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Optimal reverse Finsler-Hadwiger inequalities

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Abstract. The best constant in the reverse Finsler-Hadwiger inequality is determined. Using this result, various triangle inequalities are established

Keywords: Triangle inequalities, reverse Finsler-Hadwiger inequality.
MSC: 51M04

1. Introduction

In an article [4] recently published in this journal it is shown that in any triangle one has

$$a^2+b^2+c^2 \le 4F\sqrt{3}+3\left((a-b)^2+(b-c)^2+(c-a)^2\right), \qquad (1.1)$$

while the stronger inequality $\sum a^2 \le 4F\sqrt{3}+2\sum(a-b)^2$ holds in any acute-angled triangle. Here and troughout the article the notation is the usual one: a, b, c denote the length of sides, F the area, s the semiperimeter, r the inradius and R the circumradius of triangle ABC. The paper ends with the question to find the optimal constant k such that in any acute-angled triangle ABC, the following inequality is valid:

$$a^2+b^2+c^2 \le 4F\sqrt{3}+k\left((a-b)^2+(b-c)^2+(c-a)^2\right). \qquad (1.2)$$

The authors note:

> "In our research we found that the constant $k=\dfrac{2-\sqrt{3}}{3-2\sqrt{2}}$ is optimal and is attained for a right-angled isosceles triangle, but we do not know if the reverse Finsler-Hadwiger inequality holds true in this case."

The aim of this note is to confirm this observation by proving the following.

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Theorem 1.1. *Let* $k_0 = \dfrac{2-\sqrt{3}}{3-2\sqrt{2}}$ *and* $k > 0$*. The inequality* (1.2) *holds in any non-obtuse triangle if and only if* $k \geq k_0$*.*

The paper [4] does not address the question of the optimal value of k for which inequality (1.2) is valid in all triangles. The answer to this natural question has already been given in [3] (cf. [8]).

Theorem 1.2. *Inequality* (1.2) *holds in any triangle if and only if* $k \geq 3$*.*

The proofs are elementary and simple, provided we have the right tools at our disposal. Relevant results are presented in the next section. Armed with sufficient knowledge, we proceed to the proof of Theorems 1.1 and 1.2. The last section of the paper also contains applications of the main result.

2. Results that should be better known

The aim of this section is to gather several fundamental results from the huge literature on triangle inequalities. Their proofs are not difficult once the statements are properly enunciated. The reader will appreciate the usefulness of the facts presented below.

We start by recalling a less-familiar form of the fundamental inequality of a triangle. This characterisation of triples (s, r, R) is due to E. Rouché (1851).

Lemma 2.1. *Three positive real numbers* s*,* r*,* R *are the semiperimeter, inradius and respectively the circumradius of a triangle* ABC *if and only if*

$$s^2 \geq \phi^-(r, R) := 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R(R - 2r)} \tag{2.1}$$

and

$$s^2 \leq \phi^+(r, R) := 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R(R - 2r)}. \tag{2.2}$$

The equality holds in (2.1) *(respectively* (2.2)*) if and only if* ABC *is an isosceles triangle with top-angle greater than (respectively, less than) or equal to* $\pi/3$*.*

A triangle inequality can be treated as a problem of optimization with restrictions. Both the objective function and restrictions are nonlinear trivariate functions. There are several techniques that allow one to reduce the problem to a (sequence of) univariate optimization problem(s). This is the case, for instance, if the triangle is either isosceles or right-angled. The study of these particular classes of triangles suffices to settle the problem stated in the introduction, thanks to the following result. It has circulated especially in the Chinese literature. Our presentation follows [6], which refers to [2] as the first appearance in the literature.

Lemma 2.2. *For every acute-angled triangle ABC one can find either a right-angled triangle $A_2B_2C_2$ such that $R_2 = R$, $r_2 = r$, $s_2 < s$ or an acute isosceles triangle $A_1B_1C_1$ such that $R_1 = R$, $r_1 = r$, $s_1 \le s$, and the top-angle at least $\pi/3$.*

Proof. The desired triangles are obtained quite explicitly, by a construction that varies according to whether the circumcenter O is contained or not in incircle.

The circumcenter sits in the interior of the incircle if and only if $R < (\sqrt{2}+1)r$. If this is the case, one considers the midpoint A_1 of the minor arc BC of the circle circumscribed to $\triangle ABC$ and a chord B_1C_1 parallel to BC such that the angle $B_1A_1C_1$ have measure

$$A_1 = 2\arcsin\frac{1}{2}\left(1+\sqrt{1-\frac{2r}{R}}\right).$$

Then it is clear that $A_1B_1C_1$ is an isosceles triangle with $R_1 = R$, whose top-angle satisfies

$$\sin\frac{A_1}{2} = \frac{1}{2}\left(1+\sqrt{1-\frac{2r}{R}}\right) \ge \frac{1}{2},$$

$$\sin\frac{A_1}{2} < \frac{1}{2}\left(1+\sqrt{1-\frac{2}{\sqrt{2}+1}}\right) = \frac{1}{2}\left(1+\sqrt{2}-1\right) = \frac{1}{\sqrt{2}},$$

that is, $\dfrac{\pi}{3} \le A_1 < \dfrac{\pi}{2}$. On using formula (see, e.g., [5])

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

one readily gets

$$\begin{aligned} r_1 &= 4R_1\sin\frac{A_1}{2}\sin^2\frac{B_1}{2} = 2R\sin\frac{A_1}{2}(1-\cos B_1) = 2R\sin\frac{A_1}{2}\left(1-\sin\frac{A_1}{2}\right) \\ &= R\left(1+\sqrt{1-\frac{2r}{R}}\right)\left(1-\frac{1}{2}\left(1+\sqrt{1-\frac{2r}{R}}\right)\right) = \frac{R}{2}\left(1-(1-\frac{2r}{R})\right) = r. \end{aligned}$$

Since $A_1B_1C_1$ is an isosceles triangle with top-angle at least $\pi/3$, from the fundamental triangle inequality stated in Lemma 2.1 one obtains

$$s_1^2 = \phi^-(R_1, r_1) = \phi^-(R, r) \le s^2.$$

In case $R > (\sqrt{2}+1)r$, the circumcenter O is exterior to the incircle. A tangent to incircle drawn through O intersects the incircle in M and the circumcircle in B_2 and C_2, say. The second tangent from B_2 to incircle intersects the circumcircle again in a point A_2. Then $\sphericalangle B_2A_2C_2 = \dfrac{\pi}{2}$, $R_2 = R$, and, denoting by 2α the measure of $\sphericalangle A_2B_2C_2$, one finds

$$r_2 = \frac{1}{2}(b_2 + c_2 - a_2) = R(\sin 2\alpha + \cos 2\alpha - 1) = 2R\sin\alpha(\cos\alpha - \sin\alpha).$$

From the right-angled triangle OIM one computes

$$R^2 - 2rR = OI^2 = IM^2 + OM^2 = r^2 + (R - r\cot\alpha)^2,$$

which is readily brought to the equivalent form $r = 2R\sin\alpha(\cos\alpha - \sin\alpha)$. Thus, $r_2 = r$.

Having in view formula $4R^2\prod\cos A = s^2 - (2R+r)^2$ and the fact that $\triangle ABC$ is acute-angled, one gets $s > 2R + r = 2R_2 + r_2 = s_2$. $\square$

The result just proved has the following handy consequence, whose proof is easy now.

Lemma 2.3. *The inequality $s \geq f(r,R)$ (respectively, $s > f(r,R)$) holds for any non-obtuse triangle if and only if it holds for all right-angled triangles and all acute isosceles triangles with top-angle greater than or equal to $\pi/3$.*

3. Proof of the main result

Since the inequality of interest is homogeneous, we may select a convenient representative for each equivalence class of similar triangles. This simple yet useful remark will be helpful in the forthcoming proof.

As mentioned in [4], writing inequality (1.2) for a right-angled isosceles triangle, one gets $k \geq k_0$. Conversely, in order to verify that (1.2) with $k = k_0$ holds in all triangles, rewrite it using the familiar formulæ

$$\sum a^2 = 2(s^2 - r^2 - 4rR), \qquad F = sr, \qquad \sum ab = s^2 + r^2 + 4rR.$$

One finds successively the equivalent forms

$$s^2 - r^2 - 4rR \leq 2sr\sqrt{3} + k_0(s^2 - 3r^2 - 12rR)$$

and

$$0 \leq (k_0 - 1)s^2 + 2\sqrt{3}rs - (3k_0 - 1)(r^2 + 4rR).$$

Since the quadratic polynomial in s from the right-hand side has positive leading coefficient and negative free term, we have to prove that inequality

$$(k_0 - 1)s \geq -r\sqrt{3} + \sqrt{3r^2(3k_0 - 1)(r^2 + 4rR)} \tag{3.1}$$

holds in all non-obtuse triangles. By Lemma 2.3, it is sufficient to assume $\triangle ABC$ either right-angled or isosceles, acute, with top-angle at least $\pi/3$.

Let us first consider the particular case of a right-angled triangle ABC whose sides have length $a = 1$, $b = \sin\theta$, $c = \cos\theta$ for some positive θ with $\theta \leq \pi/4$. One readily finds that (3.1) is equivalent to

$$g(\theta) \geq 2 \quad \text{for} \quad 0 < \theta \leq \frac{\pi}{4},$$

where

$$g(\theta) := \sqrt{3}\sin 2\theta + 2k_0(2 - \sin\theta - \cos\theta - \sin\theta\cos\theta).$$

A simple computation gives

$$g'(\theta) = 2(\cos\theta - \sin\theta)[(\sqrt{3} - k_0)(\sin\theta + \cos\theta) - k_0].$$

For θ in the range of interest, the expression within parantheses is non-negative, while that within brackets is less than or equal to $\sqrt{2}(\sqrt{3}-k_0)-k_0$, which is negative. Therefore, g is a decreasing function, so that

$$g(\theta) \geq g(\frac{\pi}{4}) = 2 \quad \text{for} \quad 0 < \theta \leq \frac{\pi}{4}.$$

This means inequality (3.1) holds in all right-angled triangles.

Let us now examine the case of isosceles triangles. Without loss of generality, we may consider that sides have length 2, x, x, respectively. The top-angle of such a triangle has measure in the interval $[\pi/3, \pi/2)$ if and only if $\sqrt{2} < x < 2$. In this case, inequality (3.1) is found to be equivalent to

$$f(x)^2 \leq k_0 \quad \text{for} \quad \sqrt{2} < x < 2, \tag{3.2}$$

with

$$f : (1, +\infty) \longrightarrow \mathbb{R}, \quad f(x) := \frac{x+2}{\sqrt{x^2-1}+\sqrt{3}}.$$

This function is derivable and its first derivative is

$$f'(x) = \frac{\sqrt{3x^2-3} - 2x - 1}{(\sqrt{x^2-1}+\sqrt{3})^2\sqrt{x^2-1}}.$$

Since $\sqrt{3x^2-3} < 2x$, one has $f'(x) < 0$ for all $x > 1$. Therefore,

$$\frac{2}{\sqrt{3}} = f(2) \leq f(x) < f(\sqrt{2}) = \frac{2+\sqrt{2}}{\sqrt{3}+1}.$$

Since

$$f(2)^2 = \frac{3+2\sqrt{2}}{2+\sqrt{3}} = \frac{2-\sqrt{3}}{3-2\sqrt{2}} = k_0,$$

we conclude that relation (3.2) is true.

The proof of Theorem 1.1 is complete.

Theorem 1.2 can be established quite similarly. By [4, Theorem 1.3], it is known that inequality (1.1) holds in every triangle. It remains to show that for no $k_0 < 3$ can inequality (1.2) hold in all triangles.

Let us assume that (1.2) is true in every triangle. In particular, it is valid for the isosceles triangle whose sides have length 2, x, x, where x is restricted to $x > 1$ by the usual triangle inequality. Heron's formula yields $F = \sqrt{x^2-1}$, so that one gets

$$4 + 2x^2 \leq 4\sqrt{3x^2-3} + 2(x-2)^2 k \quad \text{for all} \quad x > 1,$$

which is equivalent to $f(x)^2 \le k$ for all $x > 1$, with f the decreasing function introduced in the previous proof. Thus one has

$$\sup_{x>1} f(x) = \lim_{x\to 1} f(x) = \sqrt{3}, \quad \inf_{x>1} f(x) = \lim_{x\to\infty} f(x) = 1,$$

whence $k \ge 3$.

4. Reformulation of the problem

It is well known (see, for instance, [5, ch. VII]) and easy to prove that the square roots of side lengths of an arbitrary triangle ABC are the lengths of the sides of an acute-angled triangle $A'B'C'$. Expressing inequality (1.2) written for $\triangle A'B'C'$ in terms of $\triangle ABC$, one gets

$$\begin{aligned} a+b+c \quad \le \quad & \sqrt{3(2ab+2bc+2ca-a^2-b^2-c^2)} \\ & + 2k(a+b+c-\sqrt{ab}-\sqrt{bc}-\sqrt{ca}). \end{aligned} \tag{4.1}$$

Therefore, if (1.2) holds in any acute-angled triangle for a certain value of k then inequality (4.1) holds in any triangle for the same value of k. Conversely, inequality (1.2) can be derived from (4.1) thanks to another classical fact – the squares of sides of an acute triangle ABC form another triangle.

These considerations justify our first result in this section.

Proposition 4.1. *For a positive number k, the following conditions are equivalent:*

(i) *inequality* (1.2) *holds in any acute-angled triangle*
(ii) *inequality* (4.1) *holds in any triangle.*

The same idea – associate to a triangle from a certain class a triangle from another class – leads to other equivalent formulations for inequality (1.2). For instance, consider the triangle $A_1B_1C_1$ formed by the medians of triangle ABC. It is a matter of simple computations to check that the next result is thus obtained.

Proposition 4.2. *Inequality*

$$8k(m_am_b+m_bm_c+m_cm_a) \le 12\sqrt{3}F + 3(2k-1)(a^2+b^2+c^2) \tag{4.2}$$

holds in all triangles if and only if $k \ge 3$.

Note that, in the optimal case $k = 3$, (4.2) becomes

$$8(m_am_b+m_bm_c+m_cm_a) \le 4\sqrt{3}F + 5(a^2+b^2+c^2).$$

Adding side by side this inequality and the equality

$$4(m_a^2+m_b^2+m_c^2) = 3(a^2+b^2+c^2)$$

results in

$$4(m_a+m_b+m_c)^2 \le 4\sqrt{3}F + 8(a^2+b^2+c^2). \tag{4.3}$$

According to the usual Finsler-Hadwiger inequality, one has

$$4\sqrt{3}F \leq 2(ab+bc+ca) - (a^2+b^2+c^2),$$

so that (4.3) is stronger than the inequality

$$2(m_a+m_b+m_c) \leq \sqrt{7(a^2+b^2+c^2)+2(ab+bc+ca)}$$

proposed as an olympiad problem in [1].

Appropriate transformations can produce asymmetric inequalities. We illustrate this idea with a sole example. Let us replace our reference triangle by a triangle $\overline{ABC}$ similar to $\triangle ACA'$, where A' is the midpoint of the line segment BC. To simplify the computations, we take $\overline{a} = 2m_a$, $\overline{b} = 2b$, $\overline{c} = a$, with m_a the length of median AA'. Since area of $\triangle ACA'$ is half the area of $\triangle ABC$, one has $\overline{F} = 2F$. Simple computations lead to the conclusion that $\triangle ACA'$ is non-obtuse if and only if $c^2 \leq a^2+b^2$, $b^2 \leq c^2$, and $a^2 \leq c^2+3b^2$.

An application of Theorem 1.1 results in a geometric inequality that looks very difficult to prove directly.

Proposition 4.3. *Let ABC be a non-obtuse triangle and c the largest of its side lengths. Then*

$$m_a \leq \frac{4\sqrt{3}F + (2k_0-1)(3b^2+c^2) - 2k_0ab}{2k_0(a+2b)}.$$

More generally, one can compose several triangle transformations. For instance, let us consider $\triangle\overline{ABC}$ resulting from $\triangle ABC$ by applying the square-root transformation after the T_{e^2}-transformation described in ch. VII, §5.1 of [5]. This puts in correspondence a pair of triangles ABC and $A'B'C'$, with $a' = a(s-a)$, $b' = b(s-b)$, $c' = c(s-c)$. Known formulæ yield

$$\begin{aligned}
16\overline{F}^2 &= 2\sum ab(s-a)(s-b) - \sum a^2(s-a)^2\\
&= 2\sum ab(-s^2+sc+ab) - \sum a^2(s^2-2sa+a^2)\\
&= -2s^2\sum ab + 6sabc + s\sum a^2b^2 - s^2\sum a^2 + 2s\sum a^3 - \sum a^4\\
&= 16F^2 - 4s^4 + 24s^2rR + 4s^2(s^2-3r^2-6rR) = 4F^2.
\end{aligned}$$

Since $\triangle\overline{ABC}$ is always acute-angled, we arrive at the following statement.

Proposition 4.4. *In any non-degenerate triangle one has*

$$\begin{aligned}
\sum\sqrt{ab(s-a)(s-b)} &< 2(r^2+4rR) + \frac{3-2\sqrt{2}}{2-\sqrt{3}}(F\sqrt{3}-r^2-4rR)\\
&\leq 2(r^2+4rR).
\end{aligned}$$

According to another classical paradigm, triangle inequalities and inequalities between three positive real numbers are essentially the same thing. Specifically, three positive numbers a, b, c are the side lengths for a triangle ABC if and only if there exist three positive numbers x, y, z such that

$a = y + z$, $b = z + x$, $c = x + y$. From Theorem 1.2 one thus gets another result.

Proposition 4.5. *The smallest real number with the property that for all positive real numbers x, y, z one has*

$$0 \leq 2\sqrt{3xyz(x+y+z)} + (k-1)(x^2+y^2+z^2) - (k+1)(xy+yz+zx)$$

is 3.

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On the stability of the second order linear recurrence

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Abstract. We prove that the second order linear recurrence

$$x_{n+2} = ax_{n+1} + bx_n, \ n \in \mathbb{N},$$

is stable in Hyers-Ulam sense if and only if its characteristic equation has no roots of module one.

Keywords: Linear recurrence, Hyers-Ulam stability.
MSC: 39B82.

1. Introduction

Hyers-Ulam stability is one of the main topics in functional equation theory. Generally, a functional equation is said to be stable in Hyers-Ulam sense if for every solution of a perturbation of the equation (called *approximate solution*) there exists a solution of the equation (called *exact solution*) near it. The first result on this topic was given by D. H. Hyers, answering to a question of S. M. Ulam [7], for the Cauchy functional equation. Recall the result of Hyers [4]:

Let ε be a positive number. Then for every function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$|f(x+y) - f(x) - f(y)| \leq \varepsilon, \ \forall \ x, y \in \mathbb{R}, \tag{1.1}$$

there exists a unique function $g : \mathbb{R} \to \mathbb{R}$, satisfying

$$g(x+y) = g(x) + g(y), \ \forall \ x, y \in \mathbb{R},$$

and

$$|f(x) - g(x)| \leq \varepsilon, \ \forall \ x \in \mathbb{R}. \tag{1.2}$$

In what follows we deal with Hyers-Ulam stability of the second order linear recurrence

$$x_{n+2} = ax_{n+1} + bx_n, \ n \in \mathbb{N}, \tag{1.3}$$

where $a, b, x_0 \in \mathbb{C}$. A recurrence is in fact a functional equation for a function defined on the set of natural numbers.

Definition 1.1. *The recurrence* (1.3) *is said to be stable in Hyers-Ulam sense if there exists $L \geq 0$ such that for every $\varepsilon > 0$ and every sequence $(x_n)_{n\geq 0}$ of complex numbers satisfying*

$$|x_{n+2} - ax_{n+1} - bx_n| \leq \varepsilon, \ n \in \mathbb{N}, \tag{1.4}$$

there exists a sequence $(y_n)_{n\geq 0}$ of complex numbers satisfying

$$y_{n+2} = ay_{n+1} + by_n \tag{1.5}$$

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$$|x_n - y_n| \le L \cdot \varepsilon,\ n \in \mathbb{N}. \tag{1.6}$$

The relation (1.4) *is a perturbation of the recurrence* (1.3)*, the sequences satisfying* (1.4) *are approximate solutions of the recurrence* (1.3) *and* $(y_n)_{n\ge 0}$ *is an exact solution of* (1.3)*.*

2. Main result

For the proof of the main result of this paper, we give a result on the stability of the first order linear recurrence.

