n}\right)$$

where $\omega$ is the primitive $n$th root of unity $e^{2\pi i/n} = \cos(2\pi/n) + i\sin(2\pi/n)$. By this relation, with $n$ replaced by $2n+1$, $p$ replaced by $2p$, and $r$ replaced by $p$, we obtain:

$$\sum_{k=1}^{n}\cos^{2p}\left(\frac{k\pi}{2n+1}\right) = -\frac{1}{2} + \frac{2n+1}{2^{2p+1}}\binom{2p}{p}, \quad p < 2n+1. \tag{1.1}$$

In this paper, we will use this finite sum of powers of cosines to prove:

**Theorem 1.** *Let $p$ and $n$ be two nonnegative integers such that $p < 2n+1$. Then*

$$\frac{2p+1}{2p+2} < \frac{1}{(2n+1)\binom{2p}{p}}\sum_{j=0}^{p}(-1)^j\binom{2p+1}{p-j}\csc\left(\frac{2j+1}{2n+1}\cdot\frac{\pi}{2}\right) < 1.$$

We note that the inequality

$$\frac{3}{4} < \frac{3}{4n+2}\csc\left(\frac{\pi}{4n+2}\right) - \frac{1}{4n+2}\csc\left(\frac{3\pi}{4n+2}\right) < 1, \quad n > 0,$$

is the case $p = 1$ in this theorem.

Taking into account that

$$\lim_{x\to 0} x\csc(x) = 1,$$

1) Department of Mathematics, University of Craiova, Craiova, 200585 Romania mircea.merca@profinfo.edu.ro

the inequality

$$\frac{2p+1}{2p+2}\cdot\frac{\pi}{2} < \frac{1}{\binom{2p}{p}}\sum_{j=0}^{p}\frac{(-1)^j}{2j+1}\binom{2p+1}{p-j} < \frac{\pi}{2} \tag{1.2}$$

is the limiting case $n\to\infty$ of Theorem 1. Because

$$\lim_{p\to\infty}\frac{1}{\binom{2p}{p}}\sum_{j=0}^{p}\frac{(-1)^j}{2j+1}\binom{2p+1}{p-j} = \frac{\pi}{2},$$

it is natural to ask about the sum

$$\sum_{j=0}^{p}\frac{(-1)^j}{2j+1}\binom{2p+1}{p-j}.$$

**Theorem 2.** *Let $p$ be a positive integers. Then*

$$\sum_{j=0}^{p}\frac{(-1)^j}{2j+1}\binom{2p+1}{p-j} = 2^{2p}\cdot\frac{(2p)!!}{(2p+1)!!}. \tag{1.3}$$

As usual, $n!!$ denotes the product of all the odd integers up to some positive integer n.

As a consequence of these theorems, we derive in the last section of this note the well-known Wallis's representation of the constant $\pi$.

## 2. **Proof of Theorem 1**

According to Jeffrey and Dai [2, eq. 9, p. 130]

$$\cos^{2p-1}(x) = \frac{1}{2^{2p-2}}\sum_{k=0}^{p-1}\binom{2p-1}{k}\cos((2p-2k-1)x), \qquad p=1,2,\dots\ ,$$

we obtain

$$\cos^{2p+1}\left(\frac{k\pi}{2n+1}\right) = \frac{1}{2^{2p}}\sum_{j=0}^{p}\binom{2p+1}{p-j}\cos\left(\frac{2j+1}{2n+1}\cdot k\pi\right). \tag{2.4}$$

The following identity

$$\sum_{k=1}^{n}\cos\left(\frac{2j+1}{2n+1}\cdot k\pi\right) = -\frac{1}{2}+\frac{(-1)^j}{2}\csc\left(\frac{2j+1}{2n+1}\cdot\frac{\pi}{2}\right) \tag{2.5}$$

is a very special case in [2, eq. 14, p. 129]

$$\sum_{k=1}^{n}\cos(kx) = \frac{\sin(nx+x/2)}{2\sin(x/2)} - \frac{1}{2}.$$

Taking into account (2.4) and (2.5), we have

$$\begin{aligned}
&\sum_{k=1}^{n} \cos^{2p+1}\left(\frac{k\pi}{2n+1}\right) = \\
&= \frac{1}{2^{2p}} \sum_{j=0}^{p} \binom{2p+1}{p-j} \sum_{k=1}^{n} \cos\left(\frac{2j+1}{2n+1}\cdot k\pi\right) \\
&= -\frac{1}{2^{2p+1}} \sum_{j=0}^{p} \binom{2p+1}{p-j} + \frac{1}{2^{2p+1}} \sum_{j=0}^{p} (-1)^j \binom{2p+1}{p-j} \csc\left(\frac{2j+1}{2n+1}\cdot\frac{\pi}{2}\right) \\
&= -\frac{1}{2} + \frac{1}{2^{2p+1}} \sum_{j=0}^{p} (-1)^j \binom{2p+1}{p-j} \csc\left(\frac{2j+1}{2n+1}\cdot\frac{\pi}{2}\right), \qquad (2.6)
\end{aligned}$$

where the last line follows from

$$\sum_{j=0}^{p} \binom{2p+1}{p-j} = 2^{2p}.$$

Considering the inequality

$$\cos^{2p+2} x < \cos^{2p+1} x < \cos^{2p} x, \quad 0 < x < \frac{\pi}{2},$$

the cosine power sum (1.1) and the identity (2.6), we get

$$\frac{2n+1}{2^{2p+3}}\binom{2p+2}{p+1} < \frac{1}{2^{2p+1}} \sum_{j=0}^{p} (-1)^j \binom{2p+1}{p-j} \csc\left(\frac{2j+1}{2n+1}\cdot\frac{\pi}{2}\right) < \frac{2n+1}{2^{2p+1}}\binom{2p}{p}$$

for $p < 2n+1$. Noting that

$$\binom{2p+2}{p+1} = \frac{2(2p+1)}{p+1}\binom{2p}{p},$$

Theorem 1 is proved. □

## 3. **Proof of Theorem 2**

By Merca and Tanriverdi [3, Theorem 2.1], we have

$$\frac{2n+1}{\pi}\cdot\frac{(2p)!!}{(2p+1)!!} - 1 < \sum_{k=1}^{n} \cos^{2p+1} \frac{k\pi}{2n+1} < \frac{2n+1}{\pi}\cdot\frac{(2p)!!}{(2p+1)!!}.$$

It is clear that

$$\lim_{n\to\infty} \frac{\pi}{2n+1} \sum_{k=1}^{n} \cos^{2p+1}\left(\frac{k\pi}{2n+1}\right) = \frac{(2p)!!}{(2p+1)!!}. \qquad (3.7)$$

On the other hand, by (2.6) we obtain

$$\lim_{n\to\infty} \frac{\pi}{2n+1} \sum_{k=1}^{n} \cos^{2p+1}\left(\frac{k\pi}{2n+1}\right) = \frac{1}{2^{2p}} \sum_{j=0}^{p} \frac{(-1)^j}{2j+1} \binom{2p+1}{p-j}. \tag{3.8}$$

Theorem 2 is proved. □

## 4. **Concluding remarks**

An infinite family of inequalities involving cosecant sums and a new combinatorial identity have been introduce in this note as applications of a Ramus's identity. Moreover, considering the inequality (1.2), Theorem 2 and the fact that

$$\binom{2p}{p} = 2^{2p} \cdot \frac{(2p-1)!!}{(2p)!!},$$

we get the inequality

$$\frac{2p+1}{2p+2} \cdot \frac{\pi}{2} < \frac{(2p)!!}{(2p-1)!!} \cdot \frac{(2p)!!}{(2p+1)!!} < \frac{\pi}{2}$$

and then the well-known Wallis's formula [1, p. 68]

$$\prod_{p=0}^{\infty} \left(\frac{2p}{2p-1} \cdot \frac{2p}{2p+1}\right) = \frac{\pi}{2}.$$

On the other hand, we found a nice limit

$$\lim_{p\to\infty} \frac{\sqrt{2p+1}}{2^{2p}} \sum_{j=0}^{p} \frac{(-1)^j}{2j+1} \binom{2p+1}{p-j} = \sqrt{\frac{\pi}{2}}.$$

Is it possible to have another proof for it?

## REFERENCES

[1] L. Berggren, J. Borwein, P. Borwein (eds.), *Pi: A Source Book*, Springer, New York, 2000.

[2] A. Jeffrey, H.-H. Dai, *Handbook of Mathematical Formulas and Integrals*, Elsevier Academic Press, 2008.

[3] M. Merca, T. Tanriverdi, An asymptotic formula of cosine power sums, *Matematiche (Catania)* **68**(1) (2013), 131–136.

[4] C. Ramus, Solution générale d'un problème d'analyse combinatoire, *J. Reine Angew. Math.* **11** (1834), 353–355.

# Again on passing to the limit under the integral sign

Tiberiu Trif[1)]

**Abstract**. We prove a Dominated Convergence Theorem for the improper Riemann integral. It enables us to provide rigorous arguments at the undergraduate level for the solutions of some contest problems (and not only).

**Keywords:** improper Riemann integral, Dominated Convergence Theorem, Bounded Convergence Theorem

**MSC:** 26A42

## 1. Introduction

The Dominated Convergence Theorem is one of the main results in Real Analysis. Its correspondent in the context of Riemann integral is the Bounded Convergence Theorem, discovered in 1885 by C. Arzelà [2] and independently in 1897 by W. F. Osgood [10]. For different proofs and historical aspects concerning the Bounded Convergence Theorem the reader is referred to the papers [3, 5, 8, 9, 11, 12].

**Theorem 1** (Bounded Convergence Theorem)**.** *Let $f : [a,b] \to \mathbb{R}$, and let $f_n : [a,b] \to \mathbb{R}$ $(n \in \mathbb{N})$ be a sequence of functions such that the following conditions are satisfied:*

(i) *$f_n$ is Riemann integrable on $[a,b]$ for all $n \in \mathbb{N}$;*

(ii) *$\lim\limits_{n\to\infty} f_n(x) = f(x)$ for all $x \in [a,b]$;*

(iii) *$f$ is Riemann integrable on $[a,b]$;*

(iv) *there exists a positive constant $M$ such that $|f_n(x)| \le M$ for all $n \in \mathbb{N}$ and all $x \in [a,b]$.*

*Then* $$\lim_{n\to\infty} \int_a^b f_n(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x.$$

The aim of the present note is to show that the Bounded Convergence Theorem easily implies a Dominated Convergence Theorem for the improper Riemann integral. Thus we are able to provide **rigorous** arguments at the undergraduate level (i.e., without referring to measure theory or Lebesgue integral) for the solutions of the following contest problems.

**Problem 1.** *a) Prove that* $$\lim_{n\to\infty} n\int_0^n \frac{\arctan \frac{x}{n}}{x(x^2+1)}\,\mathrm{d}x = \frac{\pi}{2}\,.$$

[1)]Babeş-Bolyai University, Faculty of Mathematics and Computer Science, Str. Kogălniceanu no. 1, 400084 Cluj-Napoca, Romania, `ttrif@math.ubbcluj.ro`

*b) Find* $\displaystyle \lim_{n\to\infty} n\left(n\int_0^n \frac{\arctan\frac{x}{n}}{x(x^2+1)}\,\mathrm{d}x - \frac{\pi}{2}\right).$

SEEMOUS 2014

**Problem 2.** *a) Find* $\displaystyle \lim_{n\to\infty} n\int_0^1 \left(\frac{1-x}{1+x}\right)^n \mathrm{d}x.$

*b) Given any $k\in\mathbb{N}$, find* $\displaystyle \lim_{n\to\infty} n^{k+1}\int_0^1 x^k\left(\frac{1-x}{1+x}\right)^n \mathrm{d}x.$

SEEMOUS 2012

**Problem 3.** *Compute*

$$\int_0^\infty \left(x-\frac{x^3}{2}+\frac{x^5}{2\cdot 4}-\frac{x^7}{2\cdot 4\cdot 6}+\cdots\right)\times \left(1+\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}+\frac{x^6}{2^2\cdot 4^2\cdot 6^2}+\cdots\right)\mathrm{d}x.$$

Putnam 1997, problem A3

It should be remarked that all solutions to Problem 3 published in [1, 6, 7] make use of the Monotone Convergence Theorem for the Lebesgue integral. We notice also that many problems from section 1.2 in the recent interesting book by O. Furdui [4] can be tackled by using the Dominated Convergence Theorem for the improper Riemann integral established in the next section. Among them we merely mention here the following two.

**Problem 4.** *Let $a>0$ and $b>1$ be real numbers, and let $f:[0,1]\to\mathbb{R}$ be a continuous function. Calculate* $\displaystyle \lim_{n\to\infty} n^{a/b}\int_0^1 \frac{f(x)}{1+n^a x^b}\,\mathrm{d}x.$

Problem 1.44 in [4]

**Problem 5.** *Let $k\in\mathbb{N}$. Calculate* $\displaystyle \lim_{n\to\infty}\int_0^{\sqrt[k]{n}} \left(1-\frac{x^k}{n}\right)^n \mathrm{d}x.$

Problem 1.57 in [4]

## 2. The Dominated Convergence Theorem for the improper Riemann integral

Recall first that given any non-degenerate interval $I$ of the real axis, a function $f:I\to\mathbb{R}$ is said to be *locally Riemann integrable* (on $I$) if its restriction to every compact subinterval $[u,v]$ of $I$ is Riemann integrable. Further, let $-\infty<a<b\le\infty$, and let $f:[a,b)\to\mathbb{R}$ be a locally Riemann integrable

function. The improper integral of $f$ on $[a, b)$, denoted by $\int_a^{b-} f(x)\mathrm{d}x$, is defined by

$$\int_a^{b-} f(x)\mathrm{d}x = \lim_{v \nearrow b} \int_a^v f(x)\mathrm{d}x, \tag{2.9}$$

provided that the limit exists in $\overline{\mathbb{R}}$. If the limit in (2.9) is finite, then one says that the improper integral $\int_a^{b-} f(x)\mathrm{d}x$ *converges*.

**Theorem 2** (Dominated Convergence Theorem). *Let $-\infty < a < b \leq \infty$, let $f : [a, b) \to \mathbb{R}$, and let $f_n : [a, b) \to \mathbb{R}$ $(n \in \mathbb{N})$ be a sequence of functions such that the following conditions are satisfied:*

(i) *$f_n$ is locally Riemann integrable on $[a, b)$ for all $n \in \mathbb{N}$;*

(ii) *$\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [a, b)$;*

(iii) *$f$ is locally Riemann integrable on $[a, b)$;*

(iv) *there exists a function $g : [a, b) \to [0, \infty)$ which is locally Riemann integrable on $[a, b)$ such that $\int_a^{b-} g(x)\mathrm{d}x$ converges and $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ and all $x \in [a, b)$.*

*Then all the improper integrals $\int_a^{b-} f(x)\mathrm{d}x$ and $\int_a^{b-} f_n(x)\mathrm{d}x$ $(n \in \mathbb{N})$ are convergent and* $\lim_{n\to\infty} \int_a^{b-} f_n(x)\mathrm{d}x = \int_a^{b-} f(x)\mathrm{d}x$.

*Proof.* Letting $n \to \infty$ in $|f_n(x)| \leq g(x)$ we find that $|f(x)| \leq g(x)$ for all $x \in [a, b)$. By virtue of the majorant test we conclude that all the improper integrals $\int_a^{b-} f(x)\mathrm{d}x$ and $\int_a^{b-} f_n(x)\mathrm{d}x$ $(n \in \mathbb{N})$ are absolutely convergent, hence convergent. Set

$$I := \int_a^{b-} f(x)\mathrm{d}x \quad \text{and} \quad I_n := \int_a^{b-} f_n(x)\mathrm{d}x \quad (n \in \mathbb{N}).$$

In order to prove that the sequence $(I_n)$ converges to $I$, let $\varepsilon > 0$ be arbitrarily chosen. Select $b_0 \in (a, b)$ such that

$$\int_{b_0}^{b-} g(x)\mathrm{d}x < \frac{\varepsilon}{3}\,.$$

Since $g$ is Riemann integrable on $[a, b_0]$, it is bounded on this interval. Setting $M := \sup_{x\in[a,b_0]} g(x)$, we have $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in [a, b_0]$. By Theorem 1 it follows that

$$\lim_{n\to\infty} \int_a^{b_0} f_n(x)\mathrm{d}x = \int_a^{b_0} f(x)\mathrm{d}x,$$

hence there exists $n_0 \in \mathbb{N}$ such that

$$\left| \int_a^{b_0} f_n(x)\mathrm{d}x - \int_a^{b_0} f(x)\mathrm{d}x \right| < \frac{\varepsilon}{3} \quad \text{for all } n \geq n_0.$$

Then for all $n \geq n_0$ we have

$$\begin{aligned} |I_n - I| &= \left| \int_a^{b_0} f_n(x)\mathrm{d}x + \int_{b_0}^{b-} f_n(x)\mathrm{d}x - \int_a^{b_0} f(x)\mathrm{d}x - \int_{b_0}^{b-} f(x)\mathrm{d}x \right| \\ &\leq \left| \int_a^{b_0} f_n(x)\mathrm{d}x - \int_a^{b_0} f(x)\mathrm{d}x \right| + \int_{b_0}^{b-} |f_n(x)|\mathrm{d}x + \int_{b_0}^{b-} |f(x)|\mathrm{d}x \\ &< \frac{\varepsilon}{3} + 2\int_{b_0}^{b-} g(x)\mathrm{d}x < \varepsilon. \end{aligned}$$

Therefore, $(I_n)$ converges to $I$. □

**Remark 3.** *It is obvious that similar versions of Theorem 2 hold for functions $f, f_n : (a, b] \to \mathbb{R}$ with $-\infty \leq a < b < \infty$, as well as for functions $f, f_n : (a, b) \to \mathbb{R}$ with $-\infty \leq a < b \leq \infty$.*

**Remark 4.** *Although the only ingredient used in the proof of the Dominated Convergence Theorem is Arzelà's Bounded Convergence Theorem, we were not able to localize in the literature the result for the improper Riemann integral as stated in Theorem 2.*

## 3. **Solutions to Problems 1–5**

*Solution to Problem 1.* a) The sequence of functions $f_n : (0, \infty) \to \mathbb{R}$ $(n \in \mathbb{N})$, defined by

$$f_n(x) := \frac{n \arctan \frac{x}{n}}{x(x^2+1)} \cdot \chi_{(0,n]}(x),$$

converges pointwise on $(0, \infty)$ to $f(x) := \frac{1}{1+x^2}$. Moreover, we have

$$0 < f_n(x) < f(x) \quad \text{for all } n \in \mathbb{N} \text{ and all } x \in (0, \infty)$$

because $\arctan t < t$ for all $t \in (0, \infty)$. By Theorem 2 it follows that

$$\lim_{n\to\infty} n \int_0^n \frac{\arctan \frac{x}{n}}{x(x^2+1)}\,\mathrm{d}x = \lim_{n\to\infty} \int_0^\infty f_n(x)\mathrm{d}x = \int_0^\infty f(x)\mathrm{d}x = \frac{\pi}{2}.$$

b) Letting $\quad x_n := n^2 \int_0^n \frac{\arctan \frac{x}{n}}{x(x^2+1)}\,\mathrm{d}x - n\,\frac{\pi}{2}$, we have $x_n = y_n - z_n$, where

$$y_n := n^2 \int_0^n \frac{\arctan \frac{x}{n}}{x(x^2+1)}\,\mathrm{d}x - n\int_0^n \frac{\mathrm{d}x}{x^2+1} \quad \text{and} \quad z_n := n\int_n^\infty \frac{\mathrm{d}x}{x^2+1}.$$

Since $z_n = n\left(\frac{\pi}{2} - \arctan n\right)$, it follows immediately that $\lim\limits_{n\to\infty} z_n = 1$. We also have

$$y_n = \int_0^n \frac{n^2 \arctan \frac{x}{n} - nx}{x(x^2+1)}\,\mathrm{d}x.$$

Substituting $x/n = t$ yields

$$y_n = \int_0^1 \frac{\arctan t - t}{t^3} \cdot \frac{n^2 t^2}{1 + n^2 t^2}\,\mathrm{d}t.$$

The sequence of functions $g_n : (0,1] \to \mathbb{R}$ $(n \in \mathbb{N})$, defined by

$$g_n(t) := \frac{\arctan t - t}{t^3} \cdot \frac{n^2 t^2}{1 + n^2 t^2}\,,$$

converges pointwise on $(0,1]$ to $g(t) := \dfrac{\arctan t - t}{t^3}$. In addition, we have $|g_n(t)| < -g(t)$ for all $n \in \mathbb{N}$ and all $t \in (0,1]$. Since $\lim\limits_{t \to 0+} g(t) = -\frac{1}{3}$, by Theorem 2 (or even by Theorem 1) we get

$$\begin{aligned} \lim_{n\to\infty} y_n &= \lim_{n\to\infty} \int_0^1 g_n(t)\,\mathrm{d}t = \int_0^1 g(t)\,\mathrm{d}t \\ &= \left( \frac{t - \arctan t}{2t^2} - \frac{1}{2}\arctan t \right)\Bigg|_0^1 = \frac{1}{2} - \frac{\pi}{4}\,. \end{aligned}$$

Consequently, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n - \lim\limits_{n\to\infty} z_n = -\frac{1}{2} - \frac{\pi}{4}$.

*Solution to Problem 2.* We solve directly part b) under the assumption that $k$ is a nonnegative integer. Letting

$$I_n := n^{k+1} \int_0^1 x^k \left( \frac{1-x}{1+x} \right)^n \mathrm{d}x \quad (n \in \mathbb{N}),$$

we have $I_n = \left(\frac{n}{n+1}\right)^{k+1} J_n$, where

$$J_n := \int_0^1 (n+1) \left( \frac{1-x}{1+x} \right)^n \left((n+1)x\right)^k \mathrm{d}x.$$

Substituting $\left(\frac{1-x}{1+x}\right)^{n+1} = t$, we have

$$(n+1)\left( \frac{1-x}{1+x} \right)^n \cdot \frac{-2}{(1+x)^2}\,\mathrm{d}x = \mathrm{d}t \quad \text{and} \quad x = \frac{1 - t^{\frac{1}{n+1}}}{1 + t^{\frac{1}{n+1}}}\,,$$

whence

$$\begin{aligned} J_n &= \int_1^0 -\frac{1}{2} \left( 1 + \frac{1 - t^{\frac{1}{n+1}}}{1 + t^{\frac{1}{n+1}}} \right)^2 \left( (n+1) \frac{1 - t^{\frac{1}{n+1}}}{1 + t^{\frac{1}{n+1}}} \right)^k \mathrm{d}t \\ &= 2 \int_0^1 \left( (n+1)\left(1 - t^{\frac{1}{n+1}}\right) \right)^k \frac{1}{\left(1 + t^{\frac{1}{n+1}}\right)^{k+2}}\,\mathrm{d}t. \end{aligned}$$

The sequence of functions $f_n : (0,1] \to \mathbb{R}$ $(n \in \mathbb{N})$, defined by

$$f_n(t) := \left((n+1)\left(1 - t^{\frac{1}{n+1}}\right)\right)^k \frac{1}{\left(1 + t^{\frac{1}{n+1}}\right)^{k+2}},$$

converges pointwise on $(0,1]$ to $f(t) := \frac{1}{2^{k+2}}(-\ln t)^k$. Moreover, the well known inequality $\ln t \leq t - 1$ implies

$$(n+1)\left(1 - t^{\frac{1}{n+1}}\right) \leq -\ln t \quad \text{for all } t \in (0,1],$$

whence $0 \leq f_n(t) \leq (-\ln t)^k$ for all $n \in \mathbb{N}$ and all $t \in (0,1]$. It is proved below that $\int_0^1 (-\ln t)^k \, \mathrm{d}t$ converges. Then by Theorem 2 it follows that

$$\begin{aligned} \lim_{n\to\infty} I_n &= \lim_{n\to\infty} J_n = 2 \lim_{n\to\infty} \int_0^1 f_n(t)\,\mathrm{d}t = 2\int_0^1 f(t)\,\mathrm{d}t \\ &= \frac{1}{2^{k+1}} \int_0^1 (-\ln t)^k \,\mathrm{d}t. \end{aligned}$$

Letting $L_k := \int_0^1 (-\ln t)^k \, \mathrm{d}t$, an integration by parts yields $L_k = kL_{k-1}$. Since $L_0 = 1$, an inductive argument shows that all improper integrals $L_k$ are convergent and $L_k = k!$ for all $k \geq 0$. Consequently, $\lim\limits_{n\to\infty} I_n = \frac{k!}{2^{k+1}}$.

