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Bounds for polynomial roots using powers of the generalized Frobenius companion matrix

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Abstract. Let $f \in \mathbb{C}[X]$ with $\deg f \geq 2$. If A is the generalized Frobenius companion matrix of f, we apply several matrix inequalities to A^2 and A^3 to derive new bounds for the roots of f.

Keywords: bounds, polynomial roots, companion matrix.

MSC: 12D10

1. **Introduction**

Let the polynomial

$$f_n(x) = x^n - a_1 x^{n-1} - a_2 x^{n-2} - \cdots - a_{n-1}x - a_n \in \mathbb{C}[X]$$

with $a_n \neq 0$. We suppose that there are complex numbers $b_1, b_2, \ldots b_{n-1}$, $c_1, c_2, \ldots, c_n$ such that

$$a_1 = c_1, a_2 = c_2 b_1, a_3 = c_3 b_1 b_2, \ldots, a_n = c_n b_1 b_2 \cdots b_{n-1} \tag{1.1}$$

and consider the matrix

$$A = \begin{pmatrix} 0 & b_{n-1} & 0 & \cdots & 0 \\ 0 & 0 & b_{n-2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_1 \end{pmatrix} \in M_n(\mathbb{C}). \tag{1.2}$$

Then we have the equality (see [5], p. 43)

$$f_n(x) = \det(xI_n - A),$$

which shows that the roots of the polynomial f_n are exactly the eigenvalues of matrix A.

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Type (1.1) decompositions are always possible. The simplest decomposition is obtained when we choose

$$b_1 = b_2 = \cdots = b_{n-1} = 1, \quad c_k = a_k,\ k = \overline{1,n},$$

and in this case the matrix A is the classical Frobenius companion matrix. In what follows we call A given in (1.2) *the generalized Frobenius companion matrix corresponding to decomposition* (1.1).

Using the standard notation, for $A \in M_n(\mathbb{C})$ we denote by $\sigma(A)$, $r(A)$, $\|A\|$ the spectrum, the spectral radius, and the spectral norm of A, respectively. We recall the following well-known properties of the spectrum, spectral radius, and spectral norm of A (see, e.g., [2]):

$$\begin{aligned} \sigma(A) &= \{\lambda \in \mathbb{C} : \lambda \text{ is eigenvalue of } A\}, \\ r(A) &= \max\{|\lambda| : \lambda \in \sigma(A)\}, \\ \|A\| &= \max\left\{\sqrt{\lambda} : \lambda \in \sigma(A^*A)\right\} = r(A^*A)^{1/2}, \end{aligned}$$

where A^* is the Hermitian adjoint of A. From matrix analysis we have the well known inequality

$$r(A) \leq \|A\|.$$

We need the following result due to Kittaneh ([4], p. 602):

Lemma 1. *Let $A \in M_n(\mathbb{C})$ be partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{ij} is an $n_i \times n_j$ matrix with $n_i + n_j = n$. If

$$\tilde{A} = \begin{pmatrix} \|A_{11}\| & \|A_{12}\| \\ \|A_{21}\| & \|A_{22}\| \end{pmatrix}$$

then we have the inequalities

$$r(A) \leq r(\tilde{A}),$$

$$\|A\| \leq \left\|\tilde{A}\right\|.$$

2. Main Results

In what follows we use A^m, where $m \in \{2, 3\}$, to give new bounds for the roots of f.

Case $m = 2$. Let

$$A = \begin{pmatrix} 0 & b_{n-1} & 0 & \cdots & 0 \\ 0 & 0 & b_{n-2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_1 \end{pmatrix}$$

be the generalized Frobenius companion matrix corresponding to decomposition (1.1). Calculating, for $n \geq 3$ we obtain

$$A^2 = \begin{pmatrix} 0 & 0 & b_{n-2}b_{n-1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & b_{n-3}b_{n-2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_1c_n & b_1c_{n-1} & b_1c_{n-2} & b_1c_{n-3} & b_1c_{n-4} & \cdots & b_1c_1 \\ \alpha_n & \alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} & \alpha_{n-4} & \cdots & \alpha_1 \end{pmatrix},$$

where $\alpha_k = c_1c_k + b_kc_{k+1}$, $k = \overline{1,n}$, and $b_n = c_{n+1} = 0$.

We write A^2 as a sum of three matrices

$$A^2 = R + S + T,$$

where

$$R = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \\ \alpha_n & \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \\ b_1c_n & b_1c_{n-1} & b_1c_{n-2} & \cdots & b_1c_1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$T = \begin{pmatrix} 0 & B_{n-2} \\ 0 & 0 \end{pmatrix},$$

and $B_{n-2} = \operatorname{diag}(b_{n-1}b_{n-2}, \ldots, b_2b_1) \in M_{n-2}(\mathbb{C})$.

An elementary calculation shows that we have

$$R^*S = R^*T = S^*R = S^*T = T^*R = T^*S = 0. \tag{2.3}$$

From (2.3) we find

$$\begin{aligned} \left\|A^2\right\|^2 &= \left\|\left(A^2\right)^* A^2\right\| \\ &= \left\|(R+S+T)^* \left(R+S+T\right\|\right) \\ &= \left\|R^*R + S^*S + T^*T\right\| \\ &\leq \left\|R^*R\right\| + \left\|S^*S\right\| + \left\|T^*T\right\|. \end{aligned}$$

We calculate and get

$$\|R^*R\| = |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2,$$

$$\|S^*S\| = |b_1|^2 \left(|c_1|^2 + |c_2|^2 + \cdots + |c_n|^2\right),$$

$$\|T^*T\| = \max\left(|b_1b_2|^2, |b_2b_3|^2, \ldots, |b_{n-2}b_{n-1}|^2\right).$$

Using these relations we find the inequality

$$\begin{aligned} \left\|A^2\right\|^2 & \leq \max\left(\left|b_1 b_2\right|^2, \left|b_2 b_3\right|^2, \ldots, \left|b_{n-2} b_{n-1}\right|^2\right) \\ & + \sum_{j=1}^{n}\left(\left|\alpha_j\right|^2 + \left|b_1\right|^2 \cdot \left|c_j\right|^2\right). \end{aligned} \tag{2.4}$$

Since for every root z of the polynomial f we have the inequality

$$|z| \leq \left\|A^2\right\|^{1/2}, \tag{2.5}$$

the next theorem follows from (2.4) and (2.5).

Theorem 2. *For every root z of the polynomial f we have the inequality*

$$|z| \leq \left(\max(\left|b_1 b_2\right|^2, \ldots, \left|b_{n-2} b_{n-1}\right|^2) + \sum_{j=1}^{n}\left(\left|\alpha_j\right|^2 + \left|b_1\right|^2 \cdot \left|c_j\right|^2\right)\right)^{\frac{1}{4}}.$$

Another way to establish new bounds for roots using A^2 is to partition this matrix for $n \geq 4$. We choose $b_1 = b_2 = \cdots = b_{n-1} = b > 0$ and partition A^2 as

$$A^2 = \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix},$$

where

$$A_{11} = \begin{pmatrix} 0 & 0 & b^2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & b^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ bc_n & bc_{n-1} & bc_{n-2} & bc_{n-3} & \cdots & bc_1 \end{pmatrix} \in M_{n-1}(\mathbb{C}),$$

$$A_{12} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b^2 \\ bc_1 \end{pmatrix} \in M_{n1}(\mathbb{C}),$$

$$A_{21} = (\alpha_n, \alpha_{n-1}, \ldots, \alpha_2) \in \mathbb{C}^{n-1},$$

$$A_{22} = \alpha_1 \in \mathbb{C}.$$

From Lemma 1 we obtain

$$\begin{aligned} r(A^2) & \leq r\begin{pmatrix} \|A_{11}\| & \|A_{12}\| \\ \|A_{21}\| & \|A_{22}\| \end{pmatrix} = \\ & = \frac{1}{2}\left(\|A_{11}\| + \|A_{22}\| + \sqrt{(\|A_{11}\| - \|A_{22}\|)^2 + 4\|A_{12}\| \cdot \|A_{21}\|}\right). \end{aligned} \tag{2.6}$$

Next we need to evaluate all the above norms. Obviously, we have:

$$\|A_{12}\| = b\sqrt{b^2 + |c_1|^2}, \tag{2.7}$$

$$\|A_{21}\| = \sqrt{|\alpha_2|^2 + |\alpha_3|^2 + \cdots + |\alpha_n|^2}, \tag{2.8}$$

$$\|A_{22}\| = |\alpha_1|\,. \tag{2.9}$$

We have a little bit more work to do in order to evaluate $\|A_{11}\|$. If we have a Hermitian matrix $A_1 = (a_{ij}) \in M_n(\mathbb{C})$ written in a partitioned form as follows

$$A_1 = \begin{pmatrix} \widetilde{A_1} & x \\ x^* & a_{nn} \end{pmatrix},$$

where $x \in \mathbb{C}^{n-1}$, x^* is the hermitian adjoint of x and $\widetilde{A_1} \in M_{n-1}(\mathbb{C})$, then we find

$$\det A_1 = a_{nn} \det \widetilde{A_1} - x^* \left(\operatorname{adj} \widetilde{A_1}\right) x, \tag{2.10}$$

where $\operatorname{adj} \widetilde{A_1}$ is the classical adjoint of $\widetilde{A_1}$ (see [2], p. 175).

Using successively relation (2.10) and applying recursive reasoning, we obtain

$$\begin{aligned} &\det(\lambda I_{n-1} - A_{11}A_{11}^*) = \\ &\quad = \lambda\left(\lambda - b^4\right)^{n-4} \cdot \left[\lambda^2 - b^2\left(b^2 + \alpha\right)\lambda + b^6\left(|c_{n-1}|^2 + |c_n|^2\right)\right], \end{aligned}$$

where

$$\alpha = \sum_{j=2}^{n} |c_j|^2\,.$$

From the last equation we immediately obtain

$$\|A_{11}\|^2 = \frac{1}{2}\left[b^2\left(b^2 + \alpha\right) + b^2\sqrt{\left(b^2 + \alpha\right)^2 - 4b^2\left(|c_{n-1}|^2 + |c_n|^2\right)}\right]. \tag{2.11}$$

We are able to give the next theorem.

Theorem 3. *If z is a root of f, $\beta = \|A_{11}\|$, and $\gamma = \left(\sum_{j=2}^{n} |\alpha_j|^2\right)^{1/2}$, then we have the inequality*

$$|z| \leq \left[\frac{1}{2}\left(|\alpha_1| + \beta + \sqrt{(|\alpha_1| - \beta)^2 + 4\gamma b\sqrt{b^2 + |c_1|^2}}\right)\right]^{1/2}.$$

Proof. Within relation (2.6) we replace the norms provided by (2.7), (2.8), (2.9), (2.11) and use inequality (2.5). □

Case $m = 3$. Let us now consider the third power of the matrix A. Basic computations will show that for $n \geq 4$ we have

$$A^3 = \begin{pmatrix} 0 & 0 & 0 & b_{n-3}b_{n-2}b_{n-1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & b_{n-4}b_{n-3}b_{n-2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b_1b_2c_n & b_1b_2c_{n-1} & b_1b_2c_{n-2} & b_1b_2c_{n-3} & b_1b_2c_{n-4} & \cdots & b_1b_2c_1 \\ b_1b_2\alpha_n & b_1b_2\alpha_{n-1} & b_1b_2\alpha_{n-2} & b_1b_2\alpha_{n-3} & b_1b_2\alpha_{n-4} & \cdots & b_1b_2\alpha_1 \\ \beta_n & \beta_{n-1} & \beta_{n-2} & \beta_{n-3} & \beta_{n-4} & \cdots & \beta_1 \end{pmatrix}$$

where

$$\begin{aligned} \alpha_k &= c_1c_k + b_kc_{k+1}, \\ \beta_k &= b_1c_2c_k + b_kb_{k+1}c_{k+2} + c_1\alpha_k \end{aligned}$$

for $k = \overline{1, n}$, and $b_n = b_{n+1} = c_{n+1} = c_{n+2} = 0$.

We write A^3 as a sum of four matrices

$$A^3 = M + N + P + Q, \tag{2.12}$$

where

$$M = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \beta_n & \beta_{n-1} & \beta_{n-2} & \cdots & \beta_1 \end{pmatrix} \in M_n(\mathbb{C}),$$

$$N = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ b_1\alpha_n & b_1\alpha_{n-1} & b_1\alpha_{n-2} & \cdots & b_1\alpha_1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in M_n(\mathbb{C}),$$

$$P = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \\ b_1b_2c_n & b_1b_2c_{n-1} & b_1b_2c_{n-2} & \cdots & b_1b_2c_1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in M_n(\mathbb{C}),$$

$$Q = \begin{pmatrix} 0 & T_{n-3} \\ 0 & 0 \end{pmatrix} \in M_n(\mathbb{C}),$$

and $T_{n-3} = \operatorname{diag}(b_{n-3}b_{n-2}b_{n-1}, b_{n-4}b_{n-3}b_{n-2}, \ldots, b_1b_2b_3) \in M_{n-3}(\mathbb{C})$.

We have the equalities

$$M^*N = M^*P = M^*Q = N^*P = N^*Q = P^*Q = 0. \tag{2.13}$$

Using (2.12) and (2.13), we find

$$\begin{aligned}\left\|A^3\right\|^2 &= \left\|(M+N+P+Q)^*(M+N+P+Q)\right\| \\ &= \left\|M^*M + N^*N + P^*P + Q^*Q\right\| \\ &\le \left\|M^*M\right\| + \left\|N^*N\right\| + \left\|P^*P\right\| + \left\|Q^*Q\right\|.\end{aligned}$$

Next we attempt to evaluate the four norms involving the matrices M, N, P, and Q that appear in the last inequality above. We have

$$\begin{aligned}\left\|M^*M\right\| &= |\beta_1|^2 + |\beta_2|^2 + \cdots + |\beta_n|^2, \\ \left\|N^*N\right\| &= |b_1|^2\left(|\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2\right), \\ \left\|P^*P\right\| &= |b_1b_2|^2\left(|c_1|^2 + |c_2|^2 + \cdots + |c_n|^2\right), \\ \left\|Q^*Q\right\| &= \max\left\{|b_kb_{k+1}b_{k+2}| \,:\, 1 \le k \le n-3\right\}.\end{aligned}$$

From the last four equalities we deduce now

$$\left\|A^3\right\|^2 \le \max_{1\le k\le n-3}\left\{|b_kb_{k+1}b_{k+2}|\right\} + \sum_{k=1}^{n}\left(|\beta_k|^2 + |b_1|^2|\alpha_k|^2 + |b_1b_2|^2|c_k|^2\right).$$

Since for every root z of f we have the inequality

$$|z| \le \left\|A^3\right\|^{1/3},$$

we have proved the next theorem:

Theorem 4. *For every root z of f, we have the inequality*

$$|z| \le \left(\max_{1\le k\le n-3}\left(|b_kb_{k+1}b_{k+2}|\right) + \sum_{k=1}^{n}\left(|\beta_k|^2 + |b_1|^2|\alpha_k|^2 + |b_1b_2|^2|c_k|^2\right)\right)^{1/6}.$$

3. Applications

1) Let the polynomial $f(x) = x^5 + x^4 - 2x^2 + 1$. Using the package `Mathematica`, we find that the roots of f (rounded to 6 digits) are

$$\begin{aligned}z_1 &= -0.915974 - 1.071789i, \\ z_2 &= \overline{z}_1, \\ z_3 &= -0.733892, \\ z_4 &= 0.782920 - 0.269331i, \\ z_5 &= \overline{z}_4,\end{aligned}$$

therefore $\max\{|z| : k = 1, \ldots, 5\} \approx 1.4099$. If we choose

$$b_1 = \cdots = b_4 = b = \max\left\{|a_k|^{1/k} : k = 2,3,4,5\right\} \text{ and } c_k = \frac{a_k}{b^{k-1}}$$

and use Theorem 1, we obtain that for every root z of f holds the inequality

$$|z| \leq \sqrt[4]{12.23355} \approx 1.87.$$

Applying Theorem 1 with $b_1 = b_2 = \cdots = b_{n-1} = 1$ we obtain Corollary 2.2 from [4], which, applied to f, gives the weaker bound

$$|z| \leq \sqrt[4]{18} \approx 2.0598.$$

We observe that if we apply Theorem 3 we find the bound

$$|z| \leq \sqrt[6]{19.31725} \approx 1.638,$$

which is better than the bound given by Theorem 1.

2) Let the polynomials

$$\begin{aligned}
f_1 &= x^5 + 2x^4 + 3x^3 - x - 1, \\
f_2 &= x^4 - 2x^3 + 4x^2 - x + 1, \\
f_3 &= x^6 + 2x^2 + x + 1, \\
f_4 &= x^5 - 4x^4 - 3x^3 - 2x + 1, \\
f_5 &= x^5 - 2x^4 + 3x^3 - 2x + 1, \\
f_6 &= x^4 - 2x^3 - 3x^2 - 4x + 1.
\end{aligned}$$

In the next table we denote by M_1, M_2, and M the bound (correct to 3 digits) given by Theorem 1, Theorem 3 (both applied with $b_1 = \cdots = b_{n-1} = b = \max\{|a_k|^{1/k} : k = 2,3,4,5\}$ and $c_k = \dfrac{a_k}{b^{k-1}}$) and the maximum modulus of the roots, respectively.

Polynomial	M_1	M_2	M
f_1	2.565	2.298	1.655
f_2	2.791	2.575	1.860
f_3	1.638	1.550	1.305
f_4	4.675	4.671	4.661
f_5	2.580	2.305	1.691
f_6	2.781	2.715	2.648

We remark that in the case of f_4 the bound M_2 is very close to M.

We compare now the bound M_2 with some classical bounds. Let

$$M_C = 1 + \max\left(\left|\frac{a_0}{a_n}\right|, \left|\frac{a_1}{a_n}\right|, \dots, \left|\frac{a_{n-1}}{a_n}\right|\right) \text{ (Cauchy's bound),}$$

$$M_{CM} = \left(1 + \left|\frac{a_0}{a_n}\right|^2 + \left|\frac{a_1}{a_n}\right|^2 + \dots + \left|\frac{a_{n-1}}{a_n}\right|^2\right)^{1/2} \text{ (Carmichael-Mason),}$$

$$M_W = \left(1 + \left|\frac{a_1 - a_0}{a_n}\right|^2 + \left|\frac{a_2 - a_1}{a_n}\right|^2 + \dots + \left|\frac{a_n - a_{n-1}}{a_n}\right|^2\right)^{1/2} \text{ (Williams),}$$

$$M_F = 2\max\left\{\left|\frac{a_{n-1}}{a_n}\right|, \left|\frac{a_{n-2}}{a_n}\right|^{1/2}, \dots, \left|\frac{a_1}{a_n}\right|^{1/(n-1)}, \left|\frac{a_0}{2a_n}\right|^{1/n}\right\} \text{ (Fujiwara [1]),}$$

$$M_{JLR} = \frac{1}{2}\left(1 + |a_1| + \sqrt{(|a_1| - 1)^2 + 4\delta}\right), \text{ where } \delta = \max\{|a_k| : k = \overline{1,n}\}$$

(the bound of Joyal, Labelle and Rahman from [3] which improves the classical bound of Cauchy).

We have the results given in the table below.

Polynomial	M_2	M_C	M_{CM}	M_W	M_{JLR}	M_F
f_1	2.298	4	4	3.741	3.302	
f_2	2.575	5	4.795	8.717	3.561	4
f_3	1.550	3	2.645	2.828	2	
f_4	4.671	5	5.567	7.071	4.791	
f_5	2.305	4	4.358	6.928	4.791	
f_6	2.715	5	5.567	6.164	3.561	3.561

We remark that in every case the bound M_2 is the best.

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Several discrete inequalities

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Abstract. In this paper we present new conclusions for an open problem proposed by Yu Miao and Feng Qi in [5] and obtain their results under more general conditions.

Keywords: discrete inequality, Hölder inequality.

MSC: 26D15, 28A25

1. Introduction

In [6] is considered an open question about an integral inequality. This problem was solved in different ways in [1], [3] and [7]. A complete solution for this problem can be found in [4]. But in [5], Yu Miao and Feng Qi propose a discrete form of this problem:

Open Problem 1.1. *For $n \in \mathbb{N}^*$, let $\{x_1, x_2, \ldots, x_n\}$, $\{y_1, y_2, \ldots, y_n\}$ be two sequences of positive real numbers satisfying $x_1 \geq x_2 \geq \cdots \geq x_n$, $y_1 \geq y_2 \geq \cdots \geq y_n$ and*

$$\sum_{i=1}^{m} x_i \leq \sum_{i=1}^{m} y_i \quad \text{for } 1 \leq m \leq n.$$

Under what conditions does the inequality

$$\sum_{i=1}^{n} x_i^{\alpha} y_i^{\beta} \leq \sum_{i=1}^{m} y_i^{\alpha+\beta}$$

hold for α and β?

Several answers to this open problem are presented in the same article or in [2]. In this paper we show new improvements of this discrete inequality and find the results from [5] as a consequence of our work.

2. Some useful lemmas

In this section we present and prove some useful results. First, we recall two well-known lemmas.

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Lemma 2.1. (Abel) *Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be real numbers. Then*

$$\sum_{i=1}^{n} a_i b_i = \sum_{k=1}^{n-1} \left(\sum_{i=1}^{k} a_i \right) (b_k - b_{k+1}) + (a_1 + a_2 + \cdots + a_{n-1}) \, b_n.$$

Lemma 2.2. (Cauchy) *Let x, y be two positive real numbers and $a, b \in [0, 1]$ with $a + b = 1$. Then*

$$ax + by \geq x^a y^b.$$

The next lemma is the starting point for all the results which follow in this paper.

Lemma 2.3. *Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be real numbers with*

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \quad \text{for all } k \in \{1, 2, \ldots, n\}.$$

Let $t_1, t_2, \ldots, t_n$ be some real numbers with $t_1 \geq t_2 \geq \cdots \geq t_n \geq 0$. Then

$$\sum_{i=1}^{k} a_i t_i \leq \sum_{i=1}^{k} b_i t_i \quad \text{for all } k \in \{1, 2, \ldots, n\}.$$

Proof. We evaluate the difference $\sum\limits_{i=1}^{k} a_i t_i - \sum\limits_{i=1}^{k} b_i t_i$ using Lemma 2.1. We have

$$\sum_{i=1}^{k} a_i t_i - \sum_{i=1}^{k} b_i t_i = \sum_{i=1}^{k} (a_i - b_i) \, t_i$$

$$= \sum_{j=1}^{k-1} \left(\sum_{i=1}^{j} a_i - \sum_{i=1}^{j} b_i \right) (t_j - t_{j+1}) + \left(\sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i \right) t_k \leq 0,$$

because, by hypotheses, $\sum\limits_{i=1}^{k} a_i \leq \sum\limits_{i=1}^{k} b_i$ for all $k \in \{1, 2, \ldots, n\}$ and $t_k \geq t_{k+1}$ for all $k \in \{1, 2, \ldots, n-1\}$. $\square$

The next two lemmas are consequences of the previous result.

Lemma 2.4. *Let $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ be positive real numbers with*

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \quad \text{for all } k \in \{1, 2, \ldots, n\}.$$

If $x_1 \geq x_2 \geq \cdots \geq x_n$ then

$$\sum_{i=1}^{k} x_i^{\beta} \leq \sum_{i=1}^{k} y_i^{\beta} \quad \text{for all } \beta \in [1, \infty) \text{ and all } k \in \{1, 2, \ldots, n\}.$$

Proof. For $\beta = 1$ it is clear. If $\beta \in (1, \infty)$ then there exists $\alpha \in (1, \infty)$ with
$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$
Now
$$\sum_{i=1}^{k} x_i^{\beta} = \sum_{i=1}^{k} x_i x_i^{\beta-1} \leq \sum_{i=1}^{k} y_i x_i^{\beta-1}$$
from Lemma 2.3. Moreover,
$$\sum_{i=1}^{k} y_i x_i^{\beta-1} \leq \left(\sum_{i=1}^{k} y_i^{\beta}\right)^{1/\beta} \left(\sum_{i=1}^{k} x_i^{\alpha(\beta-1)}\right)^{1/\alpha} = \left(\sum_{i=1}^{k} y_i^{\beta}\right)^{1/\beta} \left(\sum_{i=1}^{k} x_i^{\beta}\right)^{1/\alpha}$$
from Hölder inequality. We thus obtain
$$\sum_{i=1}^{k} x_i^{\beta} \leq \left(\sum_{i=1}^{k} y_i^{\beta}\right)^{1/\beta} \left(\sum_{i=1}^{k} x_i^{\beta}\right)^{1/\alpha}$$
and after simplification
$$\left(\sum_{i=1}^{k} x_i^{\beta}\right)^{1/\beta} \leq \left(\sum_{i=1}^{k} y_i^{\beta}\right)^{1/\beta},$$
whence the conclusion follows. □

Remark. This result is more general than Lemma 2.3 from [5] because we do not use the condition $y_1 \geq y_2 \geq \cdots \geq y_n$.