Lemma 2.1. *Let ε be a positive number, $p \in \mathbb{C}$, $|p| \ne 1$, and $(a_n)_{n\ge 0}$ a sequence of complex numbers. Then for every sequence of complex numbers $(x_n)_{n\ge 0}$ satisfying*

$$|x_{n+1} - px_n - a_n| \le \varepsilon,\ n \in \mathbb{N}, \tag{2.1}$$

there exists a sequence of complex numbers $(y_n)_{n\ge 0}$ with the properties

$$y_{n+1} = py_n + a_n,\ n \in \mathbb{N}, \tag{2.2}$$

$$|x_n - y_n| \le \frac{\varepsilon}{\big||p|-1\big|},\ n \in \mathbb{N}. \tag{2.3}$$

Moreover, if $|p| > 1$, then $(y_n)_{n\ge 0}$ is uniquely determined.

Proof. Existence. Suppose that $(x_n)_{n\ge 0}$ satisfies (2.1) and let

$$x_{n+1} - px_n - a_n =: b_n,\ n \ge 0.$$

By induction we get

$$x_n = p^n x_0 + \sum_{k=0}^{n-1} p^{n-k-1}(a_k + b_k),\ n \ge 1.$$

(1) Suppose that $|p| < 1$ and let $(y_n)_{n\ge 0}$ be given by (2.2) with $y_0 = x_0$. Then

$$y_n = p^n x_0 + \sum_{k=0}^{n-1} p^{n-k-1} a_k,\ n \ge 1,\ y_0 = x_0.$$

We obtain

$$\begin{aligned}|x_n - y_n| &\le \left|\sum_{k=0}^{n-1} b_k p^{n-k-1}\right| \le \sum_{k=0}^{n-1} |b_k| \cdot |p|^{n-k-1} \\ &\le \varepsilon \sum_{k=0}^{n-1} |p|^{n-k-1} = \varepsilon \frac{1-|p|^n}{1-|p|} \le \frac{\varepsilon}{1-|p|},\ n \ge 1.\end{aligned}$$

(2) Now let $|p| > 1$. Then, in view of the comparison test, the series

$$\sum_{n=1}^{\infty} \frac{b_{n-1}}{p^n}$$

is absolutely convergent, since

$$\left|\frac{b_{n-1}}{p^n}\right| \le \frac{\varepsilon}{|p|^n},\ n \ge 1,$$

and

$$\sum_{n=1}^{\infty} \frac{\varepsilon}{|p|^n} = \frac{\varepsilon}{|p|-1}.$$

Let

$$s := \sum_{n=1}^{\infty} \frac{b_{n-1}}{p^n},\ s \in \mathbb{R},$$

and define the sequence $(y_n)_{n\ge 0}$ by the recurrence (2.2) with $y_0 = x_0 + s$. We get:

$$\begin{aligned}|x_n - y_n| &= \left|-p^n s + \sum_{k=0}^{n-1} b_k p^{n-k-1}\right| = |p|^n \left|-s + \sum_{k=0}^{n-1} \frac{b_k}{p^{k+1}}\right| \\ &= |p|^n \left|\sum_{k=n}^{\infty} \frac{b_k}{p^{k+1}}\right| \le \varepsilon \sum_{n=1}^{\infty} \frac{1}{|p|^n} = \frac{\varepsilon}{|p|-1},\ n \ge 0.\end{aligned}$$

Uniqueness. Let $|p| > 1$. Suppose that there exists a sequence $(y_n)_{n\ge 0}$ satisfying (2.2) and (2.3) with $y_0 \ne x_0 + s$. Then

$$|x_n - y_n| = \left|p^n(x_0 - y_0) + \sum_{k=0}^{n-1} b_k p^{n-k-1}\right| = |p|^n \left|x_0 - y_0 + \sum_{k=0}^{n-1} \frac{b_k}{p^{k+1}}\right|$$

for all $n \ge 1$. Since

$$\lim_{n\to\infty} \left|x_0 - y_0 + \sum_{k=0}^{n-1} \frac{b_k}{p^{k+1}}\right| = |x_0 - y_0 + s| \ne 0,$$

it follows $\lim\limits_{n\to\infty} |x_n - y_n| = \infty$, contradiction with (2.3). □

Remark 2.1. *If $|p| < 1$ there exist infinitely many sequences $(y_n)_{n\ge 0}$ in Lemma 2.1 satisfying* (2.2) *and* (2.3).

Proof. Define $(y_n)_{n\ge 0}$ by (2.2), $y_0 = x_0 + u$, $|u| \le \varepsilon$. Then

$$|x_n - y_n| = \left|-p^n u + \sum_{k=0}^{n-1} b_k p^{n-k-1}\right| \le \varepsilon \sum_{k=0}^{n} |p|^k \le \frac{\varepsilon}{1-|p|}, \quad n \ge 1.$$

The results on Hyers-Ulam stability and nonstability for the recurrence (1.3) are contained in the next theorems. □

Theorem 2.1. *Let $\varepsilon > 0$ and $a, b \in \mathbb{C}$ such that the equation*

$$r^2 = ar + b$$

admits the roots r_1, r_2, $|r_1| \neq 1$, $|r_2| \neq 1$.

Then for every sequence $(x_n)_{n \geq 0}$ of complex numbers satisfying

$$|x_{n+2} - ax_{n+1} - bx_n| \leq \varepsilon,\ n \geq 0, \tag{2.4}$$

there exists a sequence $(y_n)_{n \geq 0}$ of complex numbers

$$y_{n+2} = ay_{n+1} + by_n,\ n \geq 0, \tag{2.5}$$

such that

$$|x_n - y_n| \leq \frac{\varepsilon}{|(|r_1| - 1)(|r_2| - 1)|},\ n \geq 0. \tag{2.6}$$

Moreover, if $|r_1| > 1$ and $|r_2| > 1$, then $(y_n)_{n \geq 0}$ is uniquely determined.

Proof. Existence. Let $(x_n)_{n \geq 0}$ be a sequence satisfying (2.4). Then

$$|x_{n+2} - (r_1 + r_2)x_{n+1} + r_1 r_2 x_n| \leq \varepsilon,\ n \geq 0. \tag{2.7}$$

Consider the sequence $(z_n)_{n \geq 0}$, $z_n = x_{n+1} - r_2 x_n$, $n \geq 0$. Then, according to (2.7), $(z_n)_{n \geq 0}$ satisfies the relation

$$|z_{n+1} - r_1 z_n| \leq \varepsilon,\ n \geq 0. \tag{2.8}$$

From Lemma 2.1 it follows that there exists a sequence $(w_n)_{n \geq 0}$,

$$w_{n+1} - r_1 w_n = 0,\ n \geq 0, \tag{2.9}$$

$$|z_n - w_n| \leq \frac{\varepsilon}{\big||r_1| - 1\big|},\ n \geq 0, \tag{2.10}$$

which leads to

$$|x_{n+1} - r_2 x_n - w_n| \leq \frac{\varepsilon}{\big||r_1| - 1\big|},\ n \geq 0. \tag{2.11}$$

Now, using again Lemma 2.1 and (2.11) it follows that there exists a sequence $(y_n)_{n \geq 0}$ in $\mathbb{C}$ satisfying

$$y_{n+1} - r_2 y_n - w_n = 0,\ n \geq 0, \tag{2.12}$$

$$|x_n - y_n| \leq \frac{\varepsilon}{|(|r_1| - 1)(|r_2| - 1)|},\ n \geq 0. \tag{2.13}$$

The relations (2.9) and (2.12) lead to

$$y_{n+2} - (r_1 + r_2)y_{n+1} + r_1 r_2 y_n = 0,\ n \geq 0,$$

or

$$y_{n+2} = ay_{n+1} + by_n,\ n \geq 0.$$

The existence is proved.

Uniqueness. Suppose that $|r_1| > 1$, $|r_2| > 1$, and for a sequence $(x_n)_{n\geq 0}$ satisfying (2.4) there exist two distinct sequences $(y'_n)_{n\geq 0}$, $(y''_n)_{n\geq 0}$ with the properties (2.5) and (2.6). Then

$$|y'_n - y''_n| \leq |y'_n - x_n| + |x_n - y''_n| \leq \frac{2\varepsilon}{|(|r_1|-1)(|r_2|-1)|},\ n \geq 0. \tag{2.14}$$

The sequence $(z_n)_{n\geq 0}$, $z_n = y'_n - y''_n$, $n \geq 0$, satisfies the recurrence (2.5) so its general term has the form

$$z_n = Ar_1^n + Br_2^n \ \text{ if }\ r_1 \neq r_2$$

or

$$z_n = r_1^n(An + B) \ \text{ if }\ r_1 = r_2,$$

where $A, B \in \mathbb{C}$. Obviously A, B cannot be simultaneously zero since $(y'_n)_{n\geq 0}$, $(y''_n)_{n\geq 0}$ are distinct sequences. Then $\lim\limits_{n\to\infty} |z_n| = \infty$, which contradicts (2.14) and thus the uniqueness is proved. □

Theorem 2.2. *Let $\varepsilon > 0$, $a, b \in \mathbb{C}$ such that the equation*

$$r^2 = ar + b$$

has a complex root of modulus one.

Then there exists a sequence of complex numbers $(x_n)_{n\geq 0}$ satisfying

$$|x_{n+2} - ax_{n+1} - bx_n| \leq \varepsilon,\ n \geq 0, \tag{2.15}$$

such that for every sequence of complex numbers $(y_n)_{n\geq 0}$ satisfying

$$y_{n+2} - ay_{n+1} - by_n = 0,\ n \geq 0, \tag{2.16}$$

we have $\sup\limits_{n\in\mathbb{N}} |x_n - y_n| = \infty$, i.e., the recurrence (1.3) is not stable in Hyers-Ulam sense.

Proof. Let r_1, r_2 be the roots of the equation $r^2 = ar + b$, and suppose that $|r_1| = 1$. Define the sequence $(x_n)_{n\geq 0}$ by the relation

$$x_{n+1} - r_2 x_n = \varepsilon n r_1^n,\ n \geq 0,\ x_1 \neq 0. \tag{2.17}$$

Then it is easy to check that

$$|x_{n+2} - (r_1 + r_2)x_{n+1} + r_1 r_2 x_n| = \varepsilon,\ n \geq 0,$$

therefore the relation (2.15) is satisfied.

Now let $(y_n)_{n\geq 0}$ be an arbitrary sequence satisfying (2.16). Then

$$(y_{n+2} - r_2 y_{n+1}) - r_1(y_{n+1} - r_2 y_n) = 0,\ n \geq 0,$$

therefore

$$y_{n+1} - r_2 y_n = (y_1 - r_2 y_0) r_1^n,\ n \geq 0. \tag{2.18}$$

The relations (2.17) and (2.18) lead to

$$x_{n+1} - y_{n+1} - r_2(x_n - y_n) = (\varepsilon n - y_1 + r_2 y_0) r_1^n,\ n \geq 0, \tag{2.19}$$

therefore

$$|x_{n+1} - y_{n+1} - r_2(x_n - y_n)| = |\varepsilon n - y_1 + r_2 y_0| \to \infty \tag{2.20}$$

when $n \to \infty$. We prove that $\sup\limits_{n\in\mathbb{N}} |x_n - y_n| = \infty$.

If there exists $M > 0$ such that $|x_n - y_n| \le M$ for all $n \in \mathbb{N}$, then

$$\begin{aligned}|x_{n+1} - y_{n+1} - r_2(x_n - y_n)| &\le |x_{n+1} - y_{n+1}| + |r_2||x_n - y_n| \\ &\le (1 + |r_2|)M, \ M \ge 0,\end{aligned}$$

contradiction with (2.20), and the theorem is proved. □

The results obtained in Theorem 2.1 and Theorem 2.2 lead to the following conclusion:

The linear recurrence (1.3) *is stable in Hyers-Ulam sense if and only if the roots* r_1, r_2 *of its characteristic equation satisfy the conditions* $|r_1| \neq 1$ *and* $|r_2| \neq 1$.

For more details and results on the stability of recurrences we refer the reader to [1], [2], [3], [5], [6].

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An integral inequality

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Abstract. In this article we present a general solution to an open problem posed by Q.A. Ngo et al. in [9] and other results that generalize the responses offered to the same problem by K. Boukerrioua [2] or S.S. Dragomir and Q.A. Ngo [5].

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MSC: 26D150

1. Introduction

The next question was posed in [9].

Problem 1.1. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function which satisfies*

$$\int_x^1 f(t)\,\mathrm{d}t \geqslant \int_x^1 t\mathrm{d}t,$$

for all $x \in [0,\ 1]$. Under what conditions on α and β does the inequality

$$\int_0^1 f^{\alpha+\beta}(t)\,\mathrm{d}t \geqslant \int_0^1 f^{\alpha}(t)\,t^{\beta}\mathrm{d}t$$

hold?

This problem has received several solutions and improvements in [2], [8] or [5]. In this paper, we give a more general result, which is presented in the next theorem.

Theorem 1.1. *Let $f, g : [a,b] \to [0,\infty)$ be two integrable functions, g non-decreasing, which satisfy*

$$\int_x^b f(t)\mathrm{d}t \geq \int_x^b g(t)\,\mathrm{d}t,$$

for all $x \in [a,b]$. Then the following two conclusions hold:

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(a) *For any* $\alpha \in (0,\infty)$ *and* $\beta \in [1,\infty)$ *we have*

$$\int_x^b f^{\beta}(t) g^{\alpha} \mathrm{d}t \geq \int_x^b g^{\beta+\alpha}(t)\,\mathrm{d}t,$$

for all $x \in [a,b]$;

(b) *For any* $\alpha \in (0,\infty)$ *and* $\beta \in [1,\infty)$ *we have*

$$\int_x^b f^{\beta+\alpha}(t)\mathrm{d}t \geq \int_x^b f^{\beta}(t)\, g^{\alpha}(t)\,\mathrm{d}t,$$

for all $x \in [a,b]$.

In the next sections we will give a proof of this theorem as well as similar results for nonincreasing functions. It is clear that Theorem 1.1 represents a very general solution for Problem 1.1. We also rediscover results from the bibliography as consequences of our work.

2. Two useful lemmas and some consequences

In this section we present and prove some useful results. It is easy to see that our results hold under very general conditions.

Lemma 2.1. *Let* $f, g : [a,b] \to \mathbb{R}$ *be Riemann integrable functions which satisfy*

$$\int_a^x f(t)\mathrm{d}t \geqslant \int_a^x g(t)\mathrm{d}t,$$

for all $x \in [a,b]$.

(a) *For any nonincreasing function* $h : [a,b] \to [0,\infty)$ *we have*

$$\int_a^x f(t)h(t)\mathrm{d}t \geqslant \int_a^x g(t)h(t)\mathrm{d}t,$$

for all $x \in [a,b]$;

(b) *If* f *is nonnegative,* g *is nonincreasing and nonnegative, then*

$$\int_a^x f^{\beta}(t)\mathrm{d}t \geqslant \int_a^x g^{\beta}(t)\mathrm{d}t,$$

for any $\beta \geqslant 1$ *and for all* $x \in [a,b]$.

Proof. (a) We consider the function

$$U : [a,b] \to [0,\infty), \qquad U(x) = \int_a^x [f(t) - g(t)]\mathrm{d}t.$$

This is correctly defined and continuous (see e.g. Theorem 7.33 from [1]). Since h is monotone, the Stieltjes integral

$$\int_a^x U(t)\mathrm{d}h(t)$$

exists for any $x \in [a,b]$. From the properties of Stieltjes integral (see e.g. Theorem 7.6 from [1]) we find that

$$\int_a^x U(t)\mathrm{d}h(t) = U(t)h(t)|_a^x - \int_a^x h(t)\mathrm{d}U(t). \tag{2.1}$$

As

$$\int_a^x h(t)\mathrm{d}U(t) = \int_a^x h(t)[f(t)-g(t)]\mathrm{d}t = \int_a^x h(t)f(t)\mathrm{d}t - \int_a^x h(t)g(t)\mathrm{d}t,$$

from (2.1) we obtain

$$\int_a^x h(t)f(t)\mathrm{d}t - \int_a^x h(t)g(t)\mathrm{d}t = U(x)h(x) - \int_a^x U(t)\mathrm{d}h(t) \geq 0$$

because U and h are nonnegative and $\int_a^x U(t)\mathrm{d}h(t) \leq 0$ since h is nonincreasing.

(b) For $\beta = 1$ this is true by hypothesis. If $\beta > 1$ then let be $\alpha > 0$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. The function

$$h : [a,b] \to \mathbb{R}, \quad h(t) = g^{\beta/\alpha}(t)$$

is nonnegative and nonincreasing. From part (a) and Hölder-Rogers inequality we get

$$\int_a^x g(t)\,h(t)\,\mathrm{d}t \leq \int_a^x f(t)h(t)\,\mathrm{d}t \leq \left(\int_a^x f^\beta(t)\mathrm{d}t\right)^{1/\beta} \left(\int_a^x h^\alpha(t)\,\mathrm{d}t\right)^{1/\alpha},$$

that is

$$\int_a^x g^\beta(t)\,\mathrm{d}t \leq \left(\int_a^x f^\beta(t)\mathrm{d}t\right)^{1/\beta} \left(\int_a^x g^\beta(t)\,\mathrm{d}t\right)^{1/\alpha}. \tag{2.2}$$

If $x \in [a,b]$ is such that $\int\limits_a^x g^\beta(t)\mathrm{d}t = 0$ then the fact that $\int\limits_a^x f^\beta(t)\mathrm{d}t \geq$ $\geq \int\limits_a^x g^\beta(t)\mathrm{d}t$ is clear. Otherwise, dividing (2.2) by $\left(\int\limits_a^x g^\beta(t)\,\mathrm{d}t\right)^{1/\alpha}$, we obtain

$$\left(\int\limits_a^x g^\beta(t)\,\mathrm{d}t\right)^{1/\beta} \leq \left(\int\limits_a^x f^\beta(t)\mathrm{d}t\right)^{1/\beta}$$

and the conclusion follows. □

Corollary 2.1. *Let $f, g : [a,b] \to [0,\infty)$ be two integrable functions, g nonincreasing, and $\alpha, \beta \in (0,\infty)$ with $\alpha \leq \beta$. If*

$$\int\limits_a^x f^\alpha(t)\mathrm{d}t \geq \int\limits_a^x g^\alpha(t)\,\mathrm{d}t,$$

for all $x \in [a,b]$ then

$$\int\limits_a^x f^\beta(t)\mathrm{d}t \geq \int\limits_a^x g^\beta(t)\,\mathrm{d}t,$$

for all $x \in [a,b]$.

Proof. The functions f^α and g^α verify the assumptions of the second part of Lemma 2.1. But $\frac{\beta}{\alpha} \geq 1$ so

$$\int\limits_a^x f^\beta(t)\mathrm{d}t = \int\limits_a^x (f^\alpha(t))^{\beta/\alpha}\mathrm{d}t \geq \int\limits_a^x (g^\alpha(t))^{\beta/\alpha}\mathrm{d}t = \int\limits_a^x g^\beta(t)\,\mathrm{d}t$$

for all $x \in [a,b]$. □

The next lemma is similar to Lemma 2.1, but for nondecreasing function.

Lemma 2.2. *Let $f, g : [a,b] \to \mathbb{R}$ be two Riemann integrable functions with*

$$\int\limits_x^b f(t)\mathrm{d}t \geq \int\limits_x^b g(t)\mathrm{d}t,$$

for all $x \in [a,b]$.

