First Selection Test — Solutions

Problem 1. Given an integer $n \geq 2$, let a_n, b_n, c_n be integer numbers such that $(\sqrt[3]{2}-1)^n = a_n + b_n\sqrt[3]{2} + c_n\sqrt[3]{4}$. Show that $c_n \equiv 1 \pmod 3$ if and only if $n \equiv 2 \pmod 3$.

Solution 1. The binomial expansion of $(\sqrt[3]{2}-1)^n$ yields

$$c_n = \sum_{k\equiv 2 \,(\mathrm{mod}\ 3)} (-1)^{n-k}\cdot 2^{(k-2)/3}\binom{n}{k} \equiv (-1)^n \sum_{k\equiv 2\,(\mathrm{mod}\ 3)} \binom{n}{k} \pmod 3.$$

Since

$$\sum_{k\equiv 2\,(\mathrm{mod}\ 3)} \binom{n}{k} = \frac{1}{3}\big((1+1)^n + \epsilon(1+\epsilon)^n + \epsilon^2(1+\epsilon^2)^n\big) = \frac{1}{3}\left(2^n + 2\cos(n+2)\frac{\pi}{3}\right),$$

where $1+\epsilon+\epsilon^2 = 0$, the condition $n \equiv 2 \pmod 3$ may be restated as

$$3c_n = (-1)^n\left(2^n + 2\cos(n+2)\frac{\pi}{3}\right) \equiv 3 \pmod 9.$$

Consideration of n modulo 6 yields $3c_n \equiv 3 \pmod 9$ if $n \equiv 2$ or $5 \pmod 6$, and $3c_n \equiv 0 \pmod 9$ otherwise. The conclusion follows.

Solution 2. Consider the polynomial $f = (X-1)^n - c_nX^2 - b_nX - a_n \in \mathbb{Z}[X]$. Clearly, $f(\sqrt[3]{2}) = 0$. Since X^3-2 is irreducible in $\mathbb{Z}[X]$, it follows that X^3-2 divides f in $\mathbb{Z}[X]$, so $g_n = a_n + b_nX + c_nX^2$ is the remainder of the division of $(X-1)^n$ by X^3-2 in $\mathbb{Z}[X]$. Write $n = 3q+r$, where q is a non-negative integer and $r \in \{0,1,2\}$, to get $(X-1)^n = (X^3-1)^q(X-1)^r = (X^3-2)\cdot g + (X-1)^r$ in $\mathbb{Z}_3[X]$, and deduce thereby that $g_n = (X-1)^r$ in $\mathbb{Z}_3[X]$. Consequently, $c_n \equiv 0 \pmod 3$ if $r \in \{0,1\}$, and $c_n \equiv 1 \pmod 3$ if $r = 2$. The conclusion follows.

Problem 2. Circles Ω and ω are tangent at a point P (ω lies inside Ω). A chord AB of Ω is tangent to ω at C; the line PC meets Ω again at Q. Chords QR and QS of Ω are tangent to ω. Let I, X, and Y be the incentres of the triangles APB, ARB, and ASB, respectively. Prove that $\angle PXI + \angle PYI = 90^\circ$.

Solution. Notice that a homothety centred at P mapping ω to Ω maps C to Q, and maps the line AB to the tangent to Ω at Q. Thus this tangent is parallel to AB, and hence Q is the midpoint of arc AB (not containing P). So the points I, X, and Y lie on the segments PQ, RQ, and SQ, respectively.

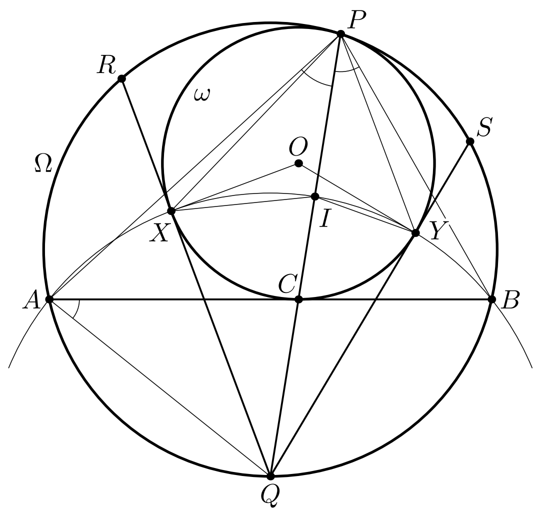
Recall that for any triangle KLM with the circumcircle Γ and incentre J, the points K, L, and J are equidistant from the midpoint of arc KL of Γ not containing M. Applying this to triangles APB, ARB, and ASB we obtain that $QA = QB = QX = QY = QI$.