*Solution to Problem 3.* Denote by $I$ the value of the integral. Since the first bracket equals $xe^{-x^2/2}$, it follows that

$$I = \int_0^\infty xe^{-x^2/2} \sum_{k=0}^\infty \frac{x^{2k}}{2^{2k}(k!)^2}\,\mathrm{d}x =: \int_0^\infty f(x)\,\mathrm{d}x.$$

The sequence of functions $f_n : [0,\infty) \to \mathbb{R}$ $(n \in \mathbb{N})$, defined by

$$f_n(x) := xe^{-x^2/2} \sum_{k=0}^n \frac{x^{2k}}{2^{2k}(k!)^2},$$

converges pointwise on $[0,\infty)$ to $f$. Moreover, we have

$$0 \leq f_n(x) \leq xe^{-x^2/2} \sum_{k=0}^\infty \frac{x^{2k}}{2^{2k}k!} = xe^{-x^2/4} =: g(x)$$

for all $n \in \mathbb{N}$ and all $x \in [0,\infty)$. Since $\int_0^\infty g(x)\,\mathrm{d}x$ converges, Theorem 2 ensures that

$$I = \lim_{n\to\infty} \int_0^\infty f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{k=0}^n \frac{1}{2^{2k}(k!)^2} \int_0^\infty e^{-x^2/2} x^{2k+1}\,\mathrm{d}x.$$

Substituting $x^2/2 = t$ we get

$$\int_0^\infty e^{-x^2/2} x^{2k+1}\,\mathrm{d}x = 2^k \int_0^\infty e^{-t} t^k \,\mathrm{d}t = 2^k \Gamma(k+1) = 2^k k!,$$

whence $I = \sum_{k=0}^{\infty} \frac{1}{2^k k!} = e^{1/2} = \sqrt{e}$.

*Solution to Problem 4.* Set $I_n := n^{a/b} \int_0^1 \frac{f(x)}{1+n^a x^b}\, \mathrm{d}x$ $(n \in \mathbb{N})$. The substitution $n^a x^b = t$ leads to

$$I_n = \frac{1}{b} \int_0^{n^a} f\left(t^{1/b}/n^{a/b}\right) \frac{t^{\frac{1}{b}-1}}{t+1}\, \mathrm{d}t.$$

The sequence of functions $f_n : (0,\infty) \to \mathbb{R}$ $(n \in \mathbb{N})$, defined by

$$f_n(t) := f\left(t^{1/b}/n^{a/b}\right) \frac{t^{\frac{1}{b}-1}}{t+1}\, \chi_{(0,n^a]}(t),$$

converges pointwise on $(0,\infty)$ to $f(t) := f(0)\, \frac{t^{\frac{1}{b}-1}}{t+1}$. In addition, we have

$$|f_n(t)| \le M\, \frac{t^{\frac{1}{b}-1}}{t+1} \quad \text{for all } n \in \mathbb{N} \text{ and all } t \in (0,\infty),$$

where $M := \max_{0 \le x \le 1} |f(x)|$. Since

$$\int_0^{\infty} \frac{t^{\frac{1}{b}-1}}{t+1}\, \mathrm{d}t = B\left(\frac{1}{b}, 1-\frac{1}{b}\right) = \Gamma\left(\frac{1}{b}\right) \Gamma\left(1-\frac{1}{b}\right) = \frac{\pi}{\sin \frac{\pi}{b}},$$

by Theorem 2 we deduce that $\lim_{n\to\infty} I_n = f(0)\, \frac{\pi}{b \sin \frac{\pi}{b}}$.

*Solution to Problem 5.* More generally, let $(a_n)$ be an arbitrary sequence in $(0,\infty)$ such that $a_n \le \sqrt[k]{n}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \infty$, and let

$$I_n := \int_0^{a_n} \left(1 - \frac{x^k}{n}\right)^n \mathrm{d}x.$$

The sequence of functions $f_n : [0,\infty) \to \mathbb{R}$ $(n \in \mathbb{N})$, defined by

$$f_n(x) := \left(1 - \frac{x^k}{n}\right)^n \chi_{[0,a_n]}(x),$$

converges pointwise on $[0,\infty)$ to $f(x) := e^{-x^k}$. Moreover, the inequality $e^t \ge 1+t$ for all $t \in \mathbb{R}$ ensures that $0 \le f_n(x) \le f(x)$ for all $n \in \mathbb{N}$ and all $x \in [0,\infty)$. Using the substitution $x^k = t$ we find that

$$\int_0^{\infty} f(x)\, \mathrm{d}x = \int_0^{\infty} e^{-t} \frac{1}{k}\, t^{\frac{1}{k}-1}\, \mathrm{d}t = \frac{1}{k}\, \Gamma\left(\frac{1}{k}\right) = \Gamma\left(1 + \frac{1}{k}\right).$$

By virtue of Theorem 2 we conclude that $\lim_{n\to\infty} I_n = \Gamma\left(1 + \frac{1}{k}\right)$.

## REFERENCES


[1] ***, 58th annual William Lowell Putnam mathematical competition, *Math. Mag.* **71** (1998), 154–160.

[2] C. Arzelà, Sulla integrazione per serie, *Atti Acc. Lincei Rend., Rome* **1** (1885), 532–537, 596–599.

[3] R. G. Bartle, Return to the Riemann integral, *Amer. Math. Monthly* **103** (1996), 625–632.

[4] O. Furdui, *Limits, Series and Fractional Part Integrals. Problems in Mathematical Analysis*, Springer, New York, 2013.

[5] R. A. Gordon, A convergence theorem for the Riemann integral, *Math. Mag.* **73** (2000), 141–147.

[6] K. S. Kedlaya, B. Poonen, R. Vakil, *The William Lowell Putnam Mathematical Competition 1985–2000. Problems, Solutions and Commentary*, The Mathematical Association of America, 2002.

[7] L. F. Klosinski, G. L. Alexanderson, L. C. Larson, The fifty-eighth William Lowell Putnam mathematical competition, *Amer. Math. Monthly* **105** (1998), 744–755.

[8] W. A. J. Luxemburg, Arzelà's dominated convergence theorem for the Riemann integral, *Amer. Math. Monthly* **78** (1971), 970–979.

[9] C. P. Niculescu, F. Popovici, The monotone convergence theorem for the Riemann integral, *Ann. Univ. Craiova, Math. Comp. Sci. Series* **38** (2) (2011), 55–58.

[10] W. F. Osgood, Non-uniform convergence and the integration of series term by term, *Amer. J. Math.* **19** (1897), 155–190.

[11] N. de Silva, A concise, elementary proof of Arzelà's bounded convergence theorem, *Amer. Math. Monthly* **117** (2010), 918–920.

[12] B. S. Thomson, Monotone convergence theorem for the Riemann integral, *Amer. Math. Monthly* **117** (2010), 547–550.

# The limit of some sequences of double integrals on the unit square

DUMITRU POPA[1)]

**Abstract**. We show that under natural hypotheses on the sequence of continuous functions $h_n : [0,1]^2 \to \mathbb{R}$, for all continuous functions $f : [0,1]^2 \to \mathbb{R}$ the following equality holds

$$\lim_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y) f(x,y)\,\mathrm{d}x\mathrm{d}y = f(1,1) \lim_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y)\,\mathrm{d}x\mathrm{d}y.$$

**Keywords:** Riemann integral, double Riemann integral, Fubini theorem, limit of sequences of integral
**MSC:** 26B15; 28A35.

## 1. Introduction

The starting point for this paper was very natural: how can we extend some well known results about the limits of Riemann integrable functions on the closed interval $[0,1]$ to the case of double Riemann integrable functions? A first question which appears is the following: with what could we replace the interval $[0,1]$ in the case of double integrals? It is obvious that in the case of double integrals, the natural analog of the interval $[0,1]$ is the unit square $[0,1]^2$, and not only! The main purpose of this paper is to indicate a way to calculate the limits of double Riemann integrable functions on the unit square. The notation and notions used and not defined in this paper are standard. For details regarding the multiple Riemann integral we recommend the reader the excellent treatment of this concept in the textbook of Nicu Boboc, see [1].

## 2. Preliminary results

We recall a well known result, see [2, Ex. 3.13, p. 53–54].

**Lemma 1.** *Let $v_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions such that the sequence $\left(\int_0^1 |v_n(x)|\,\mathrm{d}x\right)_{n\in\mathbb{N}}$ is bounded and for all $0 < u < 1$,*

$$\lim_{n\to\infty} \int_0^u |v_n(x)|\,\mathrm{d}x = 0.$$

(i) *For each continuous function $f : [0,1] \to \mathbb{R}$ the following equality holds*

$$\lim_{n\to\infty} \left( \int_0^1 v_n(x) f(x)\,\mathrm{d}x - f(1) \int_0^1 v_n(x)\,\mathrm{d}x \right) = 0.$$

[1)]Department of Mathematics Ovidius University of Constanta, Bd. Mamaia 124, 900527 Constanţa, Romania, dpopa@univ-ovidius.ro

(ii) *If* $\lim\limits_{n\to\infty}\int_0^1 v_n(x)\,\mathrm{d}x = L \in \mathbb{R}$, *then for each continuous function* $f:[0,1]\to\mathbb{R}$ *the following equality holds*

$$\lim_{n\to\infty}\int_0^1 v_n(x)f(x)\mathrm{d}x = Lf(1).$$

Now we indicate a natural way to construct examples of sequences which satisfy the hypotheses in Lemma 1.

**Proposition 2.** (i) *The sequence* $v_n:[0,1]\to\mathbb{R}$, $v_n(x)=nx^nf(x^n)$, *where* $f:[0,1]\to\mathbb{R}$ *is a continuous function, satisfies the hypotheses from Lemma 1 and* $\lim\limits_{n\to\infty}\int_0^1 v_n(x)\,\mathrm{d}x=\int_0^1 f(x)\,\mathrm{d}x$.

(ii) *The sequence* $v_n:[0,1]\to\mathbb{R}$, $v_n(x)=\frac{n^2x^n}{1+x+x^2+\cdots+x^n}$, *satisfies the hypotheses from Lemma 1 and* $\lim\limits_{n\to\infty}\int_0^1 v_n(x)\,\mathrm{d}x=\frac{\pi^2}{6}$.

*Proof.* (i) see [2, Ex. 3.13(b), p. 54]. (ii) see [2, Ex. 3.34(ii), p. 65]. □

**Proposition 3.** *Let* $h_n:[0,1]^2\to\mathbb{R}$ *be a sequence of continuous functions such that the sequence* $\left(\iint\limits_{[0,1]^2}|h_n(x,y)|\,\mathrm{d}x\mathrm{d}y\right)_{n\in\mathbb{N}}$ *is bounded and for all* $0<u<1$,

$$\lim_{n\to\infty}\iint\limits_{[0,u]\times[0,1]}|h_n(x,y)|\,\mathrm{d}x\mathrm{d}y=0.$$

*We define* $v_n:[0,1]\to\mathbb{R}$, $v_n(x)=\int_0^1 h_n(x,y)\,\mathrm{d}y$. *Then the functions* $v_n$ *are continuous for all natural numbers* $n$, *the sequence* $\left(\int_0^1|v_n(x)|\,\mathrm{d}x\right)_{n\in\mathbb{N}}$ *is bounded and for all* $0<u<1$, $\lim\limits_{n\to\infty}\int_0^u|v_n(x)|\,\mathrm{d}x=0$.

*Proof.* Since the functions $h_n$ are continuous, by a well-known result, $v_n$ are also continuous. For every natural number $n$ we have

$$|v_n(x)|\le\int_0^1|h_n(x,y)|\,\mathrm{d}y,\ \forall x\in[0,1]. \tag{2.1}$$

From (2.1) and Fubini's theorem we deduce

$$\int_0^1|v_n(x)|\,\mathrm{d}x\le\int_0^1\left(\int_0^1|h_n(x,y)|\,\mathrm{d}y\right)\mathrm{d}x=\iint\limits_{[0,1]^2}|h_n(x,y)|\,\mathrm{d}x\mathrm{d}y$$

and by the boundedness of the sequence $\left(\iint\limits_{[0,1]^2}|h_n(x,y)|\,\mathrm{d}x\mathrm{d}y\right)_{n\in\mathbb{N}}$, the boundedness of the sequence $\left(\int_0^1|v_n(x)|\,\mathrm{d}x\right)_{n\in\mathbb{N}}$ follows.

Let $0 < u < 1$. By (2.1) and Fubini's theorem we have

$$\int_0^u |v_n(x)|\,\mathrm{d}x \le \int_0^u \left(\int_0^1 |h_n(x,y)|\,\mathrm{d}y\right)\mathrm{d}x = \iint\limits_{[0,u]\times[0,1]} |h_n(x,y)|\,\mathrm{d}x\mathrm{d}y.$$

Since $\lim\limits_{n\to\infty} \iint\limits_{[0,u]\times[0,1]} |h_n(x,y)|\,\mathrm{d}x\mathrm{d}y = 0$, this shows that indeed one has $\lim\limits_{n\to\infty} \int_0^u |v_n(x)|\,\mathrm{d}x = 0$. □

## 3. The main results

We first prove an 'asymmetric' result.

**Lemma 4.** *Let $h_n : [0,1]^2 \to \mathbb{R}$ be a sequence of continuous functions such that the sequence* $\left(\iint\limits_{[0,1]^2} |h_n(x,y)|\,\mathrm{d}x\mathrm{d}y\right)_{n\in\mathbb{N}}$ *is bounded and for all* $0 < v < 1$,

$$\lim_{n\to\infty} \iint\limits_{[0,1]\times[0,v]} |h_n(x,y)|\,\mathrm{d}x\mathrm{d}y = 0.$$

(i) *For each continuous function $\varphi : [0,1]^2 \to \mathbb{R}$ with the property that $\varphi(x,1) = 0$, $\forall x \in [0,1]$ the following equality holds*

$$\lim_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y)\,\varphi(x,y)\,\mathrm{d}x\mathrm{d}y = 0.$$

(ii) *For each continuous function $f : [0,1]^2 \to \mathbb{R}$ the following equality holds*

$$\lim_{n\to\infty}\left(\iint\limits_{[0,1]^2} h_n(x,y)\,f(x,y)\,\mathrm{d}x\mathrm{d}y - \int_0^1\left(\int_0^1 h_n(x,y)\,\mathrm{d}y\right) f(x,1)\,\mathrm{d}x\right) = 0.$$

*Proof.* (i) By hypothesis there exists $A > 0$ such that

$$\iint\limits_{[0,1]^2} |h_n(x,y)|\,\mathrm{d}x\mathrm{d}y \le A, \forall n \in \mathbb{N}. \tag{3.2}$$

Since $\varphi$ is continuous on the compact set $[0,1]^2$, from the Weierstrass theorem, $\varphi$ is bounded, i.e., there exists $M > 0$ such that

$$|\varphi(x,y)| \le M, \forall (x,y) \in [0,1]^2. \tag{3.3}$$

Moreover, $\varphi$ being continuous and $[0,1]^2$ being a compact set, $\varphi$ is uniformly continuous.

Let $\varepsilon > 0$. By the uniform continuity of the function $\varphi$, there exists $\delta_\varepsilon \in (0,1)$ such that $\forall (x_1, y_1) \in [0,1]^2$ and $\forall (x_2, y_2) \in [0,1]^2$ with the property that $|x_1 - x_2| \le \delta_\varepsilon$ and $|y_1 - y_2| \le \delta_\varepsilon$ one has

$$|\varphi(x_1, y_1) - \varphi(x_2, y_2)| \le \frac{\varepsilon}{2A}. \tag{3.4}$$

Let us put $v_\varepsilon = 1 - \delta_\varepsilon \in (0,1)$ and note that the relations $y \in [0,1]$ and $|y - 1| \le \delta_\varepsilon$ are equivalent to $v_\varepsilon = 1 - \delta_\varepsilon \le y \le 1$. By using this observation from (3.4) we deduce

$$|\varphi(x, y) - \varphi(x, 1)| \le \frac{\varepsilon}{2A}, \ \forall x \in [0,1], \ \forall y \in [v_\varepsilon, 1]$$

or, since by hypothesis $\varphi(x, 1) = 0$, $\forall x \in [0,1]$

$$|\varphi(x, y)| \le \frac{\varepsilon}{2A}, \ \forall x \in [0,1], \ \forall y \in [v_\varepsilon, 1]. \tag{3.5}$$

Let $n \in \mathbb{N}$. From Fubini's theorem we have

$$\iint\limits_{[0,1]^2} h_n(x,y)\,\varphi(x,y)\,\mathrm{d}x\mathrm{d}y = \int_0^1 \left( \int_0^1 h_n(x,y)\,\varphi(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

From this we deduce

$$\left| \iint\limits_{[0,1]^2} h_n(x,y)\,\varphi(x,y)\,\mathrm{d}x\mathrm{d}y \right| \le \int_0^1 \left( \int_0^1 |h_n(x,y)|\,|\varphi(x,y)|\,\mathrm{d}y \right) \mathrm{d}x$$

$$= \int_0^1 \left( \int_0^{v_\varepsilon} |h_n(x,y)|\,|\varphi(x,y)|\,\mathrm{d}y \right) \mathrm{d}x + \int_0^1 \left( \int_{v_\varepsilon}^1 |h_n(x,y)|\,|\varphi(x,y)|\,\mathrm{d}y \right) \mathrm{d}x. \tag{3.6}$$

By using (3.3) and Fubini's theorem we have

$$\begin{aligned} \int_0^1 \left( \int_0^{v_\varepsilon} |h_n(x,y)|\,|\varphi(x,y)|\,\mathrm{d}y \right) \mathrm{d}x &\le M \int_0^1 \left( \int_0^{v_\varepsilon} |h_n(x,y)|\,\mathrm{d}y \right) \mathrm{d}x \\ &= M \iint\limits_{[0,1]\times[0,v_\varepsilon]} |h_n(x,y)|\,\mathrm{d}x\mathrm{d}y. \quad (3.7) \end{aligned}$$

From (3.5), (3.2) and Fubini's theorem we deduce

$$\begin{aligned} \int_0^1 \left( \int_{v_\varepsilon}^1 |h_n(x,y)|\,|\varphi(x,y)|\,\mathrm{d}y \right) \mathrm{d}x &\le \frac{\varepsilon}{2A} \int_0^1 \left( \int_{a\varepsilon}^1 |h_n(x,y)|\,\mathrm{d}y \right) \mathrm{d}x \\ &\le \frac{\varepsilon}{2A} \iint\limits_{[0,1]^2} |h_n(x,y)|\,\mathrm{d}x\mathrm{d}y \le \frac{\varepsilon}{2}. \quad (3.8) \end{aligned}$$

From the hypothesis there exists $n_\varepsilon \in \mathbb{N}$ such that $\forall n \ge n_\varepsilon$ we have

$$\iint\limits_{[0,1]\times[0,v_\varepsilon]} |h_n(x,y)|\,\mathrm{d}x\mathrm{d}y < \frac{\varepsilon}{2M}. \tag{3.9}$$

By using (3.7), (3.8) and (3.9), from (3.6) it follows that $\forall n \geq n_\varepsilon$ the following inequality holds

$$\left| \iint\limits_{[0,1]^2} h_n(x,y)\,\varphi(x,y)\,\mathrm{d}x\mathrm{d}y \right| < \varepsilon.$$

This ends the proof of (i).

(ii) Let $\varphi : [0,1]^2 \to \mathbb{R}$ be defined by $\varphi(x,y) = f(x,y) - f(x,1)$. Note that $\varphi$ is continuous and has the property that $\varphi(x,1) = 0$, $\forall x \in [0,1]$. From (i) we have

$$\lim_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y)\,\varphi(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{3.10}$$

From the linearity of the integral and Fubini's theorem, for each natural number $n$ we have

$$\begin{aligned} & \iint\limits_{[0,1]^2} h_n(x,y)\varphi(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ = & \iint\limits_{[0,1]^2} h_n(x,y)\,f(x,y)\mathrm{d}x\,\mathrm{d}y - \iint\limits_{[0,1]^2} h_n(x,y)f(x,1)\mathrm{d}x\,\mathrm{d}y \\ = & \iint\limits_{[0,1]^2} h_n(x,y)f(x,y)\mathrm{d}x\mathrm{d}y - \int_0^1 \left( \int_0^1 h_n(x,y)\mathrm{d}y \right) f(x,1)\mathrm{d}x. \end{aligned} \tag{3.11}$$

Replacing (3.11) in (3.10) we obtain the conclusion of (ii). □

In the following we state and prove the main result of this paper. As is expected, the "symmetry of variables" is clearly highlighted. Let us note that this result is an extension to the double integrals on the unit square of Lemma 1.

**Theorem 5.** *Let $h_n : [0,1]^2 \to \mathbb{R}$ be a sequence of continuous functions such that the sequence* $\left( \iint\limits_{[0,1]^2} |h_n(x,y)|\,\mathrm{d}x\mathrm{d}y \right)_{n\in\mathbb{N}}$ *is bounded and the following symmetric conditions are satisfied:*

$$\text{for all } 0 < u < 1,\ \lim_{n\to\infty} \iint\limits_{[0,u]\times[0,1]} |h_n(x,y)|\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{3.12}$$

*and*

$$\text{for all } 0 < v < 1,\ \lim_{n\to\infty} \iint\limits_{[0,1]\times[0,v]} |h_n(x,y)|\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{3.13}$$

(i) *For each continuous function* $f : [0,1]^2 \to \mathbb{R}$ *the following equality holds*

$$\lim_{n\to\infty}\left(\iint\limits_{[0,1]^2} h_n(x,y) f(x,y)\,\mathrm{d}x\,\mathrm{d}y - f(1,1)\iint\limits_{[0,1]^2} h_n(x,y)\,\mathrm{d}x\,\mathrm{d}y\right) = 0.$$

(ii) *If* $\lim\limits_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y)\,\mathrm{d}x\,\mathrm{d}y = A \in \mathbb{R}$, *then for each continuous function* $f : [0,1]^2 \to \mathbb{R}$ *the following equality holds*

$$\lim_{n\to\infty}\iint\limits_{[0,1]^2} h_n f(x,y) f(x,y)\,\mathrm{d}x\,\mathrm{d}y = Af(1,1).$$

*Proof.* (i) From condition (3.13) and Lemma 4 (ii) it follows that

$$\lim_{n\to\infty}\left(\iint\limits_{[0,1]^2} h_n(x,y) f(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_0^1 v_n(x) g(x)\,\mathrm{d}x\right) = 0, \tag{3.14}$$

where $v_n : [0,1] \to \mathbb{R}$, $v_n(x) = \int_0^1 h_n(x,y)\,\mathrm{d}y$, $g : [0,1] \to \mathbb{R}$, $g(x) = f(x,1)$. By using condition (3.12), from Proposition 3 it follows that the sequence $(v_n)_{n\in\mathbb{N}}$ satisfies the conditions from Lemma 1. Then, by Lemma 1 we deduce that

$$\lim_{n\to\infty}\left(\int_0^1 v_n(x) g(x)\,\mathrm{d}x - g(1)\int_0^1 v_n(x)\,\mathrm{d}x\right) = 0. \tag{3.15}$$

From (3.14) and (3.15) we obtain

$$\lim_{n\to\infty}\left(\iint\limits_{[0,1]^2} h_n(x,y) f(x,yt)\,\mathrm{d}x\,\mathrm{d}y - f(1,1)\int_0^1 v_n(x)\,\mathrm{d}x\right) = 0,$$

which, because by Fubini's theorem, $\int_0^1 v_n(x)\,\mathrm{d}x = \iint\limits_{[0,1]^2} h_n(x,y)\,\mathrm{d}x\,\mathrm{d}y$, gives us the conclusion of (i).

(ii) It follows from (i) and hypothesis. □

## 4. **Some applications**

In the sequel we give examples of sequences of functions which satisfy the hypotheses in Theorem 5.