(a) *For any nondecreasing function $h : [a,b] \to [0,\infty)$ we have*

$$\int\limits_x^b f(t)h(t)\mathrm{d}t \geq \int\limits_x^b g(t)h(t)\mathrm{d}t,$$

for all $x \in [a, b]$*;*

(b) *If* f *is nonnegative,* g *is nondecreasing and nonnegative, then*

$$\int_x^b f^\beta(t)\mathrm{d}t \geq \int_x^b g^\beta(t)\mathrm{d}t,$$

for any $\beta \geq 1$ *and for all* $x \in [a, b]$.

Proof. (a) We use the function

$$V : [a, b] \to [0, \infty),\ V(x) = \int_x^b [f(t) - g(t)]\mathrm{d}t.$$

In the same way as in Lemma 2.1, we obtain

$$\int_x^b f(t)h(t)\mathrm{d}t - \int_x^b g(t)h(t)\mathrm{d}t = V(x)h(x) + \int_x^b V(t)\mathrm{d}h(t) \geq 0$$

because V and h are nonnegative and $\int_x^b V(t)\mathrm{d}h(t) \geq 0$ since h is nondecreasing.

(b) Part (b) here follows in the same way as part (b) in Lemma 2.1. □

Remark. Lemma 2.2 provides confirmation to the conjecture posed by Ngo in [10].

Corollary 2.2. *Let* $f, g : [a, b] \to [0, \infty)$ *be two integrable funtions,* g *nondecreasing, and* $\alpha, \beta \in (0, \infty)$ *with* $\alpha \leq \beta$*. If*

$$\int_x^b f^\alpha(t)\mathrm{d}t \geq \int_x^b g^\alpha(t)\,\mathrm{d}t,$$

for all $x \in [a, b]$ *then*

$$\int_x^b f^\beta(t)\mathrm{d}t \geq \int_x^b g^\beta(t)\,\mathrm{d}t,$$

for all $x \in [a, b]$.

Proof. The proof follows similarly to the proof of Corollary 2.1, part (b). □

3. PROOF OF THEOREM 1.2. AND OTHER SIMILAR RESULTS

Using Lemma 2.2, we now give a proof for Theorem 1.1.

Proof. (a) From part (b) of Lemma 2.2 we obtain

$$\int_x^b f^{\beta}(t)\,\mathrm{d}t \geq \int_x^b g^{\beta}(t)\,\mathrm{d}t$$

for all $x \in [a,b]$. Now apply part (a) from the same lemma for $h = g^{\alpha}$ and we have the conclusion.

(b) Using the general Cauchy inequality, we have

$$\frac{\beta}{\alpha+\beta} f^{\alpha+\beta}(t) + \frac{\alpha}{\alpha+\beta} g^{\alpha+\beta}(t) \geq f^{\beta}(t)\, g^{\alpha}(t),$$

for all $t \in [a,b]$. Integrating the resulting inequality from x to b, we obtain

$$\frac{\beta}{\alpha+\beta}\int_x^b f^{\alpha+\beta}(t)\mathrm{d}t + \frac{\alpha}{\alpha+\beta}\int_x^b g^{\alpha+\beta}(t)\,\mathrm{d}t \geq \int_x^b f^{\beta}(t)\, g^{\alpha}(t)\mathrm{d}t,$$

for all $x \in [a,b]$. By part (a)

$$\int_x^b f^{\beta}(t)\, g^{\alpha}(t)\mathrm{d}t \geq \int_x^b g^{\alpha+\beta}(t)\mathrm{d}t,$$

so

$$\frac{\beta}{\alpha+\beta}\int_x^b f^{\alpha+\beta}(t)\mathrm{d}t + \frac{\alpha}{\alpha+\beta}\int_x^b f^{\beta}(t)\, g^{\alpha}(t)\mathrm{d}t \geq \int_x^b f^{\beta}(t)\, g^{\alpha}(t)\mathrm{d}t,$$

that is

$$\frac{\beta}{\alpha+\beta}\int_x^b f^{\alpha+\beta}(t)\mathrm{d}t \geq \left(1-\frac{\alpha}{\alpha+\beta}\right)\int_x^b f^{\beta}(t)\, g^{\alpha}(t)\mathrm{d}t,$$

and the conclusion follows. □

The next result is a consequence of Theorem 1.1.

Corollary 3.1. *Let $\alpha, \beta \in (0,\infty)$. Let $f, g : [a,b] \to [0,\infty)$ be two integrable functions, g nondecreasing, with*

$$\int_x^b f^{\beta}(t)\mathrm{d}t \geq \int_x^b g^{\beta}(t)\,\mathrm{d}t,$$

for all $x \in [a,b]$.

(a) *For all $x \in [a,b]$ we have*

$$\int_x^b f^{\beta}(t) g^{\alpha}\mathrm{d}t \geq \int_x^b g^{\beta+\alpha}(t)\,\mathrm{d}t;$$

(b) *For all* $x \in [a, b]$ *we have*

$$\int_x^b f^{\beta+\alpha}(t)\mathrm{d}t \geq \int_x^b f^{\beta}(t)\, g^{\alpha}(t)\, \mathrm{d}t.$$

Proof. For the first part apply Lemma 2.3(a) for the function $h = g^{\alpha}$. The proof of the second part is similar to the proof of the analogous part of Theorem 1.1. □

A similar result with Theorem 1.1 exists for descreasing function.

Theorem 3.2. *Let* $f, g : [a, b] \to [0, \infty)$ *be two integrable functions,* g *nonincreasing, which satisfy*

$$\int_a^x f(t)\mathrm{d}t \geq \int_a^x g(t)\, \mathrm{d}t,$$

for all $x \in [a, b]$*. Then the following two conclusions hold:*

(a) *For any* $\alpha \in (0, \infty)$ *and* $\beta \in [1, \infty)$*, we have*

$$\int_a^x f^{\beta}(t) g^{\alpha}\mathrm{d}t \geq \int_a^x g^{\beta+\alpha}(t)\, \mathrm{d}t,$$

for all $x \in [a, b]$*;*

(b) *For any* $\alpha \in (0, \infty)$ *and* $\beta \in [1, \infty)$*, we have*

$$\int_a^x f^{\beta+\alpha}(t)\mathrm{d}t \geq \int_a^x f^{\beta}(t)\, g^{\alpha}(t)\, \mathrm{d}t,$$

for all $x \in [a, b]$*.*

Proof. It follows in the same way as the proof of Theorem 1.1, but applying Lemma 2.1. □

Also, a consequence of Theorem 3.2 is represented by the next corollary.

Corollary 3.2. *Let* $\alpha, \beta \in (0, \infty)$*. Let* $f, g : [a, b] \to [0, \infty)$ *be two integrable functions,* g *nonincreasing, with*

$$\int_a^x f^{\beta}(t)\mathrm{d}t \geq \int_a^x g^{\beta}(t)\, \mathrm{d}t,$$

for all $x \in [a, b]$*.*

(a) *For all* $x \in [a, b]$ *we have*

$$\int_a^x f^{\beta}(t) g^{\alpha}\mathrm{d}t \geq \int_a^x g^{\alpha+\beta}(t)\, \mathrm{d}t;$$

(b) *For all* $x \in [a, b]$ *we have*

$$\int_a^x f^{\alpha+\beta}(t)\mathrm{d}t \geq \int_a^x f^{\beta}(t)\, g^{\alpha}(t)\,\mathrm{d}t.$$

Proof. For the first part, apply Lemma 2.1.(a) for the function $h = g^{\alpha}$. The proof of the second part is similar with the proof of Corollary 3.1(b). □

4. Applications

In this last section we give alternative proofs for some results from the bibliography. We start with a problem of V. Krasniqi [6]. Another solution can be found in [13].

Corollary 4.1. *Let* α *be a positive real number and* f *be a nonnegative function on* $[0, 1]$ *such that*

$$\int_x^1 (f(t))^{\alpha}\,\mathrm{d}t \geq \int_x^1 t^{\alpha}\mathrm{d}t,$$

for all $0 \leq x \leq 1$. *Prove that*

$$\int_0^1 (f(t))^{\alpha+\beta}\,\mathrm{d}t \geq \int_0^1 (f(t))^{\alpha}\, t^{\beta}\mathrm{d}t \geq \int_0^1 t^{\alpha+\beta}\mathrm{d}t,$$

for every positive β.

Proof. The first inequality is obtained from part (b) of Corollary 3.1 in the case $g(t) = t$. The second inequality represents the same case but for the part (a) of the same corollary. □

Duong Viet Thong proposed an extension of the previous problem (see [12]). We present our solution to this problem.

Corollary 4.2. *Let* f *be a nonnegative continuous function on* $[0, 1]$ *and* $k \geq 1$ *be an integer. Prove that if*

$$\int_x^1 f(t)\,\mathrm{d}t \geq \int_x^1 t^{k}\mathrm{d}t,$$

for all $0 \leq x \leq 1$, *then*

$$\int_0^1 (f(x))^{\alpha+\beta}\,\mathrm{d}x \geq \int_0^1 (f(x))^{\beta}\, x^{k\alpha} dx,$$

for all $\alpha \geq 0$ *and* $\beta \geq 1$.

Proof. We apply Theorem 1.1, part (b), in the case $g(t) = t^k$ and $[a,b] = [0,1]$. □

Another solution can be found in [7]. The author observed that this result can be extended to a more general result, which he proved by using Fubini's theorem. Next, we give our proof for this extension.

Corollary 4.3 (E. Lampakis [7])**.** *Suppose f is a nonnegative continuous function on $[0,1]$, $k \geq 1$ is an integer, $\alpha \geq 0$, and $\beta \geq 1$ are reals. If*

$$\int_x^1 f(t)\,\mathrm{d}t \geq \int_x^1 t^k \mathrm{d}t,$$

for all $0 \leq x \leq 1$, then

(a) *For all $0 \leq x \leq 1$ we have*

$$\int_x^1 (f(t))^\beta\,\mathrm{d}t \geq \int_x^1 t^{k\beta}\mathrm{d}t;$$

(b) *We have*

$$\int_0^1 (f(x))^{\alpha+\beta}\,\mathrm{d}x \geq \int_0^1 (f(x))^\beta x^{k\alpha}\mathrm{d}x \geq \int_0^1 x^{k(\alpha+\beta)}\mathrm{d}x.$$

Proof. The first part is a consequence of Lemma 2.2 for the case $g(t) = t^k$ and $[a,b] = [0,1]$. For the proof of the second part we apply Theorem 1.1 for $g(t) = t^k$ and $[a,b] = [0,1]$. □

The next two corollaries go to another type of extensions. In fact the function $g(t) = t^k$ is replaced by an arbitrary monotone function g.

Corollary 4.4 (J. Sándor [11])**.** *Let $f, g : [a,b] \to [0,\infty)$ be two continuous and nonincreasing functions. If*

$$\int_a^x f(t)\mathrm{d}t \geq \int_a^x g(t)\,\mathrm{d}t,$$

for all $x \in [a,b]$ then

$$\int_a^x f^2(t)\mathrm{d}t \geq \int_a^x g^2(t)\,\mathrm{d}t,$$

for all $x \in [a,b]$.

Proof. Apply Lemma 2.1(b) for $\beta = 2$. Note that we do not need the monotonicity of f. □

A similar result can be found in [4]. Now, we give a proof in the spirit of this paper.

Corollary 4.5 (A. Ciupan [3])**.** *Let* $f, g : [0,1] \to \mathbb{R}$ *be two functions such that* f *is continuous and* g *is nondecreasing and differentiable with* $g(0) \geq 0$*. If*

$$\int_t^1 f(x)\,\mathrm{d}x \geq \int_t^1 g(x)\,\mathrm{d}x$$

for any $t \in [0,1]$ *then*

$$\int_0^1 f^2(x)\,\mathrm{d}x \geq \int_0^1 g^2(x)\,\mathrm{d}x.$$

Proof. Evidently, g is continuous and nonnegative. We consider the function $h : [0,1] \to \mathbb{R}$ defined by $h(x) = |f(x)|$. Function h is continuous,

$$\int_t^1 h(x)\,\mathrm{d}x \geq \int_t^1 f(x)\,\mathrm{d}x$$

for any $t \in [0,1]$ and

$$\int_0^1 h^2(x)\,\mathrm{d}x = \int_0^1 f^2(x)\,\mathrm{d}x.$$

Now, we obtain the conclusion by applying Lemma 2.2(b) for the functions h, g and $\beta = 2$. □

Finally we present another recent result. This represents its authors' contribution to the development of the initial open problem.

Corollary 4.6 (S.S. Dragomir, Q.A. Ngo [5])**.** *Let* $f : [a,b] \to [0,\infty)$ *be a continuous function and* $g : [a,b] \to [0,\infty)$ *be nondecreasing and differentiable function on* (a,b)*. If for any* $\alpha > 0$ *and all* $x \in [a,b]$ *we have*

$$\int_x^b f^\alpha(t)\,\mathrm{d}t \geq \int_x^b g^\alpha(t)\,\mathrm{d}t$$

then the inequality

$$\int_x^b f^\beta(t)\,\mathrm{d}t \geq \int_x^b g^\beta(t)\,\mathrm{d}t$$

holds for any $\beta > \alpha$ *and all* $x \in [a,b]$*.*

Proof. From hypothesis, we have that g is continuous on (a,b). Also $\lim\limits_{x\searrow a} g(x)$ and $\lim\limits_{x\nearrow b} g(x)$ exist and are finite. Let h be defined on $[a,b]$ by

$$h(a)=\lim_{x\searrow a} g(x)\,,\ h(b)=\lim_{x\nearrow b} g(x)\,,\quad \text{and}\ \ h(x)=g(x)\,,\forall x\in(a,b)\,.$$

It is clear that for all $\alpha>0$ we have

$$\int_x^b g^{\alpha}(t)\,\mathrm{d}t=\int_x^b h^{\alpha}(t)\,\mathrm{d}t.$$

Now we obtain the conclusion by applying Corollary 2.2 for the functions f and h. □

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Certain inequalities for the triangle

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Abstract. The aim of this paper is to establish several inequalities between the distances from an interior point M of the triangle ABC to the vertices of the triangle and the radii of the circumcircles of the triangles MBC, MCA and MAB.

Keywords: Geometric inequalities, Euler's Inequality.

MSC: 51M04

1. INTRODUCTION

In the interior of the triangle ABC we choose a point M. Let R_a, R_b, R_c be the radii of the circumcircles of the triangles MBC, MCA and MAB, respectively. Let R_1, R_2, R_3 be the distances from M to the vertices of ABC and let r_1, r_2, r_3 be the distances from M to the sides of ABC. The well-known inequality of Erdős-Mordell is given by

$$R_1 + R_2 + R_3 \geq 2\left(r_1 + r_2 + r_3\right). \tag{1.1}$$

A generalization of the Erdős-Mordell inequality can be found in [8], namely

$$x^2R_1 + y^2R_2 + z^2R_3 \geq 2\left(yzr_1 + zxr_2 + xyr_3\right), \tag{1.2}$$

for every real numbers $x, y, z \geq 0$, and in [2] this inequality is expressed in the following way:

$$\lambda_1R_1 + \lambda_2R_2 + \lambda_3R_3 \geq 2\left(\sqrt{\lambda_2\lambda_3}r_1 + \sqrt{\lambda_3\lambda_1}r_2 + \sqrt{\lambda_1\lambda_2}r_3\right), \tag{1.3}$$

for every $\lambda_1, \lambda_2, \lambda_3 > 0$.

In [10], G. Tsintsifas showed that if $(\lambda_1, \lambda_2, \lambda_3)$ are barycentric coordinates of M and R is the circumradius of triangle ABC, then there is the inequality

$$\lambda_1R_a + \lambda_2R_b + \lambda_3R_c \geq R \geq \lambda_1R_1 + \lambda_2R_2 + \lambda_3R_3. \tag{1.4}$$

In [3], W-D Jiang proved the inequality

$$\frac{w_a}{\sqrt{R_bR_c}} + \frac{w_b}{\sqrt{R_aR_c}} + \frac{w_c}{\sqrt{R_aR_b}} \leq \frac{9}{2}, \tag{1.5}$$

where w_a, w_b, w_c are the lengths of the bisectors. This improves a result due to Liu, namely,

$$\frac{w_a}{R_b + R_c} + \frac{w_b}{R_a + R_c} + \frac{w_c}{R_a + R_b} \leq \frac{9}{4}. \tag{1.6}$$

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From [4] and [5], we have the following inequalities:

$$\lambda_1^2 R_a + \lambda_2^2 R_b + \lambda_3^2 R_c \geq \lambda_1\lambda_2\lambda_3\left(\frac{R_1}{\lambda_1}+\frac{R_2}{\lambda_2}+\frac{R_3}{\lambda_3}\right) \tag{1.7}$$

and

$$\frac{R_1}{R_b+R_c}+\frac{R_2}{R_a+R_c}+\frac{R_3}{R_a+R_b} \leq \frac{3}{2}\,. \tag{1.8}$$

In [6] and [7], we found the following inequalities:

$$R_aR_bR_c \geq R_1R_2R_3, \tag{1.9}$$

$$R_a+R_b+R_c \geq R_1+R_2+R_3, \tag{1.10}$$

and in [9] we also found the inequality

$$R_a+R_b+R_c \leq 3R, \tag{1.11}$$

where R_a, R_b, R_c are the circumradii of triangles IBC, ICA, and IAB, respectively, I is the incenter of triangle ABC, and R is the circumradius of triangle ABC.

2. Main results

Theorem 2.1. *Let $\lambda_2, \lambda_2, \lambda_3$ be some positive real numbers and M a point in the interior of the triangle ABC. If R_a, R_b, R_c are the circumradii of the triangles MBC, MCA, and MAB, then we have the following inequality:*

$$\begin{aligned} &2\left(\lambda_1 R_a \sin\frac{A}{2} + \lambda_2 R_b \sin\frac{B}{2} + \lambda_3 R_c \sin\frac{C}{2}\right) \\ &\geq \sqrt{\lambda_2\lambda_3}R_1 + \sqrt{\lambda_3\lambda_1}R_2 + \sqrt{\lambda_1\lambda_2}R_3. \end{aligned} \tag{2.1}$$

Proof. From triangles MAB and MAC, we deduce that

$$R_2 = 2R_c \sin\sphericalangle MAB \text{ and } R_3 = 2R_b\sin\sphericalangle MAC.$$

It follows that

$$\begin{aligned}\frac{R_2}{2R_c}+\frac{R_3}{2R_b} &= \sin\sphericalangle MAB + \sin\sphericalangle MAC \\ &= 2\sin\frac{A}{2}\cos\frac{\sphericalangle MAB - \sphericalangle MAC}{2} \leq 2\sin\frac{A}{2}\,.\end{aligned}$$

Therefore

$$\frac{R_2}{2R_c}+\frac{R_3}{2R_b} \leq 2\sin\frac{A}{2}\,.$$

The equality holds if and only if AM is the bisector of the angle A.

This inequality is found in [1] in the equivalent form $\sin\frac{A}{2} \geq \frac{r_2+r_3}{2R_1}$. The claimed equivalence holds because we have $r_2 = \frac{R_3R_1}{2R_b}$ and $r_3 = \frac{R_1R_2}{2R_c}$.