Since Q is the midpoint of arc AB, we get that $\angle QPA = \angle QPB = \angle QAB$. Thus the triangles QAC and QPA are similar, and $QC \cdot QP = QA^2 = QX^2$. Since QX is tangent to ω, it follows that X is their point of tangency; analogously, Y is the point of tangency of QS with ω.

Finally, from isosceles triangles QXI and QYI we get $\angle QXI = \angle QIX = 90^\circ - \angle IQX/2$ and $\angle QYI = \angle QIY = 90^\circ - \angle IQY/2$. Denoting by O the centre of ω, we obtain $\angle QIX + \angle QIY = 180^\circ - \angle XQY/2 = 180^\circ - (180^\circ - \angle XOY)/2 = 90^\circ + \angle XPY$. Thus,

$$\angle PXI + \angle PYI = \angle XIY - \angle XPY = (90^\circ + \angle XPY) - \angle XPY = 90^\circ$$

as required.



Remark. The relation $QC \cdot QP = QA^2$ also follows from the inversion of pole Q interchanging the line AB and the circle Ω,

Problem 3. Determine all injective functions f of the set of positive integers into itself satisfying the following condition: If S is a finite set of positive integers such that $\sum_{s\in S} 1/s$ is an integer, then $\sum_{s\in S} 1/f(s)$ is also an integer.

Solution. We shall prove that the identity is the unique function satisfying the conditions in the statement. Clearly, $f(1) = 1$, so $f(n) \geq 2$ if $n \geq 2$, by injectivity. We will use the following well-known result.

Egyptian fractions theorem. *For every positive rational q and positive integer N, there exists a set $\{n_1, \dots, n_k\}$ of positive integers such that $n_i > N$ for every $i = 1, 2, \dots, k$, and*

$$q = \sum_{i=1}^{k} \frac{1}{n_i}.$$

Now, consider an integer $n \geq 2$ and use the Egyptian fractions theorem to write $1 - 1/n = \sum_{s\in S} 1/s$, where S is a set of integers greater than $n(n+1)$, and get thereby

$$1 = \frac{1}{n} + \sum_{s\in S} \frac{1}{s} = \frac{1}{n+1} + \frac{1}{n(n+1)} + \sum_{s\in S} \frac{1}{s}.$$

Consequently,

$$\frac{1}{f(n)} + \sum_{s\in S} \frac{1}{f(s)} \quad \text{and} \quad \frac{1}{f(n+1)} + \frac{1}{f(n(n+1))} + \sum_{s\in S} \frac{1}{f(s)}$$

both are positive integers, so

$$\frac{1}{f(n+1)} + \frac{1}{f(n(n+1))} - \frac{1}{f(n)}$$

is an integer. Since

$$-\frac{1}{2} \leq -\frac{1}{f(n)} < \frac{1}{f(n+1)} + \frac{1}{f(n(n+1))} - \frac{1}{f(n)} < \frac{1}{f(n+1)} + \frac{1}{f(n(n+1))} \leq \frac{1}{2} + \frac{1}{2} = 1,$$

it follows that

$$\frac{1}{f(n)} = \frac{1}{f(n+1)} + \frac{1}{f(n(n+1))}.$$

In particular, f is strictly increasing, so $f(n) \geq n$.

Finally, proceed by induction on $n \geq 2$ to prove that $f(n) = n$. To show that $f(2) = 2$, simply notice that $2/f(2) = 1/f(2) + 1/f(3) + 1/f(6)$ is a positive integer not exceeding 1. To complete the proof, let $f(n) = n$ for some $n \geq 2$ and write

$$\frac{1}{n} = \frac{1}{f(n)} = \frac{1}{f(n+1)} + \frac{1}{f(n(n+1))} \leq \frac{1}{n+1} + \frac{1}{n(n+1)} = \frac{1}{n}$$

to conclude that $f(n+1) = n+1$.