**Proposition 6.** (i) *Let* $a_n$, $b_n : [0,1] \to \mathbb{R}$ *be two sequences of continuous functions which satisfy the hypotheses in Lemma 1. Then the sequence of continuous functions* $h_n : [0,1]^2 \to \mathbb{R}$, $h_n(x,y) = a_n(x) b_n(y)$, $\forall x, y \in [0,1]$, *satisfies the hypotheses in Theorem 5.*

(ii) *Let $a > 0$, $b > 0$. The sequence of continuous functions*

$$h_n : [0,1]^2 \to \mathbb{R}, \quad h_n(x,y) = n^2 \left(\frac{ax+by}{a+b}\right)^n, \quad \forall x,y \in [0,1],$$

*satisfies the hypotheses in Theorem 5.*
*Moreover,*

$$\lim_{n\to\infty} n^2 \iint\limits_{[0,1]^2} \left(\frac{ax+by}{a+b}\right)^n \mathrm{d}x\,\mathrm{d}y = \frac{(a+b)^2}{ab}.$$

*Proof.* It is left as an exercise to the reader. □

From Proposition 6 and Theorem 5 we can give some concrete examples. We state such an example and leave to the reader to state other possible examples.

**Corollary 7.** (i) *For each continuous function $f : [0,1]^2 \to \mathbb{R}$ the following limit*

$$\lim_{n\to\infty} n^4 \iint\limits_{[0,1]^2} \frac{x^n y^n}{(1+x+x^2+\cdots+x^n)(1+y+y^2+\cdots+y^n)} f(x,y)\mathrm{d}x\,\mathrm{d}y$$

*is equal to $\frac{\pi^4}{36} f(1,1)$.*

(ii) *For each $a > 0$, $b > 0$ and each continuous function $f : [0,1]^2 \to \mathbb{R}$ the following equality holds*

$$\lim_{n\to\infty} n^2 \iint\limits_{[0,1]^2} \left(\frac{ax+by}{a+b}\right)^n f(x,y)\mathrm{d}x\,\mathrm{d}y = \frac{(a+b)^2}{ab} f(1,1).$$

*Proof.* (i) From Proposition 2 (ii) the sequence of functions $v_n : [0,1] \to \mathbb{R}$, $v_n(x) = \frac{n^2 x^n}{1+x+x^2+\cdots+x^n}$, satisfies the hypotheses required in Lemma 1 and $\lim\limits_{n\to\infty} \int_0^1 v_n(x)\,\mathrm{d}x = \frac{\pi^2}{6}$. From Proposition 6 (i) the sequence of functions $h_n : [0,1]^2 \to \mathbb{R}$, $h_n(x,y) = v_n(x) v_n(y)$, satisfies the hypotheses in Theorem 5 and by Fubini's theorem

$$\lim_{n\to\infty} \iint\limits_{[0,1]^2} h_n(x,y)\,\mathrm{d}x\mathrm{d}y = \left(\lim_{n\to\infty} \int_0^1 v_n(x)\,\mathrm{d}x\right)^2 = \frac{\pi^4}{36}.$$

From Theorem 5 (ii) we obtain the statement.

(ii) From Proposition 6 (ii) the sequence of functions $h_n : [0,1]^2 \to \mathbb{R}$, $h_n(x,y) = n^2 \left(\frac{ax+by}{a+b}\right)^n$, satisfies the hypotheses in Theorem 5. From Theorem 5 (ii) we get the statement. □

## References

[1] N. Boboc, *Analiză matematică*, partea a II-a, Ed. Univ. Bucureşti, Bucureşti, 1993.
[2] D. Popa, *Exerciţii de analiză matematică*, Biblioteca S. S. M. R., Editura Mira, Bucureşti, 2007.


# Olimpiada de Matematică a Studenţilor din Sud-Estul Europei, SEEMOUS 2014

MARCEL ROMAN[1)], RADU STRUGARIU[2)]

**Abstract**. This note deals with the problems of the 8th South Eastern European Mathematical Olympiad for University Students, SEEMOUS 2014, organized in Iaşi, Romania, by the Gheorghe Asachi Technical University of Iaşi, the Mathematical Society of South Eastern Europe (MASSEE), and the Romanian Mathematical Society, between March 5 and March 9, 2014.

**Keywords:** Similar matrices, normal matrix, Hermitian matrix, determinants, eigenvalues, eigenvectors, sequences of real numbers, integral convergence, Taylor series, continuous functions

**MSC:** 1C20; 15A16; 15A18; 15A24; 33D05; 40A30; 51D20.

## 1. Introducere

Ediţia a 8-a a Olimpiadei de Matematică pentru Studenţi, SEEMOUS 2014, a fost găzduită de Universitatea Tehnică „Gheorghe Asachi" din Iaşi şi a reunit 104 studenţi reprezentând 23 de echipe din şase ţări (Bulgaria, Grecia, Iran, România, Rusia, şi Turkmenistan).

Studenţii au avut de rezolvat în 5 ore un număr de 4 probleme. În cele ce urmează vom prezenta problemele propuse în concurs şi vom comenta soluţiile. Alte detalii despre desfăşurarea olimpiadei pot fi găsite pe site-ul SEEMOUS 2014: http://math.etti.tuiasi.ro/seemous/

## 2. Probleme, soluţii, comentarii

**Problema 1.** Fie $n$ un număr natural nenul iar $f : \mathbb{R} \to \mathbb{R}\backslash\{0\}$ o funcţie satisfăcând relaţia $f(2014) = 1 - f(2013)$. Fie $x_1, \ldots, x_n$ numere reale distincte

[1)] Universitatea Tehnică „Gheorghe Asachi" din Iaşi, Departamentul de Matematică şi Informatică, `marcelroman@tuiasi.ro`

[2)] Universitatea Tehnică „Gheorghe Asachi" din Iaşi, Departamentul de Matematică şi Informatică, `rstrugariu@tuiasi.ro`

două câte două. Dacă

$$\begin{vmatrix} 1+f(x_1) & f(x_2) & f(x_3) & \dots & f(x_n) \\ f(x_1) & 1+f(x_2) & f(x_3) & \dots & f(x_n) \\ f(x_1) & f(x_2) & 1+f(x_3) & \dots & f(x_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(x_1) & f(x_2) & f(x_3) & \dots & 1+f(x_n) \end{vmatrix} = 0,$$

să se demonstreze că funcţia $f$ nu este continuă.

**Dimitris Georgiou**, Grecia

*Considerată de către juriu ca problemă cu un grad de dificultate mediu spre uşor, problema 1 a fost rezolvată în întregime de către 27 de participanţi. Diversele soluţii oferite de către candidaţi au fost diferite doar în ceea ce priveşte calculul determinantului.*

**Soluţie.** Adunând toate coloanele la prima şi scoţând factor comun,

$$\begin{vmatrix} 1+f(x_1) & f(x_2) & f(x_3) & \dots & f(x_n) \\ f(x_1) & 1+f(x_2) & f(x_3) & \dots & f(x_n) \\ f(x_1) & f(x_2) & 1+f(x_3) & \dots & f(x_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(x_1) & f(x_2) & f(x_3) & \dots & 1+f(x_n) \end{vmatrix} =$$

$$= (1 + f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n))\cdot$$

$$\begin{vmatrix} 1 & f(x_2) & f(x_3) & \dots & f(x_n) \\ 1 & 1+f(x_2) & f(x_3) & \dots & f(x_n) \\ 1 & f(x_2) & 1+f(x_3) & \dots & f(x_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & f(x_2) & f(x_3) & \dots & 1+f(x_n) \end{vmatrix}.$$

Ultimul determinant fiind 1 (lucru ce se vede scăzând prima linie din celelalte), relaţia dată devine

$$1 + f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n) = 0,$$

sau

$$f(2014) + f(2013) + f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n) = 0. \tag{2.1}$$

Presupunem că $f$ ar fi continuă. Dacă ar exista numerele reale distincte $c, d$ (cu $c < d$) astfel încât $f(c)f(d) < 0$, atunci, conform teoremei lui Bolzano, ar exista cel puţin un $e \in (c, d)$ astfel încât $f(e) = 0$, contradicţie. Aşadar, valorile lui $f$ sunt fie toate pozitive, fie toate negative.

Dacă toate valorile lui $f$ sunt pozitive, obţinem

$$f(2014) + f(2013) + f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n) > 0,$$

iar dacă toate valorile lui $f$ sunt negative,

$$f(2014) + f(2013) + f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n) < 0.$$

Ultimele două relaţii obţinute sunt în contradicţie cu relaţia (2.1).

În concluzie, $f$ nu este funcţie continuă. $\square$

**Problema 2.** Se consideră şirul $(x_n)$ dat prin

$$x_1 = 2, \qquad x_{n+1} = \frac{x_n + 1 + \sqrt{x_n^2 + 2x_n + 5}}{2}, \quad n \geq 1.$$

Să se arate că şirul $y_n = \sum\limits_{k=1}^{n} \frac{1}{x_k^2 - 1}$, $n \geq 1$, este convergent şi să se găsească limita sa.

**Pirmyrat Gurbanov**, Turkmenistan

*Această problemă a fost considerată o problemă medie spre uşoară de către juriu. Cu toate acestea, la problema 2 au fost doar patru rezolvări complete, toate în spiritul soluţiei dată de juriu. Ideea de rezolvare, aceeaşi la toţi cei patru studenţi, a fost de a utiliza relaţia* $y_n = \frac{2}{3} - \frac{1}{x_n + 1}$. *Arătând apoi că şirul* $(x_n)_n$ *are limita* $\infty$ *rezultă convergenţa şi limita şirului* $(y_n)_n$.

**Soluţie.** Mai întâi, se arată uşor prin inducţie că $x_n > 0$, $\forall n \in \mathbb{N}^*$. Apoi, folosind relaţia de recurenţă din enunţ:

$$x_{n+1} = \frac{x_n + 1 + \sqrt{x_n^2 + 2x_n + 5}}{2} > \frac{x_n + 1 + \sqrt{(x_n + 1)^2}}{2} = x_n + 1, \ \forall n \geq 1.$$

Aşadar, şirul $(x_n)_n$ este strict crescător. Presupunând că şirul ar fi convergent, fie $L = \lim\limits_{n\to\infty} x_n$. Din inegalitatea de mai sus, prin trecere la limită s-ar obţine $L \geq L + 1$, contradicţie. Rezultă că $\lim\limits_{n\to\infty} x_n = \infty$.

Pornind tot de la recurenţa din enunţ, avem succesiv

$$(2x_{n+1} - x_n - 1)^2 = x_n^2 + 2x_n + 5$$

şi

$$x_{n+1}^2 - (x_n + 1)x_{n+1} - 1 = 0.$$

Se obţine

$$\frac{1}{x_{n+1}^2 - 1} = \frac{1}{x_{n+1}(x_n + 1)}.$$

Avem

$$\frac{x_{n+1}}{x_{n+1}^2 - 1} = \frac{1}{x_n + 1} = \frac{x_n - 1}{x_n^2 - 1} = \frac{x_n}{x_n^2 - 1} - \frac{1}{x_n^2 - 1},$$

de unde putem scrie $y_n$ ca sumă telescopică:

$$y_n = \sum_{k=1}^{n} \frac{1}{x_k^2 - 1} = \sum_{k=1}^{n} \left( \frac{x_k}{x_k^2 - 1} - \frac{x_{k+1}}{x_{k+1}^2 - 1} \right) = \frac{x_1}{x_1^2 - 1} - \frac{x_{n+1}}{x_{n+1}^2 - 1} =$$
$$= \frac{2}{3} - \frac{1}{x_n + 1}.$$

În concluzie, cum şirul $(x_n)_n$ are limita $\infty$, şirul $(y_n)_n$ este convergent şi $\lim\limits_{n\to\infty} y_n = \dfrac{2}{3}$. □

**Problema 3.** Fie $A \in \mathcal{M}_n(\mathbb{C})$ şi $a \in \mathbb{C}$, $a \neq 0$, astfel încât $A - A^* = 2aI_n$, unde $A^* = (\bar{A})^t$ iar $\bar{A}$ este conjugata matricei $A$.

(a) Să se arate că $|\det A| \geq |a|^n$.

(b) Să se arate că dacă $|\det A| = |a|^n$ atunci $A = aI_n$.

**Vasile Pop**, România

*Juriul a considerat problema drept o problemă de dificultate medie. Au fost 8 studenţi care au rezolvat complet problema, toţi având abordări asemănătoare celei din soluţia 1, care este şi soluţia oficială.*

**Soluţia 1.** (a) Din relaţia dată în enunţ se obţine $A^* - A = 2\bar{a}I_n$; adunând această relaţie la prima, se obţine $a + \bar{a} = 0$. Prin urmare, există $b \in \mathbb{R}^*$ astfel încât $a = ib$.

Putem scrie relaţia din enunţ în forma $A - aI_n = A^* + aI_n$ sau $A - aI_n = (A - aI_n)^*$, de unde matricea $B = A - aI_n \in \mathcal{M}_n(\mathbb{C})$ este o matrice hermitiană, deci are toate valorile proprii numere reale $\lambda_{1B} = \alpha_1$, $\ldots$, $\lambda_{nB} = \alpha_n$.

Valorile proprii ale matricei $A$ sunt $\lambda_{1A} = \alpha_1 + ib, \ldots, \lambda_{nA} = \alpha_n + ib$ şi avem

$$|\det A| = |\lambda_{1A}| \cdots |\lambda_{nA}| = \sqrt{\alpha_1^2 + b^2} \cdots \sqrt{\alpha_n^2 + b^2} \geq |b|^n = |a|^n.$$

(b) Pentru egalitate trebuie să avem $\alpha_1 = \cdots = \alpha_n = 0$ şi cum matricea hermitiană $B$ este diagonalizabilă rezultă $B = 0$ şi în consecinţă $A = aI_n$. □

*Abordarea din soluţia ce urmează a fost propusă în juriu de Dmitro Mitin.*

**Soluţia 2.** Din relaţia din enunţ deducem $AA^* = A^2 - 2aA = A^*A$, deci matricea $A$ e normală. Prin urmare, există $U \in \mathcal{M}_n(\mathbb{C})$ unitară şi $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ astfel încât, notând $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, să avem $A = UDU^{-1}$. Obţinem

$$2aI_n = U^{-1}AU - U^{-1}A^*U = D - U^*A^*(U^{-1})^* = D - D^* = D - \overline{D},$$

de unde $\lambda_k - \overline{\lambda_k} = 2a$, deci $a$ este de forma $ib$, $b \in \mathbb{R}$, şi există numerele reale $\alpha_1, \ldots, \alpha_n$ astfel încât $\lambda_k = \alpha_k + ib$, $k \in \{1, \ldots, n\}$. Se finalizează ca la soluţia 1. □

**Problema 4.** (a) Arătaţi că

$$\lim_{n\to\infty}\int_0^n \frac{\operatorname{arctg}\frac{x}{n}}{x(x^2+1)}\,\mathrm{d}x=\frac{\pi}{2}.$$

(b) Găsiţi valoarea limitei

$$\lim_{n\to\infty} n\left(n\int_0^n \frac{\operatorname{arctg}\frac{x}{n}}{x(x^2+1)}\,\mathrm{d}x-\frac{\pi}{2}\right).$$

**Vladimir Babev**, Bulgaria

*Problema a fost considerată de către juriu drept o problemă dificilă, iar rezultatele au confirmat acest lucru, un singur candidat obţinând punctajul maxim. Prezentăm pentru început soluţia autorului.*

**Soluţia 1.** (b) (Punctul (a) este o parte a soluţiei). Observăm că

$$\int_0^n \frac{\operatorname{arctg}\dfrac{x}{n}}{x\,(x^2+1)}\,\mathrm{d}x = \int_0^1 \frac{\operatorname{arctg}x}{x\,(n^2x^2+1)}\,\mathrm{d}x=I_n.$$

Pentru orice $p\in\mathbb{N}$ şi orice $x\in(0,\,1]$ au loc relaţiile

$$\sum_{k=0}^{2p+1}\frac{(-1)^k x^{2k}}{2k+1}\le\frac{\operatorname{arctg}x}{x}\le\sum_{k=0}^{2p}\frac{(-1)^k x^{2k}}{2k+1}\ .$$

Rezultă

$$\sum_{k=0}^{2p+1}\frac{(-1)^k}{2k+1}\,J_k\ \le\ I_n\ \le\ \sum_{k=0}^{2p}\frac{(-1)^k}{2k+1}\,J_k,$$

unde

$$J_k=\int_0^1\frac{x^{2k}}{n^2x^2+1}\,\mathrm{d}x.$$

Avem

$$J_0=\frac{\operatorname{arctg}n}{n}\le 1,$$

$$J_1=\int_0^1\frac{x^2}{n^2x^2+1}\,\mathrm{d}x\ =\ \frac{1}{n^2}\,(1-\ J_0)=\ \frac{1}{n^2}\left(1-\ \frac{\operatorname{arctg}n}{n}\right),$$

iar pentru $k\ge 2$

$$J_k=\int_0^1\frac{x^{2k}}{n^2x^2+1}\mathrm{d}x=\frac{1}{n^2}\left(\frac{1}{2k-1}-\ J_{k-1}\right)=$$
$$=\frac{1}{n^2}\left(\frac{1}{2k-1}-\frac{1}{n^2}\left(\frac{1}{2k-3}-\ J_{k-2}\right)\right).$$

Cum $0 < J_m \le J_0 \le 1$ pentru orice $m \in \mathbb{N}$, relaţia anterioară conduce la

$$\left| \frac{(-1)^k}{2k+1} J_k - \frac{(-1)^k}{2k+1} \cdot \frac{1}{2k-1} \cdot \frac{1}{n^2} \right| \le \frac{2}{n^4}.$$

În concluzie,

$$\left| \sum_{k=2}^{N} \frac{(-1)^k}{2k+1} J_k - \frac{1}{n^2} \sum_{k=2}^{N} \frac{(-1)^k}{(2k-1)(2k+1)} \right| \le \frac{4}{n^3} \text{ pentru } N \in \{2n\,,\ 2n+1\}.$$

Atunci

$$-\frac{4}{n} - n\,\frac{\pi}{2} + n^2 J_0 - \frac{n^2 J_1}{3} + \sum_{k=2}^{2n+1} \frac{(-1)^k}{(2k-1)(2k+1)} \le n\left(n\,I_n - \frac{\pi}{2}\right)$$

şi

$$n\left(n\,I_n - \frac{\pi}{2}\right) \le -n\,\frac{\pi}{2} + n^2 J_0 - \frac{n^2 J_1}{3} + \sum_{k=2}^{2n} \frac{(-1)^k}{(2k-1)(2k+1)} + \frac{4}{n}.$$

Deoarece

$$\lim_{n\to\infty} \left(-n\,\frac{\pi}{2} + n^2 J_0 - \frac{n^2 J_1}{3}\right) = \lim_{n\to\infty} \left(-n \operatorname{arctg}\frac{1}{n} - \frac{1}{3}\left(1 - \frac{\operatorname{arctg} n}{n}\right)\right) = \\ = -\frac{4}{3},$$

iar

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{(2k-1)(2k+1)} = \frac{1}{2}\sum_{k=2}^{\infty} (-1)^k \left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) = \\ = \frac{1}{2}\left(2\sum_{k=2}^{\infty} \frac{(-1)^k}{2k-1} - \frac{1}{3}\right) = \left(1 - \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\right) - \frac{1}{6} = \\ = \frac{5}{6} - \frac{\pi}{4},$$

avem în final că

$$\lim_{n\to\infty} n\left(n\int_0^n \frac{\operatorname{arctg}\dfrac{x}{n}}{x\,(x^2+1)}\,\mathrm{d}x - \frac{\pi}{2}\right) = -\frac{1}{2} - \frac{\pi}{4}. \qquad \square$$

*Următoarea soluţie, destul de apropiată de baremul de corectare, a fost propusă de către Ovidiu Furdui de la Universitatea Tehnică din Cluj-Napoca.*

**Soluţia 2.** (a) Notăm

$$x_n = n \frac{\operatorname{arctg} \dfrac{x}{n}}{x(x^2+1)} \,\mathrm{d}x.$$

Avem

$$x_n = \int_0^\infty \frac{\operatorname{arctg} \frac{x}{n}}{\frac{x}{n}(x^2+1)} \,\mathrm{d}x - \int_1^\infty \frac{\operatorname{arctg} y}{y} \cdot \frac{n}{n^2y^2+1} \,\mathrm{d}y.$$

Pe de altă parte,

$$0 < \int_1^\infty \frac{\operatorname{arctg} y}{y} \cdot \frac{n}{n^2y^2+1} \,\mathrm{d}y \le \frac{1}{n} \int_1^\infty \frac{\operatorname{arctg} y}{y^3} \,\mathrm{d}y < \frac{\pi}{2n} \int_1^\infty \frac{\mathrm{d}y}{y^3} = \frac{\pi}{4n},$$

de unde rezultă

$$\lim_{n\to\infty} \int_1^\infty \frac{\operatorname{arctg} y}{y} \cdot \frac{n}{n^2y^2+1} \,\mathrm{d}y = 0.$$

Fie

$$f_n(x) = \frac{\operatorname{arctg} \dfrac{x}{n}}{\dfrac{x}{n}(x^2+1)}.$$

Observăm că $\lim\limits_{n\to\infty} f_n(x) = \dfrac{1}{x^2+1}$ şi $f_n(x) \le \dfrac{1}{x^2+1}$ pentru orice $x \ge 0$. Aceasta implică, în baza teoremei de convergenţă dominată a lui Lebesgue, că

$$\lim_{n\to\infty} \int_0^\infty \frac{\operatorname{arctg} \dfrac{x}{n}}{\dfrac{x}{n}(x^2+1)} \,\mathrm{d}x = \int_0^\infty \frac{1}{x^2+1} \,\mathrm{d}x = \frac{\pi}{2}.$$

(b) Un calcul arată că

$$\begin{aligned}
&n\left(x_n - \frac{\pi}{2}\right) = \\
&= n\left(\int_0^\infty \frac{\operatorname{arctg} \frac{x}{n}}{\frac{x}{n}(x^2+1)}\mathrm{d}x - \int_1^\infty \frac{\operatorname{arctg} y}{y} \cdot \frac{n}{n^2y^2+1}\mathrm{d}y - \int_0^\infty \frac{1}{x^2+1}\mathrm{d}x\right) = \\
&= n\int_0^\infty \frac{\operatorname{arctg} \frac{x}{n} - \frac{x}{n}}{\frac{x}{n}(x^2+1)} \,\mathrm{d}x - n^2 \int_1^\infty \frac{\operatorname{arctg} y}{y\,(n^2y^2+1)} \,\mathrm{d}y = \\
&= \int_0^\infty \frac{\operatorname{arctg} y - y}{y(y^2+\frac{1}{n^2})} \,\mathrm{d}y - \int_1^\infty \frac{\operatorname{arctg} y}{y} \cdot \frac{1}{y^2+\frac{1}{n^2}} \,\mathrm{d}y = \\
&= \int_0^1 \frac{\operatorname{arctg} y - y}{y(y^2+\frac{1}{n^2})} \,\mathrm{d}y - \int_1^\infty \frac{\mathrm{d}y}{y^2+\frac{1}{n^2}} = \\
&= \int_0^1 \frac{\operatorname{arctg} y - y}{y(y^2+\frac{1}{n^2})} \,\mathrm{d}y - n\left(\frac{\pi}{2} - \operatorname{arctg} n\right).
\end{aligned}$$

Cum
$$0 < \frac{y - \operatorname{arctg} y}{y(y^2 + \frac{1}{n^2})} < \frac{y - \operatorname{arctg} y}{y^3},$$
care este o funcţie integrabilă pe $[0,1]$, obţinem, în baza teoremei de convergenţă dominată a lui Lebesgue, că
$$\lim_{n\to\infty} \int_0^1 \frac{\operatorname{arctg} y - y}{y(y^2 + \frac{1}{n^2})}\,\mathrm{d}x = \int_0^1 \frac{\operatorname{arctg} y - y}{y^3}\,\mathrm{d}y = \frac{1}{2} - \frac{\pi}{4},$$
ultima integrală fiind calculată prin părţi.