In an analogous way, we obtain the following inequalities:

$$\frac{R_1}{2R_c}+\frac{R_3}{2R_a}\le 2\sin\frac{B}{2}\quad\text{and}\quad \frac{R_2}{2R_a}+\frac{R_1}{2R_b}\le 2\sin\frac{C}{2}\,.$$

By using the above inequalities, we estimate the sum

$$\lambda_1 R_a\sin\frac{A}{2}+\lambda_2 R_b\sin\frac{B}{2}+\ \lambda_3 R_c\sin\frac{C}{2}$$

and we deduce

$$\begin{aligned}&\lambda_1 R_a\sin\frac{A}{2}+\lambda_2 R_b\sin\frac{B}{2}+\lambda_3 R_c\sin\frac{C}{2}\\ &\ge \lambda_1 R_a\left(\frac{R_2}{4R_c}+\frac{R_3}{4R_b}\right)+\lambda_2 R_b\left(\frac{R_1}{4R_c}+\frac{R_3}{4R_a}\right)+\lambda_3 R_c\left(\frac{R_2}{4R_a}+\frac{R_1}{4R_b}\right)\\ &= R_1\left(\frac{\lambda_2 R_b}{4R_c}+\frac{\lambda_3 R_c}{4R_b}\right)+R_2\left(\frac{\lambda_1 R_a}{4R_c}+\frac{\lambda_3 R_c}{4R_a}\right)+R_3\left(\frac{\lambda_2 R_b}{4R_a}+\frac{\lambda_1 R_a}{4R_b}\right)\\ &\ge \frac{1}{2}\left(\sqrt{\lambda_2\lambda_3}R_1+\sqrt{\lambda_3\lambda_1}R_2+\sqrt{\lambda_1\lambda_2}R_3\right).\end{aligned}$$

Consequently, we obtain the inequality of the statement. □

Remark 2.1. In relation (2.1) the equality holds if and only if M is the incentre of ABC and

$$\lambda_1\sin^2\frac{A}{2}=\lambda_2\sin^2\frac{B}{2}=\lambda_3\sin^2\frac{C}{2}.$$

Corollary 2.1. *There are the following inequalities:*

$$2\left(R_a\sin\frac{A}{2}+R_b\sin\frac{B}{2}+R_c\sin\frac{C}{2}\right)\ge R_1+R_2+R_3, \tag{2.2}$$

$$2\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)\ge\frac{R_1}{\sqrt{R_bR_c}}+\frac{R_2}{\sqrt{R_cR_a}}+\frac{R_3}{\sqrt{R_aR_b}}\,, \tag{2.3}$$

$$3\ge\frac{R_1}{\sqrt{R_bR_c}}+\frac{R_2}{\sqrt{R_cR_a}}+\frac{R_3}{\sqrt{R_aR_b}}\,, \tag{2.4}$$

$$\begin{aligned}&R_a\sin A+R_b\sin B+R_c\sin C\\ &\ge R_1\sqrt{\cos\frac{B}{2}\cos\frac{C}{2}}+R_2\sqrt{\cos\frac{A}{2}\cos\frac{C}{2}}+R_3\sqrt{\cos\frac{A}{2}\cos\frac{B}{2}},\end{aligned} \tag{2.5}$$

$$\frac{R_a}{\cos\frac{A}{2}}+\frac{R_b}{\cos\frac{B}{2}}+\frac{R_c}{\cos\frac{C}{2}}\ge\frac{R_1}{\cos\frac{A}{2}}+\frac{R_2}{\cos\frac{B}{2}}+\frac{R_3}{\cos\frac{C}{2}}, \tag{2.6}$$

$$\sqrt{\frac{r}{R}}\,(R_a+R_b+R_c)\ge R_1\sqrt{\sin\frac{A}{2}}+R_2\sqrt{\sin\frac{B}{2}}+R_3\sqrt{\sin\frac{C}{2}}\,. \tag{2.7}$$

Proof. If in Theorem 2.1 we make the substitutions $\lambda_1 = \lambda_2 = \lambda_3$, then we obtain the inequality

$$2\left(R_a \sin\frac{A}{2} + R_b \sin\frac{B}{2} + R_c \sin\frac{C}{2}\right) \geq R_1 + R_2 + R_3.$$

We take $\lambda_1 = \dfrac{1}{R_a}$, $\lambda_2 = \dfrac{1}{R_b}$, and $\lambda_3 = \dfrac{1}{R_c}$ in inequality of Theorem 2.1 and we get inequality (2.3). By using Jensen's Inequality we get

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \leq \frac{3}{2},$$

and with the aid of inequality (2.3), we deduce inequality (2.4). Making the substitutions $\lambda_1 = \cos\dfrac{A}{2}$, $\lambda_2 = \cos\dfrac{A}{2}$, and $\lambda_3 = \cos\dfrac{C}{2}$ in Theorem 2.1, we obtain inequality (2.1).

Now, we prove the inequality

$$\sin B \sin C \leq \cos^2\frac{A}{2} \tag{2.8}$$

in the following way:

$$\sin B \sin C = \frac{1}{2}[\cos(B-C) - \cos(B+C)] \leq \frac{1}{2}(1+\cos A) = \cos^2\frac{A}{2}.$$

Therefore, by taking $\lambda_1 = \dfrac{1}{\sin A}$, $\lambda_2 = \dfrac{1}{\sin B}$, and $\lambda_3 = \dfrac{1}{\sin C}$ in Theorem 2.1, and using the above inequality and its permutations, we have inequality (2.6). If, in Theorem 2.1, we make the substitutions $\lambda_1 = \dfrac{1}{\sin\frac{A}{2}}$, $\lambda_2 = \dfrac{1}{\sin\frac{B}{2}}$, and $\lambda_3 = \dfrac{1}{\sin\frac{C}{2}}$, then we obtain the inequality

$$2(R_a + R_b + R_c) \geq \frac{1}{\sqrt{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}}\left(R_1\sqrt{\sin\frac{A}{2}} + R_2\sqrt{\sin\frac{B}{2}} + R_3\sqrt{\sin\frac{C}{2}}\right).$$

But there is the equality

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R},$$

so, we obtain (2.7). □

Corollary 2.2. *There is the following inequality:*

$$R_aR_bR_c \geq \frac{R}{2r}\,R_1R_2R_3. \tag{2.9}$$

Proof. By the proof of Theorem 2.1 we have the inequalities

$$\frac{R_2}{2R_c}+\frac{R_3}{2R_b}\leq 2\sin\frac{A}{2}\,,\ \frac{R_1}{2R_c}+\frac{R_3}{2R_a}\leq 2\sin\frac{B}{2}\,,\ \text{ and }\ \frac{R_2}{2R_a}+\frac{R_1}{2R_b}\leq 2\sin\frac{C}{2}\,.$$

We apply the arithmetic-geometric mean inequality and we find the following relations:

$$\sqrt{R_bR_cR_2R_3}\leq 2R_bR_c\sin\frac{A}{2}\,,\quad \sqrt{R_aR_cR_1R_3}\leq 2R_aR_c\sin\frac{B}{2}\,,$$

and

$$\sqrt{R_aR_bR_1R_2}\leq 2R_aR_b\sin\frac{C}{2}\,.$$

Taking the product of these inequalities and using equality

$$\sin\frac{A}{2}\,\sin\frac{B}{2}\,\sin\frac{C}{2}=\frac{r}{4R}\,,$$

we obtain relation (2.9). □

Remark 2.2. Applying Euler's inequality $R \geq 2r$, from relation (2.9) we deduce inequality (1.9).

According to the relations $R_a=\dfrac{R_2R_3}{2r_1}$, $R_b=\dfrac{R_1R_3}{2r_2}$, and $R_c=\dfrac{R_1R_2}{2r_3}$, inequality (2.9) becomes

$$R_1R_2R_3\sin\frac{A}{2}\,\sin\frac{B}{2}\,\sin\frac{C}{2}\geq r_1r_2r_3,$$

which is the inequality of Bottema (see [1], Theorem 12.26, p. 111).

Using Theorem 2.1 and the above relations we obtain the following inequality of Erdős-Mordell type

$$\lambda_1\frac{R_2R_3}{r_1}\sin\frac{A}{2}+\lambda_2\frac{R_1R_3}{r_2}\sin\frac{B}{2}+\lambda_3\frac{R_1R_2}{r_3}\sin\frac{C}{2}$$
$$\geq\sqrt{\lambda_2\lambda_3}R_1+\sqrt{\lambda_3\lambda_1}R_2+\sqrt{\lambda_1\lambda_2}R_3.$$

3. Applications

Application 3.1. If $M\equiv I$, where I is the incenter of triangle ABC, then we have the equalities $R_1=\dfrac{r}{\sin\dfrac{A}{2}}$, $R_2=\dfrac{r}{\sin\dfrac{B}{2}}$, $R_3=\dfrac{r}{\sin\dfrac{C}{2}}$, $R_a=2R\sin\dfrac{A}{2}$,

$R_b = 2R\sin\dfrac{B}{2}$, and $R_c = 2R\sin\dfrac{C}{2}$. By replacing these relations in the inequalities (2.1), (2.6), and (2.7), we obtain the following inequalities:

$$2R\left(\sum_{\text{cyclic}} \sin A \cdot \sin\frac{A}{2}\right) \geq r \sum_{\text{cyclic}} \frac{\sqrt{\cos\dfrac{B}{2}\cos\dfrac{C}{2}}}{\sin\dfrac{A}{2}}, \tag{3.1}$$

$$R\sum_{\text{cyclic}} \tan\frac{A}{2} \geq r \sum_{\text{cyclic}} \frac{1}{\sin A}, \tag{3.2}$$

and

$$2\sqrt{\frac{R}{r}} \sum_{\text{cyclic}} \sin\frac{A}{2} \geq \sum_{\text{cyclic}} \frac{1}{\sqrt{\sin\dfrac{A}{2}}}. \tag{3.3}$$

Application 3.2. If $M \equiv G$, where G is the centroid of triangle ABC, then we have the equalities $R_1 = \dfrac{2}{3}m_a$, $R_2 = \dfrac{2}{3}m_b$, $R_3 = \dfrac{2}{3}m_c$, $R_a = \dfrac{2}{3}\dfrac{m_b m_c}{h_a}$, $R_b = \dfrac{2}{3}\dfrac{m_c m_a}{h_b}$, and $R_c = \dfrac{2}{3}\dfrac{m_a m_b}{h_c}$, where we use the notations: m_a, m_b, m_c – the lengths of the medians and h_a, h_b, h_c – the lengths of the altitudes. By replacing these relations in the inequalities (2.2), (2.3), (2.1), (2.6), (2.7), and (2.9), we obtain the following inequalities:

$$2\sum_{\text{cyclic}} \frac{m_b m_c}{h_a} \sin\frac{A}{2} \geq m_a + m_b + m_c, \tag{3.4}$$

$$2\sum_{\text{cyclic}} \sin\frac{A}{2} \geq \sum_{\text{cyclic}} \sqrt{\frac{h_b h_c}{m_b m_c}}, \tag{3.5}$$

$$2\sum_{\text{cyclic}} \frac{m_b m_c}{h_a} \sin A \geq \sum_{\text{cyclic}} m_a \sqrt{\cos\frac{B}{2}\cos\frac{C}{2}}, \tag{3.6}$$

$$2R\sum_{\text{cyclic}} m_b m_c \sin\frac{A}{2} \geq \triangle \sum_{\text{cyclic}} \frac{m_a}{\cos\dfrac{A}{2}}, \tag{3.7}$$

where $\triangle$ is the area of triangle ABC,

$$\sqrt{\frac{r}{R}} \sum_{\text{cyclic}} \frac{m_b m_c}{h_a} \geq \sum_{\text{cyclic}} m_a \sqrt{\sin\frac{A}{2}}, \tag{3.8}$$

and

$$m_a m_b m_c \geq \frac{R}{2r} h_a h_b h_c. \tag{3.9}$$

Application 3.3. If $M \equiv K$, where K is the symmedian point of triangle ABC, then we have the equalities

$$R_1 = \frac{2bcm_a}{a^2+b^2+c^2}\,,\ R_2 = \frac{2ca}{a^2+b^2+c^2}\,,\ R_3 = \frac{2abm_c}{a^2+b^2+c^2}\,,$$

$$R_a = \frac{4Rm_bm_c}{a^2+b^2+c^2}\,,\ R_b = \frac{4Rm_cm_a}{a^2+b^2+c^2}\,,\ \text{ and }\ R_c = \frac{4Rm_am_b}{a^2+b^2+c^2}\,.$$

By replacing these relations in the inequalities (2.2), (2.3), (2.1), (2.6), and (2.7), we obtain the following inequalities:

$$2\sum_{\text{cyclic}} m_bm_c \sin\frac{A}{2} \ge \sum_{\text{cyclic}} m_ah_a\,, \tag{3.10}$$

$$2\sum_{\text{cyclic}} \sin\frac{A}{2} \ge \sum_{\text{cyclic}} \frac{h_a}{\sqrt{m_bm_c}}\,, \tag{3.11}$$

$$\sum_{\text{cyclic}} m_bm_c \sin A \ge \sum_{\text{cyclic}} \sqrt{\cos\frac{B}{2}\cos\frac{C}{2}\,m_ah_a}\,, \tag{3.12}$$

$$\sum_{\text{cyclic}} \frac{m_bm_c}{\cos\dfrac{A}{2}} \ge \sum_{\text{cyclic}} \frac{m_ah_a}{\cos\dfrac{A}{2}}\,, \tag{3.13}$$

and

$$\sqrt{\frac{r}{R}}\sum_{\text{cyclic}} m_bm_c \ge \sum_{\text{cyclic}} m_ah_a\sqrt{\sin\frac{A}{2}}\,. \tag{3.14}$$

Application 3.4. Let ABC be an acute triangle.

1) If $M \equiv O$, where O is the circumcenter of the triangle ABC, then we have the equalities $R_1 = R_2 = R_3 = R$, $R_a = \dfrac{R}{2\cos A}$, $R_b = \dfrac{R}{2\cos B}$, and $R_c = \dfrac{R}{2\cos C}$, where R is the circumradius.

By replacing these relations in the inequalities (2.1), (2.6), (2.7), and (2.9), we obtain the following inequalities:

$$\sum_{\text{cyclic}} \tan A \ge 2\sum_{\text{cyclic}} \sqrt{\cos\frac{B}{2}\cos\frac{C}{2}}\,, \tag{3.15}$$

$$\sum_{\text{cyclic}} \frac{1}{\cos A\cos\dfrac{A}{2}} \ge 2\sum_{\text{cyclic}} \frac{1}{\cos\dfrac{A}{2}}\,, \tag{3.16}$$

$$\sqrt{\frac{r}{R}}\sum_{\text{cyclic}} \frac{1}{\cos A} \ge \sum_{\text{cyclic}} \sqrt{\sin\frac{A}{2}}\,, \tag{3.17}$$

and

$$\frac{r}{4R} \geq \cos A \cos B \cos C. \tag{3.18}$$

2) If $M \equiv H$, where H is the orthocenter of the triangle ABC, then we have the equalities $R_1 = 2R\cos A$, $R_2 = 2R\cos B$, $R_3 = 2R\cos C$, and $R_a = R_b = R_c = R$.

By replacing these relations in the inequalities (2.3), (2.1), and (2.6), we obtain the following inequalities:

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \geq 1 + \frac{r}{R}, \quad \text{(see [1])} \tag{3.19}$$

$$\sum_{\text{cyclic}} \sin A \geq 2 \sum_{\text{cyclic}} \cos A \sqrt{\cos\frac{B}{2}\cos\frac{C}{2}}, \tag{3.20}$$

and

$$\sum_{\text{cyclic}} \frac{1}{\cos\frac{A}{2}} \geq 2 \sum_{\text{cyclic}} \frac{\cos A}{\cos\frac{A}{2}}. \tag{3.21}$$

We mention that relations (3.18) and (3.19) are true in any triangle.

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Asymptotic behavior of some sums associated to an infinite set of natural numbers

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Abstract. We study the asymptotic behavior for some sums associated to an infinite set of natural numbers.

Keywords: Arithmetic functions; asymptotic results; Mertens type evaluations.

MSC: 11N56, 11N37

INTRODUCTION

In this paper we study the asymptotic behavior for some sums associated to an infinite set of natural numbers. We will prove (see Proposition 1.1 and Proposition 1.2), that this behavior depends on the properties of the infinite set of natural numbers and gives relevant information on when a natural series associated to an infinite set is divergent, in which case the asymptotic behavior depends on another natural function associated to the divergent series. We will apply these results for the set of all natural numbers and the set of all prime numbers (see Corollary 1.1 and Corollary 1.2). Our method of proof was suggested by the proof of Theorem 8, pages 15–16 in G. Tenenbaum's book [7].

Let us fix some notation.

Let $a \in \mathbb{R} \cup \{-\infty\}$ and $g : (a, \infty) \to \mathbb{R}$ be such that there exists $b \geq a$ with $g(x) \neq 0$ for each $x \in (b, \infty)$. For a function $f : (a, \infty) \to \mathbb{R}$ we write $f(x) \sim g(x)$ if and only if $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$.

Let $a \in \mathbb{R} \cup \{-\infty\}$ and $g : (a, \infty) \to \mathbb{R}$ be such that there exists $b \geq a$ with $g(x) \geq 0$ for each $x \in (b, \infty)$. For a function $f : (a, \infty) \to \mathbb{R}$ we write $f(x) = O(g(x))$ if and only if there exists $M > 0$ and $c \geq b$ such that $|f(x)| \leq Mg(x)$ for each $x \geq c$.

All notation and notions used (and not defined) in this paper are standard, see e.g. [1], [7].

1. THE RESULTS

Proposition 1.1. *Let A be an infinite set of natural numbers greater than or equal to 2. Then there exists $c_A \in (0, \infty)$ such that*

$$\sum_{k\leq x,\ k\in A} \left[\ln\left(1 - \frac{1}{\sqrt{k}}\right) + \frac{1}{\sqrt{k}}\right] = -\frac{1}{2} \sum_{k\leq x,\ k\in A} \frac{1}{k} - c_A + O\left(\frac{1}{\sqrt{x}}\right).$$

In particular, the series $\sum\limits_{k\in A} \left[\ln\left(1 - \frac{1}{\sqrt{k}}\right) + \frac{1}{\sqrt{k}}\right]$ and $\sum\limits_{k\in A} \frac{1}{k}$ have the same nature.

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Proof. We will use the following inequalities whose proofs are left as an exercise to the reader:

$$0 < \ln \frac{1}{1-x} - x - \frac{x^2}{2} < \frac{x^3}{3\,(1-x)}, \; 0 < x < 1. \tag{1}$$

From (1) we get

$$0 < \ln \frac{1}{1-\frac{1}{\sqrt{k}}} - \frac{1}{\sqrt{k}} - \frac{1}{2k} < \frac{1}{3k\sqrt{k}} \cdot \frac{1}{1-\frac{1}{\sqrt{k}}}, \tag{2}$$

for all $k \in A$.

Since $\frac{1}{n^{3/2}} \sim \frac{1}{n^{3/2}\left(1-\frac{1}{\sqrt{n}}\right)}$, it follows that the series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ and $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}\left(1-\frac{1}{\sqrt{k}}\right)}$ have the same nature, hence $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}\left(1-\frac{1}{\sqrt{k}}\right)}$ is convergent. From Stolz-Cesaro theorem, the case $\frac{0}{0}$, we have

$$\sum_{k=n+1}^{\infty} \frac{1}{k^{3/2}\left(1-\frac{1}{\sqrt{k}}\right)} \sim \sum_{k=n+1}^{\infty} \frac{1}{k^{3/2}}.$$

Since $\sum_{k=n+1}^{\infty} \frac{1}{k^{3/2}} \sim \frac{2}{\sqrt{n}}$, we get

$$\sum_{k=n+1}^{\infty} \frac{1}{k^{3/2}\left(1-\frac{1}{\sqrt{k}}\right)} \sim \frac{2}{\sqrt{n}}. \tag{3}$$

From (2) and the comparison criterion for positive series we get that

$$\sum_{k\in A} \left(\ln \frac{1}{1-\frac{1}{\sqrt{k}}} - \frac{1}{\sqrt{k}} - \frac{1}{2k} \right)$$

is a convergent series. Let $c_A \in (0, \infty)$ be its sum, i.e.,

$$c_A = \sum_{k\in A} \left(\ln \frac{1}{1-\frac{1}{\sqrt{k}}} - \frac{1}{\sqrt{k}} - \frac{1}{2k} \right).$$

From (2) we also get

$$0<\sum_{k\in A,k>x}\left(\ln\frac{1}{1-\frac{1}{\sqrt{k}}}-\frac{1}{\sqrt{k}}-\frac{1}{2k}\right)<\frac{1}{3}\sum_{k\in A,k>x}\frac{1}{k^{3/2}\left(1-\frac{1}{\sqrt{k}}\right)}$$

$$\leq\frac{1}{3}\sum_{k=[x]+1}^{\infty}\frac{1}{k^{3/2}\left(1-\frac{1}{\sqrt{k}}\right)}. \tag{4}$$

(Here we used that $\{k\in\mathbb{N}\mid k>x\}\subseteq\{k\in\mathbb{N}\mid k\geq[x]+1\}$, where $[x]$ is the integral part of x.)