Lemma 2.5. *Let $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ be positive real numbers with*
$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \quad \textit{for all } k \in \{1, 2, \ldots, n\}.$$
If $y_1 \leq y_2 \leq \cdots \leq y_n$ then
$$\sum_{i=1}^{k} x_i^{\alpha} \leq \sum_{i=1}^{k} y_i^{\alpha} \quad \textit{for } \alpha \in (0, 1] \textit{ and } k \in \{1, 2, \ldots, n\}.$$

Proof. For $\alpha = 1$ it is clear. If $\alpha \in (0, 1)$ we apply Lemma 2.3 for $t_i = y_i^{\alpha-1}$ and get
$$\sum_{i=1}^{k} y_i^{\alpha} = \sum_{i=1}^{k} y_i y_i^{\alpha-1} \geq \sum_{i=1}^{k} x_i y_i^{\alpha-1}.$$
From Hölder inequality we obtain
$$\sum_{i=1}^{k} x_i^{\alpha} = \sum_{i=1}^{k} \frac{x_i^{\alpha}}{y_i^{\alpha(1-\alpha)}} y_i^{\alpha(1-\alpha)} \leq \left(\sum_{i=1}^{k} \frac{x_i}{y_i^{1-\alpha}}\right)^{\alpha} \left(\sum_{i=1}^{k} y_i^{\alpha}\right)^{1-\alpha}$$

and

$$\sum_{i=1}^{k} x_i^{\alpha} \leq \left(\sum_{i=1}^{k} \frac{x_i}{y_i^{1-\alpha}}\right)^{\alpha} \left(\sum_{i=1}^{k} y_i^{\alpha}\right)^{1-\alpha} \leq \left(\sum_{i=1}^{k} y_i^{\alpha}\right)^{\alpha} \left(\sum_{i=1}^{k} y_i^{\alpha}\right)^{1-\alpha} = \sum_{i=1}^{k} y_i^{\alpha},$$

which concludes our proof. □

3. An answer for the open problem and other improvements

Now we present some discrete inequalities as consequences of the results from the previous section.

Proposition 3.1. *Let $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ be positive real numbers with*

$$\sum_{i=1}^{k} x_i^{\alpha} \leq \sum_{i=1}^{k} y_i^{\alpha} \quad \text{for all } k \in \{1, 2, \ldots, n\}.$$

If $x_1 \geq x_2 \geq \cdots \geq x_n$ then

$$\sum_{i=1}^{k} x_i^{\beta} \leq \sum_{i=1}^{k} y_i^{\beta} \quad \text{for all } \beta \geq \alpha \text{ and all } k \in \{1, 2, \ldots, n\}.$$

Proof. We apply Lemma 2.4 for $x_i := x_i^{\alpha}, y_i := y_i^{\alpha}$ and $\beta := \dfrac{\beta}{\alpha}$. □

Proposition 3.2. *Let $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ be positive real numbers with*

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \quad \text{for all } k \in \{1, 2, \ldots, n\}.$$

If $x_1 \geq x_2 \geq \cdots \geq x_n$ then

$$\sum_{i=1}^{k} x_i^{\alpha+\beta} \leq \sum_{i=1}^{k} x_i^{\alpha} y_i^{\beta} \quad \text{for all } \beta \geq 1,\ \alpha \geq 0 \text{ and all } k \in \{1, 2, \ldots, n\}.$$

Proof. From $\sum\limits_{i=1}^{k} x_i \leq \sum\limits_{i=1}^{k} y_i$ we get $\sum\limits_{i=1}^{k} x_i^{\beta} \leq \sum\limits_{i=1}^{k} y_i^{\beta}$ by using Lemma 2.4. Now we apply Lemma 2.3 for $a_i := x_i^{\beta}$, $b_i := y_i^{\beta}$ and $t_i := x_i^{\alpha}$ and obtain the conclusion. □

Now we are ready to present a more general version of the Open Problem 1.1. We also give a proof for this result.

Proposition 3.3. *Let $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ be positive real numbers with*

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \quad \text{for all } k \in \{1, 2, \ldots, n\}.$$

If $x_1 \geq x_2 \geq \cdots \geq x_n$ *then*

$$\sum_{i=1}^{k} x_i^{\alpha} y_i^{\beta} \leq \sum_{i=1}^{k} y_i^{\alpha+\beta} \quad \textit{for all } \beta \geq 1,\ \alpha \geq 0 \ \textit{ and all } k \in \{1, 2, \ldots, n\}.$$

Proof. From $\sum\limits_{i=1}^{k} x_i \leq \sum\limits_{i=1}^{k} y_i$ we obtain $\sum\limits_{i=1}^{k} x_i^{\beta} \leq \sum\limits_{i=1}^{k} y_i^{\beta}$ by using Lemma 2.4. If we apply Lemma 2.2 we obtain

$$\frac{\alpha}{\alpha+\beta} x_i^{\alpha+\beta} + \frac{\beta}{\alpha+\beta} y_i^{\alpha+\beta} \geq x_i^{\alpha} y_i^{\beta}$$

and

$$\frac{\alpha}{\alpha+\beta} \sum_{i=1}^{k} x_i^{\alpha+\beta} + \frac{\beta}{\alpha+\beta} \sum_{i=1}^{k} y_i^{\alpha+\beta} \geq \sum_{i=1}^{k} x_i^{\alpha} y_i^{\beta}.$$

From Proposition 3.2 we have

$$\frac{\alpha}{\alpha+\beta} \sum_{i=1}^{k} x_i^{\alpha} y_i^{\beta} + \frac{\beta}{\alpha+\beta} \sum_{i=1}^{k} y_i^{\alpha+\beta}$$

$$\geq \frac{\alpha}{\alpha+\beta} \sum_{i=1}^{k} x_i^{\alpha+\beta} + \frac{\beta}{\alpha+\beta} \sum_{i=1}^{k} y_i^{\alpha+\beta} \geq \sum_{i=1}^{k} x_i^{\alpha} y_i^{\beta}$$

and

$$\frac{\beta}{\alpha+\beta} \sum_{i=1}^{k} y_i^{\alpha+\beta} \geq \left(1 - \frac{\alpha}{\alpha+\beta}\right) \sum_{i=1}^{k} x_i^{\alpha} y_i^{\beta},$$

which conclude the proof. □

Remark. Proposition 3.3 represents a more general result than Theorem 3.1 from [5] because we do not use condition $y_1 \geq y_2 \geq \cdots \geq y_n$.

Finally, we give two more results, similar with Propositions 3.1 and 3.2.

Proposition 3.4. *Let* β *and* $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ *be positive real numbers with*

$$\sum_{i=1}^{k} x_i^{\beta} \leq \sum_{i=1}^{k} y_i^{\beta} \quad \textit{for all } k \in \{1, 2, \ldots, n\}.$$

If $y_1 \leq y_2 \leq \cdots \leq y_n$ *then*

$$\sum_{i=1}^{k} x_i^{\alpha} \leq \sum_{i=1}^{k} y_i^{\alpha} \quad \textit{for all } \alpha \in (0, \beta) \ \textit{and all } k \in \{1, 2, \ldots, n\}.$$

Proof. Apply Lemma 2.5 for $x_i := x_i^{\beta}, y_i := y_i^{\beta}$ and $\alpha := \frac{\alpha}{\beta}$. □

Proposition 3.5. *Let $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ be positive real numbers with*

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \quad \textit{for all } k \in \{1, 2, \ldots, n\}.$$

If $y_1 \leq y_2 \leq \cdots \leq y_n$ then

$$\sum_{i=1}^{k} x_i^{\alpha} y_i^{\beta} \leq \sum_{i=1}^{k} y_i^{\alpha+\beta} \quad \textit{for all } \alpha \in (0,1],\ \beta \leq 0 \textit{ and all } k \in \{1, 2, \ldots, n\}.$$

Proof. We are using Lemma 2.5 and the condition $\sum\limits_{i=1}^{k} x_i \leq \sum\limits_{i=1}^{k} y_i$ to get

$$\sum_{i=1}^{k} x_i^{\alpha} \leq \sum_{i=1}^{k} y_i^{\alpha} \quad \text{for all } \alpha \in (0,1]\,.$$

For all $\beta \leq 0$ we choose $t_i := y_i^{\beta}$ in Lemma 2.3 and obtain the conclusion by applying this lemma for $x_i := x_i^{\alpha}$ and $y_i := y_i^{\beta}$. $\square$

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Problems with lattices defined by equivalence relations

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Abstract. Using some simple theoretical notions, such as equivalence relation, quotient set, lattice network, measure of a set, some difficult problems of I.M.O. type are treated in a unitary approach. Some of the problems presented here are related to Blichfeldt's Theorem (1914) and Minkowski's Convex Body Theorem (1912).

Keywords: equivalence relation, quotient set, planar and spatial lattice
MSC: 11P21

1. Introduction

In the mathematical contests of I.M.O. or I.M.C. type for students, the difficulties of the problems are not given by the complexity of the theoretical notions, but by the lack of similarity to other problems and because much creativity is needed for finding the solution. We give an example of such a problem from the selection contests of I.M.O. Romanian team in 2008.

On the real line we consider a finite number of intervals with the sum of their lengths smaller than 1. Prove that there exists a unitary division of the real line (see Definition 2.15) which has no common points with these intervals.

A nice solution is presented below: We divide the real line in segments of length 1, we cut these unit segments and we put all of them over one of them. The original intervals are, therefore, transposed on this segment and since the sum of their lengths is smaller than 1, there exist some points on the segment not covered by any interval. We choose such a point and construct the unitary division with an extremity at this point. This division has no common points with the initial intervals.

Some of the problems presented here are inspired by two theorems from the geometry of numbers.

• **Blichfeldt's Theorem** [1]. Any bounded planar region with area greater than A placed in any position of the unit square lattice can be translated so that the number of lattice points inside the region will be at least $A+1$.

• **Minkowski Convex Body Theorem** [3]. A bounded convex region symmetric about a lattice point and with area greater than 4 must contain three lattice points in the interior.

For a detailed discussion about these two theorems see [4], pp. 119–126.

The theorems can be generalized to the n-dimensional case as well:

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• Let V be a bounded region in $\mathbb{R}^n$ with volume greater than 1. Then V contains two distinct points $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ such that the point $(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$ is a lattice point in $\mathbb{R}^n$.

• If V is a bounded, convex region in $\mathbb{R}^n$ having volume greater than 2^n and is symmetric about the origin, then V contains a lattice point other than the origin.

These two theorems with proofs can be found in [2], pp. 26–28.

The goal of this paper is to present some general ideas which allow a unitary approach for some problems of I.M.O. or I.M.C. type. The theoretical aspects which are necessary for solving such problems are simple algebraic notions, such as: equivalence relation, quotient set, lattice network, measure of a set, Dirichlet principle. The complexity and the diversity of the chosen problems prove the efficiency of the proposed model.

2. Theoretical facts

Definition 2.5. *If A is a set, then the subset $\rho \subset A \times A$ is called an* equivalence relation *on A if the following conditions are satisfied:*

(r) $(a, a) \in \rho$, *for every* $a \in A$ *(reflexivity)*
(t) *if* $(a_1, a_2) \in \rho$ *and* $(a_2, a_3) \in \rho$, *then* $(a_1, a_3) \in \rho$ *(transitivity)*
(s) *if* $(a_1, a_2) \in \rho$, *then* $(a_2, a_1) \in \rho$ *(symmetry)*

In the sequel, let A be a set and ρ an equivalence relation on A. We will denote by $a_1 \,\rho\, a_2$ the fact that $(a_1, a_2) \in \rho$.

Definition 2.6. *For every $a \in A$, the set*

$$\widehat{a} = \{x \in A \mid x \,\rho\, a\}$$

is called the equivalence class *of a.*

Remark 2.7. If $a_1, a_2 \in A$ then $\widehat{a_1} = \widehat{a_2}$ or $\widehat{a_1} \cap \widehat{a_2} = \emptyset$. We can notice also that the set of equivalence classes represents a partition of the set on which the relation is defined.

Definition 2.8. *A subset $S \subset A$ is called a* complete system of representatives (c.s.r.) *of the classes of the equivalence ρ if the following conditions are satisfied:*

(a) *for every* $a \in A$, *there exists* $s \in S$ *such that* $a \in \widehat{s}$,
(b) *if* $s_1, s_2 \in S$, $s_1 \neq s_2$, *then* $\widehat{s_1} \cap \widehat{s_2} = \emptyset$.

Definition 2.9. *Let S be a c.s.r. of ρ, $B \subset A$ a subset of A and $\widehat{B} = \cup_{b \in B} \widehat{b}$. The set $S_B = S \cap \widehat{B}$ is called a* system of representatives of classes of the set B.

Definition 2.10. *Let S be a c.s.r. of ρ. The set $\{\widehat{s} \mid s \in S\}$ is called the* quotient set *of A with respect to ρ and is denoted by A/ρ.*

In sections 3 and 4 we will use the following results.

Theorem 2.11. *Let ρ be an equivalence relation on A, S a c.s.r. of ρ and $B \subset A$ a subset of A. Then the following conditions are equivalent:*

i) *There exists $a \in A$ such that $\widehat{a} \cap B = \emptyset$.*
ii) *There is no c.s.r. included in B.*
iii) $A \neq \widehat{B}$.
iv) $S \neq S_B$.

Theorem 2.12. *If A is an uncountable set and if each equivalence class is a countable set, then A/ρ is uncountable.*

Corollary 2.13. *Let A be an uncountable set with every equivalence class being a countable set and let B be a countable subset of A. Then there exists a class $\widehat{a} \subset A$ such that $\widehat{a} \cap B = \emptyset$.*

Theorem 2.14. *If B is subset of A which contains strictly a c.s.r. then there exist $b_1, b_2 \in B$, $b_1 \neq b_2$, such that $\widehat{b_1} = \widehat{b_2}$.*

Definition 2.15. *If the straight line D is identified with the real line, then every subset of the form $\{x + k \mid k \in \mathbb{Z}\}$, $x \in \mathbb{R}$, is called a* unitary division *of D.*

Definition 2.16. *If the plane P is identified with $\mathbb{R}^2$, then the set*

$$\{(x, y) \in \mathbb{R}^2 \mid (x, y) \in \mathbb{Z}^2\}$$

is called a planar lattice. *The straight lines $x = k$ and $y = k$, $k \in \mathbb{Z}$, are called* lattice lines *and the points $(x, y) \in \mathbb{Z}^2$ are called* lattice points.

Definition 2.17. *If the 3-dimensional space is identified with $\mathbb{R}^3$, then the set*

$$\{(x, y, z) \in \mathbb{R}^3 \mid (x, y, z) \in \mathbb{Z}^3\}$$

is called a spatial lattice. *The planes $x = k$ or $y = k$ or $z = k$, $k \in \mathbb{Z}$, are called* lattice planes. *The straight lines $x = k, y = p$ or $x = k, z = p$ or $y = k, z = p$, $k, p \in \mathbb{Z}$, are called* lattice lines. *The points $(x, y, z) \in \mathbb{Z}^3$ are called* lattice points.

3. Lattices determined by equivalence relations

In this section we present some equivalence relations for which the equivalence classes determine lattices.

Theorem 3.1. *The relation $\rho \subset \mathbb{R} \times \mathbb{R}$ defined by*

$$x \,\rho\, y \;\Leftrightarrow\; x - y \in \mathbb{Z}$$

for every $x, y \in \mathbb{R}$ is an equivalence relation on $\mathbb{R}$. The equivalence classes are unitary divisions of the real line and $[0, 1)$ is a c.s.r. of ρ.

Proof. The properties (r), (t), (s) are easily verified, so ρ is an equivalence relation. We notice that $x \,\rho\, y \Leftrightarrow \{x\} = \{y\}$, where $\{x\}$ denotes the decimal part of x. For $x_0 \in \mathbb{R}$ the equivalence class of x_0 can be represented by the set $\{x \in \mathbb{R} \mid \{x\} = \{x_0\}\} = \{\varepsilon_0 + k \mid k \in \mathbb{Z}\}$, which is the unitary division of $\mathbb{R}$ which contains the point ε_0, $\varepsilon_0 \in [0,1)$ being the decimal part of all the elements of this class. In every equivalence class we choose as a representative the number from the interval $[0,1)$, and so we obtain that $[0,1)$ is a c.s.r. of ρ.

Theorem 3.2. *In the plane $\mathbb{R}^2$ the relation $\rho \subset \mathbb{R}^2 \times \mathbb{R}^2$, defined by*

$$(x_1, y_1)\,\rho\,(x_2, y_2) \Leftrightarrow x_1 - x_2 \in \mathbb{Z} \text{ and } y_1 - y_2 \in \mathbb{Z}$$

is an equivalence relation. The equivalence classes are lattice points and the square $[0,1) \times [0,1)$ is a c.s.r. of ρ.

Proof. We have $(x_1, y_1)\,\rho\,(x_2, y_2) \Leftrightarrow \{x_1\} = \{x_2\}$ and $\{\, y_1 \,\} = \{\, y_2 \,\}$. The equivalence class of a point (x_0, y_0) is $\widehat{(x_0, y_0)} = \{\, (x_0 + k, y_0 + p) \mid k, p \in \mathbb{Z} \,\}$, which is the plane lattice with the point (x_0, y_0). The quotient set can be represented by the complete system of representatives choosing in every class as a representative the point $(x_0, y_0) \in [0,1) \times [0,1)$.

Theorem 3.3. *In the space $\mathbb{R}^3$ the relation $\rho \subset \mathbb{R}^3 \times \mathbb{R}^3$, defined by*

$$(x_1, y_1, z_1)\,\rho\,(x_2, y_2, z_2) \Leftrightarrow x_1 - x_2 \in \mathbb{Z},\ y_1 - y_2 \in \mathbb{Z} \text{ and } z_1 - z_2 \in \mathbb{Z}$$

is an equivalence relation. The equivalence classes are spatial lattices and one representation of the quotient set is the cube $[0,1) \times [0,1) \times [0,1)$.

Theorem 3.4. *On the unit circle $U = \{\, z \in \mathbb{C} \mid |z| = 1 \,\}$ of the complex plane, the relation $\rho \subset \mathbb{C} \times \mathbb{C}$, defined by $z_1 \,\rho\, z_2 \Leftrightarrow z_1^n = z_2^n$, is an equivalence relation for every fixed $n \in \mathbb{N}^*$. For $n \geq 3$, the equivalence classes are the vertices of regular polygons with n sides inscribed in the circle U and the arc $\{\, z = \cos t + i \sin t \mid t \in [0, 2\pi/n) \,\}$ is a c.s.r. of ρ.*

Proof. If $z_0 \in U$ then the equivalence class of z_0 is

$$\widehat{z_0} = \{\, z \in \mathbb{C} \mid z^n = z_0^n \,\} = \left\{\, z \in \mathbb{C} \mid \left(\frac{z}{z_0}\right)^n = 1 \,\right\} = \left\{\, z \in \mathbb{C} \mid \frac{z}{z_0} \in U_n \,\right\},$$

where $U_n = \{\, z \in \mathbb{C} \mid z^n = 1 \,\} = \left\{\, \varepsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \mid 0 \leq k \leq n-1 \,\right\}$. So $\widehat{z_0} = \{\, z_k = z_0 \varepsilon_k \mid 0 \leq k \leq n-1 \,\}$, which are the vertices of the regular polygon having z_0 as a vertex.

4. **Problems**

Problem 4.1. *Let $(x_n)_n$ be a sequence of real numbers. Prove that for every $r \in \mathbb{R}^*$ there is an arithmetic progression $(a_n)_n$ with ratio r such that $\{\, x_n \mid n \in \mathbb{N} \,\} \cap \{\, a_n \mid n \in \mathbb{N} \,\} = \emptyset$.*

Solution. Because $a_0 + kr \neq x_n \Leftrightarrow \frac{a_0}{r} + k \neq \frac{x_n}{r} \Leftrightarrow b_0 + k \neq y_n$, we can suppose that $r = 1$. Every arithmetic progression with ratio $r = 1$ is a subset of a unitary division of the real line. It is sufficient to prove that for every sequence $(y_n)_n$ there exists a unitary division of the real line which has no common point with the set $B = \{ y_n \mid n \in \mathbb{N} \}$. If $\{ y_n \}$ is the decimal part of y_n, then the set $\widehat{B} = \{ \{ y_n \} \mid n \in \mathbb{N} \} \subset [0, 1)$ is a countable subset of the uncountable interval [0,1). It follows that there exists $a \in [0, 1) \setminus \widehat{B}$. Using Theorem 2.11 and Corollary 2.13 we have $\widehat{a} \cap B = \emptyset$, so the progression with ratio 1 and the first term a has no common points with the sequence $(y_n)_n$.

Problem 4.2. *Let $(A_n)_n$ be a sequence of points belonging to the circle $\mathcal{C}$. Prove that for every $N \in \mathbb{N}$, $N \geq 3$, there exists a regular polygon with N vertices inscribed in the circle $\mathcal{C}$, with none of the vertices belonging to $(A_n)_n$.*

Solution. We can suppose that the circle $\mathcal{C}$ is the unit circle of the complex plane: $U = \{ z \in \mathbb{C} \mid |z| = 1 \}$. Defining the equivalence relation on U by $z_1 \, \rho \, z_2 \Leftrightarrow z_1^N = z_2^N$, we know by Theorem 3.4 that the equivalence classes of this relation are the regular polygons with N sides inscribed in U. Each class is a finite set (at most countable) and U is uncountable. Using Theorem 2.12 and Corollary 2.13 for the set $B = \{ A_n \mid n \in \mathbb{N} \}$ we get an $a \in U$ such that its class, $\widehat{a}$, a regular polygon with N sides, has no common points with B.

Problem 4.3. *On the real line we consider a finite number of intervals having the sum of lengths equal to L. Prove that if $L > 1$, then there exist two distinct numbers x_1, x_2 on these intervals such that $x_1 - x_2 \in \mathbb{Z}$. If $L < 1$, prove that there exists a unitary division of the real line which has no common points with these intervals.*

Solution. Considering the relation ρ on $\mathbb{R}$: $x \, \rho \, y \Leftrightarrow x - y \in \mathbb{Z}$, we know by Theorem 3.1 that the equivalence classes are unitary divisions of $\mathbb{R}$ and $S = [0, 1)$ is a complete system of representatives of the classes. For every interval I_k consider the set $S_k = \{ \{ x \} \mid x \in I_k \}$. Consider also $B = \cup_{k=1}^n I_k$ and $S_B = \cup_{k=1}^n S_k$. If $\mathbb{Z} \cap I_k = \emptyset$, the set $S_k \subset [0, 1)$ is formed by a single interval with a length equal to the length of I_k. But if $\mathbb{Z} \cap I_k \neq \emptyset$, then S_k is formed by a union of two intervals with the sum of lengths equal to the length of I_k, if this length is strictly less than 1. If the length of I_k is greater than 1, then $S_k = [0, 1)$.

Suppose $L > 1$. In this case there are points in $[0, 1)$ covered at least twice by the sets S_k. For such a point $\varepsilon_0 \in [0, 1)$, there are distinct points $x_1, x_2 \in B$ with $\{ x_1 \} = \{ x_2 \} = \varepsilon_0$, so $x_1 - x_2 \in \mathbb{Z}$.

Suppose $L < 1$. Because $\sum_{k=1}^n \ell(I_k) = \sum_{k=1}^n \ell(S_k) < 1 = \ell([0, 1))$, we can find $a \in [0, 1) \setminus S_B$. So $S = [0, 1) \neq S_B$. By Theorem 2.11, $\widehat{a} \cap B = \emptyset$, i.e., the unitary division $\widehat{a}$ has no common points with the intervals considered.

Problem 4.4. *On a circle of radius 1 we consider a finite numbers of arcs having the sum of lengths equal to L. Prove that:*

a) If $L > 2\pi/n$, $n \in \mathbb{N}$, $n \geq 3$, then there exists a regular polygon with n vertices, inscribed in the circle, for which at least two of its vertices lie on the given arcs.
b) If $L < 2\pi/n$, $n \in \mathbb{N}$, $n \geq 3$, prove that there exists a regular polygon with n vertices for which none of its vertices lies on the given arcs.

Solution. Consider all arcs obtained by rotations with $2k\pi/n$ with $k = 0, 1, \ldots, n-1$. We have obtained a set of arcs having the sum of lengths $L' = nL$.

If $L > 2\pi/n$ then $L' > 2\pi$. Since L' exceeds the length of the circle, there are points which are covered by two arcs. A point C with this property comes from points A and B by rotation with $2k_1\pi/n$ and $2k_2\pi/n$. A regular polygon with n vertices having C as a vertex has also A and B as vertices.

If $L < 2\pi/n$ then $L' < 2\pi$, so there is a point D which has remained uncovered by the initial arcs and their rotations. A regular polygon with n vertices having D as a vertex does not have common points with the arcs.

