Test 3 — Solutions

Problem 1. Two circles γ and γ' cross one another at points A and B. The tangent to γ' at A meets γ again at C, the tangent to γ at A meets γ' again at C', and the line CC' separates the points A and B. Let Γ be the circle externally tangent to γ, externally tangent to γ', tangent to the line CC', and lying on the same side of CC' as B. Show that the circles γ and γ' intercept equal segments on one of the tangents to Γ through A.

Solution. Invert with respect to a circle centred at A and denote by X^* the image of a point $X \neq A$ under this inversion. The circles γ and γ' invert into straight lines B^*C^* and $B^*C'^*$, and the tangents at A into lines through A, parallel to B^*C^* and $B^*C'^*$. The line CC' inverts into the circle $AC^*C'^*$, and 'CC' separating A and B' is equivalent to 'B^* lying inside circle $AC^*C'^*$.' So $AC^*B^*C'^*$ is a parallelogram, obtuse-angled at A and B^*. Draw the line through A, parallel to $C^*C'^*$, meeting the lines B^*C^* and $B^*C'^*$ at D and D', respectively. Then A, C^* and C'^* are the midpoints of the sides of the triangle B^*DD', and the line DD' is the inverse of the line through A on which γ and γ' intercept equal segments. The circle $AC^*C'^*$ is the nine-point circle of the triangle B^*DD'; by Feuerbach's theorem, it touches the incircle of that triangle. Since this incircle is the inverse of the circle Γ in the original configuration, touching γ and γ' externally and the line CC', the conclusion follows.

Remark. The statement can be slightly generalised, using the fact that the nine-point circle of a triangle is tangent not only to the incircle but also to the three excircles — the full version of Feuerbach's theorem.

Problem 2. Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be sequences of real numbers such that $a_0 > 1/2$, $a_{n+1} \geq a_n$, and $b_{n+1} = a_n(b_n + b_{n+2})$, for all non-negative integers n. Show that the sequence $(b_n)_{n\geq 0}$ is bounded.

Solution. Use the relation in the statement and the fact that no a_k is $1/2$, to write

$$b_{k+2}^2 - b_k^2 = \frac{(b_{k+1} - b_k)^2 - (b_{k+2} - b_{k+1})^2}{2a_k - 1}, \quad k \in \mathbb{N},$$

fix an integer $n > 2$, and sum over $0, 1, \ldots, n-2$, to obtain

$$\begin{aligned} b_n^2 + b_{n-1}^2 - b_1^2 - b_0^2 &= \frac{(b_1 - b_0)^2}{2a_0 - 1} - \sum_{k=0}^{n-3} \left(\frac{1}{2a_k - 1} - \frac{1}{2a_{k+1} - 1} \right) (b_{k+2} - b_{k+1})^2 - \frac{(b_n - b_{n-1})^2}{2a_{n-2} - 1} \\ &\leq (b_1 - b_0)^2/(2a_0 - 1), \end{aligned}$$

since the a_k form an increasing sequence of real numbers greater than $1/2$. Consequently, $b_n^2 \leq b_n^2 + b_{n-1}^2 \leq b_0^2 + b_1^2 + (b_1 - b_0)^2/(2a_0 - 1)$, and the conclusion follows.

Remark. Leaving aside the trivial case where the b_n are all zero, it is readily checked that the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$, where $a_n = 1$ and $b_n = \cos(n\pi/3)$, satisfy the conditions in the statement, so the hypothesis is not vacuous.

Problem 3. If k and n are positive integers, and $k \leq n$, let $M(n,k)$ denote the least common multiple of the numbers $n, n-1, \ldots, n-k+1$. Let $f(n)$ be the largest positive integer $k \leq n$ such that $M(n,1) < M(n,2) < \cdots < M(n,k)$. Prove that:

(a) $f(n) < 3\sqrt{n}$ for all positive integers n; and

(b) If N is a positive integer, then $f(n) > N$ for all but finitely many positive integers n.

Solution. **(a)** Clearly, $f(1) = 1$. Notice that

$$M(n, k+1) = \operatorname{lcm}(M(n, k), n-k), \quad 1 \le k < n. \tag{$*$}$$

Thus, $M(n, k) \le M(n, k+1)$, and equality holds if and only if $n-k$ divides $M(n, k)$. If $m > 1$, then $M(m^2, 2) = m^2(m^2-1) = (m^2-m)(m^2+m)$, so $M(m^2, 2)$ is divisible by m^2-m, and hence so is $M(m^2, m)$. By $(*)$, $M(m^2, m) = M(m^2, m+1)$, so $f(m^2) \le m$ for all positive integers m. With reference again to $(*)$, if $M(n, k) = M(n, k+1)$, then $M(n+\ell, k+\ell) = M(n+\ell, k+\ell+1)$, so $f(n+\ell) \le f(n)+\ell$ for all positive integers ℓ and n. Finally, if $m^2 \le n < (m+1)^2$, then $n = m^2+\ell$ for some non-negative integer $\ell \le 2m$, so $f(n) = f(m^2+\ell) \le f(m^2)+\ell \le m+\ell \le 3m \le 3\sqrt{n}$, where one of the last two inequalities must in fact be a strict inequality.

(b) We show that $f(n) > N$ for all integers $n > N! + N$. Refer again to $(*)$ to write

$$\begin{aligned} M(n, N+1) &= \operatorname{lcm}(M(n, N), n-N) = \frac{M(n, N) \cdot (n-N)}{\gcd(M(n, N), n-N)} \\ &\ge \frac{M(n, N) \cdot (n-N)}{\prod_{k=1}^{N} \gcd(n-k+1, n-N)} \ge \frac{M(n, N) \cdot (n-N)}{\prod_{k=1}^{N}(N-k+1)} = \frac{M(n, N) \cdot (n-N)}{N!}. \end{aligned}$$

Consequently, if $n > N! + N$, then $M(n, 1) < \cdots < M(n, N) < M(n, N+1)$, so $f(n) > N$.