Define $h_A:[2,\infty)\to(0,\infty)$ by

$$h_A(x)=\sum_{k\in A,k>x}\left(\ln\frac{1}{1-\frac{1}{\sqrt{k}}}-\frac{1}{\sqrt{k}}-\frac{1}{2k}\right).$$

Then, (4) and (3) give

$$h_A(x)=O\left(\frac{1}{\sqrt{x}}\right). \tag{5}$$

Since

$$c_A=\sum_{k\in A,\ k\leq x}\left(\ln\frac{1}{1-\frac{1}{\sqrt{k}}}-\frac{1}{\sqrt{k}}-\frac{1}{2k}\right)+\sum_{k\in A,k>x}\left(\ln\frac{1}{1-\frac{1}{\sqrt{k}}}-\frac{1}{\sqrt{k}}-\frac{1}{2k}\right)$$

we get

$$\sum_{k\in A,\ k\leq x}\left[\ln\left(1-\frac{1}{\sqrt{k}}\right)+\frac{1}{\sqrt{k}}\right]=-\frac{1}{2}\sum_{k\in A,\ k\leq x}\frac{1}{k}-c_A+h_A(x), \tag{6}$$

for all $x\geq 2$. From (5) and (6) we get the statement. □

The third and fourth limit from the next corollary were suggested by Exercise 11 from page 22 in G. Tenenbaum's book [7].

Corollary 1.1. *The following are true:*

$$\lim_{x\to\infty}\frac{\sum\limits_{2\leq k\leq x,\ k\text{ natural}}\left[\ln\left(1-\frac{1}{\sqrt{k}}\right)+\frac{1}{\sqrt{k}}\right]}{\ln x}=-\frac{1}{2},$$

$$\lim_{x\to\infty}\frac{\sum\limits_{p\leq x,\ p\text{ prime}}\left[\ln\left(1-\frac{1}{\sqrt{p}}\right)+\frac{1}{\sqrt{p}}\right]}{\ln(\ln(x))}=-\frac{1}{2},$$

$$\lim_{x\to\infty}\frac{\displaystyle\sum_{\sqrt{x}<k\le x,\ k\text{ natural}}\left[\ln\left(1-\frac{1}{\sqrt{k}}\right)+\frac{1}{\sqrt{k}}\right]}{\ln x}=-\frac{1}{4},$$

$$\lim_{x\to\infty}\sum_{\sqrt{x}<p\le x,\ p\text{ prime}}\left[\ln\left(1-\frac{1}{\sqrt{p}}\right)+\frac{1}{\sqrt{p}}\right]=-\frac{\ln 2}{2}.$$

Proof. It is well known that

$$\sum_{2\le k\le x,\ k\text{ natural}}\frac{1}{k}=\ln x+\gamma-1+O\left(\frac{1}{x}\right),$$

where γ is the Euler constant; see [7, Theorem 5, page 6].

Define $f:[2,\infty)\to\mathbb{R}$, $f(x)=\displaystyle\sum_{2\le k\le x,\ k\text{ natural}}\left[\ln\left(1-\frac{1}{\sqrt{k}}\right)+\frac{1}{\sqrt{k}}\right]$.
By Proposition 1.1 and the above evaluation we get

$$f(x)=-\frac{1}{2}\ln x-\frac{\gamma-1}{2}-c_A+O\left(\frac{1}{\sqrt{x}}\right)$$

and from here we get the first limit of the statement.

We have

$$\begin{aligned}\sum_{\sqrt{x}<k\le x,\ k\text{ natural}}\left[\ln\left(1-\frac{1}{\sqrt{k}}\right)+\frac{1}{\sqrt{k}}\right] &= f(x)-f\left(\sqrt{x}\right)\\ =-\frac{1}{2}\ln x+\frac{1}{2}\ln\sqrt{x}+O\left(\frac{1}{\sqrt{x}}\right)+O\left(\frac{1}{\sqrt[4]{x}}\right) &= -\frac{1}{4}\ln x+O\left(\frac{1}{\sqrt[4]{x}}\right),\end{aligned}$$

and the third limit follows.

For the other two limits we use the famous Mertens theorem, see [7, page 16], or [1], [3], [4], which asserts that there exists a real number B such that

$$\sum_{p\le x,\ p\text{ prime}}\frac{1}{p}=\ln(\ln x)+B+O\left(\frac{1}{\ln x}\right).$$

From Proposition 1.1 and Mertens's theorem we get

$$\sum_{p\le x,\ p\text{ prime}}\left[\ln\left(1-\frac{1}{\sqrt{p}}\right)+\frac{1}{\sqrt{p}}\right]=-\frac{1}{2}\ln(\ln x)-\frac{1}{2}B-c_A+O\left(\frac{1}{\ln x}\right)\quad(7)$$

and now we get the second limit.

From (7) we also deduce

$$\begin{aligned}\sum_{\sqrt{x}<p\le x,p\text{ prime}}\left[\ln\left(1-\frac{1}{\sqrt{p}}\right)+\frac{1}{\sqrt{p}}\right]&=-\frac{1}{2}\ln(\ln x)+\frac{1}{2}\ln\left(\ln\sqrt{x}\right)+O\left(\frac{1}{\ln x}\right)\\ &=-\frac{\ln 2}{2}+O\left(\frac{1}{\ln x}\right),\end{aligned}$$

and the last limit follows. $\square$

Proposition 1.2. *Let A be an infinite set of natural numbers greater than or equal to 2. Then there exists $b_A \in (0,\infty)$such that*

$$\sum_{k\in A,\ k\le x} \ln\left(1-\frac{\ln k}{k}\right) = -\sum_{k\in A,\ k\le x}\frac{\ln k}{k} - b_A + O\left(\frac{\ln^2 x}{x}\right).$$

In particular, the series $\sum_{k\in A}\ln\left(1-\frac{\ln k}{k}\right)$ and $\sum_{k\in A}\frac{\ln k}{k}$ have the same nature.

Proof. We use the inequalities

$$0<\ln\frac{1}{1-x}-x<\frac{x^2}{2\,(1-x)},\ \ 0<x<1. \tag{8}$$

From (8) we get

$$0<\ln\frac{1}{1-\dfrac{\ln k}{k}}-\frac{\ln k}{k}<\frac{1}{2}\cdot\frac{\left(\dfrac{\ln k}{k}\right)^2}{1-\dfrac{\ln k}{k}}, \tag{9}$$

for all $k\in A$.

Since $\dfrac{\left(\dfrac{\ln n}{n}\right)^2}{1-\dfrac{\ln n}{n}} \sim \left(\dfrac{\ln n}{n}\right)^2$ and the series $\sum\limits_{n=2}^{\infty}\left(\dfrac{\ln n}{n}\right)^2$ is convergent, it follows that the series $\sum\limits_{n=2}^{\infty}\dfrac{\left(\dfrac{\ln n}{n}\right)^2}{1-\dfrac{\ln n}{n}}$ is convergent. Further, from Stolz-Cesàro theorem, the case $\dfrac{0}{0}$, we get

$$\sum_{k=n+1}^{\infty}\frac{\left(\dfrac{\ln k}{k}\right)^2}{1-\dfrac{\ln k}{k}} \sim \sum_{k=n+1}^{\infty}\left(\frac{\ln k}{k}\right)^2.$$

By using Proposition 6(2) and Proposition 8(1) from [2, pp. 233 and 239–240], we get

$$\sum_{k=n+1}^{\infty}\left(\frac{\ln k}{k}\right)^2 \sim \int_n^{\infty}\left(\frac{\ln x}{x}\right)^2 \mathrm{d}x \sim \frac{\ln^2 n}{n}$$

and thus

$$\sum_{k=n+1}^{\infty}\frac{\left(\frac{\ln k}{k}\right)^2}{1-\frac{\ln k}{k}}\sim\frac{\ln^2 n}{n}. \tag{10}$$

Let us remark that the above evaluations can be obtained by using the results from [5] or [6, chapter V].

From (9) and the comparison criterion for positive series we get that the series $\sum_{k\in A}\left(\ln\frac{1}{1-\frac{\ln k}{k}}-\frac{\ln k}{k}\right)$ is convergent. Let $b_A\in(0,\infty)$ be its sum, i.e.,

$$b_A=\sum_{k\in A}\left(\ln\frac{1}{1-\frac{\ln k}{k}}-\frac{\ln k}{k}\right).$$

From (9) we get

$$0<\sum_{k\in A,k>x}\left(\ln\frac{1}{1-\frac{\ln k}{k}}-\frac{\ln k}{k}\right)<\frac{1}{2}\sum_{k\in A,k>x}\frac{\left(\frac{\ln k}{k}\right)^2}{1-\frac{\ln k}{k}}$$
$$\leq\frac{1}{2}\sum_{k=[x]+1}^{\infty}\frac{\left(\frac{\ln k}{k}\right)^2}{1-\frac{\ln k}{k}}. \tag{11}$$

Define $h_A:[2,\infty)\to(0,\infty)$ by $h_A(x)=\sum_{k\in A,k>x}\left(\ln\frac{1}{1-\frac{\ln k}{k}}-\frac{\ln k}{k}\right)$ and note that from (11) and (10) we deduce

$$h_A(x)=O\left(\frac{\ln^2 x}{x}\right). \tag{12}$$

Since

$$b_A=\sum_{k\in A,\ k\leq x}\left(\ln\frac{1}{1-\frac{\ln k}{k}}-\frac{\ln k}{k}\right)+\sum_{k\in A,k>x}\left(\ln\frac{1}{1-\frac{\ln k}{k}}-\frac{\ln k}{k}\right),$$

we get

$$\sum_{k\in A,\ k\leq x}\ln\left(1-\frac{\ln k}{k}\right)=-\sum_{k\in A,\ k\leq x}\frac{\ln k}{k}-b_A+h_A(x),\ \forall x\geq 2. \tag{13}$$

From (12) and (13) we get the statement. $\square$

Since
$$\sum_{2\le k\le x,\ k \text{ natural}} \frac{\ln k}{k} \sim \frac{1}{2}\ln^2 x$$
(see Proposition 6(2) and Proposition 8(1) from [2], or [5], [6, chapter V]), by Mertens theorem
$$\sum_{p\le x,\ p \text{ prime}} \frac{\ln p}{p} \sim \ln x$$
(see [7, page 14] or [1, 3, 4]) and from Proposition 1.2 we get

Corollary 1.2. *The following are true:*
$$\lim_{x\to\infty} \frac{\displaystyle\sum_{k\le x,\ k \text{ natural}} \ln\left(1-\frac{\ln k}{k}\right)}{\ln^2 x} = -\frac{1}{2};\ \lim_{x\to\infty} \frac{\displaystyle\sum_{p\le x,\ p \text{ prime}} \ln\left(1-\frac{\ln p}{p}\right)}{\ln x} = -1.$$

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NOTE MATEMATICE

Asupra vitezei de aproximare cu polinoame Bernstein

Gheorghe Stoica[1]

Abstract. Using standard binomial estimates, we obtain a rate of $n^{-1/2}$ in the approximation with Bernstein polynomials at each point where the first derivative exists.

Keywords: Bernstein polynomials, rate of convergence
MSC: 41A10, 41A25.

În 1912, S.N. Bernstein a introdus polinoamele (care astăzi îi poartă numele)

$$B_n(f,x)=\sum_{k=0}^{n}C_n^k x^k(1-x)^{n-k}f\left(\frac{k}{n}\right)$$

asociate unei funcţii $f:[0,1]\to\mathbb{R}$ şi, cu ajutorul lor, a demonstrat teorema lui Weierstrass de aproximare a funcţiilor continue (vezi [2], p. 5, Theorem 1.1.1), anume

(i) *forma locală*: dacă f este mărginită pe $[0,1]$ şi continuă într-un punct x_0, atunci

$$\lim_{n\to\infty}B_n(f,x_0)=f(x_0).$$

(ii) *forma globală*: dacă f este continuă pe $[0,1]$, atunci convergenţa din (i) este uniformă pe $[0,1]$, i.e.,

$$\lim_{n\to\infty}\sup_{x\in[0,1]}|B_n(f,x)-f(x)|=0.$$

În 1932, E.V. Voronovskaya a obţinut viteza de convergenţă în teorema de mai sus, sub forma unei formule asimptotice. Anume, pentru o funcţie f mărginită pe $[0,1]$, avem (vezi [1], p. 307, Theorem 3.1)

(i) *forma locală*: dacă $f''(x_0)$ există, atunci

$$\lim_{n\to\infty}n\Big(B_n(f,x_0)-f(x_0)\Big)=\frac{x_0(1-x_0)}{2}f''(x_0).$$

(ii) *forma globală*: dacă f'' este continuă pe $[0,1]$, atunci convergenţa din (i) este uniformă pe $[0,1]$, i.e.,

$$\lim_{n\to\infty}\sup_{x\in[0,1]}\left|n\Big(B_n(f,x)-f(x)\Big)-\frac{x(1-x)}{2}f''(x)\right|=0.$$

În acelaşi an, Bernstein a generalizat rezultatul lui Voronovskaya pentru derivatele de ordin par ale lui f. Începând cu rezultatele lui T. Popoviciu din 1935, s-au obţinut alte viteze de convergenţă în limbajul modulului de continuitate al lui f sau f'. În particular, aceste rezultate se aplică funcţiilor

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Lipschitz sau funcţiilor cu f' Lipschitz. Deşi literatura de specialitate pe acest subiect este foarte vastă, nu am întâlnit o viteză de convergenţă în punctele unde f' există, fără alte ipoteze suplimentare. Ne propunem să demonstrăm aici un astfel de rezultat folosind ingrediente deja cunoscute.

Mai întâi trebuie să menţionăm la ce ne putem aştepta: pentru funcţia $f(x) = |x - x_0|$, a cărei derivată este discontinuă în x_0, avem (cf. [2], p. 21)

$$|B_n(f, x) - f(x)| \sim \frac{1}{\sqrt{2\pi n}} \text{ când } n \to \infty.$$

Cu alte cuvinte, în ipoteza existenţei lui $f'(x_0)$, nu ne putem aştepta la o viteză de convergenţă mai bună decât $n^{-1/2}$. Într-adevăr, avem următorul rezultat:

Teoremă. *Pentru o funcţie reală f mărginită pe $[0, 1]$, avem*

(i) forma locală: *dacă $f'(x_0)$ există, atunci*

$$\lim_{n\to\infty} \sqrt{n}\, |B_n(f, x_0) - f(x_0)| = 0.$$

(ii) forma globală: *dacă f' este continuă pe $[0, 1]$, atunci convergenţa din (i) este uniformă pe $[0, 1]$, i.e.,*

$$\lim_{n\to\infty} \sup_{x\in[0,1]} \sqrt{n} |B_n(f, x) - f(x)| = 0.$$

Demonstraţie. Putem scrie

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + (x - x_0)g(x - x_0), \tag{1}$$

unde $g(x) \to 0$ când $x \to 0$.

Înlocuind (1) în definiţia polinoamelor Bernstein, obţinem

$$\begin{aligned} B_n(f, x_0) &= \sum_{k=0}^{n} C_n^k x_0^k (1 - x_0)^{n-k} f(x_0) + \\ &+ \sum_{k=0}^{n} \left(\frac{k}{n} - x_0\right) C_n^k x_0^k (1 - x_0)^{n-k} f'(x_0) + \\ &+ \sum_{k=0}^{n} \left(\frac{k}{n} - x_0\right) g\left(\frac{k}{n} - x_0\right) C_n^k x_0^k (1 - x_0)^{n-k}. \end{aligned} \tag{2}$$

Un exerciţiu simplu de combinatorică arată că prima sumă din (2) este egală cu $f(x_0)$, iar a doua este egală cu 0. În continuare vom estima a treia sumă.

Fie $\varepsilon > 0$ şi luăm $\delta > 0$ astfel ca $|g(x)| < \varepsilon$ pentru $|x| < \delta$. Împărţim a treia sumă din (2) în două subsume, notate $\sum'$ şi $\sum''$, prima luată după acei indici k care satisfac inegalitatea $\left|\frac{k}{n} - x_0\right| < \delta$, iar a doua după acei indici k care satisfac inegalitatea $\left|\frac{k}{n} - x_0\right| \geq \delta$. În continuare vom folosi două rezultate ajutătoare.

Pentru orice $n \geq 1$ avem (vezi [2], p. 15, formula (9))

$$\sum_{k=0}^{n} \left|\frac{k}{n} - x_0\right| C_n^k x_0^k (1-x_0)^{n-k} \leq C_1\sqrt{n}, \tag{3}$$

unde $C_1 > 0$ este o constantă universală (nu depinde nici de n, nici de x_0).

Pentru orice $\delta > 0$ avem (vezi [1], p. 304, formula (1.6)):

$$\sum{}'' C_n^k x_0^k (1-x_0)^{n-k} \leq \frac{C_2(\delta)}{n}, \tag{4}$$

unde $C_2(\delta) > 0$ depinde numai de δ.

Folosind relaţia (3), obţinem

$$\begin{aligned}
&\left|\sum{}' \left(\frac{k}{n} - x_0\right) g\left(\frac{k}{n} - x_0\right) C_n^k x_0^k (1-x_0)^{n-k}\right| \leq \\
\leq\ & \sum{}' \left|\frac{k}{n} - x_0\right| \left|g\left(\frac{k}{n} - x_0\right)\right| C_n^k x_0^k (1-x_0)^{n-k} \leq \\
\leq\ & \varepsilon \sum{}' \left|\frac{k}{n} - x_0\right| C_n^k x_0^k (1-x_0)^{n-k} \leq \\
\leq\ & \varepsilon \sum_{k=0}^{n} \left|\frac{k}{n} - x_0\right| C_n^k x_0^k (1-x_0)^{n-k} \leq \varepsilon \frac{C_1}{\sqrt{n}}.
\end{aligned} \tag{5}$$

Întrucât f este mărginită, există $M > 0$ astfel ca $|g(x)| \leq M$ pentru orice $x \in [0,1]$. Ţinând cont că $\left|\frac{k}{n} - x_0\right| \leq 2$, împreună cu relaţia (4), obţinem

$$\begin{aligned}
&\left|\sum{}'' \left(\frac{k}{n} - x_0\right) g\left(\frac{k}{n} - x_0\right) C_n^k x_0^k (1-x_0)^{n-k}\right| \leq \\
\leq\ & 2M \sum{}'' C_n^k x_0^k (1-x_0)^{n-k} \leq 2M \frac{C_2(\delta)}{n}.
\end{aligned} \tag{6}$$

Combinând relaţiile (2), (5) şi (6), deducem că

$$\sqrt{n}\,|B_n(f, x_0) - f(x_0)| \leq C_1\varepsilon + 2M\frac{C_2(\delta)}{\sqrt{n}}. \tag{7}$$

Facem $n \to \infty$ şi ţinem cont că $\varepsilon > 0$ este arbitrar, deci trecând la limită în (7) se demonstrează teorema în forma locală.

Să observăm că δ din formula (7) depinde de x_0. Dacă, în plus, f' este continuă pe $[0,1]$, atunci funcţia g din formula (1) converge uniform către 0 când $x \to 0$, deci formula (7) are loc cu δ independent de x_0. În particular, convergenţa către 0 a membrului stâng din (7) este uniformă în raport cu $x \in [0,1]$, deci forma globală a teoremei este demonstrată.

Observaţie. Metoda de demonstraţie de mai sus se pretează la generalizare pentru derivatele lui f de ordin impar. Invităm cititorul să furnizeze detaliile tehnice.

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$\det AB = \det A \det B$

Constantin-Nicolae Beli[1]

Abstract. În această notă dăm o demonstraţie proprietăţii referitoare la determinantul produsului a două matrice pătratice folosind definiţia determinantului.

Keywords: Matrices, determinants.

