Test 2 — Solutions

Problem 1. Given an integer a and a positive integer n, show that the sum $\sum_{k=1}^{n} a^{(k,n)}$ is divisible by n, where (x, y) denotes the greatest common divisor of the integers x and y.

Solution. Write $\sum_{k=1}^{n} a^{(k,n)} = \sum_{d|n} \phi(n/d)a^d$, where ϕ is Euler's totient function ($\phi(m)$ is the number of positive integers less than m and prime to m), and notice that, if n and n' are coprime positive integers, then

$$\sum_{k=1}^{nn'} a^{(k,nn')} = \sum_{d|n} \phi(n/d) \sum_{d'|n'} \phi(n'/d')(a^d)^{d'}.$$

Consequently, it is sufficient to prove the assertion for $n = p^m$, where p is a prime and m is a non-negative integer. In this case,

$$\begin{aligned}\sum_{k=1}^{p^m} a^{(k,p^m)} &= \sum_{k=0}^{m} \phi(p^{m-k})a^{p^k} = \sum_{k=0}^{m-1} (p^{m-k} - p^{m-k-1})a^{p^k} + a^{p^m} \\ &= p^m a + \sum_{k=1}^{m} p^{m-k}(a^{p^k} - a^{p^{k-1}}) \equiv 0 \pmod{p^m},\end{aligned}$$

since $a^{p^k} \equiv a^{p^{k-1}} \pmod{p^k}$, $k = 1, \ldots, m$, by Fermat's theorem.

Problem 2. Let ABC be a triangle. Let A' be the centre of the circle through the midpoint of the side BC and the orthogonal projections of B and C on the lines of support of the internal bisectrices of the angles ACB and ABC, respectively; the points B' and C' are defined similarly. Prove that the nine-point circle of the triangle ABC and the circumcircle of $A'B'C'$ are concentric.

Solution 1. All the angles in the solution are directed modulo π. The following notation is used throughout the proof:

2α, 2β, 2γ	measures of the angles BAC, CBA, ACB, respectively;
$2a$, $2b$, $2c$	lengths of the sides BC, CA, AB, respectively;
I	incentre of the triangle ABC;
M_A, M_B, M_C	midpoints of the sides BC, CA, AB, respectively;
X_Y	orthogonal projection of X on the line YI, for all $X, Y \in \{A, B, C\}$; and
ω_A, ω_B, ω_C	circumcircles of the triangles $M_A B_C C_B$, $M_B C_A A_C$, $M_C A_B B_A$, respectively, centred at A', B', C', respectively.

Since the angle AA_BB is right, the segments $M_CA = M_CB = M_CA_B = c$ (see Fig. 1), so $\angle M_CA_BB = \angle A_BBM_C = \angle CBA_B = \beta$; this means that the lines M_CA_B and BC are parallel, so A_B lies on the line M_BM_C. Similarly, (M_A, M_B, C_A, C_B), (M_B, M_C, A_B, A_C), (M_C, M_A, B_C, B_A) are quartets of collinear points.