Problem 4. Given two integers $h \ge 1$ and $p \ge 2$, determine the minimum number of pairs of opponents an hp-member parliament may have, if in every partition of the parliament into h houses of p member each some house contains at least one pair of opponents.

Solution. The required minimum is $(h-1) \cdot \min(p, h/2+1) + 1$. Looking upon the parliament as a graph on hp vertices, two vertices being joined by an edge if the corresponding members are opponents, the above minimum is achieved by at least one of the following two graphs: The graph obtained by adjoining $h(p-1)-1$ isolated vertices to the complete graph on $h+1$ vertices, in which case the number of edges is $h(h+1)/2 = (h-1)(h/2+1)+1$; and the graph obtained by adjoining $p-2$ isolated vertices to a star with $p(h-1)+1$ rays.

Letting $N(h, p) = (h-1) \cdot \min(p, h/2+1)$, we now proceed to prove by induction on h that if the number of edges of a graph on hp vertices does not exceed $N(h, p)$, then the graph is h-partite on p-element classes. The base case $h = 1$ is clear.

Next, let $h \ge 2$ and let $G = (V, E)$ be a graph on hp vertices which has at most $N(h, p)$ edges. If necessary, add some extra edges to obtain $|E| = N(h, p)$.

Begin by forming a p-element house V_0 of independent vertices $v_1, \ldots, v_p$ by the following p-step greedy algorithm: Start with the empty set, and at step j choose a vertex v_j of maximal degree from the set of vertices joined by an edge to no v_i, $i < j$, and different from any of these; this set is nonempty, for if each of the remaining $hp-j+1$ vertices were joined by an edge to some v_i, $i < j$, then $|E| \ge hp-j+1 \ge hp-p+1 > N(h, p)$ — a contradiction. Notice that $\deg v_1 \ge \deg v_2 \ge \cdots \ge \deg v_p$.

Let $d = \sum_{v \in V_0} \deg v$, so the subgraph G' induced by the $p(h-1)$ vertices in $V \setminus V_0$ has exactly $N(h, p) - d$ edges. If $d \ge \Delta N = N(h, p) - N(h-1, p)$, then G' is $(h-1)$-partite on p-element classes by the induction hypothesis, and the conclusion follows.

Henceforth, assume

$$d < \Delta N = \begin{cases} p, & \text{if } h \ge 2p-1, \\ h, & \text{if } h \le 2p-2, \end{cases} \tag{$*$}$$

and notice that $\Delta N \le h$ in either case, so $d \le h-1$. Let V' be the set of all vertices in $V \setminus V_0$ joined by an edge to some vertex in V_0, and notice that $|V'| \le d \le h-1$, and $\deg v \le \deg v_p$ for all vertices v outside $V_0 \cup V'$.

If $\deg v_p = 0$, then the vertices outside $V_0 \cup V'$ are all isolated. Since $|V'| \le d \le h-1$, each vertex of V' may be included in a different p-element house (other than V_0) along with $p-1$ vertices outside $V_0 \cup V'$ each, to obtain $|V'|$ more p-element houses. The remaining vertices, if any, are then arbitrarily split into p-element houses.

Finally, we rule out the case $\deg v_p \ge 1$. Suppose, if possible, that $\deg v_p \ge 1$. Then $\deg v_i \ge 1$, $i = 1, \dots, p$, and $d \ge p$, so $(*)$ yields $\Delta N = h$, $h \le 2p-2$, and $N(h,p) = (h-1)(h+2)/2$. Hence

$$p \le d \le h-1 \le 2p-3.$$

The inequality $d \le 2p-3$ forces $\deg v_p = 1$, so $\deg v \le 1$ for all vertices v outside $V_0 \cup V'$, and

$$\sum_{v \in V \setminus (V_0 \cup V')} \deg v \le hp - |V_0| = p(h-1).$$

Further on, split $V' = V_1 \sqcup \cdots \sqcup V_p$, where V_j is the set of all vertices joined by an edge to v_j, but to no v_i, $i < j$. Notice that $|V_i| \le \deg v_i$, and $\deg v \le \deg v_i$ for all vertices v in V_i. Consequently,

$$\sum_{v \in V_0 \cup V'} \deg v = \sum_{i=1}^{p} \left(\deg v_i + \sum_{v \in V_i} \deg v \right) \le \sum_{i=1}^{p} \deg v_i \, (\deg v_i + 1) = \sum_{i=1}^{p} (\deg v_i - 1)^2 + 3d - p.$$

Since $\deg v_i \ge 1$, $i = 1, \dots, p$,

$$\sum_{i=1}^{p} (\deg v_i - 1)^2 \le \left(\sum_{i=1}^{p} (\deg v_i - 1) \right)^2 = (d-p)^2,$$

so (recalling that $d \le h-1$)

$$\begin{aligned} \sum_{v \in V_0 \cup V'} \deg v &\le (d-p)^2 + 3d - p \le (h-p-1)^2 + 3(h-1) - p \\ &= (h-1)(h+2) + p(p-2h+1) = 2N(h,p) + p(p-2h+1). \end{aligned}$$

Hence, by the preceding,

$$\begin{aligned} 2N(h,p) = 2|E| = \sum_{v \in V} \deg v &= \sum_{v \in V_0 \cup V'} \deg v + \sum_{v \in V \setminus (V_0 \cup V')} \deg v \\ &\le 2N(h,p) + p(p-2h+1) + p(h-1) = 2N(h,p) + p(p-h) < 2N(h,p), \end{aligned}$$

which is a contradiction. This ends the proof.