Remark 4.5. Considering the relation on the unit circle in the complex plane: $z_1 \,\rho\, z_2 \;\Leftrightarrow\; z_1^n = z_2^n$, we know by Theorem 3.4 that the equivalence classes are regular polygons with n vertices, inscribed in the circle. We can solve this problem using Theorem 2.14 for the first part and Theorem 2.11 for the second part of the problem.

Problem 4.6. *Let $(a_n)_n$ be a sequence of real numbers. Prove that there exists $a \in \mathbb{R}$ such that $a_n - a$ is irrational for every $n \in \mathbb{N}$.*

Solution. We consider on $\mathbb{R}$ the equivalence relation: $x \,\rho\, y \Leftrightarrow x - y \in \mathbb{Q}$. The equivalence classes have the form $\widehat{a} = a + \mathbb{Q}$, so they are countable. Using Theorem 2.12 we deduce that the quotient set is uncountable. Taking the set $B = \{\, a_n \mid n \in \mathbb{N} \,\}$ it follows from Corollary 2.13 that there is a class $\widehat{a}$ such that $\widehat{a} \cap B = \emptyset$. This is equivalent with $a + q \neq a_n$ for every $q \in \mathbb{Q}$ and for every $n \in \mathbb{N}$. So $a_n - a \in \mathbb{R} \setminus \mathbb{Q}$, for every $n \in \mathbb{N}$.

Problem 4.7. *Let $(a_n)_n$ be a sequence of real numbers. Prove that there exists $a \in \mathbb{R}$ such that for every polynomial $P \in \mathbb{Q}[X]$ and for every $n \in \mathbb{N}$ we have $P(a_n - a) \neq 0$.*

Solution. A real number b is called algebraic if there is a polynomial $P \in \mathbb{Q}[X]$ such that $P(b) = 0$. The set of algebraic numbers $\mathcal{A}$ is a field which includes $\mathbb{Q}$. Because $\mathbb{Q}[X]$ is countable and each polynomial from $\mathbb{Q}[X]$ has a finite number of roots (algebraic numbers), we deduce that $\mathcal{A}$ is countable.

Let $\rho \subset \mathbb{R} \times \mathbb{R}$ be the equivalence relation defined by $x \,\rho\, y \Leftrightarrow x - y \in \mathcal{A}$. The equivalence classes have the form $\widehat{x} = x + \mathcal{A}$, so they are countable. Using Theorem 2.12 and Corollary 2.13 we find a class $\widehat{a} = a + \mathcal{A}$ which has no common points with the countable set $\{\, a_n \mid n \in \mathbb{N} \,\}$. It follows $a_n - a \notin \mathcal{A}$, for all $n \in \mathbb{N}$.

Problem 4.8. *Let $(a_n)_n$ be a sequence of nonzero real numbers. Prove that there exists $a \in \mathbb{R}$ such that for every polynomial $P \in \mathbb{Q}[X]$ and for every $n \in \mathbb{N}$ we have $P(a_n a) \neq 0$.*

Solution. Because the set of algebraic numbers $\mathcal{A}$ has the algebraic structure of a field, we deduce that $\rho \subset \mathbb{R}^* \times \mathbb{R}^*$ defined by $x \,\rho\, y \;\Leftrightarrow\; x/y \in \mathcal{A}^*$ is an equivalence relation. The equivalence classes have the form $\widehat{x} = x\mathcal{A}$, so they are countable. Using Theorem 2.12 and Corollary 2.13 we find a class $\widehat{b} = b \cdot \mathcal{A}$ which has no common points with the countable set $\{\, a_n \mid n \in \mathbb{N} \,\}$. It follows $a_n/b \notin \mathcal{A}$, for every $n \in \mathbb{N}$. Taking $a = 1/b$ the conclusion follows.

Problem 4.9. [1] *In the plane, consider a finite number of polygons with the sum of areas equal to S. Prove that:*
a) If $S > 1$, then there is a planar lattice with at least two lattice points contained in the given polygons.
b) If $S < 1$, then there is a planar lattice with all its points in the exterior of the given polygons.

Solution 1. Suppose we cut the lattice through the lattice lines. We overlap completely the obtained unit squares with the $[0,1] \times [0,1]$ square. Some of the initial polygons have been cut but they are transformed in a finite number of polygons included in $[0,1] \times [0,1]$. The sum of areas of all these new polygons is the same with the sum of areas of the original polygons.

If $S > 1$, there are in the unit square $[0,1]\times[0,1]$ points that are covered more than once by the new polygons. If (x_0, y_0) is a point with this property and it is covered by the translation of the polygons P_i and P_j, then the lattice $(x_0, y_0) + \mathbb{Z} \times \mathbb{Z}$ has at least two common points with the polygons.

If $S < 1$, there are uncovered points in $[0,1] \times [0,1]$ by the translated polygons. Let (x_0, y_0) be such a point. The lattice $(x_0, y_0) + \mathbb{Z} \times \mathbb{Z}$ has no common points with the polygons.

Solution 2. If we consider the equivalence relation from Theorem 3.2 we know that the classes of equivalence are planar lattices and a complete system of representatives of the classes is the square $[0,1]\times[0,1]$ with area 1. We apply Theorem 2.14 for a) and Theorem 2.11 for b).

Problem 4.10. [4] *Let R be a bounded convex region in $\mathbb{R}^2$ having area greater than 4. If R is symmetric about the origin then R contains a lattice point other than the origin.*

Solution. Consider the region $R' = \{x/2 \mid x \in R\}$ which is convex having area greater than 1. From Blichfeldt result (Problem 4.9) there are distinct points $\left(\dfrac{x_1}{2}, \dfrac{y_1}{2}\right)$, $\left(\dfrac{x_2}{2}, \dfrac{y_2}{2}\right)$ such that $\left(\dfrac{x_1 - x_2}{2}, \dfrac{y_1 - y_2}{2}\right) \in \mathbb{Z}^2$, $(x_1, y_1) \in R$, $(x_2, y_2) \in R$. Since R is symmetric about the origin $(-x_2, -y_2) \in R$. The fact that R is convex ensures that every point on the line segment between (x_1, y_1) and $(-x_2, -y_2)$ is in R. Therefore $\frac{1}{2}(x_1, y_1) + \frac{1}{2}(-x_2, -y_2) \in R \cap \mathbb{Z}^2$.

Problem 4.11. [5] *We consider in the plane a finite number of segments having the sum of lengths $S < \sqrt{2}$. Prove that there is a planar lattice with lattice lines not intersecting any of the given segments.*

Solution. We chose in the plane a rectangular system of coordinates. Let P_{x_i} and P_{y_i} be the projections of the segment L_i on the Ox and Oy axis. If p_{x_i}, p_{y_i} and l_i are the corresponding lengths, then $p_{x_i} = l_i \cdot |\cos\alpha_i|$ and $p_{y_i} = l_i \cdot |\sin\alpha_i|$, where α_i is the angle between the segment L_i and the axis Ox. Using the inequality $(a+b)^2 \le 2(a^2+b^2)$ we get $p_{x_i} + p_{y_i} \le \sqrt{2} \cdot l_i$. If p_x is the sum of all projections of the segments on Ox and p_y is the sum of all projections on Oy, then $p_x + p_y \le \sqrt{2} \cdot \sum l_i < \sqrt{2} \cdot \sqrt{2} = 2$.

If we rotate the rectangular system by the angle $t \in [0, \pi/2]$ and we set $p_x(t)$ and $p_y(t)$ to be the sum of the projections on the axis of the rotated system of coordinates, we have $p_x(0) = p_y(\pi/2)$ and $p_y(0) = p_x(\pi/2)$. So there is a $t \in [0, \pi/2]$ such that $p_x(t) = p_y(t) < 1$. Let $x'Oy'$ be this new system. By Problem 4.3, there exists a unitary division $x_0 + \mathbb{Z}$ of the axis Ox', having no common points with the segments $P_{x'_i}$. Similarly, there is $y_0 + \mathbb{Z}$, a unitary division of Oy', having no common points with the projections $P_{y'_i}$. The lattice $(x_0, y_0) + \mathbb{Z} \times \mathbb{Z}$ has the property that the segments aren't intersected by the lattice lines.

Problem 4.12. *Prove that for every function $f : \mathbb{R} \to \mathbb{R}$ there is a real number a such that the graph of the function $g = f - a$ does not contain points with both coordinates rational numbers.*

Solution. The set $A = \{\, f(q) \mid q \in \mathbb{Q} \,\}$ is countable, so it can be written as $A = \{\, x_n \mid n \in \mathbb{N} \,\}$. Consider $B = \cup_{n\in\mathbb{N}}(x_n + \mathbb{Q}) = A + \mathbb{Q}$, which is a countable union of countable sets, so is countable. We can choose $a \in \mathbb{R} \setminus B$. We prove that $g = f - a$ satisfies the desired property.

Suppose $(q, g(q))$ has both coordinates rational numbers. It follows $q \in \mathbb{Q}$ and $f(q) - a \in \mathbb{Q}$. Then $a \in f(q) + \mathbb{Q} \subset B$, which contradicts the choice of a.

Problem 4.13. [6] *For every point from the space which has rational coordinates it can be obtained a spatial lattice which has this point as a lattice point. Prove that no matter what direction these lattices have, we can obtain another lattice with its lines passing between the lines of the other lattices.*

Solution. The set $\mathbb{Q}^3$ is countable and for every spatial lattice the set of its lines is countable. So the set of all lines of all lattices we can draw is countable. Let $D = \{\, d_n \mid n \in \mathbb{N} \,\}$ be this set. Consider now an arbitrarily chosen lattice. We cut this lattice into cubes with side length 1 and overlap all this cubes with the unit cube $V = [0,1] \times [0,1] \times [0,1]$. The lines of D are transformed in a countable set of segments, D', from the cube V. If we prove that there is a point $M(x_0, y_0, z_0) \in V$ such that the lines parallel to lattice lines, passing through M, do not intersect the lines from D', then a lattice containing M and with lines parallel to the lines of the lattice arbitrarily chosen will not intersect the lines of D.

There exists a set, which is at most countable, having as elements perpendicular planes with Ox, containing segments from D'. But there exists an uncountably infinity of planes $x = x_0 \in [0,1]$ which do not contain segments from D'. Such a plane is intersected by segments from D' in a countable set of points. Let P_z and P_y the projections of the segments of D' on the planes $z = 0$ and $y = 0$. These projections will cut the segments $x = x_0$, $z = 0$ and $x = x_0$, $y = 0$ in countable sets of points. Let $(x_0, y_n, 0)$ and $(x_0, 0, z'_n)$ be these points.

We choose $y_0 \in [0,1] \setminus \{\, y_n \mid n \in \mathbb{N} \,\}$ such that the plane $y = y_0$ does not contain segments from D'. The projection P_x of the lines from D' on the plane $x = 0$ cuts the line $y = y_0$, $x = 0$ in a countable set of points $\{\, (0, y_0, z''_n) \mid n \in \mathbb{N} \,\}$.

Because the sets $\{\, z'_n \mid n \in \mathbb{N} \,\}$ and $\{\, z''_n \mid n \in \mathbb{N} \,\}$ are countable, there exists $z_0 \in [0,1] \setminus \{\, z'_n, z''_n \mid n \in \mathbb{N} \,\}$. The segments

$$\{\, (x_0, y_0, t) \mid t \in [0,1] \,\}, \ \{\, (x_0, t, z_0) \mid t \in [0,1] \,\} \ \text{and} \ \{\, (t, y_0, z_0) \mid t \in [0,1] \,\}$$

do not intersect segments from D'.

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Concursul naţional studenţesc de matematică „Traian Lalescu“, ediţia 2013

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Abstract. This note deals with the problems of the 2013 edition of the ,,Traian Lalescu" mathematical contest for university students, hosted by the University of Alba Iulia between May 20 and May 22, 2013.

Keywords: eigenvalue, eigenvector, continue function, minimal polynomial

MSC: 11C08; 11C20; 26A15; 26A42

În perioada 20-22 mai 2013 s-a desfăşurat la Alba Iulia etapa naţională a concursului studenţesc „Traian Lalescu".

La concurs au participat peste 60 de studenţi, reprezentând 10 universităţi din cinci centre universitare: Bucureşti, Cluj, Constanţa, Iaşi şi Timişoara.

Concursul s-a desfăşurat pe patru secţiuni: A - facultăţi de matematică, B - învăţământ tehnic, profil electric anul I, C - învăţământ tehnic, profil mecanic şi construcţii, anul I, D - învăţământ tehnic, anul II.

Subiectele au fost propuse, discutate şi alese în dimineaţa concursului de câte o comisie la fiecare secţiune, în care fiecare universitate a avut câte un reprezentant.

La organizarea concursului, pe lângă Universitatea „1 Decembrie 1918" din Alba Iulia, care a oferit condiţii optime de concurs, cazare şi masă, au contribuit Ministerul Educaţiei şi Cercetării şi Fundaţia „Traian Lalescu".

Prezentăm în cele ce urmează enunţurile şi soluţiile problemelor date la secţiunile A şi B ale concursului. Pentru soluţiile oficiale facem trimitere la http://www.uab.ro/ctl.

Secţiunea A

Problema 1. Fie $A \in \mathcal{M}_n(\mathbb{C})$ o matrice, λ o valoare proprie a matricei A^n, iar $v \in \mathbb{C}^n$ un vector propriu asociat lui λ. Să se arate că dacă vectorii $v, Av, \ldots, A^{n-1}v$ sunt liniar independenţi, atunci $A^n = \lambda I_n$.

Vasile Pop

Această problemă a fost considerată uşoară de către juriu. Prezentăm două soluţii date de studenţi în concurs.

Soluţia 1. Fie $u : \mathbb{C}^n \to \mathbb{C}^n$, $u(x) = A^n x$. Observăm că $v, Av, \ldots, A^{n-1}v$ sunt vectori proprii pentru u; fiind în număr de n şi liniar independenţi, ei constituie o bază de vectori proprii pentru u. Matricea lui u în această bază

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este λI_n. În consecinţă, există $S \in \mathrm{GL}_n(\mathbb{C})$ astfel încât $A^n = S\lambda I_n S^{-1}$, deci $A^n = \lambda I_n$. $\square$

Soluţia 2. Notând cu $P_A = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$ polinomul caracteristic al matricei A şi aplicând teorema Hamilton-Cayley obţinem

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n = 0,$$

de unde

$$A^n v + a_{n-1}A^{n-1}v + \cdots + a_1 Av + a_0 v = 0,$$

sau încă

$$\lambda v + a_{n-1}A^{n-1}v + \cdots + a_1 Av + a_0 v = 0;$$

ţinând cont de independenţa liniară din enunţ, deducem

$$a_{n-1} = a_{n-2} = \cdots = a_1 = a_0 + \lambda = 0.$$

Rezultă $P_A = X^n - \lambda I_n$, de unde, aplicând din nou teorema Hamilton-Cayley, $A^n = \lambda I_n$. $\square$

Problema 2. Să se determine funcţiile continue $f : \mathbb{R} \to \mathbb{R}$ cu proprietatea că pentru orice $x, y \in \mathbb{R}$ pentru care $x - y \in \mathbb{R} \setminus \mathbb{Q}$ avem $f(x) - f(y) \in \mathbb{R} \setminus \mathbb{Q}$.

Vasile Pop

Aceasta a fost considerată de juriu drept o problemă de dificultate medie. Studenţii care au rezolvat problema au procedat în spiritul soluţiei pe care o prezentăm mai jos.

Soluţie. Fie $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ fixat. Din condiţia dată rezultă

$$f(x + \alpha) - f(x) \in \mathbb{R} \setminus \mathbb{Q}, \ \forall\, x \in \mathbb{R}$$

deci funcţia continuă $g_\alpha : \mathbb{R} \to \mathbb{R}$, $g_\alpha(x) = f(x + \alpha) - f(x)$ ia numai valori numere iraţionale. Din continuitate rezultă că această funcţie este constantă, deci

$$g_\alpha(x) = g_\alpha(0), \ \forall\, x \in \mathbb{R} \Leftrightarrow$$
$$f(x + \alpha) - f(x) = f(\alpha) - f(0), \ \forall\, x \in \mathbb{R}, \ \forall\, \alpha \in \mathbb{R} \setminus \mathbb{Q}. \tag{1}$$

Pentru x_0 fixat şi α variabil în $\mathbb{R} \setminus \mathbb{Q}$ rezultă

$$f(x_0 + \alpha) - f(\alpha) = f(x_0) - f(0). \tag{2}$$

Funcţia $h_{x_0} : \mathbb{R} \to \mathbb{R}$, $h_{x_0}(\alpha) = f(x_0 + \alpha) - f(\alpha)$, este continuă pe $\mathbb{R}$ şi constantă pe $\mathbb{R} \setminus \mathbb{Q}$, deci constantă pe $\mathbb{R}$. Astfel, relaţia (2) are loc pentru orice $\alpha \in \mathbb{R}$, deci şi relaţia (1) este valabilă pentru orice $x \in \mathbb{R}$ şi $\alpha \in \mathbb{R}$.

Avem aşadar de determinat funcţiile $f : \mathbb{R} \to \mathbb{R}$ pentru care

$$f(x + y) - f(x) = f(y) - f(0), \ \forall\, x, y \in \mathbb{R}.$$

Funcţia $A : \mathbb{R} \to \mathbb{R}$, $A(x) = f(x) - f(0)$, verifică ecuaţia lui Cauchy

$$A(x + y) = A(x) + A(y), \ \forall\, x, y \in \mathbb{R},$$

pentru care soluţiile continue sunt $A(x) = ax$, $x \in \mathbb{R}$. Deci $f(x) = ax + b$, $\forall\, x \in \mathbb{R}$, cu care revenind obţinem $a(x - y) \in \mathbb{R} \setminus \mathbb{Q}$, $\forall\, x - y \in \mathbb{R} \setminus \mathbb{Q}$, fapt echivalent cu $a \in \mathbb{Q}$ (dacă $a \in \mathbb{R} \setminus \mathbb{Q}$, pentru $x - y = \dfrac{1}{a} \in \mathbb{R} \setminus \mathbb{Q}$ s-ar obţine $a \cdot \dfrac{1}{a} = 1 \notin \mathbb{R} \setminus \mathbb{Q}$).

Funcţiile cerute sunt prin urmare $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$, $a \in \mathbb{Q}$, $b \in \mathbb{R}$. □

Problema 3. Fie $k \in \mathbb{N}^*$. Demonstraţi că valoarea minimă a lui $n \in \mathbb{N}^*$ pentru care există matrice $A \in \mathcal{M}_n(\mathbb{Q})$ cu proprietatea $A^{2^k} = -I_n$ este 2^k.

Gabriel Mincu

Aceasta a fost considerată de juriu drept o problemă de dificultate medie. Concurenţii au dat mai multe soluţii, care au diferit însă numai la nivelul unor detalii tehnice. Soluţia prezentată mai jos urmează ideile din demonstraţiile apărute în concurs.

Soluţie. Fie $n \in \mathbb{N}^*$ pentru care există matrice $A \in \mathcal{M}_n(\mathbb{Q})$ cu proprietatea $A^{2^k} = -I_n$ şi fie A o astfel de matrice. Atunci A anulează polinomul $f = X^{2^k} + 1$, deci $\mu_A \mid f$, unde μ_A este polinomul minimal al matricei A. Cum polinomul f este ireductibil peste $\mathbb{Q}$ (lucru care se poate constata aplicând criteriul lui Eisenstein polinomului $f(X + 1)$), deducem că $\mu_A = f$. Conform teoremei lui Frobenius, P_A este o putere a lui f. Prin urmare, valoarea minimă cerută este cel puţin 2^k. Este însă uşor de văzut că matricea companion

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & -1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} \in \mathcal{M}_{2^k}(\mathbb{Q})$$

verifică $A^{2^k} = -I_{2^k}$. Aşadar valoarea minimă cerută este 2^k. □

Problema 4. Să se demonstreze că

$$\lim_{n\to\infty} \frac{n}{\ln n} \int_0^1 \frac{n^2x^2 - [nx]^2}{(1 + x^2)(1 + [nx]^2)}\, \mathrm{d}x = 1.$$

Tiberiu Trif

Aceasta a fost considerată de juriu drept o problemă dificilă. Aprecierea s-a dovedit a fi corectă, doar un singur concurent abordând problema, fără a reuşi însă finalizarea soluţiei. Abordarea sa este prezentată în soluţia 2, în timp ce soluţia 1 este cea propusă de autorul problemei.

Soluţia 1. Notăm $I_n = \int\limits_0^1 \frac{n^2x^2 - [nx]^2}{(1+x^2)(1+[nx]^2)}\,\mathrm{d}x$ şi $l = \lim\limits_{n\to\infty} \frac{n}{\ln n} I_n$. Făcând schimbarea de variabilă $nx = t$, obţinem

$$I_n = \int\limits_0^1 \frac{t^2}{n^2+t^2}\,\mathrm{d}t + n\int\limits_1^n \frac{t^2-[t]^2}{(n^2+t^2)(1+[t]^2)}\,\mathrm{d}t.$$

Notând cu J_n cea de-a doua integrală din membrul drept al relaţiei anterioare şi ţinând cont de relaţia $\int\limits_0^1 \frac{t^2}{n^2+t^2}\,\mathrm{d}t = 1 - n\operatorname{arctg}\frac{1}{n}$, constatăm că

$$l = \lim_{n\to\infty} \frac{n^2}{\ln n}\left(1 - n\operatorname{arctg}\frac{1}{n}\right) + \lim_{n\to\infty}\frac{n^2}{\ln n} J_n.$$

Întrucât $\lim\limits_{n\to\infty} n^2\left(1 - n\operatorname{arctg}\frac{1}{n}\right) = \frac{1}{3}$, deducem că

$$l = \lim_{n\to\infty}\frac{n^2}{\ln n} J_n. \tag{3}$$

Notând $J_{n.k} = \int\limits_k^{k+1} \frac{t^2-k^2}{(n^2+t^2)(k^2+1)}\,\mathrm{d}t$, avem $J_n = \sum\limits_{k=1}^{n-1} J_{n,k}$ şi

$$J_{n,k} > \frac{1}{(n^2+(k+1)^2)(k^2+1)}\int\limits_k^{k+1}(t^2-k^2)\,\mathrm{d}t = \frac{1}{n^2+(k+1)^2}\cdot\frac{3k+1}{3(k^2+1)}.$$

De aici, $J_{n,k} > \frac{1}{n^2+(k+1)^2}\cdot\frac{1}{k+1}$.

Considerăm funcţiile $f_n : [1,\infty) \to (0,\infty)$, $f_n(t) = \frac{1}{t(n^2+t^2)}$, care sunt strict descrescătoare pe $[1,\infty)$. Avem $J_{n,k} > f_n(k+1) > \int\limits_{k+1}^{k+2} f_n(t)\,\mathrm{d}t$ oricare ar fi $k \in \{1,2,\ldots,n-1\}$. Rezultă de aici că

$$J_n = \sum_{k=1}^{n-1} J_{n,k} > \int\limits_2^{n+1}\frac{\mathrm{d}t}{t(n^2+t^2)} = \frac{1}{n^2}\ln\frac{t}{\sqrt{n^2+t^2}}\Bigg|_2^{n+1},$$

deci

$$\frac{n^2}{\ln n}J_n > \frac{1}{\ln n}\left(\ln\frac{n+1}{\sqrt{2n^2+2n+1}} - \ln 2 + \ln\sqrt{n^2+4}\right). \tag{4}$$

Avem însă şi

$$J_{n,k} < \frac{1}{(n^2+k^2)(k^2+1)} \int\limits_{k}^{k+1} (t^2 - k^2)\,\mathrm{d}t = \frac{1}{n^2+k^2} \cdot \frac{3k+1}{3(k^2+1)}.$$

Deducem că

$$J_{n,k} < \frac{3k+1}{(n^2+k^2)\cdot 3k^2} = \frac{1}{k(n^2+k^2)} + \frac{1}{3k^2(n^2+k^2)},$$

adică pentru orice $k \in \{1, 2, \ldots, n-1\}$ avem $J_{n,k} < f_n(k) + g_n(k)$, unde funcţiile $g_n : [1, \infty) \to (0, \infty)$ sunt definite prin $g_n(t) = \dfrac{1}{3t^2(n^2+t^2)}$.