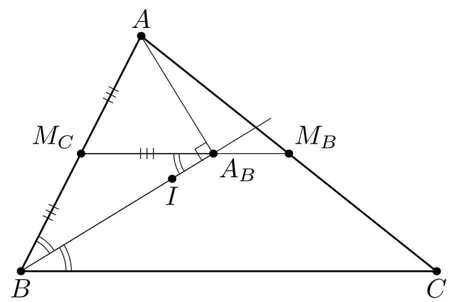

Fig. 1

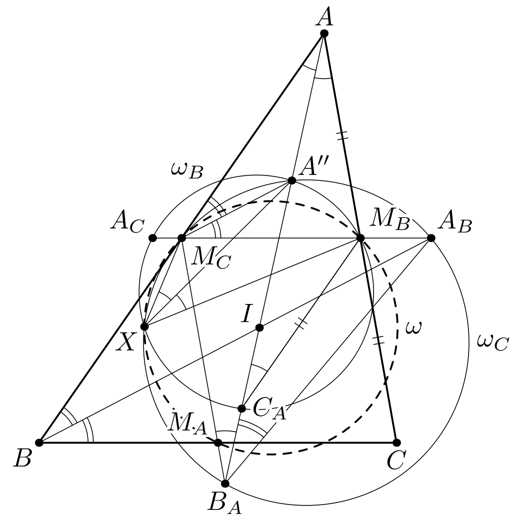

Fig. 2

Let A'' be the incentre of the triangle AM_CM_B (in other words, A'' is the midpoint of AI). We show that A'' lies on both ω_B and ω_C. For that, notice first that the points A, B, A_B, and B_A lie on the circle on diameter AB; hence $\angle A_BB_AA = \angle A_BBA = \beta$. Next, the points A_B, A_C, M_B, and M_C lie on a line parallel to BC, so $\angle A_BM_CA'' = \angle M_BM_CA'' = \beta = \angle A_BB_AA''$. This means that A'' lies on ω_C. Similarly, A'' lies on ω_B.

Let X be the second point of intersection of ω_B and ω_C. By the preceding, $\angle M_BXA'' = \angle M_BC_AA'' = \alpha$ and similarly $\angle A''XM_C = \alpha$. This yields $\angle M_BXM_C = \angle M_BXA'' + \angle A''XM_C = 2\alpha = \angle M_BM_AM_C$, which shows that X lies also on the circumcircle ω of the triangle $M_AM_BM_C$, which is the nine-poin t circle of the triangle ABC. Denote the center of ω by O'.

Now the lines $A''X$, M_BX, and M_CX are the radical axes of the circles ω_B, ω_C, and ω. Since XA'' forms equal angles with M_BX and M_CX, the triangle $O'B'C'$ formed by the centres of these circles has equal angles at B' and C'; therefore, $O'B' = O'C'$. A similar argument shows that $O'B' = O'A'$, and O' is consequently the circumcentre of the triangle $A'B'C'$.

Remark. The point X in the solution is the Feuerbach point of the triangle ABC — the point at which the incircle is internally tangent to the nine-point circle.

Solution 2. With reference to the notation in Solution 1, let O and O' be the circumcentre and the centre of the nine-point circle of the triangle ABC, respectively. As in the previous solution, usage is made of the fact that (M_A, M_B, C_A, C_B), (M_B, M_C, A_B, A_C), and (M_C, M_A, B_C, B_A) are quartets of collinear points, and $M_AB_C = M_AC_B = a$, $M_BA_C = M_BC_A = b$, and $M_CA_B = M_CB_B = c$.

We claim that $A'O' = IO/2$; similarly, $B'O' = IO/2 = C'O'$, whence the required result. To prove the claim, notice that each vector $\vec{v}$ is uniquely determined by its projections on the lines AB and AC. The signed lengths of these projections will be denoted $\mathrm{pr}_c\,\vec{v}$ and $\mathrm{pr}_b\,\vec{v}$, respectively, the rays AB and AC emanating from A being considered positive.

The points O' and A' are the circumcentres of the triangles $M_AM_BM_C$ and $M_AB_CC_B$, respectively. Project onto M_AM_C, to get $\mathrm{pr}_b\,\overrightarrow{A'O'} = \mathrm{pr}_b\big(\overrightarrow{M_AO'} - \overrightarrow{M_AA'}\big) = (-b/2 + a/2) = (a-b)/2$. Similarly, $\mathrm{pr}_c\,\overrightarrow{A'O'} = (a-c)/2$. On the other hand, $\mathrm{pr}_b\,\overrightarrow{IO} = \mathrm{pr}_b\big(\overrightarrow{AO} - \overrightarrow{AI}\big) = b-(b+c-a) = a-c$; similarly, $\mathrm{pr}_c\,\overrightarrow{IO} = a-b$. This means that the vector $\overrightarrow{A'O'}$ reflected in the bisectrix AI of the angle BAC is equal to the vector $\overrightarrow{IO}/2$, hence the claim.