Deci
$$\lim_{n\to\infty} n\left(x_n - \frac{\pi}{2}\right) = \frac{1}{2} - \frac{\pi}{4} - 1 = -\frac{1}{2} - \frac{\pi}{4}. \qquad \square$$

*Următoarea soluţie, cu diverse variaţiuni, a fost propusă de către mai mulţi membri ai juriului.*

**Soluţia 3.** (a) Se observă că, printr-o schimbare de variabilă, prima limită devine
$$\lim_{n\to\infty} \int_0^1 \frac{n \operatorname{arctg} y}{y(n^2y^2+1)}\,\mathrm{d}y.$$

Cum
$$1 - \frac{y^2}{3} \le \frac{\operatorname{arctg} y}{y} \le 1,\ \forall y \in (0,1],$$
rezultă
$$\int_0^1 \frac{n}{n^2y^2+1}\left(1 - \frac{y^2}{3}\right)\mathrm{d}y \le \int_0^1 \frac{n \operatorname{arctg} y}{y(n^2y^2+1)}\,\mathrm{d}y \le \int_0^1 \frac{n}{n^2y^2+1}\,\mathrm{d}y,$$
de unde
$$\operatorname{arctg} n - \frac{1}{3n} + \frac{\operatorname{arctg} n}{3n^2} \le \int_0^1 \frac{n \operatorname{arctg} y}{y(n^2y^2+1)}\,\mathrm{d}y \le \operatorname{arctg} n.$$

Limita de la (a) se obţine acum prin aplicarea criteriului „cleştelui”.

(b) Folosind (a), scriem limita sub forma
$$\ell = \lim_{n\to\infty} \frac{n\displaystyle\int_0^n \frac{\operatorname{arctg} \frac{x}{n}}{x(x^2+1)}\,\mathrm{d}x - \frac{\pi}{2}}{\dfrac{1}{n}}$$
şi aplicăm o regulă de tip L'Hôpital. Obţinem că aceasta este egală cu
$$\lim_{n\to\infty} \frac{\displaystyle\int_0^n \frac{\operatorname{arctg} \dfrac{x}{n}}{x(x^2+1)}\mathrm{d}x + n\left(\int_0^n \frac{1}{x(x^2+1)} \cdot \frac{1}{(\frac{x}{n})^2+1} \cdot \frac{-x}{n^2}\mathrm{d}x + \frac{\frac{\pi}{4}}{n(n^2+1)}\right)}{-\dfrac{1}{n^2}}$$
$$= -\frac{\pi}{4} - \lim_{n\to\infty}\left(n^2\int_0^n \frac{\operatorname{arctg} \frac{x}{n}}{x(x^2+1)}\,\mathrm{d}x - n\int_0^n \frac{1}{x^2+1} \cdot \frac{1}{(\frac{x}{n})^2+1}\,\mathrm{d}x\right).$$

Făcând din nou o schimbare de variabilă, rezultă

$$\ell = -\frac{\pi}{4} - \lim_{n\to\infty} \int_0^1 \left( \frac{\operatorname{arctg} y}{y} - \frac{1}{y^2+1} \right) \frac{1}{y^2 + \frac{1}{n^2}} \,\mathrm{d}y.$$

Cum

$$0 < \left( \frac{\operatorname{arctg} y}{y} - \frac{1}{y^2+1} \right) \frac{1}{y^2 + \frac{1}{n^2}} < \frac{1}{y^2} \left( \frac{\operatorname{arctg} y}{y} - \frac{1}{y^2+1} \right),$$

rezultă folosind teorema de convergenţă dominată a lui Lebesgue că

$$\lim_{n\to\infty} \int_0^1 \left( \frac{\operatorname{arctg} y}{y} - \frac{1}{y^2+1} \right) \frac{1}{y^2 + \frac{1}{n^2}} \,\mathrm{d}y = \int_0^1 \frac{1}{y^2} \left( \frac{\operatorname{arctg} y}{y} - \frac{1}{y^2+1} \right) \mathrm{d}y.$$

Se observă că

$$\frac{1}{y^2} \left( \frac{\operatorname{arctg} y}{y} - \frac{1}{y^2+1} \right) = \frac{\operatorname{arctg} y - y}{y^3} + \frac{1}{y^2+1},$$

de unde (vezi şi Soluţia 2), integrala este egală cu $\frac{1}{2} - \frac{\pi}{4} + \frac{\pi}{4} = \frac{1}{2}$. În concluzie, $\ell = -\frac{\pi}{4} - \frac{1}{2}$.

O altă metodă de calcul pentru integrala $\displaystyle\int_0^1 \frac{1}{y^2} \left( \frac{\operatorname{arctg} y}{y} - \frac{1}{y^2+1} \right) \mathrm{d}y$ este următoarea: observăm că

$$\frac{1}{y^2} \left( \frac{\operatorname{arctg} y}{y} - \frac{1}{y^2+1} \right) = \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{2n+1} - 1 \right) y^{2n-2}, \ \forall y \in (0,1).$$

Seria din membrul drept este uniform convergentă pe compacţii din $[0,1)$; integrând termen cu termen şi aplicând teorema lui Abel, rezultă

$$\int_0^1 \frac{1}{y^2} \left( \frac{\operatorname{arctg} y}{y} - \frac{1}{y^2+1} \right) \mathrm{d}y = \sum_{n=1}^{\infty} (-1)^n \frac{-2n}{(2n-1)(2n+1)} = \frac{1}{2}.$$

*Pentru următoarea soluţie, studentul Eduard Valentin Curcă a primit premiul special al juriului.*

**Soluţia 4.** Începem prin a proba următoarea

*Lemă.* Fie $f : [0,1] \to \mathbb{R}$ continuă. Atunci

$$\lim_{n\to\infty} \int_0^1 f(x) \operatorname{arctg} nx \,\mathrm{d}x = \frac{\pi}{2} \int_0^1 f(x) \,\mathrm{d}x.$$

*Demonstraţie.* Notăm $M = \sup\limits_{x\in[0,1]} |f(x)|$. Pentru $\varepsilon > 0$, $\delta = \min\left\{\frac{\varepsilon}{M\pi}, 1\right\}$ şi $n > \frac{1}{\delta} \operatorname{ctg} \frac{\varepsilon}{2M}$ avem

$$\left| \frac{\pi}{2} \int_0^1 f(x)\mathrm{d}x - \int_0^1 f(x) \operatorname{arctg} nx \,\mathrm{d}x \right| \le \int_0^\delta \left( \frac{\pi}{2} - \operatorname{arctg} nx \right) |f(x)| \,\mathrm{d}x +$$

$$+\int_{\delta}^{1}\left(\frac{\pi}{2}-\operatorname{arctg} nx\right)|f(x)|\,\mathrm{d}x<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon. \qquad \square$$

(a) Avem

$$\begin{aligned}
n\int_0^n \frac{\operatorname{arctg}\frac{x}{n}}{x(x^2+1)}\,\mathrm{d}x &= \int_0^1 \frac{n\operatorname{arctg} y}{y(n^2y^2+1)}\,\mathrm{d}y = \\
&= \frac{\operatorname{arctg} y}{y}\cdot \operatorname{arctg} ny\Big|_0^1 - \int_0^1\left(\frac{\operatorname{arctg} y}{y}\right)'\operatorname{arctg} ny\,\mathrm{d}y = \\
&= \frac{\pi}{4}\operatorname{arctg}\, n - \int_0^1\left(\frac{\operatorname{arctg} y}{y}\right)'\operatorname{arctg} ny\,\mathrm{d}y.
\end{aligned}$$

Conform lemei,

$$\begin{aligned}
\lim_{n\to\infty}\int_0^1\left(\frac{\operatorname{arctg} y}{y}\right)'\operatorname{arctg} ny\,\mathrm{d}y &= \frac{\pi}{2}\int_0^1\left(\frac{\operatorname{arctg} y}{y}\right)'\mathrm{d}y = \\
&= \frac{\pi}{2}\left(\frac{\operatorname{arctg} y}{y}\right)\Big|_0^1 = \frac{\pi}{2}\cdot\left(\frac{\pi}{4}-1\right).
\end{aligned}$$

Limita cerută rezultă imediat.

(b) Notăm

$$a_n = n\int_0^n \frac{\operatorname{arctg}\frac{x}{n}}{x(x^2+1)}\,\mathrm{d}x - \frac{\pi}{2}.$$

Trebuie calculată limita $\lim\limits_{n\to\infty}\frac{a_n}{\frac{1}{n}}$. Vom face acest lucru folosind teorema Cesàro-Stolz. Din (a), avem

$$a_n = \frac{\pi}{4}\operatorname{arctg} n - \int_0^1 f(y)\operatorname{arctg} ny\,\mathrm{d}y - \frac{\pi}{2},$$

unde $f(y)=\left(\frac{\operatorname{arctg} y}{y}\right)'$. Obţinem

$$\begin{aligned}
\frac{a_{n+1}-a_n}{\frac{1}{n+1}-\frac{1}{n}} = n(n+1)\Big(&\frac{\pi}{4}\left(\operatorname{arctg} n - \operatorname{arctg}(n+1)\right) - \\
&-\int_0^1 f(y)\left(\operatorname{arctg} ny - \operatorname{arctg}(n+1)y\right)\,\mathrm{d}y\Big).
\end{aligned}$$

Dar

$$n(n+1)\frac{\pi}{4}\left(\operatorname{arctg} n - \operatorname{arctg}(n+1)\right) = \frac{\pi}{4}\frac{\operatorname{arctg}\frac{-1}{1+n(n+1)}}{\frac{1}{n(n+1)}} \to -\frac{\pi}{4} \text{ pentru } n\to\infty.$$

Observăm că

$$\frac{f(y)}{y} = \frac{1}{y}\left(\sum_{n=0}^{\infty}(-1)^n\frac{y^{2n}}{2n+1}\right)' = \sum_{n=1}^{\infty}(-1)^n\frac{2n}{2n+1}y^{2n-2},$$

iar seria din membrul drept este uniform convergentă pe compacţii din intervalul $(-1,1)$, de unde $\dfrac{f(y)}{y}$ este continuă pe $(-1,1)$.

Notând

$$L=\lim_{n\to\infty} n(n+1)\int_0^1 f(y)\left(\operatorname{arctg} ny-\operatorname{arctg}(n+1)y\right)\,\mathrm{d}y$$

şi ţinând cont că pentru $t\geq 0$ avem $t-\frac{t^3}{3}<\operatorname{arctg} t\leq t$, obţinem

$$\lim_{n\to\infty}\int_0^1 f(y)\frac{-n(n+1)y}{1+n(n+1)y^2}\,\mathrm{d}y-\lim_{n\to\infty}\int_0^1\frac{-f(y)}{y}\frac{n(n+1)y^4}{(1+n(n+1)y^2)^3}\,\mathrm{d}y\leq L$$

şi

$$L\leq\int_0^1\frac{-f(y)}{y}\,\mathrm{d}y.$$

Dar

$$\frac{n(n+1)y^4}{(1+n(n+1)y^2)^3}<\frac{1}{n(n+1)}\ \text{pentru orice } y\in[0,1],$$

deci

$$\lim_{n\to\infty}\int_0^1\frac{-f(y)}{y}\frac{n(n+1)y^4}{(1+n(n+1)y^2)^3}\,\mathrm{d}y=0.$$

Pe de altă parte, daţi fiind $\varepsilon>0$ şi $A\in(0,1)$, există $N\in\mathbb{N}$ astfel încât $\frac{n(n+1)y^2}{1+n(n+1)y^2}>1-\varepsilon$ pentru orice $y\geq A$ şi orice $n\geq N$. Aşadar,

$$\lim_{n\to\infty}\int_0^1 f(y)\frac{-n(n+1)y}{1+n(n+1)y^2}\,\mathrm{d}y>(1-\varepsilon)\int_A^1\frac{-f(y)}{y}\,\mathrm{d}y.$$

Din această relaţie şi din continuitatea funcţiei $y\mapsto\frac{f(y)}{y}$ obţinem

$$\lim_{n\to\infty}\int_0^1 f(y)\frac{-n(n+1)y}{1+n(n+1)y^2}\,\mathrm{d}y>(1-\varepsilon)\int_0^1\frac{-f(y)}{y}\,\mathrm{d}y.$$

Cum $\varepsilon$ a fost luat arbitrar, din cele de mai sus reiese că

$$\lim_{n\to\infty} n(n+1)\int_0^1 f(y)\left(\operatorname{arctg} ny-\operatorname{arctg}(n+1)y\right)\,\mathrm{d}y=-\int_0^1\frac{f(y)}{y}\,\mathrm{d}y.$$

Pe de altă parte, folosind dezvoltarea lui $\dfrac{f(y)}{y}$ de mai sus rezultă, integrând termen cu termen (vezi şi Soluţia 3), că

$$\int_0^1\frac{f(y)}{y}\,\mathrm{d}y=\sum_{n=1}^{\infty}(-1)^n\frac{2n}{(2n-1)(2n+1)}=\frac{1}{2},$$

de unde urmează concluzia că limita căutată are valoarea $-\dfrac{\pi}{4}-\dfrac{1}{2}$. $\square$

# NOTE MATEMATICE

## An exotic limit with fractional parts. A note on the problem GMB–26856

Ovidiu Furdui[1)]

**Abstract**. In this note we generalize a problem of Mihai Piticari and Sorin Rădulescu published in Gazeta Matematică B which is about calculating the exotic limit

$$\lim_{n\to\infty}\frac{1}{n^2}\left(\left\{\frac{n}{\sqrt{1}}\right\}+\left\{\frac{n}{\sqrt{2}}\right\}+\left\{\frac{n}{\sqrt{3}}\right\}+\cdots+\left\{\frac{n}{\sqrt{n^2}}\right\}\right).$$

**Keywords:** fractional parts, limits, Riemann zeta function
**MSC:** 40A05, 11M06

### 1. Introduction and the result

In [2, Problem 26856, p. 587] Mihai Piticari and Sorin Rădulescu have proposed the calculation of the following limit

$$\lim_{n\to\infty}\frac{1}{n^2}\left(\left\{\frac{n}{\sqrt{1}}\right\}+\left\{\frac{n}{\sqrt{2}}\right\}+\left\{\frac{n}{\sqrt{3}}\right\}+\cdots+\left\{\frac{n}{\sqrt{n^2}}\right\}\right),$$

where $\{a\}$ denotes the fractional part of $a$.

In what follows we solve a more general problem and show that, if $p\geq 2$ is an integer then

$$\lim_{n\to\infty}\frac{1}{n^p}\left(\left\{\frac{n}{\sqrt[p]{1}}\right\}+\left\{\frac{n}{\sqrt[p]{2}}\right\}+\left\{\frac{n}{\sqrt[p]{3}}\right\}+\cdots+\left\{\frac{n}{\sqrt[p]{n^p}}\right\}\right)=\frac{p}{p-1}-\zeta(p),$$

where $\zeta$ denotes the Riemann zeta function and $\{a\}$ stands for the fractional part of $a$.

In particular, when $p=2$ we get that

$$\lim_{n\to\infty}\frac{1}{n^2}\left(\left\{\frac{n}{\sqrt{1}}\right\}+\left\{\frac{n}{\sqrt{2}}\right\}+\left\{\frac{n}{\sqrt{3}}\right\}+\cdots+\left\{\frac{n}{\sqrt{n^2}}\right\}\right)=2-\zeta(2).$$

Recall, the Riemann zeta function is defined by

$$\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}=1+\frac{1}{2^z}+\frac{1}{3^z}+\cdots+\frac{1}{n^z}+\cdots,\quad \Re(z)>1.$$

Let $(x_n)_{n\geq 1}$ be the sequence defined by

$$x_n=\frac{1}{n^p}\sum_{k=1}^{n^p}\left\{\frac{n}{\sqrt[p]{k}}\right\}.$$

[1)]Department of Mathematics, Technical University of Cluj-Napoca, 400114, Romania, `Ovidiu.Furdui@math.utcluj.ro`

We have,

$$x_n = \frac{1}{n^{p-1}} \sum_{k=1}^{n^p} \frac{1}{\sqrt[p]{k}} - \frac{1}{n^p} \sum_{k=1}^{n^p} \left\lfloor \frac{n}{\sqrt[p]{k}} \right\rfloor, \tag{1.1}$$

where $\lfloor a \rfloor$ denotes the floor of $a$.

A calculation shows, and the reader is invited to check, that

$$\lim_{n\to\infty} \frac{1}{n^{p-1}} \sum_{k=1}^{n^p} \frac{1}{\sqrt[p]{k}} = \frac{p}{p-1}. \tag{1.2}$$

On the other hand,

$$\left\lfloor \frac{n}{\sqrt[p]{k}} \right\rfloor = i \quad \Leftrightarrow \quad \left(\frac{n}{i+1}\right)^p < k \le \left(\frac{n}{i}\right)^p,$$

and it follows that

$$\begin{aligned}
\sum_{k=1}^{n^p} \left\lfloor \frac{n}{\sqrt[p]{k}} \right\rfloor &= \sum_{i=1}^{n} i \left( \left\lfloor \frac{n^p}{i^p} \right\rfloor - \left\lfloor \frac{n^p}{(i+1)^p} \right\rfloor \right) \\
&= \sum_{i=1}^{n} i \left( \frac{n^p}{i^p} - \frac{n^p}{(i+1)^p} \right) - \sum_{i=1}^{n} i \left( \left\{ \frac{n^p}{i^p} \right\} - \left\{ \frac{n^p}{(i+1)^p} \right\} \right) \\
&= n^p \sum_{i=1}^{n} \left( \frac{1}{i^{p-1}} - \frac{1}{(i+1)^{p-1}} + \frac{1}{(i+1)^p} \right) \\
&\quad - \sum_{i=1}^{n} i \left( \left\{ \frac{n^p}{i^p} \right\} - \left\{ \frac{n^p}{(i+1)^p} \right\} \right) \\
&= n^p \left( 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(n+1)^p} - \frac{1}{(n+1)^{p-1}} \right) \\
&\quad - \sum_{i=1}^{n} i \left( \left\{ \frac{n^p}{i^p} \right\} - \left\{ \frac{n^p}{(i+1)^p} \right\} \right).
\end{aligned} \tag{1.3}$$

Now we consider separately the cases when $p = 2$ and $p \ge 3$.

**The case $p = 2$.** We have, based on (1.3), that

$$\begin{aligned}\sum_{k=1}^{n^2}\left\lfloor\frac{n}{\sqrt{k}}\right\rfloor &= n^2\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(n+1)^2}-\frac{1}{(n+1)}\right)\\ &\quad -\sum_{i=1}^{n} i\left(\left\{\frac{n^2}{i^2}\right\}-\left\{\frac{n^2}{(i+1)^2}\right\}\right)\\ &= n^2\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(n+1)^2}-\frac{1}{(n+1)}\right)\\ &\quad -\sum_{i=1}^{n} i\left\{\left(\frac{n}{i}\right)^2\right\}+\sum_{i=1}^{n}(i+1)\left\{\left(\frac{n}{i+1}\right)^2\right\}\\ &\quad -\sum_{i=1}^{n}\left\{\left(\frac{n}{i+1}\right)^2\right\}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{n^2}\sum_{k=1}^{n^2}\left\lfloor\frac{n}{\sqrt{k}}\right\rfloor &= \left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(n+1)^2}-\frac{1}{(n+1)}\right)\\ &\quad -\frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}\left\{\left(\frac{n}{i}\right)^2\right\}+\frac{1}{n}\sum_{i=1}^{n}\frac{i+1}{n}\left\{\left(\frac{n}{i+1}\right)^2\right\}\\ &\quad -\frac{1}{n^2}\sum_{i=1}^{n}\left\{\left(\frac{n}{i+1}\right)^2\right\}.\end{aligned}$$

Letting $n \to \infty$ in the preceding equality we get that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n^2}\left\lfloor\frac{n}{\sqrt{k}}\right\rfloor = \frac{\pi^2}{6}-\int_0^1 x\left\{\frac{1}{x^2}\right\}dx+\int_0^1 x\left\{\frac{1}{x^2}\right\}dx = \frac{\pi^2}{6}. \tag{1.4}$$

Combining (1.1), (1.2) and (1.4) we have that

$$\lim_{n\to\infty}\frac{1}{n^2}\left(\left\{\frac{n}{\sqrt{1}}\right\}+\left\{\frac{n}{\sqrt{2}}\right\}+\left\{\frac{n}{\sqrt{3}}\right\}+\cdots+\left\{\frac{n}{\sqrt{n^2}}\right\}\right) = 2-\frac{\pi^2}{6}.$$

It is worth mentioning that

$$\int_0^1 x\left\{\frac{1}{x^2}\right\}dx = \frac{1-\gamma}{2},$$

and other exotic fractional part integrals as well as open problems can be found in [1]. (Here $\gamma$ denotes the Euler–Mascheroni constant.)

**The case $p \geq 3$.** We have, based on (1.3), that

$$\frac{1}{n^p}\sum_{k=1}^{n^p}\left\lfloor\frac{n}{\sqrt[p]{k}}\right\rfloor = \left(1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{(n+1)^p}-\frac{1}{(n+1)^{p-1}}\right)$$
$$-\frac{1}{n^p}\sum_{i=1}^{n} i\left(\left\{\frac{n^p}{i^p}\right\}-\left\{\frac{n^p}{(i+1)^p}\right\}\right).$$

Passing to the limit when $n \to \infty$ in the preceding equality we get that

$$\lim_{n\to\infty}\frac{1}{n^p}\sum_{k=1}^{n^p}\left\lfloor\frac{n}{\sqrt[p]{k}}\right\rfloor = \zeta(p). \tag{1.5}$$

We used that

$$\left|\frac{1}{n^p}\sum_{i=1}^{n} i\left(\left\{\frac{n^p}{i^p}\right\}-\left\{\frac{n^p}{(i+1)^p}\right\}\right)\right| \leq \frac{1}{n^p}\sum_{i=1}^{n} 2i = \frac{n+1}{n^{p-1}},$$

which implies, since $p > 3$, that

$$\lim_{n\to\infty}\frac{1}{n^p}\sum_{i=1}^{n} i\left(\left\{\frac{n^p}{i^p}\right\}-\left\{\frac{n^p}{(i+1)^p}\right\}\right) = 0.$$

Combining (1.1), (1.2) and (1.5) we have

$$\lim_{n\to\infty}\frac{1}{n^p}\left(\left\{\frac{n}{\sqrt[p]{1}}\right\}+\left\{\frac{n}{\sqrt[p]{2}}\right\}+\left\{\frac{n}{\sqrt[p]{3}}\right\}+\cdots+\left\{\frac{n}{\sqrt[p]{n^p}}\right\}\right) = \frac{p}{p-1}-\zeta(p),$$

and the problem is solved.

We close this note with the following problem which is left as an exercise to the interested reader.

**Problem.** Let $p \geq 2$ be an integer. Prove that

(a) $\displaystyle\lim_{n\to\infty}\frac{1}{n^p}\left(\left\{\frac{2n}{\sqrt[p]{2}}\right\}+\left\{\frac{2n}{\sqrt[p]{4}}\right\}+\left\{\frac{2n}{\sqrt[p]{6}}\right\}+\cdots+\left\{\frac{2n}{\sqrt[p]{(2n)^p}}\right\}\right) = 2^{p-1}\left(\frac{p}{p-1}-\zeta(p)\right)$;

(b) $\displaystyle\lim_{n\to\infty}\frac{1}{n^p}\left(\left\{\frac{2n+1}{\sqrt[p]{1}}\right\}+\left\{\frac{2n+1}{\sqrt[p]{3}}\right\}+\cdots+\left\{\frac{2n+1}{\sqrt[p]{(2n+1)^p}}\right\}\right) = 2^{p-1}\left(\frac{p}{p-1}-\zeta(p)\right)$.

## REFERENCES


[1] O. Furdui, *Limits, Series and Fractional Part Integrals. Problems in Mathematical Analysis*, Springer, London, 2013.

[2] M. Piticari, S. Rădulescu, Problem 26856, *Gazeta Matematică, Seria B*, Anul CXVIII, 12, 2013.

# A note on equal sums of fifth powers

Susil Kumar Jena[1)]

**Abstract**. In this note, we study two Diophantine equations: $X_1^5 + X_2^5 + X_3^5 + 2X_4^5 = Y_1^5 + Y_2^5 + Y_3^5 + 2Y_4^5$ and $X_5^5 + X_6^5 + 2X_7^5 = Y_5^5 + Y_6^5 + Y_7^5 + 2Y_8^5$, and give two parametric solutions related to them.