Întrucât şi funcţiile g_n sunt strict descrescătoare pe $[1, \infty)$, obţinem ca mai sus $J_{n,k} < \int\limits_{k-1}^{k} f_n(t)\,\mathrm{d}t + \int\limits_{k-1}^{k} g_n(t)\,\mathrm{d}t$ pentru orice $k \in \{2, 3, \ldots, n-1\}$. Drept urmare, are loc

$$J_n < \int\limits_{1}^{n} f_n(t)\,\mathrm{d}t + \int\limits_{1}^{n} g_n(t)\,\mathrm{d}t + \int\limits_{1}^{2} \frac{t^2-1}{2(n^2+t^2)}\,\mathrm{d}t,$$

deci

$$J_n < \frac{1}{n^2} \ln \frac{t}{\sqrt{n^2+t^2}}\Bigg|_1^n - \frac{1}{3n^2 t}\Bigg|_1^n - \frac{1}{3n^3}\operatorname{arctg}\frac{t}{n}\Bigg|_1^n + \frac{3}{2n^2},$$

de unde

$$\frac{n^2}{\ln n} J_n < \frac{\ln\sqrt{n^2+1}}{\ln n} + \frac{5}{6\ln n} + \frac{1}{3n\ln n}\operatorname{arctg}\frac{1}{n}. \tag{5}$$

Cum expresiile din membrii din dreapta ai relaţiilor (4) şi (5) tind la 1 când n tinde către infinit, concluzionăm că $l = \lim\limits_{n\to\infty} \dfrac{n^2}{\ln n} J_n = 1$. $\square$

Soluţia 2. Folosind notaţiile I_n şi l de la soluţia 1, obţinem succesiv:

$$l = \lim_{n\to\infty} \frac{n}{\ln n} \sum_{k=0}^{n-1} \int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} \frac{n^2x^2 - k^2}{(1+x^2)(1+k^2)}\,\mathrm{d}x =$$

$$= \lim_{n\to\infty} \frac{n}{\ln n} \sum_{k=0}^{n-1} \int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} \left(\frac{n^2}{1+k^2} - \frac{n^2+k^2}{(1+k^2)(1+x^2)} \right) \mathrm{d}x =$$

$$= \lim_{n\to\infty} \frac{n}{\ln n} \sum_{k=0}^{n-1} \left(\frac{n}{1+k^2} - \frac{n^2+k^2}{1+k^2} \left(\operatorname{arctg}\frac{k+1}{n} - \operatorname{arctg}\frac{k}{n} \right) \right) \mathrm{d}x =$$

$$= \lim_{n\to\infty} \frac{n}{\ln n} \sum_{k=0}^{n-1} \left(\frac{n}{1+k^2} - \frac{n^2+k^2}{1+k^2} \operatorname{arctg} \frac{n}{n^2+k^2+k} \right) dx.$$

Cum însă $x - \dfrac{x^3}{3} < \operatorname{arctg} x < x$ pentru orice $x > 0$, putem încadra l între

$$\lim_{n\to\infty} \frac{n}{\ln n} \sum_{k=0}^{n-1} \left(\frac{n}{1+k^2} - \frac{n^2+k^2}{1+k^2} \cdot \frac{n}{n^2+k^2+k} \right) \text{ şi}$$

$$\lim_{n\to\infty} \frac{n}{\ln n} \sum_{k=0}^{n-1} \left(\frac{n}{1+k^2} - \frac{n^2+k^2}{1+k^2} \cdot \frac{n}{n^2+k^2+k} + \frac{n^2+k^2}{1+k^2} \cdot \frac{n^3}{(n^2+k^2+k)^3} \right).$$

Dar $0 < \dfrac{n}{\ln n} \displaystyle\sum_{k=0}^{n-1} \frac{n^2+k^2}{1+k^2} \cdot \frac{n^3}{(n^2+k^2+k)^3} < \frac{1}{\ln n} \sum_{k=0}^{n-1} \frac{1}{1+k^2} \xrightarrow[n\to\infty]{} 0$, deci

$$l = \lim_{n\to\infty} \frac{n}{\ln n} \sum_{k=0}^{n-1} \left(\frac{n}{1+k^2} \cdot \frac{k}{n^2+k^2+k} \right).$$

Cum

$$\frac{1}{\ln n} \sum_{k=1}^{n-1} \left(\frac{n^2 k}{(1+k^2)(n^2+k^2+k)} - \frac{1}{k} \right) = -\frac{1}{\ln n} \sum_{k=1}^{n-1} \left(\frac{1}{k(1+k^2)} + \frac{1}{n^2+k^2+k} \right),$$

iar membrul drept al acestei relaţii tinde la zero când n tinde la infinit, obţinem $l = \lim\limits_{n\to\infty} \dfrac{1}{\ln n} \displaystyle\sum_{k=1}^{n-1} \frac{1}{k} = 1.$ □

Secţiunea B

Problema B1. Punctele M_1 şi M_2 se mişcă rectiliniu şi uniform pornind din $A_1(0,0,0)$, respectiv $B_1(1,0,0)$, cu vitezele $\overline{v}_1 = \overline{i} + \overline{j} + \overline{k}$ şi $\overline{v}_2 = \overline{i} + \overline{j} - \overline{k}$.

Să se determine ecuaţia suprafeţei generate de dreptele M_1M_2 şi să se precizeze forma ei.

Vasile Pop

Soluţie. La momentul arbitrar t vectorii de poziţie ai punctului $M_1(t)$ şi $M_2(t)$ sunt $\overline{r}_{M_1} = \overline{r}_{A_1} + t \cdot \overline{v}_1$ şi $\overline{r}_{M_2} = \overline{r}_{A_2} + t \cdot \overline{v}_2$, deci punctele M_1 şi M_2 au coordonatele:

$$M_1(t) = (t, t, t),\ M_2(t) = (1+t, t, -t).$$

Un punct arbitrar de pe dreapta M_1M_2 are vectorul de poziţie de forma

$$\overline{r} = (1-s)\overline{r}_{M_1} + s \cdot \overline{r}_{M_2},\ s \in \mathbb{R},$$

deci punctele suprafeţei S au coordonatele (x, y, z) date prin relaţiile:

$$S : \begin{cases} x = (1-s)t + s(1+t) = t + s, \\ y = (1-s)t + st = t, \\ z = (1-s)t - st = t - 2st, \; t, s \in \mathbb{R}, \end{cases}$$

(ecuaţiile parametrice).

Din primele două relaţii obţinem $t = y$ şi $s = x - y$; introducând aceste valori în cea de-a treia relaţie, obţinem ecuaţia implicită

$$S : \; z = y - 2y(x - y)$$

sau

$$S : \; 2y^2 - 2xy + y - z = 0,$$

care este ecuaţia unei cuadrice.

Matricea formei pătratice este

$$A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

cu valorile proprii $\lambda_1 = 1 + \sqrt{2} > 0$, $\lambda_2 = 1 - \sqrt{2} < 0$ şi $\lambda_3 = 0$. Suprafaţa este un paraboloid hiperbolic. $\square$

Deşi juriul a considerat problema uşoară, rezultatele au arătat o slabă pregătire a studenţilor în probleme de geometrie aplicată. Acelaşi lucru s-a observat şi la o problemă asemănătoare, dată la secţiunea C, pe care o lăsăm spre rezolvare:

Problema C3. Să se determine ecuaţia suprafeţei formată din toate punctele spaţiului egal depărtate de dreptele $D_1 : x = y = z$, $D_2 : x - 1 = = y = -z$, şi să se precizeze forma ei. (**Răspuns.** Suprafaţa este un paraboloid hiperbolic de ecuaţie $S : 2yz + 2xz - 2x + y + 1 = 0$.)

Problema B2. Fie V un spaţiu vectorial de dimensiune finită peste corpul K iar $F : V \times V \to K$ o formă biliniară. Definim subspaţiile:

$$V_1 = \{x \in V \mid F(x, y) = 0, \; \forall\, y \in V\}$$

$$V_2 = \{y \in V \mid F(x, y) = 0, \; \forall\, x \in V\}.$$

Să se arate că V_1 şi V_2 au aceeaşi dimensiune.

Problema este clasică, dar face apel la câteva noţiuni esenţiale cum sunt matricea unei forme pătratice, matricea unei aplicaţii liniare sau teorema dimensiunii pentru aplicaţii liniare. Ea s-a dovedit foarte bună ca problemă de concurs.

Soluţie. Alegem o bază $B = \{e_1, e_2, \ldots, e_n\}$ în V şi considerăm matricea lui F în baza B, $A = (a_{ij})_{i,j=\overline{1,n}}$, unde $a_{ij} = F(e_i, e_j) \in K$, $i = \overline{1,n}$.

Avem

$$F\left(\sum_{i=1}^{n} x_i e_i, \sum_{j=1}^{n} y_j e_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j F(e_i, e_j) = \sum_{i,j=1}^{n} a_{ij} x_i y_j$$

şi atunci

- $$x = \sum_{i=1}^{n} x_i e_i \in V_1 \Leftrightarrow \sum_{i,j=1}^{n} a_{ij} x_i y_j = 0,\ \forall\ j = \overline{1,n} \Leftrightarrow$$

$$\Leftrightarrow \sum_{i=1}^{n} a_{ij} x_i = 0,\ \forall\ j = \overline{1,n} \Leftrightarrow A^t \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

- $$y = \sum_{j=1}^{n} y_j e_j \in V_2 \Leftrightarrow \sum_{i,j=1}^{n} a_{ij} x_i y_j = 0,\ \forall\ i = \overline{1,n} \Leftrightarrow$$

$$\Leftrightarrow \sum_{j=1}^{n} a_{ij} y_j = 0,\ \forall\ i = \overline{1,n} \Leftrightarrow A \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

astfel că $V_1 = \ker A^t$ şi $V_2 = \ker A$.

Cum $\operatorname{rang} A = \operatorname{rang} A^t$ şi $\dim\ker A = n - \operatorname{rang} A$, $\dim\ker A^t = n - \operatorname{rang} A^t$ rezultă $\dim V_1 = \dim V_2$. □

Problema B3.

a) Să se arate că

$$\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots = \int_0^1 \frac{x^n}{1+x} \mathrm{d}x.$$

b) Să se calculeze

$$\sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots\right)^2.$$

Ovidiu Furdui

Problema consta iniţial doar în cerinţa de la actualul punct b), dar juriul a considerat necesară adăugarea punctului a) pentru a oferi ideea de rezolvare.

Soluţie. a) Fie $m \in \mathbb{N}$ şi fie

$$S_{2m} = \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} + \cdots - \frac{1}{n+2m}$$

suma parţială de ordin $2m$ asociată seriei din enunţ. Avem că

$$\begin{aligned}S_{2m} &= \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} + \dots - \frac{1}{n+2m} = \\ &= \int_0^1 \left(x^n - x^{n+1} + x^{n+2} - \dots - x^{n+2m-1}\right) \mathrm{d}x = \\ &= \int_0^1 x^n \cdot \frac{1-x^{2m}}{1+x} \mathrm{d}x = \int_0^1 \frac{x^n}{1+x} \mathrm{d}x - \int_0^1 \frac{x^{n+2m}}{1+x} \mathrm{d}x.\end{aligned}$$

Rezultă că

$$\left| S_{2m} - \int_0^1 \frac{x^n}{1+x} \mathrm{d}x \right| = \left| - \int_0^1 \frac{x^{n+2m}}{1+x} \mathrm{d}x \right| \leq \int_0^1 x^{n+2m} \mathrm{d}x = \frac{1}{n+2m+1}.$$

Deci

$$\lim_{m\to\infty} S_{2m} = \int_0^1 \frac{x^n}{1+x} \mathrm{d}x.$$

b) Aşadar

$$\sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \cdots \right)^2 = \sum_{n=0}^{\infty} \left(\int_0^1 \frac{x^n}{1+x} \mathrm{d}x \right) \left(\int_0^1 \frac{y^n}{1+y} \mathrm{d}y \right) =$$

$$= \sum_{n=0}^{\infty} \int_0^1 \int_0^1 \frac{(xy)^n}{(1+x)(1+y)} \mathrm{d}x\mathrm{d}y \overset{(*)}{=} \int_0^1\int_0^1 \frac{1}{(1+x)(1+y)} \left(\sum_{n=0}^{\infty} (xy)^n \right) \mathrm{d}x\mathrm{d}y =$$

$$= \int_0^1\int_0^1 \frac{1}{(1+x)(1+y)(1-xy)} \mathrm{d}x\mathrm{d}y = \int_0^1 \frac{1}{1+x} \left(\int_0^1 \frac{1}{(1+y)(1-xy)} \mathrm{d}y \right) \mathrm{d}x =$$

$$= \int_0^1 \frac{1}{1+x} \left(\frac{\ln 2 - \ln(1-x)}{1+x} \right) \mathrm{d}x =$$

$$= \left(\frac{(1-x)\ln(1-x)}{2(1+x)} + \frac{1}{2}\ln(1+x) - \frac{\ln 2}{1+x} \right) \Bigg|_0^1 = \ln 2$$

şi problema este rezolvată.

Remarcă. Egalitatea $(*)$, adică însumarea termenilor sub semnul integralei duble, este justificată în virtutea teoremei:

Dacă $(u_n)_n$ este un şir de funcţii nenegative măsurabile, atunci

$$\int \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \int u_n.$$

Problema B4.

Să se determine funcţiile continue $f : \mathbb{R} \to \mathbb{R}$ cu proprietatea

$$f(x) - f(y) \in \mathbb{R} \setminus \mathbb{Q}$$

pentru orice $x, y \in \mathbb{R}$ pentru care $x - y \in \mathbb{R} \setminus \mathbb{Q}$.

Vasile Pop

Este aceeaşi cu problema A2 şi a fost considerată pe bună dreptate de juriu cea mai grea de la secţiunea B, doar primii doi clasaţi reuşind să o rezolve parţial.

PROBLEMS

Authors should submit proposed problems to gmaproblems@rms.unibuc.ro. Files should be in PDF or DVI format. Once a problem is accepted for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of June 2014**.

PROPOSED PROBLEMS

393. Let A, B, C, D be four distinct points in a plane Π, which are not the vertices of a parallelogram. Let H be one of the halfspaces bounded by Π.

(i) In H we consider the semicircles of diameters AB and CD that are orthogonal on Π. Prove that in H there is exactly one semicircle with the diameter situated on Π that is orthogonal on the two semicircles and on Π.

We denote by $C(AB, CD)$ the semicircle from (i). Similarly we define $C(AC, BD)$ and $C(AD, BC)$.

(ii) Prove that $C(AB, CD)$, $C(AC, BD)$ and $C(AD, BC)$ pass through the same point.

(iii) Prove that $C(AB, CD)$, $C(AC, BD)$ and $C(AD, BC)$ are orthogonal on each other.

Proposed by Sergiu Moroianu, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

394. Find all polynomials $P \in \mathbb{Z}[X]$ such that $a^2+b^2+c^2 \mid f(a)+f(b)+f(c)$ for any $a, b, c \in \mathbb{Z}$.

Proposed by Vlad Matei, student, University of Wisconsin, Madison, USA.

395. Let $z_1, z_2, \ldots, z_n \geq 1$. Put $P = \prod_{i=1}^n z_i$, $P_i = \prod_{j \neq i} z_j$ $(1 \leq i \leq n)$. Prove the following inequality:

$$\sum_{i=1}^n \frac{1}{1+z_i} + \frac{n(n-2)}{1+\sqrt[n]{P}} \geq (n-1) \sum_{i=1}^n \frac{1}{1+\sqrt[n-1]{P_i}}.$$

Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, and Ştefan Spătaru, International Computer High School of Bucharest, Romania.

396. Let F be a field and let V be an F-vector space. We denote, as usual, by $T(V)$, $S(V)$ and $\Lambda(V)$ the tensor, symmetric and exterior algebras over V, respectively.

Let $I_{S'}$ be the subgroup of $T(V)$ generated by $x_1 \otimes \cdots \otimes x_n - x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$ with $x_1, \ldots, x_n \in V$ and $\sigma \in A_n$. Then $I_{S'}$ is a homogenous ideal in $T(V)$ and we denote $S'(V) = T(V)/I_{S'}$. Then $S'(V)$ is a graded algebra, $S'(V) = \bigoplus_{n\geq 0} S'^n(V)$. We denote by $\odot$ the product on $S'(V)$. Hence if $x_1, \ldots, x_n \in V$ then the image of $x_1 \otimes \cdots \otimes x_n \in T(V)$ in $S'(V) = T(V)/I_{S'}$ is $x_1 \odot \cdots \odot x_n$.

(i) For $n \geq 1$ let $\rho_{S'^n,S^n} : S'^n(V) \to S^n(V)$ be the linear map given by $x_1 \odot \cdots \odot x_n \mapsto x_1 \cdots x_n$. For $n \geq 2$ find a linear map $\rho_{\Lambda^n,S'^n} : \Lambda^n(V) \to S'^n(V)$ such that the short sequence

$$0 \to \Lambda^n(V) \xrightarrow{\rho_{\Lambda^n,S'^n}} S'^n(V) \xrightarrow{\rho_{S'^n,S^n}} S^n(V) \to 0$$

is exact.

(ii) If $F = \mathbb{F}_2$ prove that for any $n \geq 1$ there is a linear map $\rho_{S^n,\Lambda^n} : S^n(V) \to \Lambda^n(V)$ with $x_1 \cdots x_n \mapsto x_1 \wedge \cdots \wedge x_n$. If $n = 2, 3$ find a linear map $\rho_{T^{n-1},S^n} : T^{n-1}(V) \to S^n(V)$ such that the short sequence

$$0 \to T^{n-1}(V) \xrightarrow{\rho_{T^{n-1},S^n}} S^n(V) \xrightarrow{\rho_{S^n,\Lambda^n}} \Lambda^n(V) \to 0$$

is exact.

Proposed by Constantin-Nicolae Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

397. Let $n \geq 1$ be an integer and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a function with the property that the image under f of any sphere S of codimension 1 is a sphere of codimension 1 of the same radius. Prove that f is an isometry.

Proposed by Marius Cavachi, Ovidius University of Constanţa, Romania.

398. Let $A \in \mathcal{M}_n(\mathbb{Q})$ be an invertible matrix.

a) Prove that if for every positive integer k there exists $B_k \in \mathcal{M}_n(\mathbb{Q})$ such that $B_k^k = A$, then all the eigenvalues of A are equal to 1.

b) Is the converse of a) true?

Proposed by Victor Alexandru, Cornel Băeţica, Gabriel Mincu, University of Bucharest, Romania.

399. Let $n \geq 3$ and let $P = a_n X^n + \cdots + a_0 \in \mathbb{R}[X]$ with $a_i > 0$ $\forall i$ such that all the roots of P' are real. If $0 \leq a < b$ prove that

$$\frac{\int_a^b \big(P'(x)\big)^{-1} \mathrm{d}x}{\int_a^b \big(P''(x)\big)^{-1} \mathrm{d}x} \geq \frac{P'(b) - P'(a)}{P(b) - P(a)}.$$

Proposed by Florin Stănescu, Şerban Cioculescu School, Găeşti, Dâmboviţa, Romania.

400. Let $S(n) := \sum_{k=0}^{n}(-2)^k\binom{n}{k}\binom{2n-k}{n-k}$. Prove that $4(n+1)S(n)+(n+2)S(n+2)=0$ and conclude that

$$S_n = \begin{cases} (-1)^{n/2}\binom{n}{n/2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Proposed by Mihai Prunescu, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

401. Let a and b be positive integers. Prove the following identities:

$$(i) \quad \sum_{p\geq 0} p\binom{2a}{a-p}\binom{2b}{b-p} = \frac{ab}{2(a+b)}\binom{2a}{a}\binom{2b}{b},$$

$$(ii) \quad \sum_{p\geq 0}(2p+1)\binom{2a+1}{a-p}\binom{2b+1}{b-p} = \frac{(2a+1)(2b+1)}{a+b+1}\binom{2a}{a}\binom{2b}{b},$$

with the convention that $\binom{m}{n}=0$ if $n<0$ or $n>m$.

Proposed by Ionel Popescu, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

402. Let $u:[a,b]\to\mathbb{R}$ be a twice differentiable function with $u'(a)=u'(b)=0$ and let $\lambda\in\mathbb{R}$.

(1) Prove that $u''(c)=\lambda u(c)u'(c)$ for some $c\in(a,b)$.

(2) If moreover $u''(a)=0$ prove that $(d-a)u''(d)=u'(d)(1+\lambda(d-a)u(d))$ for some $d\in(a,b)$

Proposed by Cezar Lupu, University of Pittsburgh, USA.

403. A parabola $\mathcal{P}$ has the focus F at distance d from the directrix Δ. Find the maximum length of an arc of $\mathcal{P}$ corresponding to a chord of length L.

Proposed by Gabriel Mincu, University of Bucharest, Romania.

404. Let $F:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ be a function satisfying the following conditions:

1) $|F(x,y)|\geq |x|+|y|\ \forall x,y\in\mathbb{Z}$.

2) There are $m,n\geq 1$ and matrices $A=(a_{i,j})$, $B=(b_{i,j})\in M_{m,n}(\mathbb{Z})$ such that

$$F(x,y)=\max_{1\leq i\leq m}\min_{1\leq j\leq n}(a_{i,j}x+b_{i,j}y)\ \forall x,y\in\mathbb{Z}.$$

Prove that either $F(x,y)\geq 0\ \forall x,y\in\mathbb{Z}$ or $F(x,y)\leq 0\ \forall x,y\in\mathbb{Z}$. Give an example of a function F for each of these two cases.

Proposed by Şerban Basarab, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

SOLUTIONS

365. Let K be a field and let $f, g \in K[X]$, $f, g \notin K$, such that $g^n - 1 \mid f^n - 1$ for all $n \geq 1$. Then f is a power of g.

Proposed by Marius Cavachi, Ovidius University of Constanţa, Romania.

Solution by the author. We define several polynomials.

For $k \geq 0$, $n \geq 1$ we define $r_n^{(k)} \in K[X]$ by $r_n^{(0)} = \frac{f^n - 1}{g^n - 1}$ and inductively $r_n^{(k+1)} = g^{k+1} r_{n+1}^{(k)} - f r_n^{(k)}$.

For $k \geq 0$ we define $p_0 = 1$ and $p_{k+1} = (1 - g^{k+1}) f p_k$ for $k \geq 0$. Thus $p_k = (1-g) \cdots (1-g^k) f^k$.

For $k \geq 0$ we define $Q_k \in K[X, Y]$ with $\deg_Y Q_k \leq k$ by $Q_0 = -1$ and $Q_{k+1}(Y) = g^{k+1}(Y-1) Q_k(gY) - f(g^{k+1}Y - 1) Q_k(Y)$ for $k \geq 0$. (Here we regard Q_k as polynomials in the variable Y with coefficients in $K[X]$, i.e., $Q_k \in K[X][Y]$.)

By straightforward calculations one verifies by induction on k that

$$r_n^{(k)} = \frac{p_k f^n + Q_k(g^n)}{(g^{n+k} - 1)(g^{n+k-1} - 1) \cdots (g^n - 1)}.$$

Let $k \geq 0$ be large enough such that $(k+1) \deg g > \deg f$. Since also $\deg_Y Q_k \leq k$, we get

$$\left((k+1)n + \frac{k(k+1)}{2} \right) \deg g > \deg(p_k f^n + Q_k(g^n)).$$

Hence the degree of the denominator of the fraction above, which gives $r_n^{(k)}$, is larger than the degree of the numerator when n is large enough. But $r_n^{(k)} \in K[X]$ so we must have $r_n^{(k)} = 0$, i.e., $p_k f^n + Q_k(g^n) = 0$ for n large enough, say for $n \geq N_0$.

We write $Q_k = a_k Y^k + \cdots + a_0$ with $a_i \in K[X]$. Then we have

$$p_n f^n + \sum_{j=0}^{k} a_j g^{nj} = 0, \quad n \geq N_0.$$

Let $m \geq N_0$. In the equation above we take $n = m, m+1, \ldots, m+k+1$. Hence we get that the homogeneous linear system of $k+2$ equations with $k+2$ unknowns

$$f^{m+i} X_{k+1} + \sum_{j=0}^{k} g^{(m+i)j} X_j = 0, \quad 0 \leq i \leq m+1,$$

has the solution $X_{k+1} = p_k$, $X_j = a_j$ for $0 \leq j \leq k$. This solution is nontrivial since $p_k = (1-g) \cdots (1-g^k) \neq 0$. It follows that Δ, the determinant of the system, is 0. One checks that

$$\Delta = f^m \prod_{j=0}^{k} g^{mj} \Delta_1,$$

where Δ_1 is a Vandermonde determinant, namely

$$\Delta_1 = \pm \prod_{0\le j<l\le k} (g^j - g^l) \prod_{j=0}^{k} (f - g^j).$$

Since $g^j \neq g^l$ when $j \neq l$, we have $f = g^j$ for some j. □

A note from the editor. The proof of this result is very ingenious but it involves a construction, the polynomials $r_n^{(k)}$, which the reader might find very unnatural. We give a possible approach that leads to the definition of $r_n^{(k)}$ in a natural way.

The author first used this method to solve the similar problem for $\mathbb{Z}$ instead of $K[X]$: determine all $a, b \in \mathbb{Z}$ with $|a|, |b| > 1$ such that $b^n - 1 \mid a^n - 1$ $\forall n \geq 1$.