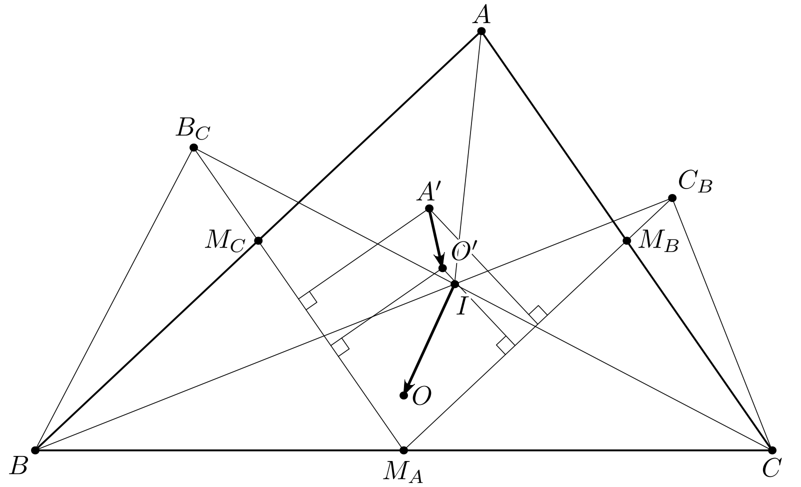

Fig. 3

Problem 3. Given a positive real number t, determine the sets A of real numbers containing t, for which there exists a set B of real numbers depending on A, $|B| \geq 4$, such that the elements of the set $AB = \{ab\colon a \in A, b \in B\}$ form a finite arithmetic progression.

Solution. The required sets are $\{t\}$, $\{-t,t\}$, $\{0,t\}$ and $\{-t,0,t\}$. It is readily checked that the elements of the Minkowski product of each of these sets and the set $\{-1,0,1,2\}$ form a finite arithmetic progression.

Now, let A and B be sets of real numbers satisfying the conditions in the statement, and let $|A| \geq 2$ (the case $|A| = 1$ is trivial). Clearly, A and B are both finite.

Let $d > 0$ be the difference of the arithmetic progression AB, consider two distinct elements of A, say x and x', and two distinct elements of B, say y and y', and notice that the elements of A, respectively B, are integral multiples of $d/(y-y')$, respectively $d/(x-x')$. Scaling A and B accordingly, we may (and will) assume that A and B are both sets of integers. Dividing, if necessary, the elements of A, respectively B, by their greatest common divisor, we may (and will) further assume that the elements of A, respectively B, are jointly coprime: $\gcd A = 1$ and $\gcd B = 1$. Further, recall that A and B are both finite and let a^*, respectively b^*, be an element of A, respectively B, of maximal absolute value. If necessary, multiply by -1 to assume $a^* > 0$ and $b^* > 0$. Under these simplifying assumptions, we will show that A is one of the sets $\{-1,1\}$, $\{0,1\}$, $\{-1,0,1\}$, whence the conclusion.

Since $\gcd B = 1$ and d divides $(x-x')y$ for all x and x' in A and all y in B, it follows that d divides the difference of any two members of A. Similarly, d divides the difference of any two members of B, and since $|B| \geq 4$, it follows that $b^* > d$.

Consider now elements a in A and b in B such that $ab = a^*b^* - d$, and notice that $ab = a^*b^* - d \geq b^* - d > 0$. Moreover, $|a| = a^*$, for otherwise $a^*b^* - d = ab = |a||b| \leq (a^*-1)b^* = a^*b^* - b^* < a^*b^* - d$ which is a contradiction.

This means that $d = a^*b^* - |a||b| = a^*(b^* - |b|) \geq a^*$. Now, since $a^* \leq d$, and the elements of A are congruent modulo d, the only possible options for A are either subsets of $\{-d,0,d\}$, or $\{-d/2, d/2\}$ if d is even, or finally sets of the form $\{a^*, a^*-d\}$, where $d > a^* > |a^*-d|$. The first two cases are covered by the answer.

To rule out the last option, notice that $a = a^*$ (since $|a| = a^* > |a^*-d|$), and therefore $d = a^*(b^*-|b|)$. This means that a^* divides d, so $a^* \leq d/2$ and $|a^*-d| \geq a^*$, in contradiction with $a^* > |a^*-d|$.

Problem 4. Consider the integral lattice $\mathbb{Z}^n$, $n \geq 2$, in the Euclidean n-space. Define a *line* in $\mathbb{Z}^n$ to be a set of the form $a_1 \times \cdots \times a_{k-1} \times \mathbb{Z} \times a_{k+1} \times \cdots \times a_n$, where k is an integer in the range 1, 2, ..., n, and the a_i are arbitrary integers. A subset A of $\mathbb{Z}^n$ is called *admissible* if it is non-empty, finite, and every line in $\mathbb{Z}^n$ which intersects A contains at least two points of A. A subset N of $\mathbb{Z}^n$ is called *null* if it is non-empty, and every line in $\mathbb{Z}^n$ intersects N in an even number of points (possibly zero).