**Keywords:** Diophantine equations, equal sums of fifth powers.

**MSC:** 11D41, 11D72.

## 1. Introduction

We are concerned in this note with the Diophantine equations

$$X_1^5 + X_2^5 + X_3^5 + 2X_4^5 = Y_1^5 + Y_2^5 + Y_3^5 + 2Y_4^5, \tag{1.1}$$

and

$$X_5^5 + X_6^5 + 2X_7^5 = Y_5^5 + Y_6^5 + Y_7^5 + 2Y_8^5, \tag{1.2}$$

where $X_i$, $1 \le i \le 7$, and $Y_j$, $1 \le j \le 8$, are the integer variables. Many papers (see [1]–[6]) relating to equal sums of fifth powers have been published, but none of them has considered these two equations which we want to discuss. The parametric solutions of (1.1) and (1.2) are given in Table 2.1 and Table 2.2 respectively, followed by some numerical examples.

## 2. The parametric solutions

We can easily verify the polynomial identity

$$\begin{aligned} &n^3\left\{(m+1)^5 + (m-1)^5 - 10m - 2m^5\right\} \\ &= (mn+1)^5 + (mn-1)^5 - 10mn - 2(mn)^5 \end{aligned} \tag{2.1}$$

for any real values of $m$ and $n$ by expanding and simplifying the LHS and RHS of (2.1). Using this identity, we find two parametric solutions of (1.1) and (1.2).

2.1. **Parametric solution of** (1.1)**.** In (2.1) take $n = p^5$ and $m = 10^4q^5$ to get

$$\begin{aligned} &p^{15}\left\{(10^4q^5+1)^5 + (10^4q^5-1)^5 - 10^5q^5 - 2(10^4q^5)^5\right\} \\ &= \left(10^4p^5q^5+1\right)^5 + \left(10^4p^5q^5-1\right)^5 - 10^5p^5q^5 - 2\left(10^4p^5q^5\right)^5; \end{aligned}$$

$$\begin{aligned} \Rightarrow\ &p^{15}(10^4q^5+1)^5 + p^{15}(10^4q^5-1)^5 - 10^5p^{15}q^5 - 2p^{15}(10^4q^5)^5 = \\ &= (10^4p^5q^5+1)^5 + (10^4p^5q^5-1)^5 - 10^5p^5q^5 - 2(10^4p^5q^5)^5; \end{aligned}$$

$$\Rightarrow (p^3(10^4q^5+1))^5 + (p^3(10^4q^5-1))^5 - (10p^3q)^5 - 2(10^4p^3q^5)^5 =$$

[1)] Professor, Department of Electronics and Telecommunication Engineering, KIIT University, Bhubaneswar 751024, Odisha, India, susil_kumar@yahoo.co.uk

$$= (10^4p^5q^5+1)^5 + (10^4p^5q^5-1)^5 - (10pq)^5 - 2(10^4p^5q^5)^5;$$

$$\begin{aligned} &\Rightarrow (p^3(10^4q^5+1))^5 + (p^3(10^4q^5-1))^5 + (10pq)^5 + 2(10^4p^5q^5)^5 \\ &= (10^4p^5q^5+1)^5 + (10^4p^5q^5-1)^5 + (10p^3q)^5 + 2(10^4p^3q^5)^5. \end{aligned} \tag{2.2}$$

Comparing (1.1) and (2.2) we get a parametric solution of (1.1) as

**Table 2.1**

| | |
|---|---|
| $X_1 = p^3(10^4q^5+1)$, | $X_2 = p^3(10^4q^5-1)$, |
| $X_3 = 10pq$, | $X_4 = 10^4p^5q^5$, |
| $Y_1 = (10^4p^5q^5+1)$, | $Y_2 = (10^4p^5q^5-1)$, |
| $Y_3 = 10p^3q$, | $Y_4 = 10^4p^3q^5$. |

**Example 1.**

$$\begin{aligned} (p,q)=(2,1):\quad & 80008^5 + 79992^5 + 20^5 + 2\times 320000^5 \\ &= 320001^5 + 319999^5 + 80^5 + 2\times 80000^5; \\ (p,q)=(3,2):\quad & 8640027^5 + 8639973^5 + 60^5 + 2\times 77760000^5 \\ &= 77760001^5 + 77759999^5 + 540^5 + 2\times 8640000^5. \end{aligned}$$

2.2. **Parametric solution of** (1.2)**.** From (2.1) we get

$$\begin{aligned} &(m+1)^5n^3 + (m-1)^5n^3 - 10mn^3 - 2m^5n^3 \\ &\quad = (mn+1)^5 + (mn-1)^5 - 10mn - 2(mn)^5; \\ &\Rightarrow (m+1)^5n^3 + (m-1)^5n^3 + 2(mn)^5 \\ &\quad = (mn+1)^5 + (mn-1)^5 + 10mn^3 - 10mn + 2m^5n^3; \end{aligned}$$

$$\begin{aligned} &\Rightarrow (m+1)^5n^3 + (m-1)^5n^3 + 2(mn)^5 \\ &\quad = (mn+1)^5 + (mn-1)^5 + 10m(n^3-n) + 2m^5n^3. \end{aligned} \tag{2.3}$$

Taking $m = 10^4(n^3-n)^4$ in (2.3) gives

$$\begin{aligned} &(10^4(n^3-n)^4+1)^5n^3 + (10^4(n^3-n)^4-1)^5n^3 \\ &\quad + 2(10^4(n^3-n)^4n)^5 \\ &= (10^4(n^3-n)^4n+1)^5 + (10^4(n^3-n)^4n-1)^5 \\ &\quad + (10(n^3-n))^5 + 2(10(n^3-n))^{20}n^3. \end{aligned} \tag{2.4}$$

Now, in (2.4) put $n = r^5$ to get

$$\begin{aligned} &(10^4(r^{15}-r^5)^4+1)^5r^{15} + (10^4(r^{15}-r^5)^4-1)^5r^{15} + 2(10^4(r^{15}-r^5)^4r^5)^5 \\ &\quad = (10^4(r^{15}-r^5)^4r^5+1)^5 + (10^4(r^{15}-r^5)^4r^5-1)^5 \\ &\qquad +(10(r^{15}-r^5))^5 + 2(10(r^{15}-r^5))^{20}r^{15}; \end{aligned}$$

$$\begin{aligned}&\Rightarrow \{(10^4(r^{15}-r^5)^4+1)r^3\}^5+\{(10^4(r^{15}-r^5)^4-1)r^3\}^5+\\&\qquad+2\{10^4(r^{15}-r^5)^4r^5\}^5\\&=\{10^4(r^{15}-r^5)^4r^5+1\}^5+\{10^4(r^{15}-r^5)^4r^5-1\}^5\\&\qquad+\{10(r^{15}-r^5)\}^5+2\{10^4(r^{15}-r^5)^4r^3\}^5.\end{aligned}\tag{2.5}$$

Comparing (1.2) and (2.5) we get a parametric solution of (1.2) as

**Table 2.2**

| | |
|---|---|
| $X_5=(10^4(r^{15}-r^5)^4+1)r^3$, | $X_6=(10^4(r^{15}-r^5)^4-1)r^3$, |
| $X_7=10^4(r^{15}-r^5)^4r^5$, | $Y_5=(10^4(r^{15}-r^5)^4r^5+1)$, |
| $Y_6=(10^4(r^{15}-r^5)^4r^5-1)$, | $Y_7=10(r^{15}-r^5)$, |
| $Y_8=10^4(r^{15}-r^5)^4r^3$. | |

**Example 2.** $r=2:$

$$\begin{aligned}&918739598204259532800008^5\\&+918739598204259532799992^5\\&+2\times 3674958392817038131200000^5\\&=3674958392817038131200001^5\\&+3674958392817038131199999^5\\&+327360^5+2\times 918739598204259532800000^5.\end{aligned}$$

## 3. Conclusion

The method used for getting the main results of this note is elementary, reminding one of the good old days. We hope there are still many easily understood, yet, difficult problems in the domain of *Number Theory* that can be probed with elementary methods and arguments.

**Acknowledgment.** I am very grateful to my family for their support and encouragement.

## References


[1] A. Bremner, A geometric approach to equal sums of fifth powers, *J. Number Theory* **13** (1981), 337–354.

[2] A. Choudhry, The Diophantine equation $X_1^5+X_2^5+2X_3^5=Y_1^5+Y_2^5+2Y_3^5$, *Ganita* **48** (1997), 115–116.

[3] A. Choudhry, On equal sums of fifth powers, *Indian J. Pure Appl. Math.* **28** (1997), 1443–1450.

[4] L.J. Lander, Geometric aspects of Diophantine equations involving equal sums of like power, *Amer. Math. Monthly* **75** (1968), 1061–1073.

[5] A. Moessner, Due Sistemi Diofantei, *Boll. Un. Mat. Ital.* **6** (1951), 117–118.

[6] H.P.F. Swinnerton-Dyer, A solution of $A^5+B^5+C^5=D^5+E^5+F^5$, *Proc. Cambridge Phil. Soc.* **48** (1952), 516–518.

# PROBLEMS

Authors should submit proposed problems to gmaproblems@rms.unibuc.ro. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of November 2014**.

## PROPOSED PROBLEMS

**405.** Let $f \in C^2([0,1])$ such that $f(0)+2f(\frac{1}{p})+2f(\frac{2}{p})+\cdots+2f(\frac{p-1}{p})+f(1) = 0$, where $p \geq 2$ is an integer. Prove that

$$\left(\int_0^1 f(x)\mathrm{d}x\right)^2 \leq \frac{1}{120p^4}\int_0^1 (f''(x))^2\mathrm{d}x.$$

Proposed by Cristian Chiser, Craiova, Romania.

**406.** Let $d$ be a positive integer. Define a $2d \times 2d$ matrix $\mathbf{M}(d)$ with entries in $\{-1,0,1\}$ as follows: For $1 \leq a \leq 2d$ and $1 \leq b \leq d$,

$$M_{a,2b-1} = \begin{cases} 1 & \text{if } a = 2b, \\ -1 & \text{if } a = 2b+2, \\ 0, & \text{otherwise} \end{cases} \quad M_{a,2b} = \begin{cases} 1 & \text{if } a = 2b-1, \\ -1 & \text{if } a = b-1, \\ 0, & \text{otherwise.} \end{cases}$$

Prove that $\det \mathbf{M}(d) = (-1)^d$.

Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

**407.** (i) Let $i, n$ be two integers with $2 \leq i \leq n-2$ and let $x_k, y_k \in \mathbb{R}$ for $k = 1, \ldots, n$ such that $x_1 \geq \cdots \geq x_n$, $y_1 \geq \cdots \geq y_n$. Prove that

$$n\left(\sum_{j=1}^{i-1} x_j y_j + x_i y_{i+1} + x_{i+1} y_i + \sum_{j=i+2}^{n} x_j y_j\right) \geq \sum_{j=1}^{n} x_j \sum_{j=1}^{n} y_j.$$

(ii) As an application, prove that if $a_k, b_k \in \mathbb{R}$ for $k = 1,2,3,4$ such that $a_1 \geq a_2 \geq a_3 \geq a_4$ and $b_1 \geq b_2 \geq b_3 \geq b_4$ then

$$\frac{a_1a_2b_1b_2 + a_2a_3b_2b_3 + a_3a_4b_3b_4 + a_4a_1b_4b_1}{4} \geq \frac{a_1a_2 + a_2a_3 + a_3a_4 + a_4a_1}{4} \times \frac{b_1b_2 + b_2b_3 + b_3b_4 + b_4b_1}{4}.$$

Proposed by Ovidiu Pop, Satu Mare, Romania.

**408.** Let $p$ be a prime and let $n \geq 2$ be an integer with $p \nmid n$. Prove that

$$\mathbb{Z}_p[X^n] \cap \mathbb{Z}_p[X^n + X^p] = \mathbb{Z}_p.$$

Proposed by Victor Alexandru, University of Bucharest, Romania.

**409.** Let $n, k$ be two integers with $n \geq 3$ and $0 < k \leq n$. We have $n$ distinct points on a circle. Determine how many ways we can choose out of these $n$ points $k$ mutually disjoint pairs of consecutive points.

Proposed by Ionel Popescu, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

**410.** Let $a, b \in (0, 1)$ and let $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ be two sequences such that $a_0 = a$, $b_0 = b$ and for any $n \geq 0$ we have either $a_{n+1} = a_n^{\alpha_n}$, $b_{n+1} = b_n^{\alpha_n}$ for some $\alpha_n \geq 1$ or $a_{n+1} = \lambda_n a_n + (1 - \lambda_n)$, $b_{n+1} = \lambda_n b_n + (1 - \lambda_n)$ for some $\lambda_n \in [0, 1]$.

Prove the following.

(i) $(a_n)$ is convergent iff $(b_n)$ is convergent.

(ii) We have $\lim_{n\to\infty} a_n = 0$ or 1 iff $\lim_{n\to\infty} b_n = 0$ or 1 accordingly.

(iii) If $(a_n)$, $(b_n)$ are divergent then $\lim_{n\to\infty} (a_n - b_n) = 0$.

Proposed by Liviu Păunescu and Constantin-Nicolae Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

**411.** Consider the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \int_0^x \cos\frac{1}{t} \cos\frac{3}{t} \cos\frac{5}{t} \cos\frac{7}{t} \mathrm{d}t.$$

Prove that $f$ is well defined, differentiable and $f'(0) = \frac{1}{8}$.

Proposed by Eugen J. Ionaşcu, Department of Mathematics, Columbus State University, Columbus, Georgia, U.S.A.

**412.** Let $[A, B, C, D]$ be an equifacial tetrahedron with the lengths of the sides $a$, $b$, $c$ and the length of the heights $h$. Prove that $h > \frac{1}{\sqrt{2}} \max\{a, b, c\}$ and

$$\frac{a^2 b^2 c^2}{(2h^2 - a^2)(2h^2 - b^2)(2h^2 - c^2)} \geq 27.$$

Proposed by Marius Olteanu, S.C. Hidroconstrucţia S.A., sucursala Muntenia, Râmnicu Vâlcea, România.

**413.** Prove that $\displaystyle\int_0^\infty \frac{\cos x - \cos(tx)}{x} \mathrm{d}x = \ln|t|$ for any $t \neq 0$.

Proposed by George Stoica, Department of Mathematical Sciences, University of New Brunswick, Canada.

**414.** Does there exist a sequence $(a_n)_{n\geq 1}$ whose terms are greater or equal than 1 such that the following conditions are satisfied:

(i) $\prod_{k=1}^{n} a_k < n^n$, for all $n \geq 1$,

(ii) the sequence $(x_n)_{n\geq 1}$ given by

$$x_n = \sum_{i=1}^{n} \frac{1}{1+a_i},$$

is bounded for all $n \geq 1$?

Proposed by Cezar Lupu, University of Pittsburgh, USA.

**415.** For any positive integer $n$ we define the polynomial

$$P_n = \sum_{\substack{1\leq k\leq n \\ (k,n)=1}} X^k.$$

Prove that $P_n$ is divisible by the cyclotomic polynomial $\Phi_n$ if and only if $n$ is not squarefree.

Proposed by Filip-Andrei Chindea, student, University of Bucharest.

**416.** Let $M_n$ be the set consisting of $\emptyset$ and all non-equivalent expressions in $n$ variables $X_1, \ldots, X_n$, which are sets, that can be obtained by using only $\cup$ and $\cap$. Two expressions $E_1, E_2 \in M_n$ are considered to be equivalent if $E_1(A_1, \ldots, A_n) = E_2(A_1, \ldots, A_n)$ for any sets $A_1, \ldots, A_n$.

E.g. $M_2 = \{\emptyset, X_1, X_2, X_1 \cup X_2, X_1 \cap X_2\}$.

(i) Describe all elements of $M_n$.

(ii) Prove that $2^{\binom{n}{[n/2]}} \leq |M_n| \leq 2^{2^n}$.

**Open problem.** By Stirling's formula we have $\binom{n}{[n/2]} \sim \sqrt{\frac{2}{\pi}} \cdot \frac{2^n}{\sqrt{n}}$, so

$$\sqrt{\frac{2}{\pi}} \cdot \frac{2^n}{\sqrt{n}} \sim \binom{n}{[n/2]} \leq \log_2 |M_n| \leq 2^n.$$

Try to determine how fast $\log_2 |M_n|$ grows or at least find tighter bounds for it.

Proposed by Constantin-Nicolae Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

## SOLUTIONS

**379.** For any prime $p$ we denote by $|\cdot|_p:\mathbb{Q}\to\mathbb{R}$ the $p$-adic norm given by $|0|=0$ and $|p^t a/b|_p=1/p^t$ if $a,b\in\mathbb{Z}$, $p\nmid ab$. The $p$-adic field $\mathbb{Q}_p$ is the completion of $(\mathbb{Q},|\cdot|_p)$.

Prove that for every prime $p$ and every $a_1,\ldots,a_n\in\mathbb{Q}_p$ we have

$$\max_{i,j}\min_{(k,l)\neq(i,j)}|(a_i-a_j)-(a_k-a_l)|_p\geq 1/4^{n-1}\min_{i\neq j}|a_i-a_j|_p.$$

`Proposed by Alexandru Zaharescu, University of Illinois at Urbana-Champaign, USA.`

*Solution by C.-N. Beli.* Let $N=n^2$ and let $\{1,\ldots,n\}\times\{1,\ldots,n\}=\{(i_1,j_1),\ldots,(i_N,j_N)\}$. For every $t$ let $(k_t,l_t)\neq(i_t,j_t)$ such that

$$|(a_{i_t}-a_{j_t})-(a_{k_t}-a_{l_t})|_p=\min_{(k,l)\neq(i_t,j_t)}|(a_{i_t}-a_{j_t})-(a_k-a_l)|_p.$$

We must prove that $\max_t|(a_{i_t}-a_{j_t})-(a_{k_t}-a_{l_t})|_p\geq 1/4^{n-1}\min_{i\neq j}|a_i-a_j|_p$, i.e., that $4^{n-1}\max_t|(a_{i_t}-a_{j_t})-(a_{k_t}-a_{l_t})|_p\geq|a_i-a_j|_p$ for some $i\neq j$.

We denote by $(\mathbb{R}^n)^*$ the dual of $\mathbb{R}^n$ and for any subspaces $V\subseteq\mathbb{R}^n$, $U\subseteq(\mathbb{R}^n)^*$ we denote by $V^0\subseteq(\mathbb{R}^n)^*$ and $U^0\subseteq\mathbb{R}^n$ their annihilators, $V^0=\{f\in(\mathbb{R}^n)^*\mid f(x)=0\ \forall x\in V\}$ and $U^0=\{x\in\mathbb{R}^n\mid f(x)=0\ \forall f\in U\}$. We have $V^{00}=V$ and $U^{00}=U$.

For $1\leq t\leq N$ and $i\neq j$ we consider the linear functions $f_t,g_{i,j}\in(\mathbb{R}^n)^*$ given by $f_t(x_1,\ldots,x_n)=(x_{i_t}-x_{j_t})-(x_{k_t}-x_{l_t})$ and $g_{i,j}(x_1,\ldots,x_n)=x_i-x_j$. Let $V=\sum\limits_t\mathbb{R}f_t$. We claim that $V^0\subseteq\cup_{i\neq j}(\mathbb{R}g_{i,j})^0$. Assume the contrary. Then let $x=(x_1,\ldots,x_n)\in V^0\setminus\cup_{i\neq j}(\mathbb{R}g_{i,j})^0$. We have $x_i-x_j=g_{i,j}(x)\neq0$ if $i\neq j$ and $(x_{i_t}-x_{j_t})-(x_{k_t}-x_{l_t})=f_t(x)=0$ $\forall t$. Since $x_i$ are mutually distinct there are unique indices $i_{\max}$ and $i_{\min}$ such that $x_{i_{\max}}=\max_i x_i$ and $x_{i_{\min}}=\min_i x_i$. We have $x_i<x_{i_{\max}}$ if $i\neq i_{\max}$ and $x_i>x_{i_{\min}}$ if $i\neq i_{\min}$. Let $t$ be the index for which $(i_t,j_t)=(i_{\max},i_{\min})$. We have $x_{i_t}-x_{k_t}\geq0$ and $x_{l_t}-x_{j_t}\geq0$ with equalities iff $k_t=i_t$ and $l_t=j_t$, respectively. Then from $(x_{i_t}-x_{k_t})+(x_{l_t}-x_{j_t})=(x_{i_t}-x_{j_t})-(x_{k_t}-x_{l_t})=0$ we get $k_t=i_t$ and $l_t=j_t$, i.e. $(k_t,l_t)=(i_t,j_t)$. Contradiction.

From $V^0\subseteq\cup_{i\neq j}(\mathbb{R}g_{i,j})^0$ we obtain that $V^0\subseteq(\mathbb{R}g_{i,j})^0$ for some $i\neq j$. (Otherwise $V^0$ could be written as a finite union of subspaces of smaller dimensions, $V^0=\cup_{i\neq j}(V^0\cap(\mathbb{R}g_{i,j})^0)$.) By taking annihilators we get $V=V^{00}\supseteq(\mathbb{R}g_{i,j})^{00}=\mathbb{R}g_{i,j}$, i.e., $g_{i,j}\in V$.

Let $e_1,\ldots,e_n$ be the canonical basis of $\mathbb{R}^n$ and let $e_1^*,\ldots,e_n^*$ be the dual basis of $(\mathbb{R}^n)^*$. We have $e_k^*(e_l)=\delta_{k,l}$, so that $e_k^*(x_1,\ldots,x_n)=x_k$. Then $f_t=(e_{i_t}^*-e_{j_t}^*)-(e_{k_t}^*-e_{l_t}^*)=\sum\limits_k a_{k,t}e_k^*$ and $g_{i,j}=e_i^*-e_j^*=\sum\limits_k b_ke_k^*$, with $a_{t,k},b_k\in\mathbb{Z}$. Note that $\sum\limits_k|a_{t,k}|\leq4$. (In fact $\sum\limits_k|a_{t,k}|=4$, with the exception of the case $\{i_t,l_t\}\cap\{j_t,k_t\}\neq\emptyset$.)

Now $f_1,\ldots,f_N$ belong to the subspace of $(\mathbb{R}^n)^*$ spanned by $e_2^*-e_1^*,\ldots,e_n^*-e_1^*$, of dimension $n-1$, so $d:=\dim V\leq n-1$. Let $1\leq\beta_1<\ldots<\beta_d\leq N$ such that $f_{\beta_1},\ldots,f_{\beta_d}$ is a basis for $V$. In particular, they are linearly independent, so the vectors $(a_{1,\beta_s},\ldots,a_{n,\beta_s})^T$ with $s=1,\ldots,d$ are linearly independent. It follows that the $n \times d$ matrix $(a_{k,\beta_s})_{k,s}$ has rank $d$ and hence it has a nonzero $d \times d$ minor, i.e., there are $1 \leq \alpha_1 < \ldots < \alpha_d \leq n$ such that $\det(a_{\alpha_r,\beta_s})_{r,s} \neq 0$.

Since $g_{i,j} \in V$, there are unique $c_1, \ldots, c_d \in \mathbb{R}$ with $g_{i,j} = \sum_s c_s f_{\beta_s}$. When we write this relation in the basis $e_1^*, \ldots, e_n^*$ we get $b_k = \sum_s a_{k,\beta_s} c_s$ for $1 \leq k \leq n$. In particular, $b_{\alpha_r} = \sum_s a_{\alpha_r,\beta_s} c_s$. Hence $X = (c_1, \ldots, c_d)^T$ is a solution of the equation $AX = b$, where $A = (a_{\alpha_r,\beta_s})_{r,s}$ and $b = (b_{\alpha_1}, \ldots, b_{\alpha_d})^T$. Since $A$ is non-degenerate we get $c_s = \Delta_s/\Delta$, where $\Delta = \det A$ and $\Delta_s$ is the determinant of the matrix obtained by replacing the $s$th column of $A$ by $b$. Since $A$, $b$ have integer entries we have $\Delta, \Delta_s \in \mathbb{Z}$. Since for any $s$ it holds $\sum_r |a_{\alpha_r,\beta_s}| \leq \sum_k |a_{k,\beta_s}| \leq 4$, we have the estimate $|\Delta| \leq \prod_s \left( \sum_r |a_{\alpha_r,\beta_s}| \right) \leq 4^d \leq 4^{n-1}$.