MSC: 11C20

Teoremă. *Dacă A, B sunt două matrice $n \times n$, atunci*

$$\det A \det B = \det AB.$$

Demonstraţie. Facem întâi o observaţie. Avem

$$\left(\sum_{j=1}^{n} x_{1j}\right) \cdots \left(\sum_{j=1}^{n} x_{nj}\right) = \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} x_{1j_1} \cdots x_{nj_n}.$$

Dacă în loc de $j_1, \ldots, j_n$ scriem $f(1), \ldots, f(n)$, relaţia de mai sus se poate scrie ca

$$\prod_{i=1}^{n} \left(\sum_{j=1}^{n} x_{ij}\right) = \sum_{f \in F_n} \prod_{i=1}^{n} x_{if(i)},$$

unde F_n este mulţimea tuturor funcţiilor $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

Fie $A = (a_{ij})_{1 \le i,j \le n}$ şi $B = (b_{ij})_{1 \le i,j \le n}$. Atunci $AB = (c_{ij})_{1 \le i,j \le n}$, unde

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

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Avem

$$\det AB = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} c_{i\sigma(i)} = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} b_{j\sigma(i)} \right) =$$

$$= \sum_{\sigma \in S_n} \varepsilon(\sigma) \sum_{f \in F_n} \prod_{i=1}^{n} a_{if(i)} b_{f(i)\sigma(i)} = \sum_{\sigma \in S_n} \varepsilon(\sigma) \sum_{f \in F_n} \prod_{i=1}^{n} a_{if(i)} \prod_{i=1}^{n} b_{f(\sigma^{-1}(i))i}.$$

(Am folosit schimbarea de indici $\sigma(i) = j$, adică $i = \sigma^{-1}(j)$, cu ajutorul căreia obţinem că $\prod_{i=1}^{n} b_{f(i)\sigma(i)} = \prod_{j=1}^{n} b_{f(\sigma^{-1}(j))j}$.)

Prin urmare $\det AB$ este suma tuturor expresiilor de forma

$$\varepsilon(\sigma) \prod_{i=1}^{n} a_{if(i)} \prod_{i=1}^{n} b_{g(i)i}$$

cu $\sigma \in S_n$ şi $f, g \in F_n$ astfel încât $g = f\sigma^{-1}$, i.e. $f = g\sigma$. Rezultă că

$$\det AB = \sum_{f,g \in F_n} \left(\sum_{\sigma \in A_{f,g}} \varepsilon(\sigma) \right) \prod_{i=1}^{n} a_{if(i)} \prod_{i=1}^{n} b_{g(i)i},$$

unde $A_{f,g} = \{\sigma \in S_n \mid f = g\sigma\}$.

Vom determina $\sum_{\sigma \in A_{f,g}} \varepsilon(\sigma)$. Avem trei cazuri:

1) $g \notin S_n$. Atunci g nu este injectivă, deci există $i, j \in \{1, \ldots, n\}$, $i \neq j$, astfel încât $g(i) = g(j)$. Notăm cu τ transpoziţia $(i\, j)$. Atunci $g\tau = g$. (Dacă $k \neq i, j$, atunci $g(\tau(k)) = g(k)$; dacă $k = i$, atunci $g(\tau(i)) = g(j) = g(i)$; dacă $k = j$, atunci $g(\tau(j)) = g(i) = g(j)$.) Rezultă că pentru orice $\sigma \in A_{f,g}$ avem $g\tau\sigma = g\sigma = f$, deci $\tau\sigma \in A_{f,g}$.

Notăm $H = \langle \tau \rangle$. Deoarece τ are ordin 2, avem $H = \{1, \tau\}$. Rezultă că $H\sigma = \{\sigma, \tau\sigma\}$ pentru orice $\sigma \in S_n$. Mulţimile $H\sigma$, $\sigma \in S_n$, sunt clasele de echivalenţă la dreapta modulo H din S_n şi realizează o partiţie a lui S_n. Am arătat că pentru orice $\sigma \in A_{f,g}$ clasa sa de echivalenţă la dreapta $H\sigma$ este conţinută în $A_{f,g}$. Rezultă că $A_{f,g}$ se scrie ca o reuniune disjunctă de astfel de clase de echivalenţă:

$$A_{f,g} = H\sigma_1 \cup \cdots \cup H\sigma_s = \{\sigma_1, \tau\sigma_1\} \cup \cdots \cup \{\sigma_s, \tau\sigma_s\}.$$

Rezultă că

$$\sum_{\sigma \in A_{f,g}} \varepsilon(\sigma) = \sum_{t=1}^{s} (\varepsilon(\sigma_t) + \varepsilon(\tau\sigma_t)).$$

Cum transpoziţia τ este impară, obţinem $\varepsilon(\tau\sigma_t) = \varepsilon(\tau)\varepsilon(\sigma_t) = -\varepsilon(\sigma_t)$, deci $\varepsilon(\sigma_t) + \varepsilon(\tau\sigma_t) = 0$. Rezultă că $\sum_{\sigma \in A_{f,g}} \varepsilon(\sigma) = 0$.

2) $g \in S_n$, $f \notin S_n$. Atunci $g\sigma \in S_n$ pentru orice $\sigma \in S_n$, deci $g\sigma \neq f$. Rezultă că $A_{f,g} = \emptyset$ şi $\displaystyle\sum_{\sigma \in A_{f,g}} \varepsilon(\sigma) = 0$.

3) $f, g \in S_n$. Atunci $f = g\sigma$ este echivalent cu $\sigma = g^{-1}f$.
Deci $A_{f,g} = g^{-1}f$ şi $\displaystyle\sum_{\sigma \in A_{f,g}} \varepsilon(\sigma) = \varepsilon(g^{-1}f) = \varepsilon(g^{-1})\varepsilon(f)$.

În consecinţă

$$\det AB = \sum_{f,g \in S_n} \varepsilon(f)\varepsilon(g^{-1}) \prod_{i=1}^{n} a_{if(i)} \prod_{i=1}^{n} b_{g(i)i} =$$
$$= \left(\sum_{f \in S_n} \varepsilon(f) \prod_{i=1}^{n} a_{if(i)} \right) \left(\sum_{g \in S_n} \varepsilon(g^{-1}) \prod_{i=1}^{n} b_{g(i)i} \right) = \det A \det B.$$

În a doua paranteză am folosit schimbările de indici $h = g^{-1}$ şi $i = h(j)$, adică $j = h^{-1}(i)$, pentru a obţine că

$$\sum_{g \in S_n} \varepsilon(g^{-1}) \prod_{i=1}^{n} b_{g(i),i} = \sum_{h \in S_n} \varepsilon(h) \prod_{i=1}^{n} b_{h^{-1}(i),i} = \sum_{h \in S_n} \varepsilon(h) \prod_{j=1}^{n} b_{j,h(j)} = \det B.$$

PROBLEMS

Authors should submit proposed problems to `gmaproblems@rms.unibuc.ro`. Files should be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail address. For this issue, solutions should arrive before **15th of March 2013**.

PROPOSED PROBLEMS

365. Let K be a field and let $f, g \in K[X]$, $f, g \notin K$, such that $g^n - 1 \mid f^n - 1$ for all $n \geq 1$. Then f is a power of g.

`Proposed by Marius Cavachi, Ovidius University of Constanţa, Constanţa, Romania.`

366. Let K be an algebraically closed field of characteristic $p > 0$. For $i \geq 0$ we define the polynomials $Q_i \in \mathbb{Q}[X]$ by $Q_0 = X$ and $Q_{i+1} = \dfrac{Q_i^p - Q_i}{p}$. If $k \geq 0$ writes in basis p as $k = c_0 + c_1 p + \cdots + c_s p^s$ with $0 \leq c_i \leq p - 1$ then we define $P_k \in \mathbb{Q}[X]$ by $P_k = Q_0^{c_0} Q_1^{c_1} \cdots Q_s^{c_s}$.

Prove that if $f = X^k + a_{k-1}X^{k-1} + \cdots + a_0 \in K[X]$ with $a_0 \neq 0$ has the roots $\alpha_1, \ldots, \alpha_s$ with multiplicities $k_1, \ldots, k_s$ then

$$V_f := \{(x_n)_{n\geq 0} : x_n \in K,\ x_{n+k} + a_{k-1}x_{n+k-1} + \cdots + a_0 x_n = 0,\ \forall\, n \geq 0\}$$

is a vector space with $\{(P_j(n)\alpha_i^n)_{n\geq 0} : 1 \leq i \leq s,\ 0 \leq j \leq k_i - 1\}$ as a basis.

(Hint: Use the note "Linear Recursive Sequences in Arbitrary Characteristics" by *C.N. Beli* from the issue 1–2/2012 of GMA.)

`Proposed by Constantin-Nicolae Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.`

367. Give examples of functions $f, g : \mathbb{R} \to \mathbb{R}$ such that: f has period $\sqrt{2}$, g has period $\sqrt{3}$, and $f + g$ has period $\sqrt{5}$.

`Proposed by George Stoica, Department of Mathematical Sciences, University of New Brunswick, Canada.`

368. Find all matrices $X_1, \ldots, X_9 \in M_2(\mathbb{Z})$ with the property that $\det X_k = 1$ $\forall k$ and $X_1^4 + \cdots + X_9^4 = X_1^2 + \cdots + X_9^2 + 18I_2$.

`Proposed by Florin Stănescu, Şerban Cioculescu School, Gaeşti, Dâmboviţa, Romania.`

369. A stick is broken at random at two points (each point is uniformly distributed relative to the whole stick) and the parts' lengths are denoted by r, s, and t. Show that the probability of the existence of a triangle encompassing three circles of radii r, s, and t, each side tangent to two of the circles, and the circles mutually externally tangent, is equal to $\dfrac{5}{27}$.

`Proposed by Eugen J. Ionaşcu, Department of Mathematics, Columbus State University, Columbus, U.S.A.`

370. Calculate the improper integral $\int\limits_0^\infty \cos^2 x \cos x^2 \mathrm{d}x$.

Proposed by Angel Plaza, Department of Mathematics, Univ. Las Palamas de Gran Canaria, Spain.

371. Let $[ABCD]$ be a Crelle tetrahedron and let M, N, P, Q, R, S be the contact points of the sphere tangent to its edges.

Prove that $V_{[MNPQRS]} \le \frac{1}{2} V_{[ABCD]}$.

(By V_X we denote the volume of the polyhedron X.)

Proposed by Marius Olteanu, S.C. Hidroconstrucţia S.A., Sucursala ,,Olt-Superior'', Rm. Vâlcea, Romania.

372. Prove that $\lim\limits_{n\to\infty} \mathrm{e}^{-n} \left(1 + n + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\right) = \frac{1}{2}$.

Proposed by George Stoica, Department of Mathematical Sciences, University of New Brunswick, Canada.

373. Let $n \ge 1$ and let $\Phi_n(X, q) = \prod\limits_{k=1}^{n} (X - q^{2k-1}) = a_0 + \cdots + a_n X^n$, with $a_i \in \mathbb{R}[q]$. Prove that

$$\frac{\sum\limits_{i=0}^{n-1} a_i a_{i+1}}{\sum\limits_{i=0}^{n} a_i^2} = \frac{-q(1 - q^{2n-1})}{1 - q^{2n+1}}.$$

Proposed by Florin Spînu, Department of Mathematics, Johns Hopkins University, Baltimore, MD, U.S.A.

374. Let $A_1, \ldots, A_n$ be some points in the 3-dimensional euclidean space. Prove that on the unit sphere S^2 there is a point P such that $PA_1 \cdot PA_2 \cdots PA_n \ge 1$.

Proposed by Marius Cavachi, Ovidius University of Constanţa.

375. Let $n \ge 3$ be an integer. Find effectively the isomorphism class of Galois group $\mathrm{Gal}\left(\mathbb{Q}\left(\cos \frac{2\pi}{n}\right) \Big/ \mathbb{Q}\right)$.

Proposed by Cornel Băeţica, Faculty of Mathematics and Informatics, University of Bucharest, Bucharest, Romania.

376. (a) Show that the probability of a point $P(x, y, z)$, chosen at random with uniform distribution in $[0, 1]^3$, to be at a distance to the origin of at most $\sqrt{2}$ is $\frac{(15 - 8\sqrt{2})\pi}{12}$.

(b) Prove that

$$\int_{0}^{\pi/4} \frac{\cos^{3/2} 2\theta}{\cos^3 \theta} \mathrm{d}\theta = \frac{(4\sqrt{2} - 5)\pi}{4}.$$

Proposed by Eugen J. Ionaşcu, Department of Mathematics, Columbus State University, Columbus, U.S.A.

377. Let $p > 2$ be a prime and let n be a positive integer. Prove that

$$p^{\left[\frac{n-1}{p-1}\right]} \left| \sum_{k=0}^{\left[\frac{n}{p}\right]} (-1)^k \binom{n}{pk} \right. .$$

Proposed by Ghiocel Groza, Technical University (TUCEB), Bucharest, Romania.

378. Let $(x_n)_{n\geq 1}$ be a sequence with $0 < x_n < 1$. Then the following are equivalent:

(i) For any convergent series of positive numbers $\sum\limits_{n\geq 1} a_n$, the series $\sum\limits_{n\geq 1} a_n^{x_n}$ is convergent, as well.

(ii) The series $\sum\limits_{n\geq 1} M^{-1/(1-x_n)}$ is convergent for some $M > 1$, large enough.

Proposed by Constantin-Nicolae Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

SOLUTIONS

Corrigendum. Unfortunately, the problems from the issue 3–4/2011 of GM-A were mistakenly counted from 323 to 336, same as in the previous issue. In fact they should have been counted from 337 to 350. Here we are correcting this mistake. At each problem we indicate in parentheses the number by which it was indexed initially.

337 (323). If m, n are given positive integers and A, B, C are three matrices of size $m \times n$ with real entries, then

$$\sum_{\text{cyclic}} (\det(AB^T))^2 \det(CC^T) \leq \prod \det(AA^T) + 2 \prod_{\text{cyclic}} |\det(AB^T)|.$$

Proposed by Flavian Georgescu, student, University of Bucharest, Bucharest and Cezar Lupu, Politehnica University of Bucharest, Bucharest, Romania.

Solution by the authors. We prove a slightly stronger result, namely

$$\sum_{\text{cyclic}} (\det(AB^T))^2 \det(CC^T) \leq \prod \det(AA^T) + 2 \prod_{\text{cyclic}} \det(AB^T).$$

Let $S = \{(j_1, \dots, j_m) \mid 1 \leq j_1 < \dots < j_m \leq n\}$. If $X \in M_{m,n}(\mathbb{R})$ and $w = (j_1, \dots, j_m) \in S$ we denote by $X(w)$ the $m \times m$ matrix whose columns are the columns $j_1, \dots, j_m$ of X.

Then if X, Y are two $m \times n$ real matrices the Cauchy-Binet formula writes $\det(XY^T) = \sum\limits_{w \in S} \det X(w) \det Y(w)$.

Hence if $x = (\det X(w))_{w \in S}$ and $y = (\det Y(w))_{w \in S}$ then $\det(XY^T) = (x, y)$, where $(\cdot, \cdot) : \mathbb{R}^{\binom{n}{m}} \times \mathbb{R}^{\binom{n}{m}} \to \mathbb{R}$ is the usual dot product.

Hence if $a = (\det A(w))_{w \in S}$, $b = (\det B(w))_{w \in S}$ and $c = (\det C(w))_{w \in S}$ our statement writes as:

$$(a, b)^2(c, c) + (b, c)^2(a, a) + (c, a)^2(b, b) \leq (a, a)(b, b)(c, c) + 2(a, b)(b, c)(c, a).$$

But the dot product $(\cdot, \cdot)$ is an inner product so the Gram matrix

$$\begin{pmatrix} (a,a) & (a,b) & (a,c) \\ (b,a) & (b,b) & (b,c) \\ (c,a) & (c,b) & (c,c) \end{pmatrix}$$

is positive semidefinite and therefore its determinant

$$(a, a)(b, b)(c, c) + 2(a, b)(b, c)(c, a) - (a, b)^2(c, c) - (b, c)^2(a, a) - (c, a)^2(b, b),$$

is non-negative. $\square$

338 (324). Let p be a prime number and a, b, c, d positive integers such that $a \geq c$ and $b, d \in \{0, 1, \dots, p-1\}$. Show that

$$\binom{ap+b}{cp+d} \equiv (a-c)\binom{a}{c}\binom{p+b}{d} + c\binom{a}{c}\binom{p+b}{p+d} - (a-1)\binom{a}{c}\binom{b}{d} (\operatorname{mod} p^2).$$

`Proposed by Marian Tetiva, Gheorghe Roşca Codreanu National College, Bârlad, Romania.`

Solution by the author. We start with the equality

$$(1+X)^{ap+b} = (1+X)^p \cdots (1+X)^p (1+X)^b,$$

where there are a factors $(1+X)^p$. We express the coeficient of X^{cp+d} (from the two sides of the equality above) in two ways to obtain

$$\binom{ap+b}{cp+d} = \sum \binom{p}{i_1} \cdots \binom{p}{i_a} \binom{b}{j},$$

where the sum in the left hand is taken over all non-negative integer values of the indices $i_1, \dots, i_a, j$ with the property that $i_1 + \dots + i_a + j = cp + d$. We use the well known result that $\binom{p}{q} \equiv 0 \pmod{p}$ for $1 \leq q \leq p-1$ and we get that all products in the sum above where at least two of the indices $i_1, \dots, i_a$ belong to $\{1, \dots, p-1\}$ are divisible by p^2. Hence

$$\binom{ap+b}{cp+d} \equiv \sum{}' \binom{p}{i_1} \cdots \binom{p}{i_a} \binom{b}{j} \pmod{p^2},$$

where the sum $\sum'$ is taken only over those $i_1, \ldots, i_a, j \geq 0$ with $i_1+\cdots+i_a+j = cp+d$ such that at most one of $i_1, \ldots, i_a$ is not 0 or p. But one notes that, due to the condition that $b, d \in \{0, 1, \ldots, p-1\}$, if s of the indices $i_1, \ldots, i_a$ are equal to p and all others except, possibly, one, are 0 we have $s = c$ or $s = c - 1$. Thus we get the congruence

$$\binom{ap+b}{cp+d} \equiv (a-c)\binom{a}{c} \sum_{i+j=d} \binom{p}{i}\binom{b}{j}+$$

$$+(a-c+1)\binom{a}{c-1} \sum_{i+j=p+d} \binom{p}{i}\binom{b}{j} - (a-1)\binom{a}{c}\binom{b}{d} \pmod{p^2}.$$

Here $(a-c)\binom{a}{c}$ is the number of ways one may choose c indices $1 \leq l_1 < \ldots < l_c \leq a$ and an index $k \in \{1, \ldots, a\} \setminus \{l_1, \ldots, l_c\}$ such that $i_l = p$ if $l \in \{l_1, \ldots, l_c\}$, $i_l = 0$ if $l \notin \{l_1, \ldots, l_c, k\}$, and $i_k = i$. Thus the first sum deals with the case when $s = c$ and, similarly, the second sum deals with the case $s = c - 1$. The two sums accurately count the products where one of $i_1, \ldots, i_a$ is not p or 0. The terms where all $i_1, \ldots, i_a$ are p or 0 or, equivalently, when $j = d$ (and so the product is $\binom{b}{d}$) are counted in the first sum $(a-c)\binom{a}{c}$ times (the case $i = 0$, $j = d$) and $(a-c+1)\binom{a}{c-1}$ times in the second (the case $i = p$, $j = d$). In total this term is counted $a\binom{a}{c}$ times. In fact it appears only $\binom{a}{c}$ times. This justifies the subtraction of the last term.

To finish the proof we use the equalities

$$\sum_{i+j=d} \binom{p}{i}\binom{b}{j} = \binom{p+b}{d}, \quad \sum_{i+j=p+d} \binom{p}{i}\binom{b}{j} = \binom{p+b}{p+d}$$

together with $(a-c+1)\binom{a}{c-1} = c\binom{a}{c}$. □

339 (325). Let p be a prime number. Show that $\displaystyle\sum_{k=1}^{p} \sqrt[p]{k + \sqrt[p]{k}}$ cannot be rational.