Remark. We do not need the full version of the Egyptian fractions theorem. In fact, all we need in the solution above is the lemma below.

Lemma. *For every integer $n \geq 2$, there exists a set S_n with $\sum_{s \in S_n} 1/s = 1$ such that $n \in S_n$, but $n+1$, $n(n+1) \notin S_n$.*

Here we present a direct proof of this Lemma.

For each $n \in \{2,3,4,5\}$ one of the the sets $\{2,3,6\}$, $\{2,4,6,12\}$, and $\{2,5,7,12,20,42\}$ fits. Now assume that $n \geq 6$ and perform the following steps, starting with the set $S = \{2,3,6\}$.

Step 1. Let $k = \max S$; if $k(k+1) \leq n$ then replace k with $\{k+1, k(k+1)\}$ and repeat this step. At the end, we arrive to a set S with $k = \max S$ such that $k \leq n < k(k+1)$. If $k = n$ then we are done; otherwise we proceed to Step 2.

Step 2. Replace k by $\{n\} \cup \{k(k+1), (k+1)(k+2), \dots, n(n-1)\}$ obtaining the set S'. Notice that $n+1 \leq k(k+1)$, $n(n+1) > \max S'$; thus, if $n+1 < k(k+1)$ then we are done. Otherwise, replace $k(k+1)$ by $\big\{k(k+1)+1, k(k+1)\big(k(k+1)+1\big)\big\}$ obtaining the desired set.

Problem 4. Let n be an integer greater than 1. The set S of all diagonals of a $(4n-1)$-gon is partitioned into k sets, $S_1, \dots, S_k$, so that, for every pair of distinct indices i and j, some diagonal in S_i crosses some diagonal in S_j; that is, the two diagonals share an interior point. Determine the largest possible value of k in terms of n.

Solution. The required maximum is $k = (n-1)(4n-1)$. Notice that $|S| = 2(n-1)(4n-1)$. Assume first that $k > (n-1)(4n-1)$. Then there exists a set S_i with $|S_i| = 1$. Let $S_i = \{d\}$, and assume that there are v vertices on one side of d; then the number of vertices on the other side is $4n-3-v$, and the total number of diagonals having a common interior point with d is $v(4n-3-v) \leq (2n-2)(2n-1)$. Since each S_j with $j \neq i$ contains such a diagonal, we obtain $k \leq (2n-2)(2n-1)+1 = (n-1)(4n-1)-(n-2) \leq (n-1)(4n-1)$ — a contradiction.

Now it remains to construct a partition with $k = (n-1)(4n-1)$. Let us enumerate the vertices $A_1, \dots, A_{4n-1}$ consecutively; we assume that the enumeration is cyclic, thus $A_{i+(4n-1)} = A_i$. Now, for every $t = 2, 3, \dots, n$ and every $i = 1, 2, \dots, 4n-1$, let us define the set $S_{t,i} = \{A_iA_{i+t}, A_{i+t-1}A_{i+2n}\}$.

It is easy to see that the $(n-1)(4n-1)$ sets $S_{t,i}$ form a partition of S; we claim that this partition satisfies the problem condition. Consider two sets $S_{t,i}$ and $S_{t',i'}$; by the cyclic symmetry we may assume that $i = 0$. One can easily observe that a diagonal d has no common interior points with the diagonals from $S_{t,0}$ if and only if its endpoints are both contained in one of the sets

$$\{A_0, A_1, \dots, A_{t-1}\}, \quad \{A_t, A_{t+1}, \dots, A_{2n}\}, \quad \{A_{2n}, A_{2n+1}, \dots, A_{4n-1}\}$$

(recall that $A_{4n-1} = A_0$); in such a case we will say that d *belongs* to the corresponding set. Now, the diagonals from $S_{t',i'}$ cannot belong to one set since this set encompasses at most $2n$ consecutive vertices. On the other hand, since these two diagonals have a common interior point they cannot belong to different sets. The claim is proved.

Remark. The solution for a $(4n-3)$-gon is almost the same; one only needs to take some care of the diagonals of the form A_iA_{i+n}.