We now consider the linear functions $\tilde{f}_t$ and $\tilde{g}_{i,j} \in (\mathbb{Q}_p^n)^*$ given by $\tilde{f}_t(x_1, \ldots, x_n) = (x_{i_t} - x_{j_t}) - (x_{k_t} - x_{l_t})$ and $\tilde{g}_{i,j}(x_1, \ldots, x_n) = x_i - x_j$. Since $\tilde{f}_t, \tilde{g}_{i,j}$ have the same integer coefficients as $f_t, g_{i,j}$ and $c_s \in \mathbb{Q}$, from $g_{i,j} = \sum_s c_s f_{\beta_s}$ we get the similar relation $\tilde{g}_{i,j} = \sum_s c_s \tilde{f}_{\beta_s}$.

Let $a = (a_1, \ldots, a_n)$. From above we get $\tilde{g}_{i,j}(a) = \sum_s c_s \tilde{f}_{\beta_s}(a)$, which implies $|\tilde{g}_{i,j}(a)|_p \leq \max_s |c_s|_p |\tilde{f}_{\beta_s}(a)|_p$. But $|c_s|_p = |\Delta_s/\Delta|_p \leq |\Delta| \leq 4^{n-1}$. It follows that $|\tilde{g}_{i,j}(a)|_p \leq 4^{n-1} \max_s |\tilde{f}_{\beta_s}(a)|_p \leq 4^{n-1} \max_t |\tilde{f}_t(a)|_p$, or, to put it differently, $|a_i - a_j|_p \leq 4^{n-1} \max_t |(a_{i_t} - a_{j_t}) - (a_{k_t} - a_{l_t})|_p$. □

**380.** Let $p$ be a prime and let $A \in M_n(\mathbb{Z})$ be a matrix such that $p \mid \operatorname{tr} A^k$ for all integers $k \geq 1$.

(i) Prove that if $n < p$ then there is some integer $m \geq 1$ such that $A^m \in pM_n(\mathbb{Z})$.

(ii) Prove that if $n = p$ then $A^p - (\det A) I_p \in pM_p(\mathbb{Z})$.

`Proposed by Vlad Matei, student, University of Wisconsin, Madison, USA.`

*Solution by Victor Makanin, Sankt Petersburg, Russia.* Let $\overline{A} \in M_n(\mathbb{Z}_p)$ be the reduced modulo $p$ of matrix $A$ (obtained by replacing each and every entry of $A$ with the corresponding class of residues from $\mathbb{Z}_p$). Let $x_1, \ldots, x_n$ be the eigenvalues of $\overline{A}$ (possibly in a field extension of $\mathbb{Z}_p$) and let $\sigma_j = \sum x_1 \cdots x_j$ be the symmetric sums of $x_1, \ldots, x_n$. The characteristic equation of $\overline{A}$ is then $x^n - \sigma_1 x^{n-1} + \cdots + (-1)^n \sigma_n = \overline{0}$, and, by hypothesis, we have

$$\overline{0} = \operatorname{tr} \overline{A}^k = x_1^k + \cdots + x_n^k$$

in $\mathbb{Z}_p$, for all positive integers $k$.

(i) Further we use Newton's formulæ to infer that $j\sigma_j = \overline{0}$ for all $1 \leq j \leq n$. As $n < p$, this implies $\sigma_j = \overline{0}$ for all $1 \leq j \leq n$, thus $\overline{A}^n = O_n$, $O_n$ being the zero matrix from $M_n(\mathbb{Z}_p)$ (because $\overline{A}$ satisfies its characteristic equation). Clearly, this means that $A^n \in pM_n(\mathbb{Z})$.

(ii) In this case, again by Newton's formulæ, we get $\sigma_j = \overline{0}$ for all $1 \leq j \leq p-1$. Consequently the characteristic equation of $\overline{A}$ is $x^p + (-1)^p \sigma_p = \overline{0}$; since $\sigma_p = \det \overline{A}$, and $(-1)^p = -1$ in $\mathbb{Z}_p$ (check this separately for $p = 2$ and for odd $p$), we find that $\overline{A}^p - \det \overline{A} I_p = O_p$ in $M_p(\mathbb{Z}_p)$. This leads to the desired conclusion. □

**381.** Prove or disprove: for any unital ring $A$ there exists a map $f : A \to Z(A)$ such that $f(1) = 1$ and $f(a+b) = f(a) + f(b)$ for all $a, b \in A$. Here $Z(A)$ denotes the *center* of $A$, $Z(A) = \{r \in A : ra = ar, \forall a \in A\}$.

`Proposed by Filip-Andrei Chindea, student, University of Bucharest, Bucharest, Romania.`

*Solution by the author.* There is a large class of rings for which the property holds, e.g., commutative rings (trivially) and rings that are $k$-algebras over a field $k$ (when we can choose $f$ to be even $k$-linear); however, the general statement is false because of the following counterexample.

Let $p$ be a prime and let $C_{p^\infty}$ denote the following subring of $\mathbb{Q}$:

$$C_{p^\infty} := \left\{ \frac{a}{p^k} : a \in \mathbb{Z}, k \geq 0 \right\} \subset \mathbb{Q}.$$

Now take $A = C_{2^\infty} \times \mathbb{Q} \times C_{3^\infty}$. Define addition componentwise and multiplication by

$$(p, q, r)(p', q', r') := (pp', pq' + qr', rr').$$

It is a straightforward exercise that the two operations thus defined make $A$ into a ring with unit $1_A = (1, 0, 1)$. We claim that $A$ is a counterexample to our statement.

First, note that if $(p, q, r) \in Z(A)$, then $(p, q, r)(1, 0, 0) = (1, 0, 0)(p, q, r)$ gives $(p, 0, 0) = (p, q, 0)$, so that $q = 0$, and $(p, q, r)(0, 1, 0) = (0, 1, 0)(p, q, r)$ yields $(0, p, 0) = (0, r, 0)$, whence $p = r$. Write $p = a/2^k$, $r = b/3^m$, $a, b \in \mathbb{Z}$. From $p = r$ it follows $a \cdot 3^m = b \cdot 2^k$, hence $3^m \mid b \cdot 2^k$ and thus $3^m \mid b$. This shows that $r \in \mathbb{Z}$, so that $(p, q, r) = (r, 0, r)$ with $r \in \mathbb{Z}$. The converse is clear, and this gives $Z(A) = \{(m, 0, m) : m \in \mathbb{Z}\}$.

By contradiction, assume that $f : A \to Z(A)$ verifies the given conditions. Let $f(1, 0, 0) = (m, 0, m)$. For $2^k > |m|$, we also have $(m, 0, m) = 2^k f(1/2^k, 0, 0) = (2^k s, 0, 2^k s)$, where $f(1/2^k, 0, 0) = (s, 0, s)$, since $f$ is additive. This gives $2^k \mid m$, and so $m = 0$. Similarly $f(0, 0, 1) = (0, 0, 0)$, whence $f(1_A) = f(1, 0, 1) = f(1, 0, 0) + f(0, 0, 1) = (0, 0, 0)$, which contradicts $f(1_A) = 1_A$. This shows that $A$ is a counterexample, completing the proof. □

**Remarks.** The ring structure defined on $A$ is not artificial, $A$ being isomorphic to one of the so-called *triangular rings* of $2 \times 2$ matrices

$$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} : r \in R, m \in M, s \in S \right\},$$

whose usual matrix operations indeed define a ring whenever $R, S$ are rings and $M$ an $(R, S)$-bimodule, as in our example with $\mathbb{Q}$ and $(C_{2^\infty}, C_{3^\infty})$. These rings were used to produce many counterexamples in ring theory, taking into account that their ideals are relatively easy to identify (see T. Y. Lam, *A First Course in Noncommutative Rings*, Springer,1991, pp. 18–19).

Note that our required properties for $f$, akin to those of the trace map on rings of matrices, are usually met by some numerical structure embedded in $R$, whence the relative difficulty of answering in the negative.

**382.** Let $a_1, b_1, \dots a_s, b_s \in \mathbb{Z}_2$ and let $f : \mathbb{Z}_2^{2s} \to \mathbb{Z}_2$,

$$f(X_1, Y_1, \dots, X_s, Y_s) = \sum_{i=1}^{s} (a_i X_i^2 + X_i Y_i + b_i Y_i^2).$$

Determine $|f^{-1}(0)|$.

Proposed by Constantin-Nicolae Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

*Solution by the author.* For $1 \leq t \leq s$ we consider the functions $f_t : \mathbb{Z}_2^{2t} \to \mathbb{Z}_2$ and $g_t : \mathbb{Z}_2^2 \to \mathbb{Z}_2$, given by $f_t(X_1, Y_1, \dots, X_t, Y_t) = \sum_{i=1}^{t} (a_i X_i^2 + X_i Y_i + b_i Y_i^2)$ and $g_t(X, Y) = a_t X^2 + XY + b_t Y^2$. In particular, $f = f_s$. We also denote $k_t = |f_t^{-1}(0)|$, $l_t = |f_t^{-1}(1)|$, $r_t = |g_t^{-1}(0)|$ and $s_t = |g_t^{-1}(1)|$. Note that $f_t(X_1, Y_1, \dots, X_t, Y_t) = f_{t-1}(X_1, Y_1, \dots, X_{t-1}, Y_{t-1}) + g_t(X_t, Y_t)$, so $f_t^{-1}(0) = f_{t-1}^{-1}(0) \times g_t^{-1}(0) \cup f_{t-1}^{-1}(1) \times g_t^{-1}(1)$ and $f_t^{-1}(1) = f_{t-1}^{-1}(0) \times g_t^{-1}(1) \cup f_{t-1}^{-1}(1) \times g_t^{-1}(0)$. It follows that $k_t = k_{t-1} r_t + l_{t-1} s_t$ and $l_t = k_{t-1} s_t + l_{t-1} r_t$.

Note that $f_1 = g_1$ implies $k_1 = r_1$ and $l_1 = s_1$, so the above recurrence relations also hold at $t = 1$ if we define $k_0 = 1$, $l_0 = 0$.

We have $k_t - l_t = (k_{t-1} - l_{t-1})(r_t - s_t)$, so, by induction,

$$k_t - l_t = (k_0 - l_0) \prod_{i=1}^{t} (r_i - s_i) = \prod_{i=1}^{t} (r_i - s_i).$$

It is easy to see that if $a_i = b_i = 1$ then $g_i^{-1}(0) = \{(0,0)\}$, whence $|g_i^{-1}(0)| = 1$, and if $a_i$ or $b_i = 0$ then $|g_i^{-1}(0)| = 3$. It follows that $r_i - s_i = 1 - 3 = -2$ if $a_i = b_i = 0$ and $r_i - s_i = 3 - 1 = 2$ otherwise. For short, $r_i - s_i = (-1)^{a_i b_i} 2$. (The value of $(-1)^x$ depends only on $x$ mod 2 so $x \mapsto (-1)^x$ can be defined on $\mathbb{Z}_2$.)

In conclusion, $k_s - l_s = \prod_{i=1}^{s} (-1)^{a_i b_i} 2 = (-1)^S 2^s$, where $S = \sum_{i=1}^{s} a_i b_i$. On the other hand $k_s + l_s = |\mathbb{Z}_2^{2s}| = 2^{2s}$. Hence $|f^{-1}(0)| = k_s = 2^{2s-1} + (-1)^S 2^{s-1}$. □

**Editors' note.** An even dimensional non-degenerate quadratic form over a field $K$ of characteristic two can be written in the form $f = \sum_{i=1}^{2s} (a_i X_i^2 + X_i Y_i + b_i Y_i^2)$. When $K = \mathbb{Z}_2$ the sum $\sum_{i=1}^{s} a_i b_i$ which appears in the answer to our problem, is called the Arf invariant of $f$, $\mathrm{Arf} f$. The mathematician William Browder called this invariant *the democratic invariant* because it is the value which is taken most often by the quadratic form. (If $\mathrm{Arf} f = 0$ then $|f^{-1}(0)| = 2^{2s-1} + 2^{s-1} > 2^{2s-1} - 2^{s-1} = |f^{-1}(1)|$, while if $\mathrm{Arf} f = 1$ then $|f^{-1}(1)| = 2^{2s-1} + 2^{s-1} > 2^{2s-1} - 2^{s-1} = |f^{-1}(0)|$.)

The invariant $\mathrm{Arf} f = \sum_i a_i b_i$ can be defined when $K$ is an arbitrary field of characteristic two, but it takes value not in $K$, but in $K/\wp(K)$, where $\wp : K \to K$ is the Artin-Schreier map, $\wp(x) = x^2 + x$. (The map $\wp$ is additive, i.e., $(x+y)^2 + (x+y) =$

$(x^2+x)+(y^2+y)$, so $\wp(K)$ is a subgroup of $(K,+)$ and we may consider the quotient group $K/\wp(K)$.)

If $K$ is finite, i.e., $K=\mathbb{F}_q$ with $q=2^n$, then the invariant Arf determines uniquely $f$ up to an isometry. We now generalize our problem to arbitrary finite fields of characteristic two, i.e., we will determine $|f^{-1}(\alpha)|$ for any $\alpha\in K$. Our result will be in terms of $\operatorname{Arf} f$.

First note that $\ker\wp=\{x\in K\mid x^2+x=0\}=\{0,1\}$, so that $|\wp(K)|=|K|/\ker\wp|=|K|/2$. It follows that $|K/\wp(K)|=2$. So there is an isomorphism $\chi$ between $(K/\wp(K),+)$ and $(\{\pm1\},\cdot\}$. We have $\chi(a)=1$ iff $a=0$ in $K/\wp(K)$, that is iff $a=\wp(x)=x^2+x$ for some $x\in K$. Otherwise $\chi(a)=-1$.

For any $\alpha\in K$ we have $f(\alpha X_1,\alpha Y_1,\dots,\alpha X_s,\alpha Y_s)=\alpha^2 f(X_1,Y_1,\dots,X_s,Y_s)$. If $a\in K^*$ we have $a=a^q=(a^{q/2})^2$. Hence the mapping $(X_1,Y_1,\dots,X_s,Y_s)\mapsto(a^{q/2}X_1,a^{q/2}Y_1,\dots,a^{q/2}X_s,a^{q/2}Y_s)$ is a bijection between $f^{-1}(a)$ and $f^{-1}(1)$, so $|f^{-1}(a)|=|f^{-1}(1)|$ and $q^{2s}=|K^{2s}|=\sum\limits_{a\in K}|f^{-1}(a)|=|f^{-1}(0)|+(q-1)|f^{-1}(1)|$.

**Lemma 1.** *Keep the above notation. If $g(X,Y)=aX^2+XY+bY^2$ then:*

(i) *If $\chi(ab)=1$ then $|g^{-1}(0)|=2q-1$ and $|g^{-1}(1)|=q-1$.*

(ii) *If $\chi(ab)=-1$ then $|g^{-1}(0)|=1$ and $|g^{-1}(1)|=q+1$.*

(iii) *$|g^{-1}(0)|-|g^{-1}(1)|=\chi(ab)q$.*

*Proof.* Note that $|g^{-1}(0)|+(q-1)|g^{-1}(1)|=q^2$, so if $|g^{-1}(0)|=2q-1$ then $|g^{-1}(1)|=q-1$, and if $|g^{-1}(0)|=1$ then $|g^{-1}(1)|=q+1$. So it is enough to determine $|g^{-1}(0)|$.

(i) If $\chi(ab)=1$ then $ab=\wp(t)=t^2+t$ for some $t\in K$. We assume first that $a\neq0$. We have $t^2+t+ab=0$ so the roots of $T^2+T+ab\in K[T]$ are $t,t+1$. Hence $T^2+T+ab=(T+t)(T+t+1)$.

Assume first that $a\neq0$. Then $ag(X,Y)=a^2X^2+aXY+abY^2=(aX+tY)(aX+(t+1)Y)$, so $g^{-1}(0)=A_1\cup A_2$, where $A_1=\{(x,y)\mid ax+ty=0\}=\{(\frac{t}{a}y,y)\mid y\in K\}$, $A_1=\{(x,y)\mid ax+ty=0\}=\{(\frac{t+1}{a}y,y)\mid y\in K\}$. We have $|A_1|=|A_2|=q$ and $|A_1\cap A_2|=|\{(0,0)\}|=1$, so

$$|g^{-1}(0)|=|A_1\cup A_2|=|A_1|+|A_2|-|A_1\cap A_2|=2q-1.$$

Similarly when $b\neq0$. If $a=b=0$ then $g(X,Y)=XY$, so $g^{-1}(0)=A_1\cup A_2$, where $A_1=\{(0,y)\mid y\in K\}$, $A_2=\{(x,0)\mid y\in K\}$. Again $A_1\cap A_2=\{(0,0)\}$, so $|g^{-1}(0)|=|A_1\cup A_2|=|A_1|+|A_2|-|A_1\cap A_2|=2q-1$.

(ii) If $\chi(ab)=-1$ then $ab$ cannot be written as $t^2+t$ with $t\in K$. In particular, $ab\neq0$. Let $(x,y)\in g^{-1}(0)$. We have $ax^2+xy+by^2=0$. If $y=0$ then $ax^2=0$, so $x=0$. If $y\neq0$ then $0=\frac{a}{y^2}(ax^2+xy+by^2)=t^2+t+ab$, so $ab=t^2+t$ with $t=\frac{ax}{y}$. Contradiction. It remains $g^{-1}(0)=\{(0,0)\}$ and $|g^{-1}(0)|=1$.

(iii) follows imediately from (i) and (ii). $\square$

We now use the notation $f_t$, $g_t$, $k_t$, $l_t$, $r_t$, $s_t$ with the meaning from the case $K=\mathbb{Z}_2$. Note that for any $a\in K^*$ we have $|f_t^{-1}(a)|=|f_t^{-1}(1)|=l_t$ and $|g_t^{-1}(a)|=|g_t^{-1}(1)|=s_t$. Also by (iii) of Lemma 1 we have $r_t-s_t=\chi(a_tb_t)$.

From $f_t(X_1,Y_1,\dots,X_t,Y_t)=f_{t-1}(X_1,Y_1,\dots,X_{t-1},Y_{t-1})+g_t(X_t,Y_t)$ we get $f_t^{-1}(0)=\cup_{a\in K}f_{t-1}^{-1}(a)\times g_t^{-1}(a)$ and $f_t^{-1}(1)=\cup_{a\in K}f_{t-1}^{-1}(a)\times g_t^{-1}(a+1)$, so that $k_t=\sum\limits_{a\in K}|f_{t-1}^{-1}(a)|\cdot|g_t^{-1}(a)|$ and $l_t=\sum\limits_{a\in K}|f_{t-1}^{-1}(a)|\cdot|g_t^{-1}(a+1)|$. If $a\neq0$ then

$f_{t-1}^{-1}(a) \times g_t^{-1}(a) = l_{t-1}s_t$, while if $a = 0$ then $f_{t-1}^{-1}(a) \times g_t^{-1}(a) = k_{t-1}r_t$. Hence $k_t = (q-1)l_{t-1}s_t + k_{t-1}r_t$. If $a \neq 0, 1$ then $f_{t-1}^{-1}(a) \times g_t^{-1}(a+1) = l_{t-1}s_t$, if $a = 0$ then $f_{t-1}^{-1}(a) \times g_t^{-1}(a+1) = k_{t-1}s_t$, and if $a = 1$ then $f_{t-1}^{-1}(a) \times g_t^{-1}(a+1) = l_{t-1}r_t$. Hence $k_t = (q-2)l_{t-1}s_t + k_{t-1}s_t + l_{t-1}r_t$.

Like in the case $K = \mathbb{Z}_2$, we have $k_1 = r_1$, $l_1 = s_1$, so the relations above hold at $t = 1$ if we put $k_0 = 1$, $l_0 = 0$. It follows that

$$k_t - l_t = l_{t-1}s_t + k_{t-1}r_t - k_{t-1}s_t - l_{t-1}r_t = (k_{t-1} - l_{t-1})(r_{t-1} - s_{t-1}),$$

from which we conclude

$$k_s - l_s = (k_0 - l_0)\prod_{i=0}^{s}(r_i - s_i) = \prod_{i=1}^{t}(\chi(a_ib_i)q) = \chi(\sum_{i=1}^{s} a_ib_i)q^s = \chi(\mathrm{Arf} f)q^s.$$

Together with $(q-1)l_s + k_s = q^{2s}$, this implies that $|f^{-1}(0)| = k_s = q^{2s-1} + \chi(\mathrm{Arf} f)(q-1)q^{s-1}$ and for $a \neq 0$ we have $|f^{-1}(a)| = l_s = q^{2s-1} - \chi(\mathrm{Arf} f)q^{s-1}$. $\square$

**383.** Let $C = \mathbb{R}^2 \times \mathbb{R}_{>0}$ be the parameter space of all plane circles. Let $n > 0$ and denote by $H_k$ the subset of $C^n$ parameterizing the configurations of $n$ plane circles whose interiors contain a fixed point $Q$, such that any two circles intersect and no $k$ circles pass through the same point. Show that $H_4$ is path connected.

`Proposed by Marius Cavachi, Ovidius University of Constanţa, Constanţa, Romania.`

*Solution by the author.* Consider two configurations $A_i$, $B_i$ ($i \in \{1, \ldots, n\}$) from $H_4$ as in the hypothesis and assume that all of them contain $Q$ as an interior point. The aim is to deform $A_i$ continuously into $B_i$, preserving the $H_4$ configuration.

We make several steps.

*Step 1.* We reduce to the case when every two circles intersect in two points, i.e., no two circles are tangent to each other and no two circles coincide. If two circles are tangent to each other they can only be tangent internally, for both contain the point $Q$ inside. Suppose that there are two circles $A_i$ and $A_j$ that are tangent to each other and, moreover, $A_i$ has minimal radius among all circles with this property. When we increase slightly the radius of $A_i$ the $H_4$ configuration will be preserved, but $A_i$ and $A_j$ will no longer be tangent to each other. By repeating this procedure, we eventually ensure no tangencies between circles. If two circles $A_i$ and $A_j$ coincide then we translate slightly $A_i$ in some direction. We do so till no two circles coincide.

We do the same thing with the circles $B_i$.

The reduction from *Step 1* ensures that any small perturbations of the circles $A_i$ and $B_i$ preserve the $H_4$ structure.

Consider a sphere $\mathcal{S}$ of center $O$, tangent to the plane $\mathcal{P}$ of the two configurations at $Q$, and let $P$ be the antipodal of $Q$. Denote by $\pi : \mathcal{S} \setminus \{P\} \to \mathcal{P}$ the stereographic projection from $P$ and denote by $C_i$ and $D_i$ the inverse images via $\pi$ of $A_i$ and $B_i$, respectively. Let $O_i$ and $O_i'$ be the centers of $C_i$ and $D_i$.

*Step 2.* By slightly moving $C_i$'s and $D_i$'s (so $A_i$'s and the $B_i$'s) we may assume that:

(a) No three of the centers $O_i$ and $O_i'$ lie in the same plane with $O$.

In particular, $O \neq O_i, O_i'$ $\forall i$ and no two of the $O_i$'s and $O_i'$'s are colinear with $O$.

(b) The intersection of any four of the planes containing four of the circles $A_i$ or four of the circles $B_i$ is empty.

Note that the condition that $A_i$ contain $Q$ inside is equivalent to the fact that $PQ$ intersects the interior of $A_i$ and also to the fact that $P$ and $Q$ lie in the opposite calottes separated by $A_i$. In particular, the plane supporting $A_i$ is not parallel to $PQ$. Similarly for $B_i$'s.

*Step 3.* We move the circles $C_i$ on the sphere by translating the supporting planes along the vectors $\overrightarrow{O_iO}$. More precisely, for $t \in [0,1]$ we define $C_i(t)$ as the circle on the sphere lying on a plan parallel to that of $C_i$ and centered at $O_i(t)$ defined by $\overrightarrow{OO_i'} = (1-t)\overrightarrow{OO_i}$. In particular, $O_i(1) = O$, so $C_i(1)$ is a big circle. We define $A_i(t) = \pi(C_i(t))$. Similarly we define $O_i'(t)$, $D_i(t)$ and $B_i(t)$.

We prove that $A_i(t)$ form an $H_4$ configuration for $t \in [0,1)$.