The idea is to find some linear combination of $r_n := \frac{a^n-1}{b^n-1} \in \mathbb{Z}$ with coefficients in $\mathbb{Z}$ that is less than 1 in absolute value. Since such linear combination is an integer, it must be zero. This way we obtain algebraic relations between a and b.

We have $r_n = r'_n + r''_n$, with $r'_n = \frac{a^n}{b^n-1}$ and $r''_n = \frac{-1}{b^n-1}$. Since $|r''_n| \ll 1$ when $n \gg 0$ we will focus on r'_n. We have $b^n - 1 = b^n(1 + O(\frac{1}{b^n}))$, so $r'_n = \frac{a^n}{b^n}(1 + O(\frac{1}{b^n}))$. Similarly, $r'_{n+1} = \frac{a^{n+1}}{b^{n+1}}(1 + O(\frac{1}{b^n}))$, so $r'_{n+1} = \frac{a}{b} r'_n (1 + O(\frac{1}{b^n}))$. Therefore if $r'^1_n := br_{n+1} - ar_n$ then $r'^1_n = O(\frac{1}{b^n} r_n) = O(\frac{a^n}{b^{2n}})$. This way the order of magnitude was decreased by a factor of b^n. One calculates $r'^{(1)}_n = \frac{(1-b)a^{n+1}}{(b^n-1)(b^{n+1}-1)}$. We have $r'^{(1)}_n = \frac{(1-b)a^{n+1}}{b^{2n+1}}(1+O(\frac{1}{b^n}))$. Similarly, $r'^{(1)}_{n+1} = \frac{(1-b)a^{n+2}}{b^{2n+3}}(1 + O(\frac{1}{b^n}))$, so $r'^{(1)}_{n+1} = \frac{a}{b^2} r'^{(1)}_n (1 + O(\frac{1}{b^n}))$. Hence, if $r'^{(2)}_n = b^2 r'^{(1)}_{n+1} - a r'^{(1)}_n$ then $r'^{(2)}_n = O(\frac{1}{b^n} r'^{(2)}_n) = O(\frac{a^n}{b^{3n}})$, so the order of magnitude decreased again by b^n.

We see a pattern. For $k \geq 0$ we define $r'^{(k)}_n$ recursively by $r'^{(0)}_n = r'_n = \frac{a^n}{b^n-1}$ and $r'^{(k+1)}_n = b^{n+1} r'^{(k)}_{n+1} - a r'^{(k)}_n$. And we show inductively that $r'^{(k)}_n = \frac{p_k a^n}{(b^n-1)\cdots(b^{n+k}-1)}$, where $p_k \in \mathbb{Z}$ is some constant satisfying $p_{k+1} = (1-b^{k+1})ap_k$ so that $p_k = (1-b)\cdots(1-b^k)a^k$. The definition $r'^{(k+1)}_n = b^{n+1} r'^{(k)}_{n+1} - a r'^{(k)}_n$ is justified by the following. We have

$$r'^{(k)}_n = p_k \frac{a^n}{b^{kn+k(k+1)/2}}(1 + O(\frac{1}{b^n}))$$

and similarly

$$r'^{(k)}_n = p_k \frac{a^{n+1}}{b^{k(n+1)+k(k+1)/2}}(1 + O(\frac{1}{b^n})).$$

Hence $r'^{(k)}_{n+1} = \frac{a}{b^{k+1}} r'^{(k)}_n (1 + O(\frac{1}{b^n}))$. Therefore when we take $r'^{(k+1)}_n = b^{n+1} r'^{(k)}_{n+1} - a r'^{(k)}_n$ we have $r'^{(k+1)}_n = O(\frac{1}{b^n} r'^{(k)}_n)$. Hence at each step the order of magnitude decreases by a factor of b^n.

We now apply the same linear transformations to r_n, i.e., we define $r^{(0)}_n = r_n$ and $r^{(k+1)}_n = b^{k+1} r^{(k)}_{n+1} - a r^{(k)}_n$. Since $r_n \in \mathbb{Z}$ $\forall n$ we have $r^{(k)}_n \in \mathbb{Z}$ $\forall k, n$. Since $r_n = r'_n + r''_n$ we have $r^{(k)}_n = r'^{(k)}_n + r'^{(k)}_n$, where $r''^{(0)}_n = r''_n = \frac{-1}{b^n - 1}$ and $r''^{(k+1)}_n = b^{k+1} r''^{(k)}_{n+1} - a r''^{(k)}_n$. It is easy to see, by induction, that $r''^{(k)}_n$ is a linear combination with integer coefficients of $r''_n, \ldots, r''_{n+k}$, $r''^{(k)}_n = c_0 r''_n + \cdots + c_k r''_{n+k}$. Since $r''_{n+i} = O(\frac{1}{b^n})$ for $0 \leq i \leq k$, we have $r''^{(k)}_n = O(\frac{1}{b^n})$. Hence, $|r''^{(k)}_n| \ll 1$ for $n \gg 0$. Also note that $r'^{(k)}_n = O(\frac{a^n}{b^{(k+1)n}})$. Then if we take k large enough such that $|b|^{k+1} > |a|$ we have $|r'^{(k)}_n| \ll 1$ when $n \gg 0$. Hence for $n \gg 0$ we have $|r^{(k)}_n| \leq |r'^{(k)}_n| + |r''^{(k)}_n| < 1$. As $r^{(k)}_n \in \mathbb{Z}$, we get $r^{(k)}_n = 0$.

Now

$$
\begin{aligned}
r''^{(k)}_n &= \sum_{i=0}^{k} c_i \frac{-1}{b^{n+i} - 1} = \frac{\sum_{i=0}^{n} -c_i (b^n - 1) \cdots (\widehat{b^{n+i} - 1}) \cdots (b^{n+k} - 1)}{(b^n - 1) \cdots (b^{n+k} - 1)} \\
&= \frac{Q_k(b^n)}{(b^n - 1) \cdots (b^{n+k} - 1)},
\end{aligned}
$$

where $Q_k(X) = \sum_{i=0}^{n} -c_i (X - 1) \cdots (\widehat{b^i X - 1}) \cdots (b^k X - 1)$. (One can prove that Q_k are given by $Q_0 = -1$ and $Q_{k+1}(X) = b^{k+1}(X - 1) Q_k(bX) - a(b^{k+1} X - 1) Q_k(X)$, same as in author's proof.) Together with $r''^{(k)}_n = \frac{p_k a^n}{(b^n - 1) \cdots (b^{n+k} - 1)}$ this implies $0 = r^{(k)}_n = \frac{p_k a^n + Q_k(b^n)}{(b^n - 1) \cdots (b^{n+k} - 1)}$, so from here the proof follows as above, with a, b replacing f, g.

The same reasoning may be applied to $f, g \in K[X]$ instead of a, b. The degree function $\deg : K[X] \to \mathbb{Z}_{\geq 0} \cup \{-\infty\}$ extends to $\deg : K(X) \to \mathbb{Z} \cup \{-\infty\}$ by defining $\deg \frac{P}{Q} = \deg P - \deg Q$. This extended degree satisfies the usual properties of the degree: $\deg AB = \deg A + \deg B$ and $\deg(A + B) \leq \max\{\deg A, \deg B\}$ $\forall A, B \in K(X)$. Then we define a norm $|\cdot| : K(X) \to \mathbb{R}_{\geq 0}$ by $|A| = 2^{\deg A}$. We have $|A| = 0$ iff $\deg A = -\infty$, i.e., iff $A = 0$. Then $|\cdot|$ is a non-archimedian norm, i.e., $|A|\,|B| = |AB|$ and the triangle inequality $|A + B| \leq |A| + |B|$ is replaced by the stronger inequality $|A + B| \leq \max\{|A|, |B|\}$. The completion of $(K(X), |\cdot|)$ is $K((\frac{1}{X}))$.

Also, same as for $\mathbb{Z}$, if $A \in K[X]$ and $|A| < 1$ then $A = 0$. (Otherwise $\deg A \geq 0$ so $|A| = 2^{\deg A} \geq 1$.) The notation $A = O(B)$ means that $|A| \leq c|B|$ for some constant $c > 0$, which is equivalent to $\deg A \leq c' + \deg B$ for some constant $c' \in \mathbb{R}$ (actually, $c' = \log_2 c$).

With this definition of the norm on $K(X)$, the proof follows almost verbatim from that on $\mathbb{Z}$, but with a, b replaced by f, g. Note that the condition $|a| < b^{k+1}$ translates as $|f| < |g|^{k+1}$, which is equivalent to $\deg f \leq (k+1)\deg g$.

We give a possible approach that leads essentially to the same solution. We will produce linear combinations of the polynomials $\frac{f^n-1}{g^n-1}$ with coefficients in $K[f,g] \in K[X]$ that have negative degree, so they must be zero.

Let $a = \deg f$, $b = \deg g$. Let $k \geq 1$ with $(k+1)b > a$. Then for any $n \geq 1$ we have the identity $\frac{1}{g^n-1} = \frac{1}{g^n} + \cdots + \frac{1}{g^{kn}} + \frac{1}{g^{kn}(g^n-1)}$. (Note that $\frac{1}{g^n} + \cdots + \frac{1}{g^{kn}}$ is the beginning of the expansion $\frac{1}{g^n-1} = \sum_{i\geq 1} \frac{1}{g^{in}}$, which holds in $K((\frac{1}{X}))$.) It follows that

$$\frac{f^n-1}{g^n-1} = \left(\frac{f}{g}\right)^n + \cdots + \left(\frac{f}{g^k}\right)^n + \frac{f^n}{g^{kn}(g^n-1)} - \frac{1}{g^n-1}.$$

Note that $\deg \frac{f^n}{g^{kn}(g^n-1)} = -((k+1)b-a)n$ and $\deg \frac{1}{g^n-1} = -bn$ are $\ll 0$ when $n \gg 0$. Therefore in order that a linear combination of the $\frac{f^n-1}{g^n-1}$ with $n \gg 0$ have negative degree it is enough that in this linear combination the terms of the type $\left(\frac{f}{g^k}\right)^n$ cancel each other. To do this we employ the usual technique from linear recurrence sequences.

Now $\frac{f}{g}, \ldots, \frac{f}{g^k}$ are the roots of $A_kY^k + \cdots + A_0 = (gY-f)\cdots(g^kY-f) \in K(X)[Y]$. Note that $A_i \in K[f,g] \subseteq K[X]$. In particular, $A_0 = (-f)^n \neq 0$. Then the sequence $x_n = \left(\frac{f}{g}\right)^n + \cdots + \left(\frac{f}{g^k}\right)^n$ satisfies the linear recurrence $A_0x_n + \cdots + A_kx_{m+k} = 0$. It follows that

$$\sum_{j=0}^k A_k \frac{f^{n+k}-1}{g^{n+k}-1} = \sum_{j=0}^k A_j \left(\left(\frac{f}{g}\right)^{n+j} + \cdots + \left(\frac{f}{g^k}\right)^{n+j} + \frac{f^{n+j}}{g^{k(n+j)}(g^{n+j}-1)} \right.$$
$$\left. - \frac{1}{g^{n+j}-1}\right) = \sum_{j=0}^k A_j \left(\frac{f^{n+j}}{g^{k(n+j)}(g^{n+j}-1)} - \frac{1}{g^{n+j}-1}\right).$$

But for $n \gg 0$ and $0 \leq j \leq k$ we have $\deg A_j \frac{f^{n+j}}{g^{k(n+j)}(g^{n+j}-1)} = \deg A_j - ((k+1)b-a)(n+j) < 0$ and $\deg A_j \frac{1}{g^{n+j}-1} = \deg A_j - (n+j)b < 0$. Hence the degrees of all the terms in the sum above are negative, thus we have $\deg(\sum_{j=0}^k A_j \frac{f^{n+j}-1}{g^{n+j}-1}) < 0$, which implies $\sum_{j=0}^k A_j \frac{f^{n+j}-1}{g^{n+j}-1} = 0$. By multiplying with $(g^n-1)\cdots(g^{n+k}-1)$ one gets $0 = \sum_{j=0}^k A_j(g^n-1)\cdots(f^{n+j}-1)\cdots(g^{n+k}-1) = P_k(g^n, f^k)$, with $P_k \in K[f,g][Y,Z] \subseteq K[X][Y,Z]$ given by

$$P_k(Y,Z) = \sum_{j=0}^k A_j(Y-1)\cdots(f^jZ-1)\cdots(g^kY-1).$$

(Here f^jZ-1 replaces the factor g^jY-1 in the product $(Y-1)\cdots(g^kY-1)$.) Since all terms of $P_k(Y,Z)$ but the one corresponding to $j=0$ contain the factor $Y-1$, we have $Q_k(1,Z)=A_0(Z-1)(g-1)\cdots(g^k-1)\neq 0$, so $P_k(Y,Z)\neq 0$. (Recall that $A_0\neq 0$.)

We have $\deg_Y P_k(Y,Z)=k$ and $\deg_Z P_k(Y,Z)=1$, so we write

$$P_k(Y,Z)=\sum_{i=0}^{1}\sum_{j=0}^{k}B_{i,j}Z^iY^j.$$

Take n sufficiently large such that $P_k(g^n,f^n)=\cdots=P_k(g^{n+2k+1},f^{n+2k+1})=0$. Then the linear system of $2k+2$ equations with $2k+2$ unknowns

$$\sum_{i=0}^{1}\sum_{j=0}^{k}X_{i,j}f^{i(n+s)}g^{j(n+s)}=0,\quad 0\le s\le 2k+1,$$

has nontrivial solution, so its determinant is 0. One sees that the determinant is $\Delta=\prod_{j=0}^{k}g^{jn}\prod_{j=0}^{k}f^ng^{jn}\Delta_1$, where Δ_1 is the Vandermonde determinant

$$\Delta_1=\pm\prod_{0\le s<t\le k}(g^t-g^s)\prod_{0\le s<t\le k}(fg^t-fg^s)\prod_{0\le s,t\le k}(g^t-fg^s).$$

It follows that $g^t-fg^s=0$ for some s,t and so $f=g^{t-s}$. $\square$

Our sum $\sum_{j=0}^{k}A_j\frac{f^{n+j}-1}{g^{n+j}-1}$ coincides with $r_n^{(k)}$ defined by the author. If $(gY-f)\cdots(g^kY-f)=A_0^{(k)}+\cdots+A_k^{(k))}$ then, by using the relations $A_0^{(k+1)}+\cdots+A_{k+1}^{(k+1)}=(A_0^{(k)}+\cdots+A_k^{(k)})(g^{k+1}Y-f)$ and $r_n^{(k+1)}=g^{k+1}r_{n+1}^{(k)}-fr_n^{(k)}$, one proves by induction on k that $\sum_{j=0}^{k}A_j^{(k)}\frac{f^{n+j}-1}{g^{n+j}-1}=r_n^{(k)}$.

Also one may prove that $B_{1,0}=p_k$ and $B_{1,1}=\cdots=B_{1,k}=0$, so $P_k(Y,Z)$ can be written as $p_kZ+Q_k(Y)$ for some $Q_k\in K[X][Y]$. Therefore $P(g^n,f^n)=p_kf^n+Q_k(g^n)$, same as in the author's proof.

366. Let K be an algebraically closed field of characteristic $p>0$.

For $i\ge 0$ we define the polynomials $Q_i\in\mathbb{Q}[X]$ by $Q_0=X$ and $Q_{i+1}=\frac{Q_i^p-Q_i}{p}$. If $k\ge 0$ writes in basis p as $k=c_0+c_1p+\cdots+c_sp^s$ with $0\le c_i\le p-1$ then we define $P_k\in\mathbb{Q}[X]$ by $P_k=Q_0^{c_0}Q_1^{c_1}\cdots Q_s^{c_s}$.

Prove that if $f=X^k+a_{k-1}X^{k-1}+\cdots+a_0\in K[X]$ with $a_0\neq 0$ has the roots $\alpha_1,\ldots,\alpha_s$ with multiplicities $k_1,\ldots,k_s$ then

$$V_f:=\{(x_n)_{n\ge 0}:x_n\in K,\ x_{n+k}+a_{k-1}x_{n+k-1}+\cdots+a_0x_n=0\ \forall n\ge 0\}$$

is a vector space with $\{(P_j(n)\alpha_i^n)_{n\ge 0}:1\le i\le s,\ 0\le j\le k_i-1\}$ as a basis.

(Hint: Use the note "Linear Recursive Sequences in Arbitrary Characteristics" by C.N. Beli from the issue 1-2/2012 of GMA.)

Proposed by Constantin-Nicolae Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

Solution by the author. From [B] we know that a basis of V_F is given by $\{\binom{n}{j}\alpha_i^n)_{n\geq 0} : 1 \leq i \leq s,\ 0 \leq j \leq k_i - 1\}$. It is enough to prove that for every i $(\binom{n}{j}\alpha_i^n)_{n\geq 0}$ with $0 \leq j \leq k_i - 1$ and $(P_j(n)\alpha_i^n)_{n\geq 0}$ with $0 \leq j \leq k_i - 1$ are bases for the same K-vector space. Therefore we reduce our problem to proving that for any $k \geq 0$ $\binom{X}{0}, \ldots, \binom{X}{k}$ and $P_1, \ldots, P_k$ are bases for the same K-vector space. Here $\binom{X}{l}$ and P_l are regarded as polynomial functions from $\mathbb{Z}$ (or merely $\mathbb{Z}_{\geq 0}$) to K. In fact they are functions from $\mathbb{Z}$ to $\mathbb{Z}$ but they take values in K when composed to the left with the ring morphism $\mathbb{Z} \to K$.

We need some preliminary results.

Lemma. Q_i is a polynomial of degree i with leading coefficient $p^{-\frac{p^i-1}{p-1}}$ and $Q_i(\mathbb{Z}) \subseteq \mathbb{Z}$, for all $i \geq 0$.

Proof. We use induction on i. For $i = 0$ the three statements are obvious. We prove the induction step $i \to i+1$. Since $Q_{i+1} = \frac{Q_i^p - Q_i}{p}$ and $\deg Q_i > 0$, we have $\deg Q_{i+1} = p \deg Q_i = p^{i+1}$. Also, if a, b are the leading coefficients of Q_i and Q_{i+1}, respectively, then $b = \frac{1}{p}a^p$. Since $a = p^{-\frac{p^i-1}{p-1}}$, we have $b = p^{-\frac{p^{j+1}-1}{p-1}}$. Finally, if $n \in \mathbb{Z}$ then by the induction hypothesis $Q_i(n) \in \mathbb{Z}$, so $Q_{i+1}(n) = \frac{Q_i(n)^p - Q_i(n)}{p} \in \mathbb{Z}$, and therefore $Q_{i+1}(\mathbb{Z}) \subseteq \mathbb{Z}$. $\square$

Corollary. P_k is a polynomial of degree k with leading coefficient $p^{-e_p(k!)}$ and $P_k(\mathbb{Z}) \subseteq \mathbb{Z}$, for all $k \geq 0$. (Here by $e_p(a)$ we mean the biggest power of p dividing a.)

Proof. We write k in base p as $k = \sum_{j\geq 0} c_j p^j$ with $0 \leq c_j \leq p-1$. Then $P_k = \prod_{j\geq 0} Q_j^{c_j}$. It follows that $\deg P_k = \sum_{j\geq 0} c_j \deg Q_j = \sum_{j\geq 0} c_j p^j = k$ and, since $Q_j(\mathbb{Z}) \subseteq \mathbb{Z}$, we have $P_k(\mathbb{Z}) \subseteq \mathbb{Z}$. For the second statement recall that $e_p(k!) = \frac{k - s_p(k)}{p-1}$, where $s_p(k)$ is the sum of digits of k written in base p. Since the leading coefficient of Q_j is $p^{-\frac{p^j-1}{p-1}}$, the leading coefficient of P_k will be $\prod_{j\geq 0}\left(p^{-\frac{p^j-1}{p-1}}\right)^{c_j} = p^{-S}$, where

$$S = \sum_{j\geq 0} c_j \frac{p^j - 1}{p-1} = \frac{\sum_{j\geq 0} c_j p^j - \sum_{j\geq 0} c_j}{p-1} = \frac{k - s_p(k)}{p-1} = e_p(k!),$$

so we get the desired result. $\square$

We now start the proof. We already know from [B] that $\binom{X}{0}, \ldots, \binom{X}{k}$ are linearly independent, so they are the basis of a K-vector space V of dimension $k+1$. Therefore it is enough that W, the k-vector space spanned by $P_0, \ldots, P_k$, coincide with V, i.e., that $\binom{X}{l} \in W$ and $P_l \in V$ for $0 \leq l \leq k$.

Let $a \mapsto \hat{a}$ be the morphism of rings $\mathbb{Z} \to K$. Since K has characteristic p, the image of this morphism is $\mathbb{Z}_p$ and it can be extended to $\mathbb{Z}_{(p)} := \{\frac{a}{b} : a, b \in \mathbb{Z},\ p \nmid b\}$ as $\widehat{\frac{a}{b}} = \hat{a}\hat{b}^{-1}$.

Let M be the $\mathbb{Z}$-module generated by $\binom{X}{0},\dots,\binom{X}{k}$ and let N be the $\mathbb{Z}_{(p)}$-module generated by $P_0,\dots,P_k$. We claim that $P_l \in M$ and $\binom{X}{l} \in N$ for $0 \le l \le k$. (Here $\binom{X}{l}$ and P_l are just polynomials in $\mathbb{Q}[X]$.)

We know that $\binom{X}{l}$, $l \ge 0$, are a $\mathbb{Z}$-basis for the module of all integral valued polynomials in $\mathbb{Q}[X]$. That is, $M = \{P \in \mathbb{Q}[X] : P(\mathbb{Z}) \subseteq \mathbb{Z}\}$. Then $P_l \in M$ follows from the Corollary.

For the other statement we use induction on k. When $k = 0$ we have $\binom{X}{0} = P_0 = 1$, so our statement is trivial. We now prove the induction step $k-1 \to k$. If $l < k$ then by the induction hypothesis $\binom{X}{l}$ belongs to the $\mathbb{Z}_{(p)}$-module generated by $\binom{X}{0},\dots,\binom{X}{k-1}$ and therefore to N, so we still have to prove that $\binom{X}{k} \in N$. We have $k! = p^{e_p(k!)}a$ with $p \nmid a$. Since $\binom{X}{k}$ and P_k have the same degree k and their leading coefficients $\frac{1}{k!} = \frac{1}{e_p(k!)a}$ and $\frac{1}{e_p(k!)}$, respectively, the polynomial $a\binom{X}{k} - P_k$ has degree less than k and it is also integral valued. Therefore it belongs to the $\mathbb{Z}$-module generated by $\binom{X}{0},\dots,\binom{X}{k-1}$. Since $\binom{X}{0},\dots,\binom{X}{k-1} \in N$, we get $a\binom{X}{k} - P_k \in N$. But $p \nmid a$, so $a^{-1} \in \mathbb{Z}_{(p)}$. It follows that $\binom{X}{k} = a^{-1}P_k + a^{-1}(a\binom{X}{k} - P_k) \in N$.

Let $0 \le l \le k$. We write $P_l = \sum_{l=0}^{k} \alpha_l \binom{X}{l}$ and $\binom{X}{k} = \sum_{l=0}^{k} \beta_l P_l$, with $\alpha_l \in \mathbb{Z}$ and $\beta_l \in \mathbb{Z}_{(p)}$. When we regard $\binom{X}{l}$ and P_l as polynomial functions $\mathbb{Z} \to K$ we obtain $P_l = \sum_{l=0}^{k} \hat{\alpha}_l \binom{X}{l} \in V$ and $\binom{X}{k} = \sum_{l=0}^{k} \hat{\beta}_l P_l \in W$. □

References

[B] C.N. Beli, Linear recursive sequences in arbitrary characteristics, *Gaz. Matem. Seria A* **30(109)** (2012), 32–36.

367. Give examples of functions $f, g : \mathbb{R} \to \mathbb{R}$ such that: f has period $\sqrt{2}$, g has period $\sqrt{3}$ and $f + g$ has period $\sqrt{5}$.

`Proposed by George Stoica, Department of Mathematical Sciences, University of New Brunswick, Canada.`

Solution by the author. Note that $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ are linearly independent over $\mathbb{Q}$. In particular, if $l, m, n \in \mathbb{Z}$ are such that $l\sqrt{2}+m\sqrt{3}+n\sqrt{5} = 0$, then $l = m = n = 0$.

Let us consider

$$A := \{l\sqrt{2} + m\sqrt{3} + n\sqrt{5} \,:\, l, m, n \in \mathbb{Z}\},$$

and define:

$$f(x) = m\sqrt{3} + n\sqrt{5}, \quad g(x) = l\sqrt{2} - n\sqrt{5} \ \text{ for } x = l\sqrt{2} + m\sqrt{3} + n\sqrt{5} \in A,$$
$$\text{and } f(x) = g(x) = 0 \text{ for } x \notin A.$$

Note that a non-zero value of f determines uniquely m and n. Thus, for a fixed pair of integers m and n, not both of which are 0, we have $f(x) = m\sqrt{3} + n\sqrt{5}$ only at the points $x = l\sqrt{2} + m\sqrt{3} + n\sqrt{5}$, for arbitrary $l \in \mathbb{Z}$.