`Proposed by Marius Cavachi, Ovidius University of Constanţa, Constanţa, Romania.`

Solution by the author. First of all let us summarise some theoretical points. An algebraic integer is a complex number that is a root of a monic polynomial with integer coefficients. The algebraic integers form a ring, let us denote it with $\mathbf{A}$. Also if a is an algebraic integer then $\sqrt[n]{a} \in \mathbf{A}$. Another property we shall make use of is $\mathbf{A} \cap \mathbb{Q} = \mathbb{Z}$.

From the above we can see that $\sqrt[p]{k+\sqrt[p]{k}}$ is an algebraic integer so if we denote with $s = \sum_{k=1}^{p} \sqrt[p]{k+\sqrt[p]{k}}$, we have that $s \in A$. We argue by contradiction, so assume $s \in \mathbb{Q}$. It follows from the above that $s \in \mathbb{Z}$, and since it is positive $s \in \mathbb{N}$.

For $p = 2$ we have

$$s^4 = \left(\sqrt{2} + \sqrt{2+\sqrt{2}}\right)^4 = \left(4+\sqrt{2}+2a\right)^2 = 18 + 2b = 2d,$$

while for p odd we have

$$\begin{aligned} s^{p^2} &= \left[\left(\sum_{k=1}^{p} \sqrt[p]{k+\sqrt[p]{k}}\right)^p\right]^p = \left(\sum_{k=1}^{p} k + \sum_{k=1}^{p} \sqrt[p]{k} + pa\right)^p \\ &= \left(p\left(a+\frac{p+1}{2}\right) + \sum_{k=1}^{p} \sqrt[p]{k}\right)^p = \left(pb + \sum_{k=1}^{p} \sqrt[p]{k}\right)^p \\ &= pc + \sum_{k=1}^{p} k = p\left(c + \frac{p+1}{2}\right) = pd, \end{aligned}$$

where a, b, c, d are algebraic integers. Now if we look at d it is rational since s is rational, and since it is an algebraic integer it follows using the property recalled in the first paragraph that it is an integer. So $s^{p^2} = pd$, where $d \in \mathbb{Z}$, thus $p \mid s$.

To finish the proof let us note that $1 < \sqrt[p]{k+\sqrt[p]{k}} < 2$. The left inequality is obvious, while for the second one it is sufficient to prove that $\sqrt[p]{p+\sqrt[p]{p}} < 2$, which is equivalent to $\sqrt[p]{p} < 2^p - p$. This is easily seen to be true when $p = 2$. Now for $p \geq 3$ we have $2^p > p + 2$, so that $\sqrt[p]{p} < 2 < 2^p - p$. Summing up, we get $p < s < 2p$, and this cannot happen since $p \mid s$. This is the desired contradiction, so our assumption was false, thus $s \notin \mathbb{Q}$. $\square$

340 (326). Let $A, B \in M_n(\mathbb{R})$ diagonalisable in $M_n(\mathbb{R})$ such that $\exp(A) = \exp(B)$. Show that $A = B$.

`Proposed by Moubinool Omarjee, Jean Lurçat High School, Paris, France.`

Solution by the author. Let $X = \mathrm{sp}(A) \cup \mathrm{sp}(B) = \{\lambda_1, \dots, \lambda_q\}$, where $\lambda_k \in \mathbb{R}$ are pairwise distinct. This implies that $e^{\lambda_1}, \dots, e^{\lambda_q}$ are pairwise distinct as well.

Let P be the Legendre interpolation polynomial with $P(e^{\lambda_k}) = \lambda_k \ \forall k$.

Since A is diagonalisable we may write

$$Q^{-1}AQ = D = \mathrm{diag}(a_1, \dots, a_n) = \mathrm{diag}(P(e^{a_1}), \dots, P(e^{a_n}))$$

for some invertible matrix Q. (We have $a_1, \dots, a_n \in \{\lambda_1, \dots, \lambda_q\}$ so $P(e^{a_k}) = a_k$.) This implies that $D = P(e^D)$ so $Q^{-1}AQ = Q^{-1}P(e^A)Q$ and $A = P(e^A)$. By the same reasoning $B = P(e^B)$. But $e^A = e^B$ so $A = P(e^A) = P(e^B) = B$. $\square$

341 (327). Let N be the $n \times n$ matrix with all its elements equal to $\dfrac{1}{n}$ and $A \in M_n(\mathbb{R})$, $A = (a_{ij})_{1\le i,\, j\le n}$, such that $A^k = N$ for some positive integer k. Show that

$$\sum_{1\le i,j\le n} a_{ij}^2 \ge 1.$$

`Proposed by Lucian Ţurea, Bucharest, Romania.`

Solution by the author. For any $X \in \mathcal{M}_n(\mathbb{R})$ put

$$m(X) = \sum_{1\le i,j\le n} x_{ij}^2.$$

We first prove that for any two matrices $X, Y \in \mathcal{M}_n(\mathbb{R})$ we have the inequality

$$m(X)m(Y) \ge m(XY).$$

Let $XY = Z = \big(z_{ij}\big)$. Then, by using the Cauchy-Bunyakovsky-Schwarz inequality, we have

$$z_{ij}^2 = \left(\sum_{k=1}^n x_{ik}y_{kj}\right)^2 \le \left(\sum_{k=1}^n x_{ik}^2\right)\left(\sum_{k=1}^n y_{kj}^2\right).$$

We sum over j, and we get

$$\begin{aligned}\sum_{j=1}^n z_{ij}^2 &\le \sum_{j=1}^n \left(\sum_{k=1}^n x_{ik}^2\right)\left(\sum_{k=1}^n y_{kj}^2\right)\\ &= \left(\sum_{k=1}^n x_{ik}^2\right)\left(\sum_{j=1}^n\sum_{k=1}^n y_{kj}^2\right) = \left(\sum_{k=1}^n x_{ik}^2\right)\cdot m(Y).\end{aligned}$$

We sum over i and we get

$$\sum_{i=1}^n\sum_{j=1}^n z_{ij}^2 \le \sum_{i=1}^n \left(\sum_{k=1}^n x_{ik}^2\right)\cdot m(Y) = m(X)m(Y).$$

So we have proved our claim.

To end the proof we note that $\big(m(A)\big)^k \ge m(A^k) = m(N) = 1$ and since $m(A) \ge 0$ we obtain $m(A) \ge 1$, i.e. the conclusion. $\square$

342 (328). Given $a > 0$, let f be a real-valued continuous function on $[-a, a]$ and twice differentiable on $(-a, a)$. Show that for all $|x| < a$, there exists $|\xi| < x$ such that

$$f(x) + f(-x) - 2f(0) = x^2 f''(\xi).$$

`Proposed by George Stoica, University of New Brunswick in Saint John, NB, Canada.`

Solution by the author. Let $L(t)$ be the Lagrange interpolation polynomial with $L(-x) = f(-x)$ and $L(x) = f(x)$, i.e.,

$$L(t) = f(-x)\frac{t-x}{-x-x} + f(x)\frac{t-(-x)}{x-(-x)}.$$

For any $t \in [-a, a]$, $t \notin \{-x, x\}$, we have the Lagrange interpolation formula with remainder:

$$f(t) = L(t) + \frac{1}{2} f''(\xi)(t+x)(t-x),$$

where ξ belongs to an interval (c, d) such that $-x, x, t \in [c, d]$.

In particular, if $t = 0$ then $\xi \in (-x, x)$ and the interpolation formula above becomes

$$f(0) = L(0) - \frac{1}{2} f''(\xi)x^2,$$

i.e.,

$$f(0) = \frac{1}{2} f(-x) + \frac{1}{2} f(x) - \frac{1}{2} f''(\xi)x^2.$$

□

Solution by Angel Plaza. Let x be such that $|x| < a$. We may assume that $0 < x$. By the Taylor Theorem there exist $c \in (0, x)$ and $d \in (-x, 0)$ such that

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(c)}{2} x^2, \\ f(-x) &= f(0) - f'(0)x + \frac{f''(d)}{2} x^2, \end{aligned}$$

from where by summing up it is obtained

$$f(x) + f(-x) - 2f(0) = x^2 \frac{f''(c) + f''(d)}{2}.$$

Note that $\dfrac{f''(c) + f''(d)}{2}$ is a value between $f''(c)$ and $f''(d)$. Then by Darboux's intermediate value theorem there exists ξ between c and d such that $f''(\xi) = \dfrac{f''(c) + f''(d)}{2}$, and the problem is done. □

343 (329). Let ABC be a triangle and let P be a point in its interior with pedal triangle DEF. Suppose that the lines DE and DF are perpendicular. Prove that the isogonal conjugate of P is the orthocenter of triangle AEF.

`Proposed by Cosmin Pohoaţă, Princeton University, Princeton, NJ, U.S.A.`

Solution by the author. We denote by M the isogonal conjugate of P. Let X, Y, Z be the reflections of P into the sidelines BC, CA, and AB, respectively. It is known that the isogonal conjugate of a point is the circumcenter of the reflection triangle of the point with respect to the sides. In our case P is the circumcenter of the triangle XYZ. Now the triangles XYZ and DEF are similar. Since DE and DF are perpendicular to each other, so are XY and XZ. Hence M, the circumcenter of XYZ, is the midpoint of YZ. Since M, E, F are the midpoints of the sides of the triangle PZY, we have $EM \| PZ$ and $FM \| PY$. But $PZ \perp AF$ and $PY \perp AE$. We conclude that $EM \perp AF$ and $FM \perp AE$, so M is the orthocenter of the triangle AEF. □

344 (330). It is well-known that for $p > 1$ prime, the number

$$N = \frac{2^{p-1} - 1}{p}$$

is an integer. When is N a natural power of an integer?

Proposed by Ion Cucurezeanu, Ovidius University of Constanţa, Constanţa, Romania.

Solution by the author. Let $2^{p-1} - 1 = py^n$, $n > 1$. We distinguish two cases:

a) n is even. The equation writes as $2^{p-1} - 1 = pz^2$.

Since $\left(2^{\frac{p-1}{2}} - 1, 2^{\frac{p-1}{2}} + 1\right) = 1$, we have $2^{\frac{p-1}{2}} \pm 1 = u^2$, $2^{\frac{p-1}{2}} \mp 1 = pv^2$, with $uv = z$. When the $\pm$ sign is $+$ we have $(u-1)(u+1) = 2^{\frac{p-1}{2}}$, which implies $u = 3$, so $p = 7$. When the sign is $-$, if $p > 3$ then $u^2 \equiv -1 \pmod 4$, which is impossible. Thus $p = 3$.

b) n is odd, $n > 1$. As in case a) one has

$$2^{\frac{p-1}{2}} \pm 1 = u^n, \quad 2^{\frac{p-1}{2}} \mp 1 = pv^n, \text{ with } uv = z.$$

From the first equation we get $2^{\frac{p-1}{2}} = (u \pm 1)\dfrac{u^n \pm 1}{u \pm 1}$. But $\dfrac{u^n \pm 1}{u \pm 1}$ is an odd integer, so it must be 1, which implies $u = 1$, so the $\pm$ sign is $+$ and $p = 3$.

In conclusion, $p = 3$ or 7. □

345 (331). Let $f \in \mathbb{Z}[X]$ be a monic polynomial of degree $n+2$, with $f(0) \neq 0$, $n \in \mathbb{N}$, $n \geq 1$. Show that there are only finitely many positive integers a such that $f(X) + aX^n$ is reducible over $\mathbb{Z}[X]$.

Proposed by Vlad Matei, student, University of Cambridge, Cambridge, UK.

Solution by the author. We will prove the following

Claim. *Let $g(X) = a_{n+2}X^{n+2} + a_{n+1}X^{n+1} + a_nX^n + \cdots + a_0$ a polynomial in $\mathbb{Z}[X]$, such that $|a_n| > |a_{n+2}| + |a_{n+1}| + |a_{n-1}| + \cdots + |a_0|$, with $n \in \mathbb{N}^*$. Then g has n roots inside the unit disc and two roots outside the unit disc.*

Proof. We have that for $|z| = 1$, $|g(z) - a_nz^n| = |a_{n+2}z^{n+2} + a_{n+1}z^{n+1} + a_{n-1}z^{n-1} + \cdots + a_0| \leq |a_{n+2}| + |a_{n+1}| + |a_{n-1}| + \cdots + |a_0| < |a_n| = |a_nz^n|$, so according to Rouché's theorem we have that g has the same number of roots inside the unit disc as the polynomial a_nX^n. Thus g has n roots inside the unit disc and the other two are outside the unit disc.

Coming back to our problem, let us show that for a sufficiently large $f(X) + aX^n$ has no real roots outside the interval $[-1, 1]$.

Let us remark that $\displaystyle\lim_{x\to\infty} \frac{f(x)}{x^n} = \lim_{x\to-\infty} \frac{f(x)}{x^n} = \infty$. Thus looking on the compact set $[-n, -1] \cup [1, n]$ the function $h(x) = \dfrac{f(x)}{x^n}$ is continuous, thus it has an infimum, and for $n \to \pm\infty$ it has ∞ limit, thus looking at h on $(-\infty, -1] \cup [1, \infty)$ we obtain that is bounded from below. Thus for a natural sufficiently large, $h(x) = -a$ has no solution in $(-\infty, -1] \cup [1, \infty)$. Also, let us note that there is also a value for a such that from that point on the polynomial $f(X) + aX^n$ satisfies the conditions of the claim.

Thus, from a point onwards, we have that $f(X) + aX^n$ has no real roots in $(-\infty, -1] \cup [1, \infty)$ and has exactly two complex roots outside the unit disc. Let us remark that the latter roots are complex conjugate, so if $f(X) + aX^n$ would be reducible they would belong to the same irreducible polynomial in the decomposition, and thus a polynomial in the decomposition would be monic and with all roots of modulus < 1. It follows that the constant term of this polynomial is < 1 in absolute value. But this constant term is an integer and it cannot be 0 as $f(0) \neq 0$, so $X \nmid F(x) + aX^n$. Contradiction. Thus $f(X) + aX^n$ is irreducible from a point onwards, and the proof ends. $\square$

346 (332). The cells of a rectangular $2011 \times n$ array are colored using two colors, so that for any two columns the number of pairs of cells situated on a same row and bearing the same color is less than the number of pairs of cells situated on a same row and bearing different colors.

i) Prove that $n \leq 2012$ (a model for the extremal case $n = 2012$ does indeed exist, but you are not asked to exhibit one);

ii) Prove that for a square array (i.e. $n = 2011$) each of the colors appears at most $1006 \cdot 2011$ (and thus at least $1005 \cdot 2011$) times.

`Proposed by Dan Schwarz, Bucharest, Romania.`

Solution by the author.

i) We will more generally work with $2m - 1 = 2011$. Denote by a_i the number of cells of the first color, and by $b_i = n - a_i$ the number of cells of the second color, situated on row $1 \leq i \leq 2m - 1$. Also denote by $A = \sum_{i=1}^{2m-1} a_i$ the total number of cells of the first color, and by N the total number of pairs of cells situated on a same row and bearing different colors. We will proceed by the trusted double counting method.

On one hand, on each row $1 \leq i \leq 2m - 1$ we have exactly $a_i b_i = a_i(n - a_i)$ such pairs, so

$$N = \sum_{i=1}^{2m-1} a_i(n - a_i) \leq (2m-1)\left(\frac{\sum_{i=1}^{2m-1} a_i}{2m-1}\right)\left(n - \frac{\sum_{i=1}^{2m-1} a_i}{2m-1}\right)$$

by Jensen's inequality, since the function $f(x) = x(n - x)$ is concave.

On the other hand, the number of such pairs for any two columns is at least m (versus at most $m - 1$ pairs of cells situated on a same row and bearing the same color), so $N \geq m\binom{n}{2} = \frac{1}{2}mn(n-1)$.

Putting together the two inequalities from above yields

$$\frac{A(n(2m-1) - A)}{2m-1} \geq N \geq \frac{1}{2}mn(n-1),$$

thus $A^2 - n(2m-1)A + \frac{1}{2}(2m-1)mn(n-1) \leq 0$.

The discriminant of this trinomial is

$$\Delta = n^2(2m-1)^2 - 2(2m-1)mn(n-1) = n(2m-1)(2m-n).$$

In order for the inequality to be possible we need $\Delta \geq 0$, thus $n \leq 2m$, which in our particular case means $n \leq 2012$.

It is interesting that we achieved the right bound for n, as it will be seen in the sequel, which will thus provide an alternative proof (based on linear algebra techniques).

ii) The bounds for the number of apparitions of a color are given by the roots of the above trinomial, which are

$$\frac{\sqrt{n(2m-1)}\left(\sqrt{n(2m-1)} \pm \sqrt{2m-n}\right)}{2}.$$

For $n = 2m-1 = 2011$ this yields the required bounds. $\square$

347 (333). Prove that for any $m, n \geq 3$ there is an $m \times n$ matrix of rank 2 with entries distinct primes.

Proposed by Nicolae Constantin Beli, Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

Solution by the author. We prove that a more general result holds. Namely, if $S \subset [2, \infty)$ is a set of mutually prime integers such that

$$\lim_{x\to\infty} \frac{\#\{a \in S \mid a \leq x\}}{x^u} = \infty \quad \forall\, u < 1,$$

then for any $m, n \geq 3$ there is an $m \times n$ matrix of rank 2 with entries distinct elements of S. Then we take S to be the set of all primes to solve the problem. (We have $\lim\limits_{x\to\infty} \dfrac{\pi(x)}{x/\log x} = 1$, so $\lim\limits_{x\to\infty} \dfrac{\pi(x)}{x^u} = \infty$, $\forall\, u < 1$.)

Suppose that S is a set as above. Then for any $v > 1$ and any $C > 0$ we have $\#\{a \in S \mid x < a \leq x^v\} \geq Cx$ when x is large enough. Indeed, since $x^v \to \infty$ as $x \to \infty$ and $1/v < 1$, we have

$$\lim_{x\to\infty} \frac{\#\{a \in S \mid a \leq x^v\}}{x} = \lim_{x\to\infty} \frac{\#\{a \in S \mid a \leq x^v\}}{(x^v)^{1/v}} = \infty.$$

Thus for x large enough we have $\#\{a \in S \mid a \leq x^v\} \geq (C+1)x$, which, together with $\#\{a \in S \mid a \leq x\} \leq x$, implies $\#\{a \in S \mid x < a \leq x^v\} \geq Cx$.

Let $1 < v < \sqrt{\dfrac{4n-7}{4n-8}}$, so that $4n - 7 - (4n-8)v^2 > 0$. Then there is some N such that for any $x \geq N$ we have $\#\{a \in S \mid x < a \leq x^v\} \geq 2x$ and

$$\#\{a \in S \mid x^{2v} < a \leq x^{2v^2}\} \geq (m-2)x^{2v} > (m-2)x^2.$$

Let $x \geq \max\{N, n^{\frac{1}{4m-7-(4m-8)v^2}}\}$ be an integer. Then S contains some elements $a_1, \ldots, a_x$, $b_1, \ldots, b_x$, and $c_{i,j}$ with $1 \leq i \leq m-2$ and $1 \leq j \leq x^2$ with $x < a_1 < \ldots < a_x < b_1 < \ldots < b_x \leq x^v$ and

$$x^{2v} < c_{1,1} < \ldots < c_{1,x^2} < \ldots < c_{m-2,1} < \ldots < c_{m-2,x^2} \leq x^{2v^2}.$$

(For short $c_{i,j} < c_{k,l}$ when $(i,j) < (k,l)$ in the lexicographic order.)

Since $x^v < x^{2v}$, all elements $a_h, b_h, c_{i,j}$ of S are different from each other and so mutually prime.

We use the following result:

If a, b, c are three integers, $a, b > 0$, $(a, b) = 1$ and $c > ab - a - b$ then there are two integers $\alpha, \beta \geq 0$ such that $\alpha a + \beta b = c$. Indeed, the integral solutions of this equation are $\alpha = \alpha_0 - bt$, $\beta = \beta_0 + at$, where (α_0, β_0) is a particular solution. We take $t = [\alpha_o/b]$, so that $0 \leq \alpha \leq b - 1$. If $\beta < 0$ then

$$c = \alpha a + \beta b \leq (b - 1)a + (-1)b = ab - a - b.$$

Contradiction. So $\alpha, \beta \geq 0$, as claimed.