Note that all the planes supporting $C_i(t)$ are parallel to the plane of $C_i$, which is not parallel to $PQ$. In particular, the big circle $C_i(1)$ will not contain $P$ and $Q$. Hence $P$ and $Q$ lie in the opposite open hemispheres separated by $C_i(1)$. Also $P$ and $Q$ lie in the opposite calottes separated by $C_i$. Since $C_i(t)$ lies in the segment of sphere bounded by $C_i$ and $C_i(1)$, it will have the same property. It follows that $A_i(t) = \pi(C_i(t))$ contains $Q$ inside.

For $t \in [0,1)$ the plane supporting $C_i(t)$ is obtained from the plane supporting $C_i$ through a homothety of center $O$ and ratio $1-t$. Since by *Step 2* (b) any four of the planes supporting circles $C_i$ have an empty intersection, the intersection of any four planes containing circles $C_i(t)$ is empty as well. It follows that any four circles $C_i(t)$ are disjoint. It follows that any four circles $A_i(t)$ are disjoint.

For any $i \neq j$ we have $|A_i \cap A_j| = 2$, so $|C_i \cap C_j| = 2$. It follows that if $\ell$ is the intersection of the planes supporting $C_i$ and $C_j$ then $|\ell \cap \mathcal{S}| = |C_i \cap C_j| = 2$, so the distance $d(O,\ell)$ is smaller than the radius $R$ of $\mathcal{S}$. The intersection of the planes supporting $C_i(t)$ and $C_j(t)$ is the line $\ell(t)$ obtained from $\ell$ by a homothety of center $O$ and ratio $1-t$. It follows that $d(O,\ell(t)) = (1-t)d(O,\ell) < R$, which implies that $|C_i(t) \cap C_j(t)| = |\ell(t) \cap \mathcal{S}| = 2$, so that $|A_i(t) \cap A_j(t)| = 2$.

In conclusion, $A_i(t)$ form an $H_4$ structure for $t \in [0,1)$. Similarly for $B_i(t)$. So we are left to prove that there is a path in $H_4$ between $(A_i(1))$ and $(B_i(1))$.

Suppose that $E_1, \ldots, E_n$ are big circles of $\mathcal{S}$ not containing $P, Q$ and $OO_i''$ is perpendicular on the plane containing $E_i$. For any $i$ we have $P, Q \notin E_i$, so $\pi(E_i)$ contains $Q$ inside. Also for any indices $i \neq j$ we have $E_i \cap E_j \neq \emptyset$, so $\pi(E_i) \cap \pi(E_j) \neq \emptyset$. Hence $\pi(E_i)$ satisfy two of the conditions of $H_4$. Assume that $\pi(E_{i_1}) \cap \cdots \cap \pi(E_{i_k})$ is non-empty. Then $E_{i_1} \cap \cdots \cap E_{i_k} \neq \emptyset$. If $S \in E_{i_1} \cap \cdots \cap E_{i_k}$ then the planes supporting $E_{i_1}, \ldots, E_{i_k}$ contain the line $OS$. It follows that $OO_{i_j}'' \perp OS$ for all $j$. Hence $O$ and $O_{i_1}'', \ldots, O_{i_k}''$ lie in the plane perpendicular to $OS$ passing through $O$. Hence in order that $\pi(E_i)$ are an $H_4$ configuration it is sufficient that no four of $O_i''$ lie in the same plane with $O$.

Let $0 \leq l \leq n$. Let $E_l(t)$ with $t \in [0,1]$ be a continuous deformation of $C_l(1)$ into $D_l(1)$ through big circles on $\mathcal{S}$ not containing $P$ and $Q$. If we put $E_i(t) = C_i(1)$ for $i < l$ and $E_i(t) = D_i(1)$ for $i > l$ then $(E_1(t), \ldots, E_n(t))$ is a continuous deformation of $(C_1(1), \ldots, C_l(1), D_{l+1}(1), \ldots, D_n(1))$ into $(C_1(1), \ldots, C_{l-1}(1), D_l(1), \ldots, D_n(1))$ and so $(\pi(E_1(t)), \ldots, \pi(E_n(t)))$ is a continuous path from $(A_1(1), \ldots, A_l(1), B_{l+1}(1), \ldots, B_n(1))$ to $(A_1(1), \ldots, A_{l-1}(1), B_l(1), \ldots, B_n(1))$. Fix $t \in [0,1]$.

Let $OO''_i$ be perpendicular on the plane of $E_i(t)$. For $i < l$ we may take $O''_i = O_i$ and for $i > l$ we may take $O''_i = O'_i$. By *Step 2* (a) there are no three of $O''_i$ with $i \neq l$ that lie in the same plane with $O$. It follows that there are no four of $O''_i$ with $1 \leq i \leq n$ that lie in the same plane with $O$. Therefore $(\pi(E_1(t)), \ldots, \pi(E_n(t))) \in H_4$.

So we have found a path in $H_4$ between $(A_1(1), \ldots, A_l(1), B_{l+1}(1), \ldots, B_n(1))$ and $(A_1(1), \ldots, A_{l-1}(1), B_l(1), \ldots, B_n(1))$. By putting together these paths for $l = n, n-1, \ldots, 1$ we get a path in $H_4$ between $(A_1(1), \ldots, A_n(1))$ and $(B_1(1), \ldots, B_n(1))$. This completes the proof. $\square$

**384.** Let $M \in \mathcal{M}_2(\mathbb{C})$ be an invertible matrix and let $k > 1$ be an integer.

a) Show that if $M$ is not scalar, then for any two matrices $A, B \in \mathcal{M}_2(\mathbb{C})$ with $A^k = B^k = M$ there exists $E \in \mathcal{M}_2(\mathbb{C})$ such that $B = AE = EA$, $E^k = I_2$, and $|\operatorname{tr}(E)| \leq 2$.

b) Is the converse of a) true?

Proposed by Cornel Băeţica, University of Bucharest, Bucharest, Romania

*Solution by the author.* a) Let $J_A$ and $J_B$ be the Jordan canonical forms of $A$ and $B$ respectively, and $U, V \in GL_2(\mathbb{C})$ such that $A = UJ_AU^{-1}$ and $B = VJ_BV^{-1}$.

Since $M$ is not scalar, we have to investigate two cases:

**Case I.** $M$ has two distinct eigenvalues. Then so do $A$ and $B$. Let $J_A = \operatorname{diag}(\lambda_1, \lambda_2)$. Then, $\operatorname{diag}(\lambda_1^k, \lambda_2^k) = J_{A^k} = J_{B^k}$, so there exist $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$, $\varepsilon_1^k = \varepsilon_2^k = 1$, such that $J_B = \operatorname{diag}(\lambda_1\varepsilon_1, \lambda_2\varepsilon_2)$. Therefore, $J_B = E'J_A = J_AE'$, where $E' = \operatorname{diag}(\varepsilon_1, \varepsilon_2)$.

Since $A^k = B^k$, we get $UJ_{A^k}U^{-1} = VJ_{B^k}V^{-1}$, so $UJ_A^kU^{-1} = VJ_B^kV^{-1}$, and therefore $V^{-1}UJ_A^k = J_A^kV^{-1}U$. Consequently, there exist $a, b \in \mathbb{C}$ such that $V^{-1}U = \operatorname{diag}(a, b)$. (Diagonal matrices with distinct elements on the diagonal only commute with diagonal matrices.) We thus obtain $A = V \cdot \operatorname{diag}(a, b) \cdot J_A \cdot \operatorname{diag}(a^{-1}, b^{-1}) \cdot V^{-1}$, which means $A = VJ_AV^{-1}$.

Let $E = VE'V^{-1}$. Then $B = VJ_BV^{-1} = VE'J_AV^{-1} = VE'V^{-1}VJ_AV^{-1} = EA$ and $B = VJ_BV^{-1} = VJ_AE'V^{-1} = VJ_AV^{-1}VE'V^{-1} = AE$. Since $AE = EA$, we get $B^k = E^kA^k$, so $E^k = I_2$; obviously, $\operatorname{tr}(E) = \operatorname{tr}(E')$, so $|\operatorname{tr}(E)| = |\varepsilon_1 + \varepsilon_2| \leq 2$.

**Case II.** $M$ is not diagonalizable. Then neither of $A$ and $B$ is diagonalizable. Therefore, there exist $a, \varepsilon \in \mathbb{C}\backslash\{0\}$, $\varepsilon^k = 1$, such that

$$J_A = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix} \text{ and } J_B = \begin{pmatrix} a\varepsilon & 0 \\ 1 & a\varepsilon \end{pmatrix}.$$

Then $J_B = E'J_A = J_AE'$, where

$$E' = \begin{pmatrix} \varepsilon & 0 \\ \frac{1-\varepsilon}{a} & \varepsilon \end{pmatrix}.$$

Since $A^k = B^k$, we derive $UJ_A^kU^{-1} = VJ_B^kV^{-1}$, i.e. $V^{-1}UJ_A^k = VJ_B^kV^{-1}U$, so

$$V^{-1}U \begin{pmatrix} a^k & 0 \\ ka^{k-1} & a^k \end{pmatrix} = \begin{pmatrix} a^k & 0 \\ k\varepsilon^{k-1}a^{k-1} & a^k \end{pmatrix} V^{-1}U.$$

If $V^{-1}U = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ then the equation above is equivalent to $\beta = 0$ and $\alpha = \varepsilon\delta$, so there exist $z, t \in \mathbb{C}$, $t \neq 0$, such that

$$V^{-1}U = \begin{pmatrix} \varepsilon t & 0 \\ z & t \end{pmatrix}, \text{ or } U = V \begin{pmatrix} \varepsilon t & 0 \\ z & t \end{pmatrix}.$$

We get $A = UJ_AU^{-1} = V(V^{-1}U)J_A(V^{-1}U)^{-1}V^{-1}$, i.e.,

$$A = V \begin{pmatrix} \varepsilon t & 0 \\ z & t \end{pmatrix} \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix} \begin{pmatrix} \frac{1}{\varepsilon t} & 0 \\ -\frac{z}{\varepsilon t^2} & \frac{1}{t} \end{pmatrix} V^{-1} = V \begin{pmatrix} a & 0 \\ \frac{1}{\varepsilon} & a \end{pmatrix} V^{-1}.$$

Therefore, $A = V(\frac{1}{\varepsilon}J_B)V^{-1} = \frac{1}{\varepsilon}VJ_BV^{-1} = \frac{1}{\varepsilon}B$. We put $E = \varepsilon I_2$; it is clear that $B = AE = EA$, $E^k = I_2$, and $|\mathrm{tr}(E)| = |2\varepsilon| \leq 2$ (in fact, the last inequality is an equality in this case).

b) The converse of a) is also true:

Suppose $M$ was a scalar matrix, i.e., $M = zI_2$ for some $z \in \mathbb{C}^*$. Let also $w_1, w_2 \in \mathbb{C}^*$ such that $w_1 \neq w_2$ and $w_1^k = w_2^k = z$,

$$A = \begin{pmatrix} w_1 & 1 \\ 0 & w_2 \end{pmatrix} \text{ and } B = \begin{pmatrix} w_1 & 0 \\ 1 & w_2 \end{pmatrix}.$$

Now the eigenvalues of both $A$ and $B$ are $w_1, w_2$ and from $w_1 \neq w_2$ it follows $A \approx B \approx \mathrm{diag}(w_1, w_2)$. (Here $\approx$ means similar.) This in turn implies that $A^k \approx B^k \approx \mathrm{diag}(w_1^k, w_2^k) = zI_2$, so $A^k = B^k = zI_2 = M$

We also note that

$$AB = \begin{pmatrix} w_1^2 + 1 & w_2 \\ w_2 & w_2^2 \end{pmatrix} \neq \begin{pmatrix} w_1^2 & w_1 \\ w_1 & w_2^2 + 1 \end{pmatrix} = BA.$$

In conclusion, $A^k = B^k = M$ but $AB \neq BA$. If there is $E$ with the desired properties then $A^{-1}B = E = BA^{-1}$, whence $AB = BA$, a contradiction.

**385.** Let $x \in (0, \pi)$ and $f(x) = \dfrac{1}{\tan x} - \dfrac{1}{x}$. Prove that $f^{(n)}(x) < 0$ for $n = 0, 1, \ldots$

`George Stoica, Department of Mathematical Sciences, University of New Brunswick, Canada`

*Solution by the author.* Consider Euler's product formula:

$$\sin x = x \prod_{k=1}^{\infty} \left[1 - \left(\frac{x}{k\pi}\right)^2\right] \text{ for any } x.$$

Take logarithms on both sides and differentiate the resulting equation, to obtain

$$\frac{1}{\tan x} = \frac{1}{x} + \sum_{k=1}^{\infty} \left(\frac{1}{k\pi + x} - \frac{1}{k\pi - x}\right), \quad x \in (0, \pi).$$

Each of the functions in the right hand side brackets can be expressed as a geometric series, and the order of summation can be interchanged. We deduce that

$$f(x) = \frac{1}{\tan x} - \frac{1}{x} = -2 \sum_{k=0}^{\infty} a_k x^{2k+1}, \quad x \in (0, \pi),$$

where

$$a_k = \sum_{m=1}^{\infty} \left(\frac{1}{m\pi}\right)^{2k+2}, \quad k = 0, 1, \ldots$$

As $a_k > 0$ for $k = 0, 1, \ldots$, all derivatives $f^{(n)}(x)$, $n = 0, 1, \ldots$, can be obtained from the above series representation, and will be strictly negative for $x \in (0, \pi)$. □

*Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.* For $x \in (0, \pi)$ one has $x < \tan x$ and therefore $\frac{1}{\tan x} - \frac{1}{x} < 0$, which proves the property for $n = 0$.

The conclusion of the problem follows inmediately by considering the Laurent expansion at $x = 0$ of $\cot x$:

$$\cot x = \frac{1}{x} - \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} x^{2n-1},$$

where $B_n$ are the Bernoulli numbers. All the coefficients, except the first one, are non-positive.

Let us define function $g(x)$ as $g(x) = f(x)$ for $x \neq 0$, with $g(0) = 0$. Then the Taylor expansion of $g$ at $x = 0$ has its coefficients all non-positive because $g(x) = -\frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \ldots$.

For $x \in (0, \pi)$ and $n \geq 1$, $f^{(n)}(x) = g^{(n)}(x) = \sum_{k=0}^{\infty} n! \frac{g^{(n+k)}(0)}{(n+k)!} x^k < 0$. □

**Editors' note.** We have $\sum_{m\geq 1} \left(\frac{1}{m\pi}\right)^{2k+2} = \frac{\zeta(2k+2)}{\pi^{2k+2}}$. Hence in his solution the author basically proves and uses the formula $\frac{1}{\tan x} = \frac{1}{x} - 2\sum_{k\geq 1} \frac{\zeta(2k)}{\pi^{2k}} x^{2k-1}$, which is well known.

On the other hand, Ángel Plaza's solution uses the formula

$$\frac{1}{\tan x} = \sum_{n\geq 0} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} x^{2n-1}$$

and the fact that the sign of $B_{2n}$ is $(-1)^{n-1}$ when $n \geq 1$. But the sign of $B_{2n}$ is deduced form the formula $B_{2n} = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n)$ when $n \geq 1$, which in turn follows form the relation $\sum_{n\geq 0} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} x^{2n-1} = \frac{1}{\tan x} = \frac{1}{x} - 2\sum_{n\geq 1} \frac{\zeta(2n)}{\pi^{2n}} x^{2n-1}$.

So, indirectly, Ángel Plaza uses the same formula for $\frac{1}{\tan x}$ as the author.

**386.** Let $n \geq 1$ be an integer. Find the minimum of

$$f(\sigma) := \sum_{1\leq i\leq n/2} \sigma(i) + \sum_{n\geq i>n/2} \sigma^{-1}(i)$$

taken over all permutations $\sigma \in S_n$. Determine an explicit permutation $\sigma$ that realizes this minimum.

`Proposed by Filip-Andrei Chindea, student, University of Bucharest, Bucharest, Romania.`

*Solution by the author.* The answer is $\lfloor (3n^2+4n+4)/8 \rfloor$. Examples of $\sigma$ that realize this minimum will appear during the proof. The idea is to make changes in the decomposition in disjoint cycles of $\sigma$ which do not increase the value of $f$.

Let $A = [1, n/2] \cap \mathbb{Z}$ and $B = (n/2, n] \cap \mathbb{Z}$, so $f(\sigma) = \sum\limits_{i \in A} \sigma(i) + \sum\limits_{i \in B} \sigma^{-1}(i)$. We have $|A| = \lfloor n/2 \rfloor$ and $|B| = n - \lfloor n/2 \rfloor = \lfloor (n+1)/2 \rfloor$, so that $0 \le |B| - |A| \le 1$.

A permutation $\sigma$ can be written as a product of mutually disjoint cycles such that every element from $\{1, \ldots, n\}$ is involved in one cycle, i.e., the fixed points of $\sigma$ will be considered as cycles of length 1.

Note that there are two types of cycles. The first is of the form $(a_1, \ldots, a_s)$ with all $a_i$ in $A$ or all in $B$. The second type are the cycles $(c_1, \ldots, c_m)$ with some of the $c_i$'s in $A$ and some in $B$. By rotating the elements of the cycle we may assume that $c_1 \in A$ and $c_m \in B$. Then $c_1, \ldots, c_m$ is obtained by putting together some sequences of the form $a_1, \ldots, a_s, b_1, \ldots, b_t$, with $a_i \in A$, $b_i \in B$. The contribution of a cycle of the first form to the sum is $a_1 + \cdots + a_s$. For the second type of cycles the contribution of each sequence $a_1, \ldots, a_s, b_1, \ldots, b_t$ to the sum is

$$a_2 + \cdots + a_{s-1} + 2a_s + 2b_1 + b_2 + \cdots + b_{t-1}.$$

Indeed, every $a_i$ with $2 \le i \le s$ contributes to $\sum\limits_{i \in A} \sigma(i)$ and $a_s$ contributes to $\sum\limits_{i \in B} \sigma^{-1}(i)$ as well. Similarly, every $b_i$ with $1 \le i \le t-1$ contributes to $\sum\limits_{i \in B} \sigma^{-1}(i)$ and $b_1$ contributes to $\sum\limits_{i \in A} \sigma(i)$ as well.

(If $s = 1$ the sum $a_2 + \cdots + a_{s-1} + 2a_s$ means $a_1$. This is obvious when we write $a_2 + \cdots + a_{s-1} + 2a_s = (a_1 + \cdots + a_s) + a_s - a_1$. Similarly, $2b_1 + b_2 \cdots + b_{t-1}$ means $b_1$ if $t = 1$.)

To simplify, we may remove every cycle $(a_1, \ldots, a_s)$ of the first type from the decomposition of $\sigma$, i.e., we care replace it by the product of cycles of length one $(a_1) \cdots (a_s)$. This change does not affect $f(\sigma)$. Further we may break every cycle $(c_1, \ldots, c_m)$ of the second type into the cycles $(a_1, \ldots, a_s, b_1, \ldots, b_t)$ from which the sequence $c_1, \ldots, c_m$ is made of. Again this doesn't change $f(\sigma)$. Furthermore $f(\sigma)$ is not changed if, whenever $s \ge 3$, we replace $(a_1, \ldots, a_s, b_1, \ldots, b_t)$ by $(a_1 a_s, b_1, \ldots, b_t)(a_2) \cdots (a_{s-1})$ or if, whenever $t \ge 3$, we replace $(a_1, \ldots, a_s, b_1, \ldots, b_t)$ by $(a_1 \ldots, a_s, b_1, b_t)(b_2) \cdots (b_{t-1})$. Also if $s = t = 1$ we may replace the cycle $(a_1, b_1)$ by $(a_1)(b_1)$.

We end up with a permutation which decomposes as a product of cycles of the form $(a_1, a_2, b_1, b_2)$, $(a_1, b_1, b_2)$, $(a_1, a_2, b_1)$ and cycles of length one. Of all $\sigma$ of this form that realise the minimum of $f$ we choose one such that it has the maximum number of cycles of length 4. We make some remarks:

(1) We can ensure that cycles of each type $(a_1, b_1, b_2)$ and $(a_1, a_2, b_1)$ are not simultaneously present. If $(a_1, b_1, b_2)$, $(a_1', a_2', b_1')$ are such cycles then we may replace them by $(a_1, a_2, b_1', b_2')(b_1)(a_1')$, which doesn't change $f(\sigma)$.

(2) If there is a cycle of the type $(a_1, a_2, b_1)$ then there are no cycles of length one $(b)$ with $b \in B$. The contribution of these to $f(\sigma)$ is $2a_2 + b_1 + b$. But if we replace them with the cycle of length 4 $(a_1, a_2, b_1', b_2')$ with $\{b_1', b_2'\} = \{b_1, b\}$ and $b_1' < b_2'$ then we get a new partition $\sigma'$ with $f(\sigma') < f(\sigma)$. (The contribution of

$(a_1, a_2, b'_1, b'_2)$ to $f(\sigma')$ is $2a_2 + 2b'_1 < 2a_2 + b_1 + b$. Similarly, if there is a cycle of the type $(a_1, b_1, b_2)$ then there are no cycles of length one $(a)$ with $a \in A$.

(3) If $\sigma$ has cycles of length 3 then it has only one, which is of the type $(a_1, b_1, b_2)$, and there are no cycles of length one. Indeed, let $k \geq 0$ be the number of cycles of length 4, $l > 0$ the number of cycles of length 3 and $m \geq 0$ the number of cycles of length one. By (1) all cycles of length 3 are of the same type. Assume that they are of type $(a_1, a_2, b_1)$. By (2) all the cycles of length one are of the form $(a)$ with $a \in A$. Then $|A| = 2k + 2l + m > 2k + l > |B|$. Contradiction. So all cycles of length 3 are of the form $(a_1, b_1, b_2)$ and by (2) they are of the form $(b)$ with $b \in B$. Then $2k + 2l + m = |B| \leq |A| + 1 = 2k + l$, i.e., $l + m \leq 1$ which implies $l = 1$, $m = 0$, as claimed.

(4) If $\sigma$ has no cycles of length 3 then let $k$ be the number of cycles of length 4 and $l, m$ the numbers of cycles of the type $(a)$ with $a \in A$ and $(b)$ with $b \in B$, respectively. We claim that $0 \leq l \leq m \leq 1$. We have $|A| = 2k + l$ and $|B| = 2k + m$, so $0 \leq |B| - |A| \leq 1$ implies $l \leq m \leq l + 1$. Suppose that $l \geq 2$. Then $\sigma$ has some cycles $(a_1), (a_2), (b_1), (b_2)$ with $a_i \in A$, $b_i \in B$ and $a_1 > a_2$, $b_1 < b_2$. When we replace $(a_1), (a_2), (b_1), (b_2)$ in $\sigma$ by $(a_1, a_2, b_1, b_2)$ the sum $f(\sigma)$ decreases, which contradicts the minimality of $f(\sigma)$. (The contribution of $(a_1), (a_2), (b_1), (b_2)$ in $f(\sigma)$ is $a_1 + a_2 + b_1 + b_2$ and the contribution of $(a_1, a_2, b_1, b_2)$ is $2a_2 + 2b_2 < a_1 + a_2 + b_1 + b_2$.) So $l \leq 1$ and $m \leq l + 1$. Suppose that $(l, m) = (1, 2)$. Then let $(a_1), (b_1)(b_2)$ be the cycles of length one of $\sigma$ with $a_1 \in A$, $b_1, b_2 \in B$ and $b_1 < b_2$. When we replace $(a_1), (b_1), (b_2)$ in $\sigma$ by $(a_1, b_1, b_2)$ the sum $f(\sigma)$ decreases, which contradicts the minimality. (The contribution of $(a_1), (b_1), (b_2)$ in $f(\sigma)$ is $a_1 + b_1 + b_2$ and the contribution of $(a_1, b_1, b_2)$ is $a_1 + 2b_2 < a_1 + b_1 + b_2$.) Hence we must have $0 \leq l \leq m \leq 1$.