Since $f(x) = f(x + \sqrt{2})$ for any x in the complement of A, we conclude that f has period $\sqrt{2}$.

Similar arguments show that g has period $\sqrt{3}$ and $(f + g)(x) = l\sqrt{2} + m\sqrt{3}$ $(x \in A)$ has period $\sqrt{5}$. $\square$

368. Find all matrices $X_1, \ldots, X_9 \in M_2(\mathbb{Z})$ with the property that $\det X_k = 1$ for all k and $X_1^4 + \cdots + X_9^4 = X_1^2 + \cdots + X_9^2 + 18I_2$.

Proposed by Florin Stănescu, Şerban Cioculescu School, Găeşti, Dâmboviţa, Romania.

Solution by Victor Makanin, Sankt Petersburg, Russia. For $X \in M_2(\mathbb{Z})$ having determinant 1 and trace $a \in \mathbb{Z}$ we have $X^2 - aX + I_2 = 0_2$, which implies $X^4 - X^2 - 2I_2 = (a^3 - 3a)X - a^2 I_2$, therefore the trace of $X^4 - X^2 - 2I_2$ is

$$\operatorname{Tr}(X^4 - X^2 - 2I_2) = (a^3 - 3a)\operatorname{Tr}(X) - a^2 \operatorname{Tr}(I_2) = a^4 - 5a^2.$$

Now let $a_k = \operatorname{Tr}(X_k) \in \mathbb{Z}$ (for all $k \in \{1, \ldots, 9\}$). Since the given equality can be also written as

$$X_1^4 - X_1^2 - 2I_2 + \cdots + X_9^4 - X_9^2 - 2I_2 = 0,$$

we infer that

$$\operatorname{Tr}(X_1^4 - X_1^2 - 2I_2) + \cdots + \operatorname{Tr}(X_9^4 - X_9^2 - 2I_2) = 0,$$

and, by the above observation,

$$\sum_{k=1}^{9}(a_k^4 - 5a_k^2) = 0 \Leftrightarrow \sum_{k=1}^{9}(2a_k^2 - 5)^2 = 225.$$

Now we have the number 225 written as a sum of nine squares of numbers of the form $2a^2 - 5$, with integer a. One easily sees that the numbers $(2a_k^2 - 5)^2$ smaller than 225 can only be 5^2 (for $a_k = 0$), 3^2 (for $a_k = \pm 1$, or $a_k = \pm 2$), or 13^2 (when $a_k = \pm 3$). The last one is easily eliminated (if one square is 13^2, the remaining eight would be either 3^2, or 5^2, with sum $225 - 169 = 56$), so all of them need to be either 3^2, or 5^2 — and they are nine, which sum to 225. Obviously only the possibility $(2a_k^2 - 5)^2 = 5^2 \Leftrightarrow a_k = 0$ for all $k \in \{1, \ldots, 9\}$ remains, so each of the matrices X_k has trace 0. Conversely, if this happens, then $X_k^2 = -I_2$ and $X_k^4 = I_2$ for all k, and the condition from the enounce is fulfilled.

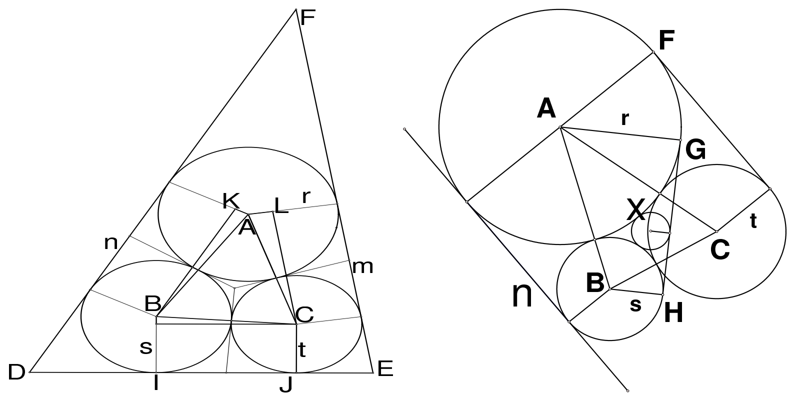
We conclude that $X_1, \ldots, X_9$ can be any integer matrices with determinant 1 and trace 0. One finds that this means that some integers a_k, b_k, c_k do exist, fulfilling $a_k^2 + b_k c_k = -1$, and such that

$$X_k = \begin{pmatrix} a_k & b_k \\ c_k & -a_k \end{pmatrix} \text{ for all } \ k \in \{1, \ldots, 9\}.$$

369. A stick is broken at random at two points (each point is uniformly distributed relative to the whole stick) and the parts' lengths are denoted by r, s, and t. Show that the probability of the existence of a triangle encompassing three circles of radii r, s and t each side tangent to two of the circles and the circles are mutually externally tangent, is equal to $\frac{5}{27}$.

Eugen J. Ionaşcu, Department of Mathematics, Columbus State University, Columbus, Georgia, U.S.A.

Solution by the author. We are beginning with the simple observation that a triangle with the sides $r+s$, $s+t$ and $t+r$ always exists. So, the three circles externally tangent of radii r, s, and t can be always constructed. Without loss of generality we may assume that $r > s > t > 0$ (the probability that two of the numbers or all three to be equal is zero) and $t+s+r=\sqrt{3}$. To account for the other possible orders, we will multiply the probability we obtain in the end by 6. We are denoting the center of the biggest circle by A, the next smaller circle's center by B and C for the center of the smallest circle. Then, the external tangent lines to each two of the circles exist. Basically we need to characterize when three of them can form a triangle with the circles in the interior (Figure 1a). So, let us start with one of the tangent lines, the one tangent to the smaller circles which does not intersect the big circle. We denote it by $\overleftrightarrow{DE}$ and let I and J the two points of tangency as in the Figure 1a.



(a) $r \geq s \geq t > 0$ (b) $s < r$, $s < 4t$

Figure 1. *The three circles and the enclosing triangle; exceptional situation*

We consider a parallel line to $\overleftrightarrow{DE}$ through C and form a rectangle and a right triangle by splitting the trapezoid $BIJC$ into two parts. The Pythagorean Theorem gives us that the length of the common tangent line segment to both of the smaller circles is equal to: $IJ = \sqrt{(s+t)^2-(s-t)^2} = 2\sqrt{st}$. Similarly, the tangent line segment to the circles centered A and C has length $2\sqrt{rt}$ and the third tangent segment is of length $2\sqrt{rs}$.

First let us show that $\overleftrightarrow{DE}$ always intersects the tangent line, m, to the circles centered at C and A (we let E be this point of intersection as a result). The order between r, s and t tells us that the angle $\sphericalangle ACB$ is the biggest angle of the triangle ABC and so it is more than $60°$. The angle between these tangent lines, say ω, is then more than $60°$ and less than $180° + 60° = 240°$ (including the reflex possibility). In order to have a triangle DEF containing the three circles we need to limit ω to be less than $180°$ which insures the existence of E. Let us observe (Figure 1 (b)) that $\omega \geq 180°$ if t is smaller than the radius x of a circle tangent to the bigger circles and their common tangent line. By what we have observed above $2\sqrt{sx} + 2\sqrt{rx} = 2\sqrt{rs}$ which means $x = \dfrac{rs}{(\sqrt{s}+\sqrt{r})^2} = \dfrac{s}{\left(\sqrt{\dfrac{s}{r}}+1\right)^2} > \dfrac{s}{4}$. So, the first restriction we need to have on these numbers is

$$t > \frac{rs}{(\sqrt{s}+\sqrt{r})^2} > \frac{s}{4}, \quad \text{or} \quad r < \frac{st}{(\sqrt{s}-\sqrt{t})^2}. \tag{6}$$

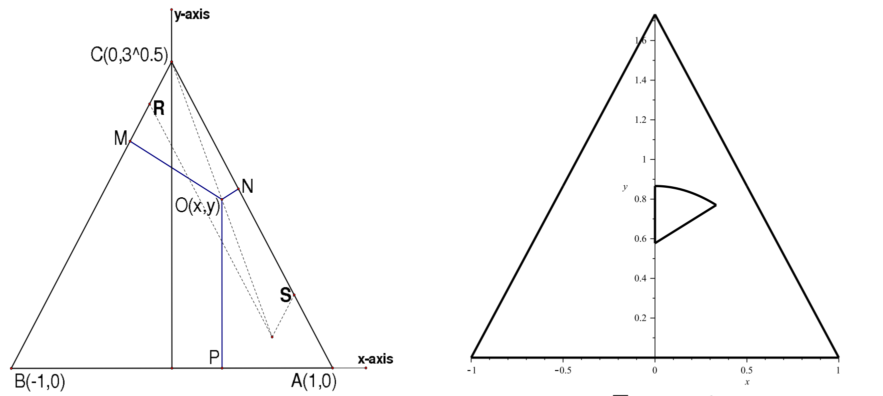
We observe that the third tangent, denoted in Figure 1 by n, is insured by (6) to intersect $\overleftrightarrow{DE}$ so we will let D be the point of intersection. Let L be the point of intersection of the parallel to m through C with the radius corresponding to the tangency point on m and similarly on the other side we let K be that point.

Finally, to insure that m and n intersect, on the same side of $\overleftrightarrow{DE}$ as the circles we need to have

$$\mathrm{m}(\sphericalangle KBA) + \mathrm{m}(\sphericalangle ABC) + \mathrm{m}(\sphericalangle BCA) + m(\angle ACL) < 180°,$$

by the original Euclidean fifth postulate. This is equivalent to

$$\arcsin\left(\frac{r-s}{r+s}\right) + \arcsin\left(\frac{r-t}{r+t}\right) < \mathrm{m}(\sphericalangle BAC).$$



(a) $0 < t < s < r$ (b) $0 < x < \dfrac{1}{3}$, $y < \dfrac{\sqrt{3}(1-x^2)}{2}$, $y > \dfrac{1+x}{\sqrt{3}}$

Figure 2. $A(1,0)$, $B(-1,0)$ *and* $C(0,\sqrt{3})$, $ON = t$, $OM = s$, $OP = r$

Because $u \to \cos u$ is a decreasing function for $u \in [0, 180°]$, using the law of cosines in the triangle ABC and the formula $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, this last inequality translates into

$$\frac{(r+s)^2+(r+t)^2-(s+t)^2}{2(r+s)(r+t)} < \left(\frac{2\sqrt{rs}}{r+s}\right)\left(\frac{2\sqrt{rt}}{r+t}\right) - \frac{(r-s)(r-t)}{(r+s)(r+t)}.$$

After some algebra, one can reduce this to

$$r < 2\sqrt{st}. \tag{7}$$

Let us observe that $2\sqrt{st} < \dfrac{st}{\left(\sqrt{s}-\sqrt{t}\right)^2}$ is equivalent to $2s+2t-5\sqrt{st} < 0$ or $\left(2\sqrt{\dfrac{s}{t}}-1\right)\left(\sqrt{\dfrac{s}{t}}-2\right) < 0$. This is true under the necessary condition $s < 4t$. So, the existence of an encompassing triangle around the three circles of radii r, s, t satisfying $t < s < r$ is given by (7), and $s < 4t$.

From a probabilistic point of view it turns out that we can look at choosing r, s and t as being the distances of a point $O(x, y)$ inside an equilateral triangle ABC, Figure 2, to the sides of the triangle as in Figure 2 (a). One can easily find that $r = y$ and $s = \dfrac{\sqrt{3}(1+x)-y}{2}$, and $t = \dfrac{\sqrt{3}(1-x)-y}{2}$.

The condition $t < s$ is equivalent to $0 < x$ and the inequality $s < r$ implies $y > \dfrac{1+x}{\sqrt{3}}$ (Figure 2 (b)). The restriction (7) is the same as $y < \dfrac{\sqrt{3}}{2}(1-x^2)$. Also, let us observe that the last restriction $s < 4t$ is equivalent to $y < \dfrac{3-5x}{\sqrt{3}}$. It turns out that $\dfrac{\sqrt{3}}{2}\left(1-x^2\right) < \dfrac{3-5x}{\sqrt{3}}$ is satisfied if $x < \dfrac{1}{3}$ which is a restriction already given by the the other inequalities we have (Figure 2 (b)). This gives

$$\begin{aligned} P &= \frac{6}{\sqrt{3}}\int_0^{\frac{1}{3}}\left(\frac{\sqrt{3}}{2}(1-x^2)-\frac{1+x}{\sqrt{3}}\right)\mathrm{d}x = \int_0^{\frac{1}{3}}\left(1-2x-3x^2\right)\mathrm{d}x \\ &= \left(t-t^2-t^3)\right)\Big|_0^{1/3} = \frac{5}{27}. \end{aligned}$$

370. Calculate the improper integral $\int_0^\infty \cos^2 x \cos x^2 \mathrm{d}x$.

`Proposed by Ángel Plaza, Department of Mathematics, Univ. de Las Palmas de Gran Canaria, Spain.`

Solution by Santiago de Luxán, Fraunhofer Heinrich-Hertz-Institute, Berlin (Germany). We will calculate the more general integral

$$\int_0^\infty \cos^2(ax)\cos\left(bx^2\right)\mathrm{d}x, \text{ where } a, b \in \mathbb{R} \text{ and } b > 0.$$

First we replace the squared cosine with a non-squared expression

$$I = \int_0^\infty \cos^2(ax)\cos\left(bx^2\right)\mathrm{d}x = \int_0^\infty \frac{1+\cos(2ax)}{2}\cos\left(bx^2\right)\mathrm{d}x.$$

Setting $x = t/\sqrt{b}$, we get that

$$I = \frac{1}{2\sqrt{b}}\left(\int_0^\infty \cos t^2 \mathrm{d}t + \int_0^\infty \cos\left(2\frac{a}{\sqrt{b}}t\right)\cos t^2 \mathrm{d}t\right).$$

Now taking into account that $\cos x \cos y = \frac{1}{2}\left(\cos(x+y)+\cos(y-x)\right)$, we derive the following result:

$$\begin{aligned}\cos\left(2\frac{a}{\sqrt{b}}t\right)\cos t^2 &= \frac{1}{2}\left(\cos\left(2\frac{a}{\sqrt{b}}t + t^2\right) + \cos\left(t^2 - 2\frac{a}{\sqrt{b}}t\right)\right)\\ &= \frac{1}{2}\left(\cos\left(\left(t+\frac{a}{\sqrt{b}}\right)^2 - \frac{a^2}{b}\right) + \cos\left(\left(t-\frac{a}{\sqrt{b}}\right)^2 - \frac{a^2}{b}\right)\right).\end{aligned}$$

On the other hand, $\cos(x-y) = \cos x\cos y + \sin x \sin y$. Therefore

$$\cos\left(\left(t+\frac{a}{\sqrt{b}}\right)^2 - \frac{a^2}{b}\right) = \cos\left(t+\frac{a}{\sqrt{b}}\right)^2\cos\frac{a}{b^2} + \sin\left(t+\frac{a}{\sqrt{b}}\right)^2\sin\frac{a}{b^2}.$$

Applying the same rule to $\cos\left(\left(t-\frac{a}{\sqrt{b}}\right)^2 - \frac{a^2}{b}\right)$ and simplifying we get that

$$\begin{aligned}I &= \frac{1}{2\sqrt{b}}\int_0^\infty \cos t^2 \mathrm{d}t\\ &+ \frac{1}{4\sqrt{b}}\cos\frac{a^2}{b}\left(\int_0^\infty \cos\left(t+\frac{a}{\sqrt{b}}\right)^2\mathrm{d}t + \int_0^\infty \cos\left(t-\frac{a}{\sqrt{b}}\right)^2\mathrm{d}t\right)\\ &+ \frac{1}{4\sqrt{b}}\sin\frac{a^2}{b}\left(\int_0^\infty \sin\left(t+\frac{a}{\sqrt{b}}\right)^2\mathrm{d}t + \int_0^\infty \sin\left(t-\frac{a}{\sqrt{b}}\right)^2\mathrm{d}t\right)\\ &= \frac{1}{2\sqrt{b}}C_0 + \frac{1}{4\sqrt{b}}\left(\cos\frac{a^2}{b}\left(C_1+C_2\right) + \sin\frac{a^2}{b}\left(S_1+S_2\right)\right).\end{aligned}$$

Now, setting $t + \frac{a}{\sqrt{b}} = \alpha$ and $t - \frac{a}{\sqrt{b}} = \beta$ we get that

$$\begin{aligned}C_1 &= \int_0^\infty \cos\left(t+\frac{a}{\sqrt{b}}\right)^2\mathrm{d}t = -\int_0^{\frac{a}{\sqrt{b}}}\cos\alpha^2\mathrm{d}\alpha + \int_0^\infty \cos\alpha^2\mathrm{d}\alpha,\\ S_1 &= \int_0^\infty \sin\left(t+\frac{a}{\sqrt{b}}\right)^2\mathrm{d}t = -\int_0^{\frac{a}{\sqrt{b}}}\sin\alpha^2\mathrm{d}\alpha + \int_0^\infty \sin\alpha^2\mathrm{d}\alpha,\\ C_2 &= \int_0^\infty \cos\left(t-\frac{a}{\sqrt{b}}\right)^2\mathrm{d}t = \int_{-\frac{a}{\sqrt{b}}}^0 \cos\beta^2\mathrm{d}\beta + \int_0^\infty \cos\beta^2\mathrm{d}\beta,\end{aligned}$$

$$S_2 = \int_0^\infty \sin\left(t - \frac{a}{\sqrt{b}}\right)^2 \mathrm{d}t = \int_{-\frac{a}{\sqrt{b}}}^0 \sin\beta^2 \mathrm{d}\beta + \int_0^\infty \sin\beta^2 \mathrm{d}\beta.$$

Since $\cos x^2$ and $\sin x^2$ are even functions, $C_1 + C_2 = 2\int_0^\infty \cos\gamma^2 \mathrm{d}\gamma$ and $S_1 + S_2 = 2\int_0^\infty \sin\gamma^2 \mathrm{d}\gamma$. Therefore,

$$\begin{aligned} I &= \frac{1}{2\sqrt{b}} \int_0^\infty \cos\gamma^2 \mathrm{d}\gamma + \frac{1}{2\sqrt{b}} \left(\cos\frac{a^2}{b} \int_0^\infty \cos\gamma^2 \mathrm{d}\gamma + \sin\frac{a^2}{b} \int_0^\infty \sin\gamma^2 \mathrm{d}\gamma\right) \\ &= \frac{1}{2\sqrt{b}} \left(C(\infty)\left(1 + \cos\frac{a^2}{b}\right) + S(\infty)\sin\frac{a^2}{b}\right), \end{aligned}$$

where $C(\infty) = \int_0^\infty \cos\gamma^2 \mathrm{d}\gamma = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ and $S(\infty) = \int_0^\infty \sin\gamma^2 \mathrm{d}\gamma = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ are the Cosine and Sine Fresnel Integrals, respectively, evaluated at ∞. Hence,

$$I = \frac{1}{4}\sqrt{\frac{\pi}{2b}} \left(1 + \cos\frac{a^2}{b} + \sin\frac{a^2}{b}\right)$$

and the proof is completed. □

371. Let $[ABCD]$ be a Crelle tetrahedron and let M, N, P, Q, R, S be the contact points of the sphere tangent to its edges. Prove that $V_{[MNPQRS]} \leq \frac{1}{2}V_{[ABCD]}$. (By V_X we denote the volume of the polyhedron X.)

`Proposed by Marius Olteanu, S.C. Hidroconstrucţia S.A., Sucursala Olt-Superior, Rm. Vâlcea, Romania.`

Solution by the author. We denote

$$AN = AP = AQ = x, \quad BM = BP = BR = y,$$
$$CM = CN = CS = z, \quad DQ = DR = DS = t.$$

Since $[MNPQRS]$ is obtained from $[ABCD]$ by removing four smaller tetrahedra, the inequality we want to prove is equivalent to

$$V_{[ANPQ]} + V_{[BMPR]} + V_{[CMNS]} + V_{[DQRS]} \geq \frac{1}{2}V_{[ABCD]}.$$

The tetrahedra $[ABCD]$ and $[ANPQ]$ share the same solid angle at A, so

$$\frac{V_{[ANPQ]}}{V_{[ABCD]}} = \frac{AP \cdot AN \cdot AQ}{AB \cdot AC \cdot AD} = \frac{x^3}{(x+y)(x+z)(x+t)}.$$

Hence we must prove that

$$\sum_{\text{cyc}} \frac{x^3}{(x+y)(x+z)(x+t)} \geq \frac{1}{2}.$$

(Here $\sum_{\text{cyc}}$ denotes the sum of all terms obtained by applying a cyclic permutation to expression under the sum sign.)

By multiplying both sides with $2(x+y)(x+z)(x+t)(y+z)(y+t)(z+t)$, the result to prove becomes

$$2\sum x^3(y+z)(y+t)(z+t) \geq (x+y)(x+z)(x+t)(y+z)(y+t)(z+t).$$

The left hand side equals $2\sum x^3(y^2z+y^2t+z^2y+z^2t+t^2y+t^2z+2yzt) = 2\sum x^3y^2z+4\sum x^3yzt$. In the right hand side no variable appears at a power >3 and we don't have terms similar to x^3y^3. One proves that the right hand side equals $\sum x^3y^2z + 2\sum x^3yzt + 2\sum x^2y^2z^2 + 4\sum x^2y^2zt$. (It is obvious that x^3y^2z appears only once in the right hand side. The term x^3yzt appears the same number of times as in $x^3(y+z)(y+t)(z+t)$, i.e., the same number of times yzt appears in $(y+z)(y+t)(z+t)$, which is 2. The term $x^2y^2z^2$ appears the same number of times as in $(x+y)(x+z)x(y+z)yz$, i.e., the same number of times xyz appears in $(x+y)(x+z)(y+z)$, which is 2. The sum $\sum x^3y^2z+2\sum x^3yzt+2\sum x^2y^2z^2$ has $24+2\cdot 4+2\cdot 4 = 40$ terms. But the right hand side has $2^6 = 64$ terms. The remaing 24 terms are of the form x^2y^2zt. Since there are 6 such products, each will appear 4 times.)

In conclusion, we must prove that $2\sum x^3y^2z + 4\sum x^3yzt \geq \sum x^3y^2z + 2\sum x^3yzt + 2\sum x^2y^2z^2 + 4\sum x^2y^2zt$, i.e.,

$$\sum x^3y^2z + 2\sum x^3yzt \geq 2\sum x^2y^2z^2 + 4\sum x^2y^2zt.$$

In the following we denote $S_1 = \sum x^3y^2z$, $S_2 = \sum x^3yzt$, $S_3 = \sum x^2y^2z^2$ and $S_4 = \sum x^2y^2zt$.

We have $x^3y^2z + z^3y^2x \geq 2\sqrt{x^3y^2z\cdot z^3y^2x} = x^2y^2z^2$, so $\sum_{\text{sym}}(x^3y^2z + z^3y^2x) \geq 2\sum_{\text{sym}} x^2y^2z^2$. Both sides of this equality are symmetric polynomials which evaluated at $(x,y,z,t) = (1,1,1,1)$ give 48. The left hand side contains only terms of the form x^3y^2z and the right hand side only terms of the form $x^2y^2z^2$, so they can be written as aS_1 and bS_3, respectively. But $S_1(1,1,1,1) = 24$ and $S_2(1,1,1,1) = 4$, so $a = 48/24 = 2$ and $b = 48/4 = 12$. Hence $2S_1 \geq 12S_3$, i.e., $S_1 \geq 6S_3$.

Similarly, $x^3y^2z + z^3t^2x \geq 2\sqrt{x^3y^2z\cdot z^3t^2x} = 2x^2z^2yt$. It follows that $\sum_{\text{sym}}(x^3y^2z + z^3t^2x) \geq 2\sum_{\text{sym}} x^2z^2yt$. Since $S_1(1,1,1,1) = 24$ and $S_4(1,1,1,1) = 6$ this writes as $2S_1 \geq 8S_4$, i.e., $S_1 \geq 4S_4$.

Finally, $x^3yzt + y^3xzt \geq 2\sqrt{x^3yzt\cdot y^3xzt} = 2x^2y^2zt$. It follows that $\sum_{\text{sym}}(x^3yzt + y^3xzt) \geq 2\sum_{\text{sym}} x^2y^2zt$. Since we have $S_2(1,1,1,1) = 4$ and $S_4(1,1,1,1) = 6$, this writes as $12S_1 \geq 8S_4$, i.e., $S_1 \geq \frac{2}{3}S_4$.