(a) Prove that every admissible set in $\mathbb{Z}^2$ contains a null set.

(b) Exhibit an admissible set in $\mathbb{Z}^3$ no subset of which is a null set.

Solution. **(a)** Let A be an admissible set in $\mathbb{Z}^2$, choose a point $\mathbf{a}_0$ of A, and, for each positive integer k, choose a point $\mathbf{a}_k$ of A different from $\mathbf{a}_{k-1}$, having the same first coordinate as the latter if k is odd, and the same second coordinate if k is even. Eventually we must choose an $\mathbf{a}_n = \mathbf{a}_m$, $m < n$. Assume $\mathbf{a}_n$ is the first point to duplicate a preceding point. If m and n have like parities, then $\mathbf{a}_m$, $\mathbf{a}_{m+1}$, ..., $\mathbf{a}_{n-1}$ form a null set, and if they have opposite parities, then $\mathbf{a}_{m+1}$, ..., $\mathbf{a}_{n-1}$ do.

(b) We exhibit a minimal admissible set A in $\mathbb{Z}^3$ which is not itself null. Here and hereafter, minimality refers to the fact that no proper subset is admissible. Since every null finite set is admissible, the conclusion follows. The set A is a set of lattice points in the parallelepiped $[0,3] \times [0,3] \times [0,4]$. We describe it by successive horizontal cross-sections:

$$A = A_0 \times 0 \cup A_1 \times 1 \cup A_2 \times 2 \cup A_3 \times 3 \cup A_4 \times 4,$$

where $A_0 = \{0,3\} \times \{0,3\}$, $A_1 = \{0,1\} \times \{2,3\} \cup \{1,2\} \times \{0,1\}$, $A_2 = \{0,1\} \times \{1,2\} \cup \{2,3\} \times \{2,3\}$, $A_3 = \{1,2\} \times \{2,3\} \cup \{2,3\} \times \{0,1\}$, and $A_4 = \{0,1\} \times \{0,1\} \cup \{2,3\} \times \{1,2\}$. Notice that, for $k = 1,2,3$, the configuration A_{k+1} is obtained from A_k by a clockwise rotation through $\pi/2$ about the centre of the square $[0,3] \times [0,3]$.

The set A is admissible, since each horizontal cross-section $A_k \times k$ is admissible in $\mathbb{Z}^2 \times k$, and the perpendicular in $\mathbb{Z}^3$ to any horizontal cross-section through any one of its points meets at least one other horizontal cross-section.

To prove minimality, we exhibit a connected geometric lattice graph G on A such that the line of support of each edge of G is a line in $\mathbb{Z}^3$ stabbing A at exactly two points, namely, the end points of that edge. The existence of such a graph implies minimality, since removal of any one point in A entails removal of all its neighbours in G, and eventually removal of all of A.

Begin by noticing that each of the verticals $i \times j \times \mathbb{Z}$ through a point of A, where either i or j is in $\{0,3\}$, stabs exactly two horizontal cross-sections of A. Join the corresponding points of A by the vertical segment they determine.

Next, consider the generic planar lattice paths $\alpha_1 = (1 \times 0)(2 \times 0)(2 \times 1)(1 \times 1)$ and $\alpha_1' = (1 \times 2)(0 \times 2)(0 \times 3)(1 \times 3)$, and, for $k = 1,2,3$, let α_{k+1} and α_{k+1}' be obtained from α_k and α_k', respectively, by a clockwise rotation through $\pi/2$ about the centre of the square $[0,3] \times [0,3]$. The edges of the lattice paths $\alpha_k \times k$ and $\alpha_k' \times k$, joining points in $A_k \times k$, $k = 1,2,3,4$, along with the vertical edges in the previous paragraph form the desired connected geometric lattice graph G on A.

Finally, the set A is not null, for the vertical $1 \times 1 \times \mathbb{Z}$ stabs exactly three horizontal cross-sections of A, namely, $A_1 \times 1$, $A_2 \times 2$ and $A_4 \times 4$; in fact, each of the lines $i \times j \times \mathbb{Z}$, $i,j \in \{1,2\}$, stabs exactly three horizontal cross-sections of A.

Remark. Examples in any higher dimension can be constructed from the example in part **(b)**. The configuration in this example consists of 36 points, but there are 24-point admissible configurations in $\mathbb{Z}^3$ containing no null proper subconfiguration.