In conclusion we have 4 possibilities:

$$\begin{aligned}
\sigma &= (a_1^1, a_2^1, b_1^1, b_2^1) \cdots (a_1^k, a_2^k, b_1^k, b_2^k), \\
\sigma &= (a_1^1, a_2^1, b_1^1, b_2^1) \cdots (a_1^k, a_2^k, b_1^k, b_2^k)(b), \\
\sigma &= (a_1^1, a_2^1, b_1^1, b_2^1) \cdots (a_1^k, a_2^k, b_1^k, b_2^k)(a)(b), \\
\sigma &= (a_1^1, a_2^1, b_1^1, b_2^1) \cdots (a_1^k, a_2^k, b_1^k, b_2^k)(a_1, b_1, b_2),
\end{aligned}$$

corresponding to $n = 4k + r$, where $r = 0, 1, 2, 3$, respectively. Note that if $r = 0, 1$ then $A = \{1, \ldots, 2k\}$ and $B = \{2k+1, \ldots, n\}$, and if $r = 2, 3$ then $A = \{1, \ldots, 2k+1\}$ and $B = \{2k + 2, \ldots, n\}$.

When $n = 4k$ we have $f(\sigma) = 2(a_2^1 + \cdots + a_2^k) + 2(b_1^1 + \cdots + b_1^k)$ and the minimum value possible is $2(1 + \cdots + k) + 2(2k + 1 + \cdots + 3k) = 6k^2 + 2k = (3n^2 + 4n)/8$.

When $n = 4k + 1$ we have $f(\sigma) = 2(a_2^1 + \cdots + a_2^k) + 2(b_1^1 + \cdots + b_1^k) + b$ and the minimum value possible is $2(1 + \cdots + k) + 2(2k + 1 + \cdots + 3k) + 3k + 1 = 6k^2 + 5k + 1$ $= (3n^2 + 4n + 1)/8$.

When $n = 4k + 2$ we have $f(\sigma) = 2(a_2^1 + \cdots + a_2^k) + a + 2(b_1^1 + \cdots + b_1^k) + b$ and the minimum value possible is $2(1 + \cdots + k) + k + 1 + 2(2k + 2 + \cdots + 3k + 1) + 3k + 2$ $= 6k^2 + 8k + 3 = (3n^2 + 4n + 4)/8$.

When $n = 4k + 3$ we have $f(\sigma) = 2(a_2^1 + \cdots + a_2^k) + a_1 + 2(b_1^1 + \cdots + b_1^k) + 2b_1$ and the minimum value possible is $2(1 + \cdots + k) + k + 1 + 2(2k + 2 + \cdots + 3k + 2)$ $= 6k^2 + 11k + 5 = (3n^2 + 4n + 4)/8$. □

**387.** Let $a > 0$ and let $(a_n)_{n\geq 0}$ be the sequence defined by $a_0 = 0$ and $a_{n+1} = \sqrt{a + a_n}$ for all $n \geq 1$. Prove that the set of all $n$ such that $a_n \in \mathbb{Q}$ is finite.

`Proposed by Marius Cavachi, Ovidius University of Constanţa, Constanţa, Romania.`

*Solution by the author.* We have $a_n = \sqrt{a + \sqrt{a + \cdots + \sqrt{a}}}$, where the number of radicals is $n$. Assume that there is some infinite set $M \subseteq \mathbb{N}^*$ such that $a_n \in \mathbb{Q}$ $\forall n \in M$.

Let $n_0 \in M$. Then $a$ is a root of the polynomial $(\ldots(a_{n_0}^2 - X)^2 - X)^2 \ldots)^2 - X$, of degree $2^{n_0}$, with rational coefficients, so it is algebraic. Then $a$ writes as $a = \frac{b}{q^2}$, where $q \in \mathbb{N}^*$ and $b$ is an algebraic integer. If $b_n = a_n q$, i.e., $a_n = \frac{b_n}{q}$, then $b_1 = \sqrt{b}$ and $b_{n+1} = \sqrt{b + qb_n}$ so $b_n$ is an algebraic integer by induction on $n$.

If $n \in M$ then $a_n \in \mathbb{Q}$, so $b_n \in \mathbb{Q}$ and since $b_n$ is algebraic we have $b_n \in \mathbb{Z}$. But the sequence $(a_n)_{n\geq 1}$ is strictly increasing an bounded and so $(b_n = qa_n)_{n\in M}$ is a sequence of integers that is both strictly increasing and bounded. But this is impossible. $\square$

**388.** Let $a, b, c \in (0, 1)$ real numbers. Prove the following inequality

$$\sum_{\text{cyc}} \frac{1}{1 - a^4} + \sum_{\text{cyc}} \frac{1}{1 - a^2bc} \geq \sum_{\text{sym}} \frac{1}{1 - a^3b}.$$

`Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, USA, and Ştefan Spătaru, International Computer High School of Bucharest, Bucharest, Romania.`

*Solution by V. Makanin, Sankt Petersburg, Russia.* The inequality

$$\sum_{\text{cyc}} x^4 + \sum_{\text{cyc}} x^2yz \geq \sum_{\text{sym}} x^3y$$

is well-known as Schur's fourth degree inequality (it is equivalent to $\sum_{\text{cyc}} x^2(x-y)(x-z) \geq 0$, in which form it can be easily proved). Now replace $x$, $y$, and $z$ with $a^n$, $b^n$, and $c^n$, respectively, then sum over all integers $n \geq 0$. Finally use the development

$$\sum_{n\geq 0} t^n = \frac{1}{1-t}$$

for every $t \in (0, 1)$ in order to obtain the desired inequality. $\square$

**389.** Let $\mathcal{F}$ be the real vector space of the continuous functions $f : [0, 1] \to \mathbb{R}$. We consider on $\mathcal{F}$ the distance $\mathrm{d}(f, g) = \int_0^1 |f(x) - g(x)|\mathrm{d}x$.

a) Show that the intersection of any affine line directed by a nowhere zero function with any sphere consists of at most two points.

b) Does this property hold for every affine line in $\mathcal{F}$?

`Proposed by Gabriel Mincu, University of Bucharest, Bucharest, Romania.`

*Solution by the author.* a) Let $l = \{h - \lambda g : \lambda \in \mathbb{R}\} \subset \mathcal{F}$ be the affine line directed by the nowhere zero function $g : [0,1] \to \mathbb{R}$ and containing the "point" $h$. We may, without loss of generality, consider that $g(x) > 0$ for all $x \in [0,1]$.

Let $\mathcal{S}$ be the sphere $\{f \in \mathcal{F} :\ \mathrm{d}(f, \varphi) = r\}$. The intersection $l \cap \mathcal{S}$ consists of the functions $h - \lambda g$ with the property $\int_0^1 |(h(x) - \lambda g(x)) - \varphi(x)|\mathrm{d}x = r$; putting $f = h - \varphi$, it is enough to prove that $\int_0^1 |f(x) - \lambda g(x)|\mathrm{d}x$ cannot take the value $r$ for more than two values of $\lambda$.

Let $F : \mathbb{R} \to \mathbb{R}$, $F(\lambda) = \int_0^1 |f(x) - \lambda g(x)|\,\mathrm{d}x$. $F$ is obviously continuous.

Let $m = \inf\limits_{x\in[0,1]} \frac{f(x)}{g(x)}$ and $M = \sup\limits_{x\in[0,1]} \frac{f(x)}{g(x)}$. Then for $\lambda \le m$ we have

$$F(\lambda) = \int_0^1 (f(x) - \lambda g(x))\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x - \lambda \int_0^1 g(x)\,\mathrm{d}x,$$

so $F$ is strictly decreasing on $(-\infty, m]$.

On the other hand, if $\lambda \ge M$ then

$$F(\lambda) = \int_0^1 (\lambda g(x) - f(x))\,\mathrm{d}x = \lambda \int_0^1 g(x)\,\mathrm{d}x - \int_0^1 f(x)\,\mathrm{d}x,$$

so $F$ is strictly increasing on $[M, \infty)$.

Moreover, given $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\alpha \in [0,1]$,

$$F(\alpha\lambda_1 + (1-\alpha)\lambda_2) = \int_0^1 |f(x) - (\alpha\lambda_1 + (1-\alpha)\lambda_2)g(x)|\,\mathrm{d}x$$

$$= \int_0^1 |\alpha(f(x) - \lambda_1 g(x)) + (1-\alpha)(f(x) - \lambda_2 g(x))|\,\mathrm{d}x \le \alpha F(\lambda_1) + (1-\alpha)F(\lambda_2),$$

so $F$ is convex.

We claim that if $m \le \lambda_1 < \lambda_2 \le M$ then the above inequality is strict, so that $F$ is strictly convex on $[m, M]$. To do so it is enough to prove that there is a subinterval of $[0,1]$ where $f(x) - \lambda_1 g(x)$ and $f(x) - \lambda_2 g(x)$ have opposite signs, so that the inequality $|\alpha(f(x) - \lambda_1 g(x)) + (1-\alpha)(f(x) - \lambda_2 g(x))| \ge |\alpha(f(x) - \lambda_1 g(x))| + |(1-\alpha)(f(x) - \lambda_2 g(x))|$ is strict.

If $m = M$, we have nothing to prove. If $m < M$, let $m \le \lambda_1 < \lambda_2 \le M$, $\alpha \in [0,1]$, and $A = \sup\limits_{x\in[0,1]} \{f(x) - \lambda_1 g(x)\} > 0$. (We have $\lambda_1 < M = \sup_{x\in[0,1]} \frac{f(x)}{g(x)}$.) Since $f - \lambda_1 g$ is continuous, it reaches its bounds on [0,1]. Consequently, there exists $a \in [0,1]$ such that $f(a) - \lambda_1 g(a) = A > 0$. Since $\lambda_1 \in [m, M)$, there are points $x \in [0,1]$ such that $f(x) - \lambda_1 g(x) = 0$. Let us suppose for instance there is $x > a$ such that $f(x) - \lambda_1 g(x) = 0$. Let then $b = \inf\{x > a : f(x) - \lambda_1 g(x) = 0\}$. Continuity implies $f(b) - \lambda_1 g(b) = 0$, and we have $f(x) - \lambda_1 g(x) > 0$ for all $x \in [a, b)$.

But $f(b) - \lambda_2 g(b) < f(b) - \lambda_1 g(b) = 0$, so $f(x) - \lambda_2 g(x)$ will be negative on a whole open interval $J$ that contains $b$. Consequently, $f(x) - \lambda_1 g(x)$ and $f(x) - \lambda_2 g(x)$ have opposite signs on $J \cap [a, b)$, as desired.

Hence $F$ is convex on $\mathbb{R}$, strictly deacreasing on $(-\infty, m]$, strictly convex on $[m, M]$ and strictly increasing on $[M, \infty)$. It follows that $F$ is strictly decreasing on the interval $(-\infty, T]$ and strictly increasing on the interval $[T, \infty)$ for some $T \in [m, M]$. Therefore, no real value can be taken more than twice by $F$, and this closes our argument.

b) No, it doesn't. Take for instance the line $L$ passing through the constant function $f(x)=1$ and directed by $g(x)=\sin 2\pi x$. Then

$$\int_0^1 (f(x)-\lambda g(x))\,\mathrm{d}x = 1+\lambda\int_0^1 \sin 2\pi x\,\mathrm{d}x = 1 \quad \text{for all} \quad \lambda\in[-1,1].$$

This means that the intersection of $L$ with the unit sphere $S(0,1)$ contains the whole line segment of $L$ corresponding to $\lambda\in[-1,1]$. □

**390.** Let $a_0,\dots,a_n\in\mathbb{C}$ be pairwise different. Solve the linear system of equations

$$\sum_{j=0}^{n} a_j^k z_j = \begin{cases} 0 & \text{if } k=0,\dots,n-1,\\ 1 & \text{if } k=n.\end{cases}$$

`George Stoica, Department of Mathematical Sciences, University of New Brunswick, Canada`

*Solution by the author.* The vector $(z_1,\dots,z_n)\in\mathbb{C}^{n+1}$ is uniquely determined since the Vandermonde determinant does not vanish for pairwise different $a_j$. Furthermore, if

$$P(z)=\prod_{j=0}^{n}(z-a_j),$$

then the residue theorem implies that

$$\sum_{j=0}^{n} P'(a_j)^{-1}a_j^k = \frac{1}{2\pi i}\lim_{M\to\infty}\int_{|z|=M}\frac{z^k}{P(z)}\,\mathrm{d}z = \begin{cases} 0 \text{ if } k=0,\dots,n-1,\\ 1 \text{ if } k=n.\end{cases}$$

Hence we have the solution,

$$z_j = P'(a_j)^{-1} = \prod_{k=0,k\neq j}^{n}(a_j-a_k)^{-1}.$$

*Solution by V. Makanin, Sankt Petersburg, Russia.* Just apply Cramer's formulæ to write $z_j$ as a fraction which has $\det(a_k^l)_{0\le k,l\le n}$ as a denominator and at the numerator a determinant which developed after the column $(0,\dots,0,1)$ gives $(-1)^{n+j}\det(a_k^l)_{k\neq j,l\le n-1}$. We have two Vandermonde determinants of dimensions $n+1\times n+1$ and $n\times n$, respectively. After simplifying one gets

$$z_j = \frac{(-1)^{n+j}}{\prod\limits_{0\le k<j}(a_j-a_k)\prod\limits_{j<l\le n}(a_l-a_j)} = \frac{1}{\prod\limits_{0\le i\le n,\, i\neq j}(a_j-a_i)}, \quad j=0,\dots,n.$$

Or, check that these are the solutions, namely verify that the identities

$$\sum_{j=0}^{n}\frac{a_j^k}{\prod\limits_{0\le i\le n,\, i\neq j}(a_j-a_i)} = \begin{cases} 0, & \text{if } k=0,\dots,n-1,\\ 1, & \text{if } k=n,\end{cases}$$

hold true, then use the uniqueness of the solution of the system (which has a Vandermonde nonzero determinant). For example, one could write down Lagrange's interpolation formula for the polynomial $x^k$ $(0 \le k \le n)$, namely

$$x^k = \sum_{j=0}^{n} a_j^k \prod_{0\le i\le n,\ i\ne j} \frac{x-a_i}{a_j-a_i},$$

then notice that

$$\sum_{j=0}^{n} \frac{a_j^k}{\prod\limits_{0\le i\le n,\ i\ne j}(a_j-a_i)}$$

is precisely the coefficient of $x^n$ in $x^k$ (that is, 0 for $k<n$, and 1 for $k=n$). Anyway, the above identities are well-known (sometimes they are called Euler's identities, if I am not mistaken). □

**391.** Let $x_n$ be the sequence defined by $x_1 = 1$, $x_{n+1} = p_{x_n}$, where $p_n$ is the $n$th prime number. Determine the assimptotic behaviour of $\sqrt[n]{x_n}$, i.e., find a function $f$ such that $\sqrt[n]{x_n} \sim f(n)$ as $n \to \infty$.

`Proposed by Constantin-Nicolae Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.`

*Solution by the author.* We first make some easy remarks. Assume that $a_n, b_n \to \infty$ as $n\to\infty$. If $a_n - b_n \to 0$ then $\frac{a_n}{b_n} - 1 \to 0$, so $a_n \sim b_n$. Also if $a_n \sim b_n$, i.e., if $\frac{a_n}{b_n} \to 1$, then $\log a_n - \log b_n = \log\frac{a_n}{b_n} \to 0$, which implies $\log a_n \sim \log b_n$.

Obviously the sequence $x_n$ is increasing, so that $x_n \to \infty$ as $n\to\infty$. From $x_{n+1} = p_{x_n}$ we get $x_n = \pi(x_{n+1})$, which, by the Prime Number Theorem, implies $x_n \sim x_{n+1}/\log x_{n+1}$. By taking logaritms one gets $\log x_n - (\log x_{n+1} - \log\log x_{n+1}) \to 0$. Hence if $y_n := \log x_n$ then $y_{n+1} - y_n - \log y_{n+1} \to 0$. This implies $y_{n+1} - y_n \sim \log y_{n+1}$ and also $y_{n+1} \sim y_{n+1} - \log y_{n+1} \sim y_n$.

Let $f(x) = \frac{x}{\log x}$. We have $f'(x) = \frac{1}{\log x} - \frac{1}{\log^2 x} \sim \frac{1}{\log}x$. By the Mean Value Theorem there is some $\xi_n \in (y_n, y_{n+1})$ such that $f(y_{n+1}) - f(y_n) = (y_{n+1}-y_n)f'(\xi_n)$. But $y_{n+1} \sim y_n$, so $\xi_n \in (y_n, y_{n+1})$ implies that $y_{n+1} \sim \xi_n$ and so $\log y_{n+1} \sim \log \xi_n$. In conclusion,

$$f(y_{n+1}) - f(y_n) = (y_{n+1}-y_n)f'(\xi_n) \sim \log y_{n+1} \cdot \frac{1}{\log \xi_n} \sim 1.$$

By the Stolz-Cesáro theorem, from $\frac{f(y_{n+1})-f(y_n)}{n+1-n} \to 1$ we get $\frac{f(y_n)}{n} \to 1$, i.e. $y_n/\log y_n = f(y_n) \sim n$. This implies that $\log y_n \sim \log y_n - \log\log y_n = \log(y_n/\log y_n) \sim \log n$. It follows that $y_n/\log n \sim y_n/\log y_n \sim n$, so $y_n \sim n\log n$.

We now use the root criterion for sequences to calculate $\lim\limits_{n\to\infty} \frac{\sqrt[n]{x_n}}{n\log n}$. We have $\frac{\sqrt[n]{x_n}}{n\log n} = \sqrt[n]{z_n}$, where $z_n = \frac{x_n}{n^n \log^n n}$.

We have

$$\frac{z_{n+1}}{z_n} = \frac{x_{n+1}}{x_n} \frac{n^n \log^n n}{(n+1)^{n+1}\log^{n+1}(n+1)}.$$

As $x_n \sim \frac{x_{n+1}}{\log x_{n+1}}$, it follows $\frac{x_{n+1}}{x_n} = \log x_{n+1} = y_{n+1} \sim (n+1)\log(n+1)$. Hence $\frac{z_{n+1}}{z_n} \sim \left(\frac{n}{n+1}\right)^n \left(\frac{\log n}{\log(n+1)}\right)^n$. But $\left(\frac{n}{n+1}\right)^n \to \frac{1}{e}$ and $\lim\limits_{n\to\infty}\left(\frac{\log n}{\log(n+1)}\right)^n = e^L$, where $L = \lim\limits_{n\to\infty} n\log\left(\frac{\log n}{\log(n+1)}\right)$. We have $\frac{\log(n+1)}{\log n} = 1+\varepsilon_n$ where $\varepsilon_n = \frac{\log(n+1)-\log n}{\log n}$. But $\log(n+1) - \log n = \log\left(1+\frac{1}{n}\right) \sim \frac{1}{n}$, so $\varepsilon_n \sim \frac{1}{n\log n}$. Since $\varepsilon_n \to 0$ we have

$\log(1+\varepsilon_n) \sim \varepsilon_n \sim \frac{1}{n\log n}$. It follows that $n\log\left(\frac{\log n}{\log(n+1)}\right) = -n\log(1+\varepsilon_n) \sim -\frac{1}{\log n}$. Thus $L = \lim\limits_{n\to\infty} n\log\left(\frac{\log n}{\log(n+1)}\right) = 0$, whence $\left(\frac{\log n}{\log(n+1)}\right)^n \to e^0 = 1$.

In conclusion,

$$\lim_{n\to\infty} \frac{\sqrt[n]{x_n}}{n\log n} = \lim_{n\to\infty} \sqrt[n]{z_n} = \lim_{n\to\infty} \frac{z_{n+1}}{z_n} = \frac{1}{e}\cdot 1 = \frac{1}{e},$$

whence $\sqrt[n]{x_n} \sim \frac{n\log n}{e}$. □

**392.** We consider on $\mathbb{R}^n$ the following norm

$$x = (x_1, x_2, \ldots, x_n) \mapsto \|x\| = \sum_{k=1}^{n} |x_k|.$$

a) Prove that if $n = 2$ then the norm $\|\cdot\|$ has the property

**(M)** For all $z, z', d \in \mathbb{R}^2$, $\|z\| > \|z-d\|$ and $\|z'\| > \|z'-d\|$ imply $\|z+z'\| > \|z+z'-d\|$.

b) Does the norm $\|\cdot\|$ have the property (M) if $n = 3$?

Proposed by Gheorghiţă Zbăganu, University of Bucharest, Bucharest, Romania.

*Solution by the author.* a) For $d \in \mathbb{R}^2$ we set $H_d = \{z \in \mathbb{R}^n | \ \|z\| > \|z-d\|\}$.

Let $z = (x, y)$ and $d = (a, b) \in \mathbb{R}^2$. Then $z \in H_d \Leftrightarrow |x| + |y| > |x-a| + |y-b|$.

If $b = 0$ then $z \in H_d$ iff $|x| > |x-a|$. When $a > 0$ this is equivalent to $x \in \left(\frac{a}{2}, +\infty\right)$, while if $a < 0$ this is equivalent to $x \in \left(-\infty, \frac{a}{2}\right)$.

If, for instance, $a > 0$ then $z = (x, y), z' = (x', y') \in H_d$ implies $x, x' > \frac{a}{2}$ and so $x + x' > a > \frac{a}{2}$. Hence $z + z' \in H_d$.

The case $a < 0$ is similar. If $a = 0$, then clearly one has $H_d = \emptyset$.

Thus, (M) holds whenever $d = (a, 0)$ with $a \in \mathbb{R}$. With a similar proof, (M) holds when $d = (0, b)$ with $b \in \mathbb{R}$.

Assume now that $a \neq 0$, $b \neq 0$. We write $x = as$ and $y = bt$. Then $z \in H_d$ iff $|as| + |bt| > |as-a| + |bt-b|$, which can be written as $|t| - |t-1| + \lambda(|s| - |s-1|) > 0$, where $\lambda = \left|\frac{a}{b}\right|$.

Let $f : \mathbb{R} \to \mathbb{R}$, $f(s) = |s| - |s-1|$. Note that $f$ is nondecreasing and $|f(s)| \leq 1$ for all $s \in \mathbb{R}$.

Set $A_\lambda = \{(s,t) \in \mathbb{R}^2 | f(t) + \lambda f(s) > 0\}$. Then, $z = (as, bt) \in H_d$ if and only if $(s, t) \in A_\lambda$.

Now let $z = (x, y), z' = (x', y') \in H_d$, $x = as, y = bt, x' = as', y' = bt'$. We then have $(s,t), (s',t') \in A_\lambda$, so $f(t) + \lambda f(s) > 0$ (thus $f(t) > 0$ or $f(s) > 0$) and $f(t') + \lambda f(s') > 0$ (thus $f(t') > 0$ or $f(s') > 0$).

**Case I.** $f(t) > 0$ and $f(t') > 0$. Then $t > \frac{1}{2}$, $t' > \frac{1}{2}$, $f(t+t') \geq f(t)$ and $f(t+t') \geq f(t')$. If $s \geq 0$, then $f(s+s') \geq f(s')$. In this case we get $f(t+t') + \lambda f(s+s') \geq f(t') + \lambda f(s') > 0$. Analogously we get $f(t+t') + \lambda f(s+s') \geq f(t) + \lambda f(s) > 0$ for $s' \geq 0$. In the remaining case $s < 0$ and $s' < 0$ we have $f(s) = f(s') = f(s+s') = -1$, and therefore $f(t) > \lambda$, $f(t') > \lambda$. It follows now that $f(t+t') \geq f(t) > \lambda$, so $(s+s', t+t') \in A_\lambda$ and consequently $z + z' \in H_d$.

**Case II.** $f(s) > 0$ and $f(s') > 0$. We put $f(\tau) + \lambda f(\sigma) = \lambda\left(f(\sigma) + \frac{1}{\lambda} f(\tau)\right)$ and apply Case I.

**Case III.** $f(t) > 0$ and $f(s') > 0$. If $f(t') \neq -1$ or $f(s) \neq -1$, then $t' > 0$ or $s > 0$ and we may again apply the reasoning in Case I to get the conclusion. If $f(t') = -1 = f(s)$, then $f(t) > \lambda$ and $f(s') > \frac{1}{\lambda}$, so $f(t)f(s') > 1$, which is impossible.

**Case IV.** $f(s) > 0$ and $f(t') > 0$. This case is similar to Case III.

Thus the relation (M) is now completely proved.

b) The following example shows that relation (M) does not hold on $\mathbb{R}^3$.

For $w = (1, -1, 1), w' = (1, 1, -1)$ and $a = (1, 1, 1) \in \mathbb{R}^3$ we clearly have $\|w\| = 3 > 2 = \|w - a\|$; $\|w'\| = 3 > 2 = \|w' - a\|$, but $\|w + w'\| = 2 < 3 = \|w + w' - a\|$. $\square$