We get $S_1+2S_2 = \frac{1}{3}S_1+\frac{2}{3}S_1+2S_2 \geq \frac{1}{3}\cdot 6S_3+\frac{2}{3}\cdot 4S_4+2\cdot\frac{2}{3}S_4 = 2S_3+4S_4$, i.e., $\sum x^3y^2z + 2\sum x^3yzt \geq 2\sum x^2y^2z^2 + 4\sum x^2y^2zt$. $\square$

Editor's note. Here is a shorter proof for inequality

$$\sum_{\text{cyc}} \frac{x^3}{(x+y)(x+z)(x+t)} \geq \frac{1}{2}.$$

Denote $x+y+z+t = 2s$. The arithmetic mean-geometric mean inequality gives $(x+y)(x+z)(x+t) \leq \left(\frac{2+2s}{3}\right)^3$, so it remains to prove

$\sum_{\text{cyc}} \frac{x^3}{(x+s)^3} \geq \frac{4}{27}$. This results by noting that one has

$$\sum_{\text{cyc}} \frac{x^3}{(x+s)^3} \geq \frac{4x-s}{27s}$$

because the numerator of the expression obtained by subtracting the right side from the left side is $(2x-s)^2(s^2+3sx-x^2)$, which is nonnegative since $x \leq 2s$. $\square$

It is clear that the equality holds if and only if $x = y = z = t$.

Yet another proof is given on p. 134 of [H].

References

[H] P.K. Hung, *Secrets in inequalities*, vol. 1, Gil, Zalău, 2007.

372. Prove that $\lim\limits_{n\to\infty} e^{-n}\left(1+n+\dfrac{n^2}{2!}+\cdots+\dfrac{n^n}{n!}\right) = \dfrac{1}{2}$.

Proposed by George Stoica, Department of Mathematical Sciences, University of New Brunswick, Canada.

Solution by the author. For every $n \geq 1$, let X_n be Poisson random variable with parameter n. The characteristic function (Fourier transform) of $\dfrac{X_n - n}{\sqrt{n}}$ is equal to

$$\mathbb{E}\exp\Big(it(X_n-n)/\sqrt{n}\Big) = \exp\Big(n(e^{it/\sqrt{n}}-1)-it\sqrt{n}\Big) = \exp\Big(-t^2/2+o(1)\Big)$$

as $n \to \infty$. Hence $\dfrac{X_n-n}{\sqrt{n}}$ approaches the normal distribution $N(0,1)$ as $n \to \infty$. In particular,

$$P\left(\frac{X_n-n}{\sqrt{n}} \leq 0\right) \to \Phi(0) \text{ as } n \to \infty,$$

where $\Phi(x)$ is the $N(0,1)$ distribution function. Then take into account that

$$P(X_n \leq n) = e^{-n}\left(1+n+\frac{n^2}{2!}+\cdots+\frac{n^n}{n!}\right) \text{ and } \Phi(0) = \frac{1}{2}.$$

Solution by Victor Makanin, Sankt Petersburg, Russia. Problem 11353 from *The American Mathematical Monthly* says that the function

$$g(s) = \int_0^\infty \left(1+\frac{x}{s}\right)^s e^{-x}\mathrm{d}x - \sqrt{\frac{s\pi}{2}}$$

decreases on $(0,\infty)$ and maps this interval onto $(2/3, 1)$. We are interested only in the inequalities

$$\frac{2}{3} < \int_0^\infty \left(1+\frac{x}{s}\right)^s e^{-x}\mathrm{d}x - \sqrt{\frac{s\pi}{2}} < 1, \quad \forall s > 0,$$

that follow from this enounce, and, actually, we will use their consequence

$$\int_0^\infty \left(1+\frac{x}{s}\right)^s e^{-x}\mathrm{d}x \sim \sqrt{\frac{s\pi}{2}}, \quad s\to\infty$$

($u(s) \sim v(s)$ for $s\to\infty$ means that $u(s)/v(s)$ has limit 1 when s goes to infinity). Now the equality

$$\int_0^\infty \left(1+\frac{x}{s}\right)^s e^{-x}\mathrm{d}x = \left(\frac{e}{s}\right)^s \int_s^\infty t^s e^{-t}\mathrm{d}t$$

can be obtained easily, by changing the variable with $x+s=t$; putting together these two results we obtain

$$\int_s^\infty t^s e^{-t}\mathrm{d}t \sim \left(\frac{s}{e}\right)^s \sqrt{\frac{s\pi}{2}}, \quad s\to\infty.$$

On the other hand, by Stirling's formula for the Gamma function, we have

$$\int_0^\infty t^s e^{-t}\mathrm{d}t \sim \left(\frac{s}{e}\right)^s \sqrt{2s\pi}, \quad s\to\infty,$$

therefore

$$\lim_{s\to\infty}\frac{\displaystyle\int_s^\infty t^s e^{-t}\mathrm{d}t}{\displaystyle\int_0^\infty t^s e^{-t}\mathrm{d}t} = \frac{1}{2},$$

which is equivalent to

$$\lim_{s\to\infty}\frac{\displaystyle\int_0^s t^s e^{-t}\mathrm{d}t}{\displaystyle\int_0^\infty t^s e^{-t}\mathrm{d}t} = \frac{1}{2}.$$

For $s=n$ (a natural variable) the last equality reads

$$\lim_{n\to\infty}\frac{1}{n!}\int_0^n t^n e^{-t}\mathrm{d}t = \frac{1}{2};$$

but, by repeatedly using integration by parts (or by Taylor's formula for the exponential with the remainder in integral form), one finds that

$$\frac{1}{n!}\int_0^n t^n e^{-t}\mathrm{d}t = 1 - e^{-n}\left(1+n+\frac{n^2}{2!}+\cdots+\frac{n^n}{n!}\right)$$

and the result from the enounce follows as required. □

Editor's note. Makanin's proof relies on the relation

$$\lim_{n\to\infty}\frac{1}{n!}\int_0^n t^n e^{-t}\mathrm{d}t = \frac{1}{2}$$

which he proves by using a result from AMM. There is however a direct proof.

Let $f(t) = t^n e^{-t}$. Since $f'(t) = (nt^{n-1} - t^n)e^{-t}$ is positive on $(0,n)$, f increases on $[0,n]$. Also note that by Stirling's theorem $n! \sim \sqrt{2\pi n} f(n)$.

If $0 \le x \ll n^{\frac{2}{3}}$ then $f(n-x) = (n-x)^n e^{x-n} = f(n)\left(1-\frac{x}{n}\right)^n e^x$. Now

$$\log\left(1-\frac{x}{n}\right)^n = n\left(-\frac{x}{n}-\frac{x^2}{2n^2}+O\left(\frac{x^3}{n^3}\right)\right) = -x-\frac{x^2}{2n}+O\left(\frac{x^3}{n^2}\right).$$

It follows that $f(n-x) = f(n)e^{-x-\frac{x^2}{2n}+O\left(\frac{x^3}{n^2}\right)}e^x = f(n)e^{-\frac{x^2}{2n}}\left(1+O\left(\frac{x^3}{n^2}\right)\right)$. (We have $x \ll n^{-\frac{2}{3}}$, so $\frac{x^3}{n^2} \ll 1$.)

In particular, if $0 < t \le n-\sqrt{2n\log n}$ we get

$$f(t) \le f(n-\sqrt{2n\log n}) = f(n)e^{-\log n}\left(1+O\left(\frac{\log^{\frac{3}{2}}}{n^{\frac{1}{2}}}\right)\right) = \frac{1}{n}f(n)(1+o(1)).$$

If n is large enough then $0 < f(t) < \frac{2}{n}f(n)$. It follows that

$$0 < \int_0^{n-\sqrt{2n\log n}} f(t)\mathrm{d}t < (n-\sqrt{2n\log n})\frac{2}{n}f(n) < 2f(n).$$

Since $n! \sim \sqrt{2\pi n}f(n)$, we get $\lim_{n\to\infty}\frac{1}{n!}\int_0^{n-\sqrt{2n\log n}} f(t)\mathrm{d}t = 0$.

On the other hand $\int_{n-\sqrt{2n\log n}}^n f(t)\mathrm{d}t = \int_0^{\sqrt{2n\log n}} f(n-x)\mathrm{d}x$. But for $0 < x < \sqrt{2n\log n}$ we have $f(n-x) = f(n)e^{-\frac{x^2}{2n}}\left(1+O\left(\frac{x^3}{n^2}\right)\right) = f(n)e^{-\frac{x^2}{2n}}\left(1+O\left(\frac{\log^{\frac{3}{2}} n}{n^{\frac{1}{2}}}\right)\right)$.

It follows that $f(n-x) \sim f(n)e^{-\frac{x^2}{2n}}$ uniformly on $[0,\sqrt{2n\log n}]$ and so

$$\begin{aligned}\frac{1}{n!}\int_0^{\sqrt{2n\log n}} f(n-x)\mathrm{d}x &\sim \frac{1}{\sqrt{2\pi n}f(n)}\int_0^{\sqrt{2n\log n}} f(n)e^{-\frac{x^2}{2n}}\mathrm{d}x\\ &= \frac{1}{\sqrt{2\pi n}}\int_0^{\sqrt{2n\log n}} e^{-\frac{x^2}{2n}}\mathrm{d}x.\end{aligned}$$

We take $x = \sqrt{2n}y$ and we get

$$\frac{1}{\sqrt{2\pi n}}\int_0^{\sqrt{2n\log n}} e^{-\frac{x^2}{2n}}\mathrm{d}x = \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\log n}} e^{-y^2}\mathrm{d}y \sim \frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-y^2}\mathrm{d}y = \frac{1}{2}.$$

Hence the conclusion. $\square$

373. Let $n \ge 1$ and let $\Phi_n(X,q) = \prod_{k=1}^n (X - q^{2k-1}) = a_0 + \cdots + a_n X^n$, with $a_i \in \mathbb{R}[q]$. Prove that

$$\frac{\sum_{i=0}^{n-1} a_i a_{i+1}}{\sum_{i=0}^n a_i^2} = \frac{-q(1-q^{2n-1})}{1-q^{2n+1}}.$$

Proposed by Florin Spînu, Department of Mathematics, Johns Hopkins University, Baltimore, MD, USA.

Solution by Constantin-Nicolae Beli. Let Ψ_n be the reciprocal of Φ_n regarded as a polynomial in X, that is, $\Psi_n(X,q) = \prod_{k=1}^n (1 - q^{2k-1}X) = b_0 + \cdots + b_n X^n$, where $b_i = a_{n-i}$. Let $\Phi_n \Psi_n = c_0 + \cdots + c_{2n} X^{2n}$. Then $c_k = \sum_{i+j=k} a_i b_j$. Hence $\sum a_i^2 = \sum a_i b_{n-i} = c_n$ and $\sum a_i a_{i+1} = \sum a_i b_{n-1-i} = c_{n-1}$. Thus we must prove that

$$\frac{c_{n-1}}{c_n} = \frac{-q(1-q^{2n})}{1-q^{2n+2}}.$$

The relation above is an equality of rational functions in the variable q. Therefore it is enough to prove it for an infinite number of values of $q \in \mathbb{C}$. We will prove it for q with $|q| = 1$, i.e., $q = e^{si}$ with $s \in \mathbb{R}$. Then $\frac{-q(1-q^{2n})}{1-q^{2n+2}} = -\frac{q^n - q^{-n}}{q^{n+1} - q^{-n-1}} = -\frac{\sin ns}{\sin(n+1)s}$, so we must prove that $c_n \sin ns + c_{n-1} \sin(n+1)s = 0$.

We have

$$c_k = \frac{1}{2\pi i} \int_C \frac{\Phi_n \Psi_n(z,q)}{z^k} \frac{\mathrm{d}z}{z},$$

where C is the unit cicle. Then the relation $c_n \sin ns + c_{n-1} \sin(n+1)s = 0$ writes as

$$\frac{1}{2\pi i} \int_C \frac{\Phi_n \Psi_n(z,s)}{z^n} (\sin ns + z \sin(n+1)s) \frac{\mathrm{d}z}{z}.$$

We consider the parametrization of the unit circle $\gamma : [-\pi, \pi] \to \mathbb{C}$, $\gamma(t) = e^{ti} = \cos t + i \sin t$.

By using the formula $e^{ai} - e^{bi} = e^{\frac{a+b}{2}i}(e^{\frac{a-b}{2}i} - e^{\frac{b-a}{2}i}) = 2ie^{\frac{a+b}{2}i} \sin \frac{a-b}{2}i$ for $z = \gamma(t) = e^{ti}$ and $q = e^{si}$ we get:

$$\begin{aligned}
\frac{\Phi_n \Psi_n(z,q)}{z^n} &= \prod_{k=1}^n \frac{(z - q^{2k-1})(1 - zq^{2k-1})}{e^{ti}} \\
&= \prod_{k=1}^n \frac{(e^{ti} - e^{(2k-1)si})(1 - e^{t+(2k-1)s})}{e^{ti}} \\
&= \prod_{k=1}^n \frac{2ie^{\frac{t+(2k-1)s}{2}i} \sin \frac{t-(2k-1)s}{2} \cdot 2ie^{\frac{t+(2k-1)s}{2}i} \sin \frac{-t-(2k-1)s}{2}}{e^{ti}} \\
&= \prod_{k=1}^n 2e^{(2k-1)si} \sin \frac{t-(2k-1)s}{2} \sin \frac{t+(2k-1)s}{2} = \alpha f(t),
\end{aligned}$$

where $\alpha = 2^n e^{n^2 si}$ and

$$f(t) = \prod_{k=1}^n \sin \frac{t-(2k-1)s}{2} \sin \frac{t+(2k-1)s}{2} = \prod_{k=-n+1}^n \sin \frac{t+(2k-1)s}{2}.$$

Since also $\frac{\mathrm{d}\gamma(t)}{\gamma(t)} = i\mathrm{d}t$, the statement we want to prove is equivalent to

$$\int_{-\pi}^{\pi} f(t)(\sin ns + \sin(n+1)s(\cos t + i\sin t))\mathrm{d}t = 0.$$

Note that f is an even function and, same as the mapping

$$t \mapsto \frac{\Phi_n(e^{ti}, e^{si})\Psi_n(e^{ti}, e^{si})}{(e^{ti})^n},$$

it has a period of 2π. Since $f(t)$ is even and $\sin t$ is odd, $f(t)\sin t$ is odd and therefore $\int_{-\pi}^{\pi} f(t)\sin t\mathrm{d}t = 0$. So we are left with proving that

$$\int_{-\pi}^{\pi} f(t)(\sin ns + \sin(n+1)s\cos t)\mathrm{d}t = 0.$$

By using the formulas $\sin a\cos b = \frac{1}{2}(\sin(a+b)+\sin(a-b))$ and $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$ we get

$$\begin{aligned}\sin ns + \sin(n+1)s\cos t &= \sin ns + \frac{1}{2}(\sin(t+(n+1)s) + \sin((n+1)s - t)) \\ &= \frac{1}{2}(\sin ns + \sin(t+(n+1)s)) + \frac{1}{2}(\sin ns + \sin((n+1)s-t)) \\ &= \sin\frac{t+(2n+1)s}{2}\cos\frac{t+s}{2} - \sin\frac{t-(2n+1)s}{2}\cos\frac{t-s}{2}.\end{aligned}$$

It follows

$$\int_{-\pi}^{\pi} f(t)(\sin ns + \sin(n+1)s\cos t)\mathrm{d}t = \int_{-\pi}^{\pi}(g(t) - h(t))\mathrm{d}t,$$

where

$$g(t) = f(t)\sin\frac{t+(2n+1)s}{2}\cos\frac{t+s}{2} = \cos\frac{t+s}{2}\prod_{k=-n+1}^{n+1}\sin\frac{t+(2k-1)s}{2}$$

and

$$h(t) = f(t)\sin\frac{t-(2n+1)s}{2}\cos\frac{t-s}{2} = \cos\frac{t+s}{2}\prod_{k=-n}^{n}\sin\frac{t+(2k-1)s}{2}.$$

Note that $g(t) = \cos\frac{t+s}{2}\prod_{k=-n}^{n}\sin\frac{t+(2k+1)s}{2} = h(t+2s)$. Also note that $f(t)$ has period 2π and so does $\sin\frac{t-(2n+1)s}{2}\cos\frac{t-s}{2} = \frac{1}{2}(\sin(t-(n+1)s) - \sin ns)$. Hence $h(t)$ has period 2π. It follows that $\int_{-\pi}^{\pi} g(t)\mathrm{d}t = \int_{-\pi+2s}^{\pi+2s} h(t)\mathrm{d}t = \int_{-\pi}^{\pi} h(t)\mathrm{d}t$, whence $\int_{-\pi}^{\pi}(g(t)-h(t))\mathrm{d}t = 0$, as claimed. □

374. Let $A_1, \ldots, A_n$ be some points in the 3-dimensional Euclidean space. Prove that on the unit sphere S^2 there is a point P such that

$$PA_1 \cdot PA_2 \cdots PA_n \geq 1.$$

`Proposed by Marius Cavachi, Ovidius University of Constanţa.`

Solution by the author. We may assume that $A_k \notin S^2$ for all k. Indeed, if we prove our statement for points not lying on S^2 then for $m \geq 1$ there are points $A_{1,m}, \dots, A_{n,m} \notin S^2$ with $\lim_{m\to\infty} A_{k,m} = A_k$ $\forall k$ and for each m there is some $P_m \in S^2$ such that $P_m A_{1,m} \cdots P_m A_{n,m} \geq 1$. Now there is a subsequence of $(P_m)_{m\geq 1}$ which is convergent to some $P \in S^2$. When we take limit over this subsequence we obtain the inequality $PA_1 \cdots PA_n \geq 1$.

The idea is to prove that for any $A \notin S^2$ we have $\int_{P\in S^2} (\log PA)\mathrm{d}s \geq 0$. By taking $A = A_k$ and adding over $1 \leq k \leq n$ one gets

$$\int_{P\in S^2} \log(PA_1 \cdots PA_n)\mathrm{d}s \geq 0.$$

It follows that $\log(PA_1 \cdots PA_n) \geq 0$, so $PA_1 \cdots PA_n \geq 1$ for some $P \in S^2$.

By choosing a suitable coordinate system we may assume that A has coordinates $(0, 0, a)$ for some $a \geq 0$, $a \neq 1$. We use the cylindrical coordinates ρ, ϕ, z given by $x = \rho\cos\phi$, $y = \rho\sin\phi$. For any $P \in S^2$ of cylindrical coordinates (ρ, ϕ, z) we have $\rho^2+z^2 = 1$, so $PA^2 = \rho^2+(z-a)^2 = 1+a^2-2az$. Since also $\mathrm{d}s = \mathrm{d}\phi\mathrm{d}z$, one gets

$$E(a) := \int_{P\in S^2} \log PA\mathrm{d}s = \int_{-1}^{1}\int_{0}^{2\pi} \frac{1}{2}\log(1 + a^2 - 2az)\mathrm{d}\phi\mathrm{d}z$$
$$= \pi\int_{-1}^{1} \log(1 + a^2 - 2az)\mathrm{d}z.$$

Obviously $E(0) = 0$, so we may assume that $a > 0$. We use the linear substitution $u = 1 + a^2 - 2az$ and we get

$$E(a) = -\frac{\pi}{2a}\int_{(1+a)^2}^{(1-a)^2} \log u\mathrm{d}u = -\frac{\pi}{-2a}(h\log u - u)\Big]_{(1+a)^2}^{(1-a)^2}$$
$$= \frac{\pi}{a}((1 + a)^2 \log(1 + a) - (1 - a)^2 \log|1 - a| - 2a).$$

We consider separately the cases $a < 1$ and $a > 1$.

If $a < 1$ then we use the Taylor expansions of $\log(1 \pm a)$ and we get

$$\frac{a}{\pi}E(a) = (1 + a)^2(a - \frac{a^2}{2} + \frac{a^3}{3} - \cdots) - (1 - a)^2(-a - \frac{a^2}{2} - \frac{a^3}{3} - \cdots) - 2a$$
$$= (2 + 2a^2)(a + \frac{a^3}{3} + \frac{a^5}{5} + \cdots) - 4a(\frac{a^2}{2} + \frac{a^4}{4} + \frac{a^6}{6} + \cdots) - 2a$$
$$= (2 + 2a^2)(\frac{a^3}{3} + \frac{a^5}{5} + \cdots) - 4a(\frac{a^4}{4} + \frac{a^6}{6} + \cdots).$$

But $2 + 2a^2 \geq 4a$, so $\frac{a}{\pi}E(a) \geq 4a(\frac{a^3}{3} - \frac{a^4}{4} + \frac{a^5}{5} - \frac{a^6}{8} + \cdots) > 0$.

If $a > 1$ then

$$\frac{a}{\pi}E(a) = (a+1)^2\log(a+1)-(a-1)^2\log(a-1)-2a = a^2\frac{a}{\pi}E(\frac{1}{a})+4a\log a > 0.$$

As $\frac{1}{a} < 1$, we get $E(\frac{1}{a}) \geq 0$. $\square$

Solution by Victor Makanin, Sankt Petersburg, Russia. We start by proving that the similar property holds in the Euclidian (2-dimensional) plane. Namely, let $C_1, \ldots, C_n$ be points in the same plane with a unit circle $\mathcal{C}$. We show that there exists some point P on this circle such that $PC_1 \cdots PC_n \geq 1$. In order to do that, let $c_1, \ldots, c_n$ be the (complex) affixes of $C_1, \ldots, C_n$ respectively in a Cartesian system of coordinates with origin in the center of $\mathcal{C}$, and consider the complex polynomial $f(z) = (z - c_1) \cdots (z - c_n) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Consider yet the polynomial $g(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + 1$ (with coefficients in reverse order). Observe that $g(0) = 1$, hence, by the Maximum Modulus Principle, the maximum of g on $\mathcal{C}$ is at least 1. But, if $z \neq 0$, $f(z) = z^n g(1/z)$, hence

$$\max_{|z|=1} |f(z)| = \max_{|z|=1} |z^n g(1/z)| = \max_{|z|=1} |g(1/z)| = \max_{|z|=1} |g(z)| \geq 1.$$

Thus, there exists z with $|z| = 1$ and $|f(z)| \geq 1$; if P is the point with affix z, P belongs to $\mathcal{C}$ and $PC_1 \cdots PC_n \geq 1$, which we wanted to prove.

Now we solve the problem. Let π be any plane through the origin (the center of S^2), and let $\mathcal{C}$ be the great circle of S^2 obtained as its intersection with π. Let $C_1, \ldots, C_n$ be the orthogonal projections of $A_1, \ldots, A_n$ on π. By the above proved statement (applied in the plane π) there exists $P \in \mathcal{C}$ such that $PC_1 \cdots PC_n \geq 1$. But we have $PA_k \geq PC_k$ for all $k \in \{1, \ldots, n\}$, therefore $PA_1 \cdots PA_n \geq 1$ follows (and, of course, P belongs to S^2), finishing the proof.

Note that we proved more than it was required, namely that in any plane passing through the center of the unit sphere there exists a point P such that $PA_1 \cdots PA_n \geq 1$. □

375. Let $n \geq 3$ be an integer. Find effectively the isomorphism class of the Galois group $\mathrm{Gal}(\mathbb{Q}(\cos \frac{2\pi}{n})/\mathbb{Q})$.

`Proposed by Cornel Băeţica, Faculty of Mathematics and Informatics, University of Bucharest, Romania.`

Solution by Constantin-Nicolae Beli. Let $\zeta = \zeta_n$ be the primitive nth root of unity, $\zeta = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$. Then $\cos \frac{2\pi}{n} = \zeta + \zeta^{-1}$, so we must determine $\mathrm{Gal}(\mathbb{Q}(\zeta + \zeta^{-1})/\mathbb{Q})$. Since $\zeta + \zeta^{-1} \in \mathbb{Q}(\zeta)$ is invariant under the automorphism $\zeta \mapsto \zeta^{-1}$, we have $\mathbb{Q}(\zeta)^{\langle \zeta \mapsto \zeta^{-1} \rangle} \subseteq \mathbb{Q}(\zeta + \zeta^{-1}) \subseteq \mathbb{Q}(\zeta)$. We have $[\mathbb{Q}(\zeta) : \mathbb{Q}(\zeta)^{\langle \zeta \mapsto \zeta^{-1} \rangle}] = |\langle \zeta \mapsto \zeta^{-1} \rangle| = 2$. Since $\mathbb{Q}(\zeta + \zeta^{-1}) \neq \mathbb{Q}(\zeta)$ (we have $\mathbb{Q}(\zeta + \zeta^{-1}) \subset \mathbb{R}$ but $\mathbb{Q}(\zeta) \not\subset \mathbb{R}$), we must have $\mathbb{Q}(\zeta + \zeta^{-1}) = \mathbb{Q}(\zeta)^{\langle \zeta \mapsto \zeta^{-1} \rangle}$ and so $\mathrm{Gal}(\mathbb{Q}(\zeta + \zeta^{-1})/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})/\langle \zeta \mapsto \zeta^{-1} \rangle$. But $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong U(\mathbb{Z}_n)$ and under this isomorphism $\zeta \mapsto \zeta^{-1}$ corresponds to -1. Therefore $\mathrm{Gal}(\mathbb{Q}(\cos \frac{2\pi}{n})/\mathbb{Q}) \cong U(\mathbb{Z}_n)/\langle -1 \rangle$.

