Fifth Selection Test — Solutions

Problem 1. Let n be a positive integer and let x_1, …, x_n be positive real numbers. Show that

$$\begin{aligned}\min\left(x_1, \frac{1}{x_1}+x_2, \dots, \frac{1}{x_{n-1}}+x_n, \frac{1}{x_n}\right) &\le 2\cos\frac{\pi}{n+2}\\ &\le \max\left(x_1, \frac{1}{x_1}+x_2, \dots, \frac{1}{x_{n-1}}+x_n, \frac{1}{x_n}\right).\end{aligned}$$

Solution. We shall actually prove that

$$\begin{aligned}&\max_{x_1>0,\dots,x_n>0} \min\left(x_1, \frac{1}{x_1}+x_2, \dots, \frac{1}{x_{n-1}}+x_n, \frac{1}{x_n}\right) =\\ &\min_{x_1>0,\dots,x_n>0} \max\left(x_1, \frac{1}{x_1}+x_2, \dots, \frac{1}{x_{n-1}}+x_n, \frac{1}{x_n}\right) = 2\cos\frac{\pi}{n+2}.\end{aligned}$$

To this end, let U denote the set of all n-tuples of positive real numbers, and, for $\mathbf{x} = (x_1, \dots, x_n)$ in U, let

$$m(\mathbf{x}) = \min\left(x_1, \frac{1}{x_1}+x_2, \dots, \frac{1}{x_{n-1}}+x_n, \frac{1}{x_n}\right)$$

and

$$M(\mathbf{x}) = \max\left(x_1, \frac{1}{x_1}+x_2, \dots, \frac{1}{x_{n-1}}+x_n, \frac{1}{x_n}\right).$$

The first step consists in assuming that $m(\mathbf{a}) = M(\mathbf{a})$ for some $\mathbf{a} = (a_1, \dots, a_n)$ in U and showing that $m(\mathbf{x}) \le m(\mathbf{a}) = M(\mathbf{a}) \le M(\mathbf{x})$ for all $\mathbf{x}$ in U. Clearly, the condition $m(\mathbf{a}) = M(\mathbf{a})$ is equivalent to

$$a_1 = \frac{1}{a_1} + a_2 = \dots = \frac{1}{a_{n-1}} + a_n = \frac{1}{a_n}. \tag{1}$$

Suppose, if possible, that $m(\mathbf{x}) > m(\mathbf{a})$ for some $\mathbf{x} = (x_1, \dots, x_n)$ in U. Then $x_1 \ge m(\mathbf{x}) > m(\mathbf{a}) = a_1$; $1/x_k + x_{k+1} \ge m(\mathbf{x}) > m(\mathbf{a}) = 1/a_k + a_{k+1}$, $k = 1, \dots, n-1$; and $1/x_n \ge m(\mathbf{x}) > m(\mathbf{a}) = 1/a_n$. The first n inequalities imply recursively that $x_k > a_k$, $k = 1, \dots, n$; in particular, $x_n > a_n$, in contradiction with $1/x_n > 1/a_n$. Consequently, $m(\mathbf{x}) \le m(\mathbf{a})$ for all $\mathbf{x}$ in U. Similarly, $M(\mathbf{x}) \ge M(\mathbf{a})$ for all $\mathbf{x}$ in U.

To show the existence of an $\mathbf{a}$ in U satisfying (1), let a denote the common value in (1) and notice that $a_k = b_k/b_{k-1}$, $k = 1, \dots, n$, where the b_k are defined by

$$b_0 = 1, \quad b_1 = a, \quad \text{and} \quad b_k = ab_{k-1} - b_{k-2}, \quad k \ge 2. \tag{2}$$

Since $1/a_n = a$, it follows that $b_{n-1} = ab_n$ which is equivalent to $b_{n+1} = 0$. Notice further that $a < 2$. Otherwise, $a_1 = a \ge 2$ and $a_k = a - 1/a_{k-1}$, $k = 2, \dots, n$, would recursively imply that $a_k \ge 1 + 1/k$, $k = 1, \dots, n$; in particular, $a_n \ge 1 + 1/n$, in contradiction with $1/a_n = a \ge 2$. We may therefore write $a = 2\cos\alpha$, for some α in the open interval $(0, \pi/2)$, to deduce that the unique solution of (2) is $b_k = \sin(k+1)\alpha/\sin\alpha$. Since b_1, …, b_n are all positive, the condition $b_{n+1} = 0$ yields $\alpha = \pi/(n+2)$ and the conclusion follows.

