Test 2 — Solutions

Problem 1. Given positive integers k and m, show that m and $\binom{n}{k}$ are coprime for infinitely many integers $n \geq k$.

Solution. Let $n = k + lmk!$, where l is an arbitrary nonnegative integer, let p be any prime factor of m, and let p^h be the highest power of p that divides $k!$ — that is, p^h divides $k!$ but p^{h+1} does not. Notice that $n \equiv k$ modulo p^{h+1}, to deduce that $n(n-1)\cdots(n-k+1) \equiv k!$ modulo p^{h+1}, so p^h is also the highest power of p that divides the product $n(n-1)\cdots(n-k+1)$. Consequently, p does not divide $\binom{n}{k}$, so m and $\binom{n}{k}$ are indeed coprime.

Problem 2. Let ABC be an acute triangle, and let M be the midpoint of the side AC. A circle through B and M meets the sides AB and BC again at P and Q, respectively. The reflection T of B across the midpoint of the segment PQ lies on the circle ABC. Evaluate the ratio BT/BM.

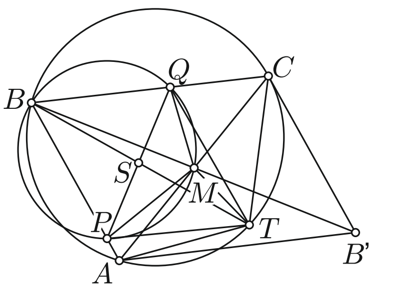
Solution. Te required ratio equals $\sqrt{2}$. To prove this, let S be the midpoint of the segment PQ, and let B' be the reflection of B across M. Clearly, $ABCB'$ is a parallelogram, $\angle ABB' = \angle PQM$, and $\angle BB'A = \angle B'BC = \angle MPQ$, so the triangles ABB' and MQP are similar. Since AM and MS are corresponding medians in these triangles,

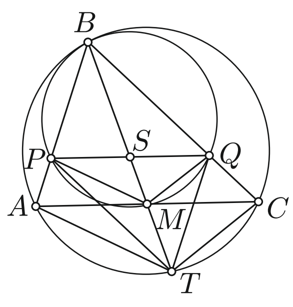
$$\angle SMP = \angle B'AM = \angle BCA = \angle BTA. \tag{1}$$

Next, $\angle ACT = \angle PBT$ and $\angle TAC = \angle TBC = \angle BTP$, so the triangles TCA and PBT are similar. Since TM and PS are corresponding medians in these triangles,

$$\angle MTA = \angle TPS = \angle BQP = \angle BMP. \tag{2}$$

If S does not lie on the segment BM, we may and will assume that S and A both lie on the same side of the line BM, since the configuration is symmetric in A and C. By (1) and (2), $\angle BMS = \angle BMP - \angle SMP = \angle MTA - \angle BTA = \angle MTB$, so the triangles BSM and BMT are similar, whence $BM^2 = BS \cdot BT = BT^2/2$; that is, $BT/BM = \sqrt{2}$.





If S lies on the segment BM, then (2) shows that $\angle BCA = \angle MTA = \angle BMP = \angle BQP$, so (PQ, AC) and (PM, AT) are pairs of parallel lines. Consequently, $BS/BM = BP/BA = BM/BT$, so $BT^2 = 2BM^2$, and again $BT/BM = \sqrt{2}$.

Problem 3. Prove that:

(a) If $(a_n)_{n\geq 1}$ is a strictly increasing sequence of positive integers such that $(a_{2n-1} + a_{2n})/a_n$ is constant as n runs through all positive integers, then this constant is an integer greater than or equal to 4; and

(b) Given an integer $N \geq 4$, there exists a strictly increasing sequence $(a_n)_{n\geq 1}$ of positive integers such that $(a_{2n-1} + a_{2n})/a_n = N$ for all indices n.

Solution. (a) Clearly, $K = (a_{2n-1} + a_{2n})/a_n$ is a positive rational number. In fact, K must be integral. To prove this, write $K = p/q$ in lowest terms to deduce that the a_n are all divisible by q. Divide them all by q to obtain a new sequence whose corresponding ratios are again K. Repetition of the process to the new sequence and its successors shows that the a_n are all divisible by arbitrarily large powers of q, so $q = 1$ and K is indeed integral.

Since the a_n form a strictly increasing sequence, it follows that $K > 2$, and since the latter is integral, it is at least 3.

To rule out the case $K = 3$, we consider the positive integers $b_n = a_{n+1} - a_n$, show that for every index m there exists an index $n > m$ such that $b_n < b_m$ and reach thereby a contradiction. Indeed, if $K = 3$, then $3b_n = b_{2n-1} + 2b_{2n} + b_{2n+1}$, so at least one of the three b's in the right-hand member must be less than b_n. Consequently, $K \geq 4$.

(b) If $N = 4$, let $a_n = 2n - 1$; in this case, the verifications are obvious. If $N \geq 5$, set $a_1 = 1$ and let $a_{2n-1} = \lfloor (Na_n - 1)/2 \rfloor$ and $a_{2n} = \lfloor Na_n/2 \rfloor + 1$. This sequence satisfies the required ratio condition, a_{2n-1} is obviously less than a_{2n}, and it is sufficient to prove that $a_{2n} < a_{2n+1}$. This can be done by noticing that $a_2 < a_3$, and showing that if $a_n < a_{n+1}$, then $a_{2n} < a_{2n+1}$. Indeed, $a_{2n+1} - a_{2n} \geq (Na_{n+1} - 2)/2 - (Na_n/2 + 1) = N(a_{n+1} - a_n)/2 - 2 \geq N/2 - 2 \geq 1/2$.

Problem 4. Given any positive integer n, prove that:

(a) Every n points in the closed unit square $[0, 1] \times [0, 1]$ can be joined by a path of length less than $2\sqrt{n} + 4$; and

(b) There exist n points in the closed unit square $[0, 1] \times [0, 1]$ that cannot be joined by a path of length less than $\sqrt{n} - 1$.

Solution. (a) Let C be an n-point configuration in the closed unit square $[0, 1] \times [0, 1]$, let $m = \lfloor \sqrt{n} \rfloor$, and consider the snake going horizontally from 0×0 to 1×0, then vertically up from 1×0 to $1 \times 1/m$, then horizontally back from $1 \times 1/m$ to $0 \times 1/m$, vertically up from $0 \times 1/m$ to $0 \times 2/m$, horizontally over to $1 \times 2/m$, and so on and so forth all the way up to 1×1 or 0×1, depending on whether m is even or odd. The length of the snake is $m + 1 + m \cdot 1/m = m + 2 \leq \sqrt{n} + 2$. Of course, the snake does not necessarily pass through any point in C, but it comes within $1/(2m)$ of C. Thus, in tracing the snake, visit each point of C by darting out, if necessary, to the nearest points in C abreast within $1/(2m)$, and then dart back. This increases the length by at most $n \cdot 2 \cdot 1/(2m) = n/m < \sqrt{n} + 2$, so the length of the visiting path is certainly less than $2\sqrt{n} + 4$.

(b) Let again $m = \lfloor \sqrt{n} \rfloor$, and consider an n-point subconfiguration C of the lattice $\{i/m \times j/m : i, j = 0, 1, \ldots, m\}$. Since any two distinct points in the lattice are at least $1/m$ distance apart, the length of a path through all of C is at least $(n - 1) \cdot 1/m \geq (n - 1)/\sqrt{n} \geq \sqrt{n} - 1$.

Remark. The problem shows that the order of magnitude of the longest shortest path through n points in a unit square is $\sqrt{n}$.