Let $n = 2^\alpha p_1^{\alpha_1} \cdots p_s^{\alpha_s}$. Then

$$U(\mathbb{Z}_n) \cong U(\mathbb{Z}_{2^\alpha}) \times U(\mathbb{Z}_{p_1^{\alpha_1}}) \times \cdots \times U(\mathbb{Z}_{p_s^{\alpha_s}})$$

and under this isomorphism -1 corresponds to $(-1,-1,\dots,-1)$. Thus

$$U(\mathbb{Z}_n)/\langle -1\rangle \cong (U(\mathbb{Z}_{2^\alpha})\times U(\mathbb{Z}_{p_1^{\alpha_1}})\times\cdots\times U(\mathbb{Z}_{p_s^{\alpha_s}}))/\langle(-1,\dots,-1)\rangle.$$

Note that if $\alpha\le 1$ then $U(\mathbb{Z}_{2^\alpha})$ is the trivial group, so it can be dropped in the product above. If $\alpha\ge 2$ then we have an isomorphism $U(\mathbb{Z}_{2^\alpha})\cong\mathbb{Z}_2\times\mathbb{Z}_{2^{\alpha-2}}$ given by $(-1)^a5^b\mapsto(a,b)$. This isomorphism maps -1 to $(1,0)$. For $1\le i\le s$ we have $U(\mathbb{Z}_{p_i^{\alpha_i}})\cong\mathbb{Z}_{p_i^{\alpha_i-1}(p_i-1)}\cong\mathbb{Z}_{2_i^a}\times\mathbb{Z}_{p_i^{\alpha_i}(p_i-1)/2^{a_i}}$, where $2^{a_i}\| p_i-1$. This isomorphism maps -1 to the only element of $\mathbb{Z}_{2_i^a}\times\mathbb{Z}_{p_i^{\alpha_i}(p_i-1)/2^{a_i}}$ of order 2, namely $(2^{a_i-1},0)$.

In conclusion, if $\alpha\le 1$ then $U(\mathbb{Z}_n)/\langle -1\rangle$ is isomorphic to

$$(\mathbb{Z}_{2^{a_1}}\times\mathbb{Z}_{p_1^{\alpha_i}(p_1-1)/2^{a_i}}\times\cdots\times\mathbb{Z}_{2^{a_s}}\times\mathbb{Z}_{p_s^{\alpha_s}(p_s-1)/2^{a_s}})/\langle(2^{a_1-1},0,\dots,2^{a^s-1},0)\rangle$$

which is also isomorphic to

$$(\mathbb{Z}_{2^{a_1}}\times\cdots\times\mathbb{Z}_{2^{a_s}})/\langle(2^{a_1-1},\dots,2^{a_s-1})\rangle\times\mathbb{Z}_{p_1^{\alpha_1-1}(p_1-1)/2^{a_1}}\times\cdots\times\mathbb{Z}_{p_s^{\alpha_s-1}(p_s-1)/2^{a_s}}$$

Similarly, when $\alpha\ge 2$ we get $U(\mathbb{Z}_n)/\langle -1\rangle$ isomorphic to

$$\begin{aligned}&(\mathbb{Z}_2\times\mathbb{Z}_{2^{a_1}}\times\cdots\times\mathbb{Z}_{2^{a_s}})/\langle(1,2^{a_1-1},\dots,2^{a_s-1})\rangle\\&\qquad\times\mathbb{Z}_{2^{\alpha-2}}\times\mathbb{Z}_{p_1^{\alpha_1-1}(p_1-1)/2^{a_1}}\times\cdots\times\mathbb{Z}_{p_s^{\alpha_s-1}(p_s-1)/2^{a_s}}.\end{aligned}$$

We need the following result:

Lemma. If $1\le a_1\le\cdots\le a_s$ are integers then

$$(\mathbb{Z}_{2^{a_1}}\times\cdots\times\mathbb{Z}_{2^{a_s}})/\langle(2^{a_1-1},\dots,2^{a_s-1})\rangle\cong\mathbb{Z}_{2^{a_1-1}}\times\mathbb{Z}_{2^{a_2}}\times\cdots\times\mathbb{Z}_{2^{a_s}}.$$

Proof. We consider the mapping

$$f:\mathbb{Z}^s\to(\mathbb{Z}_{2^{a_1}}\times\cdots\times\mathbb{Z}_{2^{a_s}})/\langle(2^{a_1-1},\dots,2^{a_s-1})\rangle$$

given by $(x_1,\dots,x_s)\mapsto(x_1,x_2+2^{a_2-a_1}x_1,\dots,x_s+2^{a_s-a_1}x_1)$. Obviously, f is linear and onto.

If $(x_1,\dots,x_s)\in\ker f$ then there is some $t\in\mathbb{Z}$ such that $(x_1,x_2+2^{a_2-a_1}x_1,\dots,x_s+2^{a_s-a_1}x_1)-t(2^{a_1-1},\dots,2^{a_s-1})=0$ in $\mathbb{Z}_{2^{a_1}}\times\cdots\times\mathbb{Z}_{2^{a_s}}$, i.e., such that $2^{a_1}\mid x_1-2^{a_1-1}t$ and $2^{a_i}\mid x_i+2^{a_i-a_1}x_1-2^{a_i-1}t=x_i+2^{a_i-a_1}(x_1-2^{a_1-1}t)$. The first condition implies that $2^{a_1-1}\mid x_1$ and for $i>1$ we have $2^{a_i}\mid 2^{a_i-a_1}(x_1-2^{a_1-1}t)$, which, together with $2^{a_i}\mid x_i+2^{a_i-a_1}(x_1-2^{a_1-1}t)$, implies $2^{a_i}\mid x_i$. Thus $(x_1,\dots,x_s)\in 2^{a_1-1}\mathbb{Z}\times 2^{a_2}\mathbb{Z}\times\cdots\times 2^{a_s}\mathbb{Z}$.

Conversely, if $(x_1,\dots,x_s)\in 2^{a_1-1}\mathbb{Z}\times 2^{a_2}\mathbb{Z}\times\cdots\times 2^{a_s}\mathbb{Z}$ then write $x_1=2^{a_1-1}y_1$ and $x_i=2^{a_i}y_i$ for $i>1$. Then

$$\begin{aligned}f(x_1,\dots,x_n)&=x_1(1,2^{a_2-a_1},\dots,2^{a_s-a_1})+(0,x_2,\dots,x_s)\\&=y_1(2^{a_1-1},2^{a_2-1},\dots,2^{a_s-1})+(0,2^{a_2}y_2,\dots,2^{a_s}y_s),\end{aligned}$$

which is 0 in $(\mathbb{Z}_{2^{a_1}}\times\cdots\times\mathbb{Z}_{2^{a_s}})/\langle(2^{a_1-1},\dots,2^{a_s-1})\rangle$. Thus $(x_1,\dots,x_s)\in\ker f$.

In conclusion, $\ker f = 2^{a_1-1}\mathbb{Z} \times 2^{a_2}\mathbb{Z} \times \cdots \times 2^{a_s}\mathbb{Z}$, which implies that

$$(\mathbb{Z}_{2^{a_1}} \times \cdots \times \mathbb{Z}_{2^{a_s}})/\langle(2^{a_1-1}, \ldots, 2^{a_s-1})\rangle \cong \mathbb{Z}^s/\ker f \cong \mathbb{Z}_{2^{a_1-1}} \times \mathbb{Z}_{2^{a_2}} \times \cdots \times \mathbb{Z}_{2^{a_s}}.$$

If $\alpha \leq 1$ then let $a_k = \min_i a_i$. By the Lemma we get

$$(\mathbb{Z}_{2^{a_1}} \times \cdots \times \mathbb{Z}_{2^{a_s}})/\langle(2^{a_1-1}, \ldots, 2^{a_s-1})\rangle \cong \mathbb{Z}_{2^{a_1}} \times \cdots \times \mathbb{Z}_{2^{a_k-1}} \times \cdots \times \mathbb{Z}_{2^{a_s}}.$$

It follows that

$$U(\mathbb{Z}_n)/\langle -1\rangle \cong \mathbb{Z}_{p_1^{\alpha_1-1}(p_1-1)} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k-1}(p_k-1)/2} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s-1}(p_s-1)}.$$

If $\alpha \geq 2$ then by the Lemma

$$(\mathbb{Z}_2 \times \mathbb{Z}_{2^{a_1}} \times \cdots \times \mathbb{Z}_{2^{a_s}})/\langle(1, 2^{a_1-1}, \ldots, 2^{a_s-1})\rangle \cong \mathbb{Z}_{2^{a_1}} \times \cdots \times \mathbb{Z}_{2^{a_s}}.$$

It follows that in this case one has

$$U(\mathbb{Z}_n)/\langle -1\rangle \cong \mathbb{Z}_{2^{\alpha-2}} \times \mathbb{Z}_{p_1^{\alpha_1-1}(p_1-1)} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s-1}(p_s-1)}.$$

376. (a) Show that the probability of a point $P(x, y, z)$, chosen at random with uniform distribution in $[0,1]^3$, to be at a distance to the origin of at most $\sqrt{2}$ is $\frac{(15-8\sqrt{2})\pi}{12}$.
(b) Prove that

$$\int_0^{\pi/4} \frac{\cos^{3/2} 2\theta}{\cos^3\theta} \mathrm{d}\theta = \frac{(4\sqrt{2}-5)\pi}{4}.$$

Eugen J. Ionaşcu, Department of Mathematics, Columbus State University, Columbus, Georgia, U.S.A.

Solution by Cristo M. Jurado (student) and Ángel Plaza, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain. (a) The probability is equal to the volume of the subset of the sphere with radius $\sqrt{2}$ inside the unit cube having one of its vertices at the center of the sphere. We may consider the sphere centered at the origin of coordinates, and the cube at the first octant. The volume may be obtained also considering the first octant of the sphere with radius $\sqrt{2}$, and taking out the part of this octant that is out of the cube, which is equivalent by symmetry three times the volume V of the part of the sphere in the first octant over the plane of equation $z = 1$. The volume V will be calculated by using the spherical coordinates (ρ, α, β), $x = \rho\cos\beta\cos\alpha$, $y = \rho\cos\beta\sin\alpha$, $z = \rho\sin\beta$. We are interested in the region $\rho \leq \sqrt{2}$, $z = \rho\sin\beta \geq 1$, which implies that $\sin\beta \geq 1/\sqrt{2}$, i.e., $\pi/4 \leq \beta \leq \pi/2$, and $\rho \geq 1/\sin\beta$. We find

$$
\begin{aligned}
V &= \int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_{1/\sin\beta}^{\sqrt{2}} \rho^2\cos\beta \mathrm{d}\rho\mathrm{d}\beta\mathrm{d}\alpha = \frac{\pi}{2}\int_{\pi/4}^{\pi/2}\int_{1/\sin\beta}^{\sqrt{2}} \rho^2\cos\beta \mathrm{d}\rho\mathrm{d}\beta \\
&= \frac{\pi}{2}\int_{\pi/4}^{\pi/2}\left(\frac{2\sqrt{2}\cos\beta}{3} - \frac{\cos\beta}{3\sin^3\beta}\right)\mathrm{d}\beta = \frac{\pi}{2}\left(\frac{2\sqrt{2}\sin\beta}{3} - \frac{1}{6\sin^2\beta}\right)\Bigg]_{\pi/4}^{\pi/2} \\
&= \frac{\pi}{2}\left(\frac{2\sqrt{2}}{3} - \frac{5}{6}\right) = \frac{\pi\sqrt{2}}{3} - \frac{5\pi}{12}.
\end{aligned}
$$

Therefore, the probability we are looking for is $P = \dfrac{4\pi(\sqrt{2})^3}{3\cdot 8} - 3V = \dfrac{(15-8\sqrt{2})\pi}{12}$.

(b) The value of the proposed integral is obtained by calculating the probability of part (a) by subtracting from the unit cube the part outside to the sphere of radius $\sqrt{2}$. This time we use the polar coordinates (ρ, θ, z), where $x = \rho\cos\theta$, $y = \rho\sin\theta$. The points of the unit cube outside the sphere of radius $\sqrt{2}$ are characterized by $\rho^2 + z^2 > 2$. Since also $0 \le z \le 1$, we have $\rho > 1$ and $\sqrt{2-\rho^2} < z \le 1$.

Note that the region we are interested in is symmetric about the half-plane $\{\theta = \pi/4\}$, which divides the cube into two congruent triangular prisms. So the volume we want to calculate is twice the volume contained between the half-planes $\{\theta = 0$ and $\{\theta = \pi/4\}$. This region is characterized by $0 \le \theta \le \pi/4$, $x = \rho\cos\theta \le 1$, $\rho > 1$, and $\sqrt{2-\rho^2} < z \le 1$. Its volume W is

$$
\begin{aligned}
W &= \int_0^{\pi/4}\int_1^{1/\cos\theta}\int_{\sqrt{2-\rho^2}}^{1} \rho\mathrm{d}z\mathrm{d}\rho\mathrm{d}\theta \\
&= \int_0^{\pi/4}\int_1^{1/\cos\theta}\left(1 - \sqrt{2-\rho^2}\right)\rho\mathrm{d}\rho\mathrm{d}\theta \\
&= \int_0^{\pi/4}\left(\frac{\rho^2}{2} + \frac{1}{3}\left(2-\rho^2\right)^{3/2}\right)\Bigg]_1^{1/\cos\theta}\mathrm{d}\theta \\
&= \int_0^{\pi/4}\left(\frac{1}{2\cos^2\theta} - \frac{1}{2}\right)\mathrm{d}\theta - \frac{1}{3}\int_0^{\pi/4}\left(\left(2 - \frac{1}{\cos^2\theta}\right)^{3/2} - 1\right)\mathrm{d}\theta \\
&= \frac{1}{2}\tan\theta\Big]_0^{\pi/4} - \frac{\pi}{8} + \frac{1}{3}\int_0^{\pi/4}\frac{\cos^{3/2}2\theta}{\cos^3\theta}\mathrm{d}\theta - \frac{\pi}{12} \\
&= \frac{1}{2} - \frac{5\pi}{24} + \frac{1}{3}\int_0^{\pi/4}\frac{\cos^{3/2}2\theta}{\cos^3\theta}\mathrm{d}\theta.
\end{aligned}
$$

This implies

$$\frac{(15-8\sqrt{2})\pi}{12} = 1 - 2W = \frac{5\pi}{12} - \frac{2}{3}\int_0^{\pi/4} \frac{\cos^{3/2} 2\theta}{\cos^3\theta}\,\mathrm{d}\theta,$$

and so $\displaystyle\int_0^{\pi/4} \frac{\cos^{3/2} 2\theta}{\cos^3\theta}\,\mathrm{d}\theta = \frac{(4\sqrt{2}-5)\pi}{4}$. $\square$

377. Let $p > 2$ be a prime and let n be a positive integer. Prove that

$$p^{\left\lfloor \frac{n-1}{p-1} \right\rfloor} \mid \sum_{k=0}^{\left\lfloor \frac{n}{p} \right\rfloor} (-1)^k \binom{n}{pk}.$$

`Proposed by Ghiocel Groza, Technical University (TUCEB), Bucharest, Romania.`

Solution by Constantin-Nicolae Beli. Let $\zeta = \zeta_p = e^{2\pi i/p}$. For any integer k we have

$$\sum_{l=0}^{p-1} \zeta^{kl} = \begin{cases} p & \text{if } p \mid k \\ 0 & \text{otherwise} \end{cases}$$

so if $f = a_0 + \cdots + a_n X^n \in \mathbb{C}[X]$ then

$$\sum_{l=0}^{p-1} f(\zeta^l) = \sum_{l=0}^{p-1}\sum_{k=0}^{n} a_k \zeta^{kl} = \sum_{k=0}^{\left\lfloor \frac{n}{p} \right\rfloor} p a_{pk}.$$

In particular, if $f = (1-X)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} X^k$ we get $\sum_{l=0}^{p-1} (1-\zeta^l)^n = pS$, where $S := \sum_{k=0}^{\left\lfloor \frac{n}{p} \right\rfloor} (-1)^k \binom{n}{pk}$.

Now the prime p of $\mathbb{Q}$ totally ramifies in $\mathbb{Q}(\zeta)$. The only prime of $\mathbb{Q}(\zeta)$ lying over p is $P = (1-\zeta)$. We denote by $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ and $w_P : \mathbb{Q}(\zeta) \to \mathbb{Z} \cup \{\infty\}$ the valuation maps corresponding to p and P. Then $e_{P/p} = p-1$, so $w_P(a) = (p-1)v_p(a)\ \forall a \in \mathbb{Q}$.

We have to prove that $v_p(S) \geq \left\lfloor \frac{n-1}{p-1} \right\rfloor$. But for any l we have $(1-\zeta^l)^n \in P^n$, so $pS \in P^n$. It follows that

$$(p-1)(1+v_p(S)) = (p-1)v_p(pS) = w_P(pS) \geq n > n-1.$$

Hence, $1 + v_p(S) > \frac{n-1}{p-1} \geq \left\lfloor \frac{n-1}{p-1} \right\rfloor$, so $v_p(S) \geq \left\lfloor \frac{n-1}{p-1} \right\rfloor$. $\square$

Editor's note. An alternative proof which does not use the arithmetics of $\mathbb{Q}(\zeta)$ was proposed by Victor Makanin. He denotes $S_n := \sum_{k=0}^{\lfloor n/p \rfloor} (-1)^k \binom{n}{pk}$ and proves the same formula $S_n = \frac{1}{p}\sum_{l=1}^{p-1} (1-\zeta^l)^n$. (At $l = 0$ the term

$(1-\zeta^l)^n$ is 0 so it can be ignored.) As $\zeta,\dots,\zeta^{p-1}$ are the roots of $\Phi_p(X)=\frac{X^p-1}{X-1}$, it follows that $1-\zeta,\dots,1-\zeta^{p-1}$ are the roots of

$$\Phi_p(1-X)=\frac{(1-X)^p-1}{1-X-1}=X^{p-1}-\binom{p}{1}X^{p-2}+\binom{p}{2}X^{p-3}-\cdots+\binom{p}{p-1}.$$

It follows that the sequence $(S_n)_{n\geq 1}$ satisfies the linear recurrence

$$S_n-\binom{p}{1}S_{n-1}+\binom{p}{2}S_{n-2}-\cdots+\binom{p}{p-1}S_{n-p+1}=0$$

for $n\geq p$. From here he uses induction on n.

If $n=1,\dots,p-1$ the statement is trivial, as $\left\lfloor\frac{n-1}{p-1}\right\rfloor=0$. If $n\geq p$ we use the recurrence relation above. For $1\leq j\leq p-1$ we have $p\mid\binom{p}{j}$ and, by the induction step, $p^{\left\lfloor\frac{n-j-1}{p-1}\right\rfloor}\mid S_{n-j}$. It follows that $p^{\left\lfloor\frac{n-j-1}{p-1}\right\rfloor+1}\mid S_{n-j}$. But for $1\leq j\leq p-1$ we have $\left\lfloor\frac{n-j-1}{p-1}\right\rfloor+1=\left\lfloor\frac{n+p-j-2}{p-1}\right\rfloor\geq\left\lfloor\frac{n-1}{p-1}\right\rfloor$. Therefore $p^{\left\lfloor\frac{n-1}{p-1}\right\rfloor}$ divides all the terms of the sum in the right side of equation

$$S_n=\binom{p}{1}S_{n-1}-\binom{p}{2}S_{n-2}+\cdots-pp-1S_{n-p+1}$$

and hence it divides S_n. □

378. Let $(x_n)_{n\geq 1}$ be a sequence with $0<x_n<1$. Then the following are equivalent.

(i) For any convergent series of positive numbers $\sum_{n\geq 1}a_n$ the series $\sum_{n\geq 1}a_n^{x_n}$ is convergent, as well.

(ii) The series $\sum_{n\geq 1}M^{-1/(1-x_n)}$ is convergent for some $M>1$, large enough.

Proposed by Constantin-Nicolae Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

Solution by the author. (ii)⇒(i) We have

$$\sum_{n\geq 1}a_n^{x_n}=\sum_{n\in A}a_n^{x_n}+\sum_{n\in B}a_n^{x_n},$$

where $A=\{n\mid a_n<M^{-1/(1-x_n)}\}$ and $B=\{n\mid a_n\geq M^{-1/(1-x_n)}\}$.

Let $S_1=\sum_{n\geq 1}a_n$. If $n\in B$ then $a_n^{1-x_n}\geq M^{-1}$, that is, $a_n^{x_n}\leq Ma_n$, and so $\sum_{n\in B}a_n^{x_n}\leq\sum_{n\geq 1}Ma_n=MS_1$.

Let $S_2=\sum_{n\geq 1}M^{-1/(1-x_n)}$. If $n\in A$ then $a_n^{x_n}<M^{-x_n/(1-x_n)}=M\cdot M^{-1/(1-x_n)}$, so

$$\sum_{n\in A}a_n^{x_n}<\sum_{n\geq 1}M\cdot M^{-1/(1-x_n)}=MS_2.$$

In consequence, $\sum_{n\geq 1}a_n^{x_n}<M(S_1+S_2)<\infty$.

(i)⇒(ii) We prove that if $\sum_{n\geq 1} M^{-1/(1-x_n)} = \infty$ for any $M > 1$ then there is a sequence $(a_n)_{n\geq 1}$ of positive numbers such that $\sum_{n\geq 1} a_n < \infty$ yet $\sum_{n\geq 1} a_n^{x_n} = \infty$.

If $M > 1$ then $\sum_{n\geq 1} M^{-1/(1-x_n)} = \infty$, so for any integer $m \geq 0$ and any $C > 0$ there is an integer $m' > m$ such that $\sum_{n=m+1}^{m'} M^{-1/(1-x_n)} \geq C$. Therefore we may construct recursively an integer sequence $0 = m_1 < m_2 < m_3 < \cdots$ such that

$$S_i := \sum_{n=m_i+1}^{m_{i+1}} i^{-1/(1-x_n)} \geq \frac{1}{i^2} \ \forall i \geq 1.$$

It follows that $i^2 S_i \geq 1$.

We define the sequence $(a_n)_{n\geq 1}$ by

$$a_n = \frac{1}{i^2 S_i} i^{-1/(1-x_n)} \ \text{ if } \ m_i + 1 \leq n \leq m_{i+1}.$$

We have

$$\sum_{n=m_i+1}^{m_{i+1}} a_i = \frac{1}{i^2 S_i} \sum_{n=m_i+1}^{m_{i+1}} i^{-1/(1-x_n)} = \frac{1}{i^2 S_i} \cdot S_i = \frac{1}{i^2}.$$

It follows that $\sum_{n\geq 1} a_n = \sum_{i\geq 1} \frac{1}{i^2} < \infty$.

If $m_i + 1 \leq n \leq m_{i+1}$ then $i^2 S_i \geq 1$ and $x_n < 1$, so $(i^2 S_i)^{x_n} \leq i^2 S_i$. Since also $i^{-x_n/(1-x_n)} = i \cdot i^{-1/(1-x_n)}$, we get

$$a_n^{x_n} = \frac{1}{(i^2 S_i)^{x_n}} i^{-x_n/(1-x_n)} \geq \frac{1}{i^2 S_i} i^{-x_n/(1-x_n)} = \frac{1}{i S_i} i^{-1/(1-x_n)},$$

whence

$$\sum_{n=m_i+1}^{m_{i+1}} a_i \geq \frac{1}{i S_i} \sum_{n=m_i+1}^{m_{i+1}} i^{-1/(1-x_n)} = \frac{1}{i S_i} \cdot S_i = \frac{1}{i}.$$

It follows that $\sum_{n\geq 1} a_n^{x_n} \geq \sum_{i\geq 1} \frac{1}{i} = \infty$. □

Note. This result generalizes Problem 364 proposed by Cristian Ghibu in the issue 1-2/2012 of GMA. Unlike our condition (ii), his condition

$$\limsup_{n\to\infty} (1 - x_n) \log n < \infty,$$

is only sufficient but not necessary for (i) to happen. It is not hard to see that his condition implies (ii) above. If $L = \limsup_{n\to\infty} (1 - x_n) \log n$ then for n large enough we have $(1 - x_n) \log n \leq 2L$, so $2 \log n \leq 4L/(1 - x_n)$. It follows that $n^2 \leq e^{\frac{4L}{1-x_n}}$. Therefore if we take $M = e^{4L}$ then, for n large enough, we have $M^{-1/(1-x_n)} \leq \frac{1}{n^2}$. Hence $\sum_{n\geq 1} M^{-1/(1-x_n)}$ is convergent.