Now for any $1 \leq h \leq x$, $1 \leq i \leq m - 2$, and $1 \leq j \leq x^2$ we have $(a_h, b_h) = 1$ and $a_h b_h - a_h - b_h \leq x^v x^v - x^v - x^v < x^{2v} < c_{i,j}$, so there are some integers $\alpha_{h,i,j}, \beta_{h,i,j} \geq 0$ with $\alpha_{h,i,j} a_h + \beta_{h,i,j} b_h = c_{i,j}$.

Note that $\alpha_{h,i,j} x \leq \alpha_{h,i,j} a_h \leq c_{i,j} \leq x^{2v^2}$ implies $\alpha_{h,i,j} \leq x^{2v^2-1}$ and similarly $\beta_{h,i,j} \leq x^{2v^2-1}$. Also $\alpha_{h,i,j}, \beta_{h,i,j}$ cannot be zero since $(a_h, c_{i,j}) = (b_h, c_{i,j}) = 1$, so they are positive.

For any $1 \leq h \leq x$ we define

$$T_h := \{(\alpha_{h,1,j_1} \beta_{h,1,j_1}, \ldots, \alpha_{h,m-2,j_{m-2}} \beta_{h,m-2,j_{m-2}}) \mid 1 \leq j_i \leq x^2 \ \forall i\}.$$

We have $|T_h| = (x^2)^{m-2} = x^{2m-4}$ and $T_h \subset ([1, x^{2v^2-1}] \cap \mathbb{Z})^{2m-4}$. Since $|T_1| + \cdots + |T_x| = x^{2n-3}$ and $T_1, \ldots, T_x$ are contained in the set $([1, x^{2v^2-1}] \cap \mathbb{Z})^{2m-4}$ of cardinality $[x^{2v^2-1}]^{2m-4}$, there is an element

$$(\alpha_1, \beta_1, \ldots, \alpha_{m-2} \beta_{m-2}) \in ([1, x^{2v^2-1}] \cap \mathbb{Z})^{2m-4}$$

belonging to at least $\dfrac{x^{2m-3}}{[x^{2v^2-1}]^{2m-4}} \geq x^{4m-7-(4m-8)v^2} \geq n$ of the sets $T_1, \ldots, T_x$. (Recall, $x \geq n^{\frac{1}{4m-7-(4m-8)v^2}}$.) Say $(\alpha_1, \beta_1, \ldots, \alpha_{m-2}, \beta_{m-2}) \in T_{h_1}, \ldots, T_{h_n}$, where $1 \leq h_1 < \ldots < h_n \leq x$.

We define $A := (a_{h_1}, \ldots, a_{h_n})$, $B := (b_{h_1}, \ldots, b_{h_n})$ and we prove that the $m \times n$ matrix

$$M = (m_{i,j})_{i,j} := \begin{pmatrix} A \\ B \\ \alpha_1 A + \beta_1 B \\ \vdots \\ \alpha_{m-2} A + \beta_{m-2} B \end{pmatrix}$$

has the required properties.

First note that $a_{h_1} < \ldots < a_{h_n}$, $b_{h_1} < \ldots < b_{h_n}$, and for $1 \leq i \leq m - 2$ we have $\alpha_i, \beta_i > 0$, so the entries of $\alpha_i A + \beta_i B$ are $\alpha_i a_{h_1} + \beta_i b_{h_1} < \ldots < \alpha_i a_{h_n} + \beta_i b_{h_n}$. Hence $m_{i,j} < m_{i,l}$ whenever $j < l$.

If $h = h_j$ then $(\alpha_1, \beta_1, \ldots, \alpha_{m-2}, \beta_{m-2}) \in X_h$, so there are $j_1, \ldots, j_{m-2}$ such that

$$(\alpha_1, \beta_1, \ldots, \alpha_{m-2}, \beta_{m-2}) = (\alpha_{h,1,j_1}, \beta_{h,1,j_1}, \ldots, \alpha_{h,m-2,j_{m-2}}, \beta a_{h,m-2,j_{m-2}}).$$

Then for any $1 \leq i \leq m - 2$ we have

$$\alpha_i a_{h_j} + \beta_i b_{h_j} = \alpha_{h,i,j_i} a_h + \beta_{h,i,j_i} b_h = c_{i,j_i}.$$

In conclusion, the entries of A belong to $\{a_1, \ldots, a_x\}$, the entries of B belong to $\{b_1, \ldots, b_x\}$, and for any $1 \le i \le m-2$ the entries $\alpha_i a_{h_j} + \beta_i b_{h_j}$ of $\alpha_i A + \beta_i B$ belong to $\{c_{i,1}, \ldots, c_{i,x}\}$. It follows that $m_{i,j} \in S\ \forall i, j$. Also since $a_h < b_{h'} < c_{i,j}$ $\forall h, h', i, j$ and $c_{i,j} < c_{k,l}$ when $i < k$ we have $m_{i,j} < m_{k,l}$ when $i < k$. Together with $m_{i,j} < m_{i,l}$ when $j < l$, this implies that $m_{i,j} < m_{k,l}$ whenever $(i,j) < (k,l)$ in the lexicographic order. Thence the entries $m_{i,j}$ of M are distinct from each other. $\square$

Note. This result is a generalization of a problem proposed by late Professor Nicolae Popescu in the 1980s in an issue of Gazeta Matematica Seria B. The problem was to prove that for any $n \ge 3$ there is an $n \times n$ matrix of determinant 0 with all entries distinct prime numbers.

There is an alternative solution by Marian Tetiva using a very strong result of Terrence Tao, which states that there are arbitrarily long arithmetic progressions of prime numbers. Thus we may take an $m \times n$ matrix with all entries distinct prime numbers such that all lines are arithmetic progressions. Then if $C_1, \ldots, C_n$ are the columns of the matrix we have

$$C_j = C_1 + (j-1)(C_2 - C_1) = (j-1)C_2 - (j-2)C_1.$$

So C_1, C_2 span the vector space generated by the columns. Since the vector space generated by the columns has dimmension 2 the rang of the matrix is 2.

348 (334). Define $\mathcal{F} = \{f : [0,1] \to [0,1] : \exists\, A, B \subset [0,1],\ A \cap B = \emptyset,\ A \cup B = [0,1],\ f(A) \subset B,\ f(B) \subset A\}$. Prove that $\mathcal{F}$ contains functions with Darboux property (a function f has the Darboux property if $f(I)$ is an interval whenever I is an interval).

`Proposed by Benjamin Bogoşel, student, West University of Timişoara, Timişoara, Romania.`

Solution by the author. It is well known that the dimension of $\mathbb{R}$ considered as a vector space over $\mathbb{Q}$ is $\aleph = \operatorname{card} \mathbb{R}$. Define $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$. This is obviously an equivalence relation and for any $x \in \mathbb{R}$ we will denote $[x] = \{y \in \mathbb{R} : y \sim x\}$ the equivalence class which contains x. It is obvious that $\mathbb{R} = \bigcup\limits_{x \in \mathbb{R}} [x]$ and $y \notin [x] \Rightarrow [x] \cap [y] = \emptyset$. In the following, we will denote $\mathcal{A} = \{[x] : x \in \mathbb{R}\}$ the set of the equivalence classes, and we will find its cardinal number. We choose a basis $\mathcal{B}$ of $\mathbb{R}$ over $\mathbb{Q}$ which contains 1 (this is possible because any linearly independent set can be extended to a basis and $\{1\}$ is such a set). If two different numbers, say x and y, from $\mathcal{B}$ would be in the same equivalence class, then we have $x - y = q \cdot 1$, $q \in \mathbb{Q}$, so 1, x and y are linearly dependent. Contradiction.

Therefore, any two different elements of $\mathcal{B}$ are in different equivalence classes, and we can define a one-to-one mapping $\phi : \mathcal{B} \to \mathcal{A}$, $\phi(b) = [b]$, $\forall\, b \in \mathcal{B}$. This proves that $\operatorname{card} \mathbb{R} = \operatorname{card} \mathcal{B} \le \operatorname{card} \mathcal{A}$. Since it is obvious that $\operatorname{card} \mathcal{A} \le \operatorname{card} \mathbb{R}$, we conclude that $\operatorname{card} \mathcal{A} = \operatorname{card} \mathbb{R}$.

We choose a bijection $\phi : \mathcal{A} \to \mathbb{R}$ and we denote $Y = \phi^{-1}((-\infty, 0])$, $Z = \phi^{-1}((0, \infty))$. Take $A = \{x \in [0,1] : [x] \in Y\}$, $B = \{x \in [0,1] : [x] \in Z\}$. Note that A and B are disjoint and non-void because Y and Z are disjoint and non-void. We have $Y \cup Z = \mathcal{A}$, so $A \cup B = \{x \in [0,1] : [x] \in \mathcal{A}\} = [0,1]$. Because A

and B contain all the elements from $[0,1]$ which are in the same equivalence class, A and B are dense in $[0,1]$.

From their definitions, card Y = card Z = card A = card $B = \aleph$. Therefore, we can find bijections $\mu : Y \to B$ and $\nu : Z \to A$.

We define the function $f : [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} \mu([x]), & x \in A, \\ \nu([x]), & x \in B. \end{cases}$$

From the definition of f and sets A, B, Y, Z we find that $f(A) \subset B$ and $f(B) \subset A$. Let's prove that f has the Darboux property. We take I an interval from $[0,1]$. Then I intersects all the classes from $\mathcal{A}$ (because any of these is dense in $\mathbb{R}$), which means that I intersects all the classes from Y and Z. Hence $f(I) = \mu(Y) \cup \nu(Z) = B \cup A = [0,1]$. Therefore, $f \in \mathcal{F}$ and f has the Darboux property. $\square$

349 (335). Let $f : [0,1] \to \mathbb{R}$ be an integrable function with $\int_0^1 f(x)\mathrm{d}x = 0$. Prove that $\int_0^1 f^2(x)\mathrm{d}x \geq 12 \left(\int_0^1 xf(x)\mathrm{d}x \right)^2$.

`Proposed by Cezar Lupu, Politehnica University of Bucharest, Bucharest, Romania, and Tudorel Lupu, Decebal High School, Constanţa, Romania.`

Solution by Angel Plaza. Let F be defined by $F(x) = \int_0^x f(t)\mathrm{d}t$. F is a differentiable function and $F'(x) = f(x)$. By the Cauchy-Schwarz inequality,

$$\left(\int_0^{1/2} xf(x)\mathrm{d}x \right)^2 \leq \left(\int_0^{1/2} x^2\mathrm{d}x \right) \left(\int_0^{1/2} f^2(x)\mathrm{d}x \right) = \frac{1}{24} \int_0^{1/2} f^2(x)\mathrm{d}x,$$

$$\left(\int_{1/2}^1 \left(x - \frac{1}{2} \right) f(x)\mathrm{d}x \right)^2 \leq \left(\int_{1/2}^1 \left(x - \frac{1}{2} \right)^2 \mathrm{d}x \right) \left(\int_{1/2}^1 f^2(x)\mathrm{d}x \right) = \frac{1}{24} \int_{1/2}^1 f^2(x)\mathrm{d}x,$$

whence

$$\frac{1}{24} \int_0^1 f^2(x)\mathrm{d}x \geq \left(\int_0^{1/2} xf(x)\mathrm{d}x \right)^2 + \left(\int_{1/2}^1 \left(x - \frac{1}{2} \right) f(x)\mathrm{d}x \right)^2.$$

Integration by parts yields

$$\int_0^{1/2} xf(x)\mathrm{d}x = \frac{1}{2} \int_0^{1/2} f(x)\mathrm{d}x - \int_0^{1/2} F(x)\mathrm{d}x,$$

$$\int_{1/2}^{1}\left(x-\frac{1}{2}\right)f(x)\mathrm{d}x=\frac{1}{2}\int_{1/2}^{1}f(x)\mathrm{d}x-\int_{1/2}^{1}F(x)\mathrm{d}x.$$

Consequently,

$$\frac{1}{24}\int_{0}^{1}f^2(x)\mathrm{d}x\geq$$

$$\geq\left(\frac{1}{2}\int_{0}^{1/2}f(x)\mathrm{d}x-\int_{0}^{1/2}F(x)\mathrm{d}x\right)^2+\left(\frac{1}{2}\int_{1/2}^{1}f(x)\mathrm{d}x-\int_{1/2}^{1}F(x)\mathrm{d}x\right)^2\geq$$

$$\geq\frac{1}{2}\left(\frac{1}{2}\int_{0}^{1/2}f(x)\mathrm{d}x-\int_{0}^{1/2}F(x)\mathrm{d}x+\frac{1}{2}\int_{1/2}^{1}f(x)\mathrm{d}x-\int_{1/2}^{1}F(x)\mathrm{d}x\right)^2=$$

$$=\frac{1}{2}\left(\int_{0}^{1}F(x)\mathrm{d}x\right)^2.$$

The conclusion follows, since, integrating by parts,

$$\begin{aligned}\int_{0}^{1}F(x)\mathrm{d}x &= \int_{0}^{1}\int_{0}^{x}f(t)\mathrm{d}t\mathrm{d}x=x\int_{0}^{x}f(t)\mathrm{d}t\Big|_0^1-\int_{0}^{1}xf(x)\mathrm{d}x=\\ &=\int_{0}^{1}f(t)\mathrm{d}t-\int_{0}^{1}xf(x)\mathrm{d}x=-\int_{0}^{1}xf(x)\mathrm{d}x.\end{aligned}$$

□

350 (336). Given a function $f:\mathbb{R}\to\mathbb{R}$, denote by f^n its n^{th} iterate. It is also given that $|f(x)-f(y)|\leq|x-y|$ for all $x,y\in\mathbb{R}$ (f is Lipschitzian, and non-expansive), and that $f^N(0)=0$ for some $N\in\mathbb{N}^*$.

i) Prove that if N is odd, then $|f(x)|\leq|x|$ for all $x\in\mathbb{R}$;

ii) Prove that if N is even, then $|f(f(x))|\leq|x|$ for all $x\in\mathbb{R}$, but not necessarily $|f(x)|\leq|x|$.

`Proposed by Dan Schwarz, Bucharest, Romania.`

Solution by the author. i) Our purpose will be to show that $f(0)=0$ (f has 0 as fixed point)[1)].

Then $|f(x)|=|f(x)-0|=|f(x)-f(0)|\leq|x-0|=|x|$ for all $x\in\mathbb{R}$.

For $1\leq k\leq N-1$ one has

$$\begin{aligned}\left|f^k(0)\right|&=\left|f^N(0)-f^k(0)\right|\leq\left|f^{N-1}(0)-f^{k-1}(0)\right|\leq\cdots\leq\left|f^{N-k}(0)-0\right|\\ &=\left|f^{N-k}(0)\right|=\left|f^N(0)-f^{N-k}(0)\right|\leq\left|f^{N-1}(0)-f^{N-k-1}(0)\right|\end{aligned}$$

[1)]Notice that f is continuous, but Sharkovsky's theorem is of no avail, since it only warrants the existence of some fixed point for f.

$$\leq \cdots \leq \left| f^k(0) - 0 \right| = \left| f^k(0) \right| .$$

This summarizes into relations

$$\left| f^k(0) \right| = \left| f^{N-k}(0) \right| = \left| f^{m+k}(0) - f^m(0) \right|$$

for all $1 \leq k \leq N-1$, $0 \leq m \leq N-k$.

Suppose moreover that N is the least odd for which it is yet unknown that from the above relations must follow $f(0) = 0$[2]. If for some $1 \leq k < \dfrac{N}{2}$ we would have $f^k(0) = f^{N-k}(0)$, then $\left| f^{N-2k}(0) \right| = \left| f^{N-k}(0) - f^k(0) \right| = 0$, for odd $N-2k$, and so by the minimality of N it follows $f(0) = 0$.

Therefore we must assume for all $1 \leq k < \dfrac{N}{2}$ that $f^k(0) = -f^{N-k}(0)$, so $\left| f^{2k}(0) \right| = \left| f^{N-2k}(0) \right| = \left| f^{N-k}(0) - f^k(0) \right| = 2 \left| f^k(0) \right|$.

Let us model the following directed graph G. The vertices are the numbers v (associated to $|f^v(0)|$), for all $1 \leq v \leq N-1$. The edges are made by red arrows from $2k$ towards k and black arrows from $N-2k$ towards $2k$, for $1 \leq k < \dfrac{N}{2}$. Then for each vertex the out-degree is exactly 1 (from an even vertex leaves a red arrow, while from an odd vertex leaves a black arrow).

Start with any vertex v and build the unique directed path determined by moving on the directed edges (on a black arrow from an odd vertex and on a red arrow from an even vertex). Since the graph is finite, the path will eventually reach a vertex w already passed through, thereby creating a cycle. Now, that means $|f^w(0)| = 2^r |f^w(0)|$, where $r \geq 1$ is the number of red arrows in the cycle, therefore $f^w(0) = 0$. But we also have $|f^v(0)| = 2^\rho |f^w(0)|$, where $\rho \geq 0$ is the number of red arrows in the path from vertex v to vertex w, therefore $f^v(0) = 0$. For $v = 1$ we thus have $f(0) = 0$.

ii) Let $N = 2m$ be the least even N for which it is yet unknown that from the above relations must follow $f^2(0) = 0$. As above, get for any $1 \leq k < \dfrac{N}{2}$ that $f^k(0) = -f^{N-k}(0)$, so

$$\left| f^{2k}(0) \right| = \left| f^{N-2k}(0) \right| = \left| f^{N-k}(0) - f^k(0) \right| = 2 \left| f^k(0) \right|$$

(otherwise some $\left| f^{N-2k}(0) \right| = \left| f^{N-k}(0) - f^k(0) \right| = 0$, and so by the minimality of N follows $f^2(0) = 0$).

But then

$$2 \left| f^k(0) \right| = \left| f^{2k}(0) \right| = \left| f^{N-2k}(0) \right| = \left| f^{2(m-k)}(0) \right| = 2 \left| f^{m-k}(0) \right| ,$$

hence $\left| f^k(0) \right| = \left| f^{m-k}(0) \right| = \left| f^{N-(m-k)}(0) \right| = \left| f^{m+k}(0) \right|$.

Since $|f^m(0)| = |f^{m+k}(0) - f^k(0)| = |f^{2m-k}(0) - f^{m-k}(0)|$, it follows we must have $f^{m+k}(0) = -f^k(0)$ and $f^{m-k}(0) = -f^{2m-k}(0) = f^k(0)$, otherwise $f^m(0) = 0$. But then $|f^{m-2k}(0)| = |f^{m-k}(0) - f^k(0)| = 0$, so $f^{m-2k}(0) = 0$ for all $1 \leq k < m/2$, and so by the minimality of N follows $f^2(0) = 0$.

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[2] An extremal argument, similar to finite recursion, or strong induction.

(The only case where this argumentation does not apply is $m = 2$. Then $\left|f^2(0)\right| = 2|f(0)|$, so either $f^2(0) = 2f(0)$, when

$$|f(0)| = \left|f^3(0) - f^2(0)\right| = 3|f(0)|,$$

hence $f(0) = 0$, or $f^2(0) = -2f(0)$, when

$$|f(0)| = \left|f^2(0) - f(0)\right| = 3|f(0)|,$$

hence $f(0) = 0$. This case thus leads to $f(0) = 0$.)[1)]

A simple counterexample for even N is sufficient. Take $f(x) = a - x$ for some real $a \neq 0$. Then all even iterates of f are $\mathrm{id}_{\mathbb{R}}$, thus have 0 as a fixed point, $|f(x) - f(y)| = |x - y|$ for all $x, y \in \mathbb{R}$, but $|f(0)| = |a| > 0$. $\square$

Mihail Bălună has given a different, rather more analytic solution, which also brings some light on the counterexamples for even N.

[1)] Alternatively, for $n = 2^\alpha\beta$, with odd β, we may use point i) to reduce to $f^{2^\alpha}(0) = 0$, and so only have to consider the even n's which are powers of 2. But from there on, we must continue as above.