Problem 2. Let K be a convex quadrangle and let ℓ be a line through the point of intersection of the diagonals of K. Show that the length of the segment of intersection $\ell \cap K$ does not exceed the length of (at least) one of the diagonals of K.

Solution. Let $\mathcal{L}$ be the pencil of lines through the point of intersection of the diagonals of K. We shall prove that that the length of each segment of intersection $\ell \cap K$, $\ell \in \mathcal{L}$, does not exceed the length of the longest diagonal of K.

To begin, notice that no intersection segment has a length greater than the diameter of K, so $\sup_{\ell\in\mathcal{L}} |\ell \cap K|$ is finite, where $|s|$ denotes the length of the line-segment s.

Suppose, if possible, that $\sup_{\ell\in\mathcal{L}} |\ell \cap K|$ is greater than the length of the longest diagonal of K. Let δ and δ' denote the diagonals of K and consider a line ℓ_0 in $\mathcal{L}$ such that

$$|\ell_0 \cap K| > \frac{\sup_{\ell\in\mathcal{L}} |\ell \cap K| + \max(|\delta|, |\delta'|)}{2} > \frac{|\delta| + |\delta'|}{2}.$$

Recall that the length of the internal bisectrix of an angle of a triangle is smaller than the arithmetic mean of the lengths of the sides forming that angle, to infer that ℓ_0 does not bisect the corresponding angle formed by the diagonals; say, the angle formed by ℓ_0 and δ is smaller than the angle formed by ℓ_0 and δ', both angles being, of course, those in the wedge containing ℓ_0.

Finally, reflect the line of support of δ in the line ℓ_0 to obtain a line ℓ_1 in $\mathcal{L}$ such that

$$\frac{|\ell_1 \cap K| + \max(|\delta|, |\delta'|)}{2} \geq \frac{|\ell_1 \cap K| + |\delta|}{2} > |\ell_0 \cap K| > \frac{\sup_{\ell\in\mathcal{L}} |\ell \cap K| + \max(|\delta|, |\delta'|)}{2},$$

and thereby reach a contradiction. (The inequality in the middle expresses the above mentioned fact about the length of an internal bisectrix in atriangle.) The conclusion follows.

Problem 3. Given a positive integer n, consider a triangular array with entries a_{ij} where i ranges from 1 to n and j ranges from 1 to $n-i+1$. The entries of the array are all either 0 or 1, and, for all $i > 1$ and any associated j, a_{ij} is 0 if $a_{i-1,j} = a_{i-1,j+1}$, and a_{ij} is 1 otherwise.

Let S denote the set of binary sequences of length n, and define a map $f\colon S \to S$ via $f\colon (a_{11}, a_{12}, \ldots, a_{1n}) \mapsto (a_{n1}, a_{n-1,2}, \ldots, a_{1n})$. Determine the number of fixed points of f.

Solution 1. The required number is $2^{\lfloor (n+1)/2 \rfloor}$. To prove this, we establish a bijection between the fixed points of f and the binary palindromes of length n. More precisely, we show that the assignment

$$(a_{11}, a_{12}, \ldots, a_{1n}) \mapsto (a_{11}, a_{21}, \ldots, a_{n1})$$

is bijective, on one hand, and that $(a_{11}, a_{12}, \ldots, a_{1n})$ is fixed by f if and only if $(a_{11}, a_{21}, \ldots, a_{n1})$ is a palindrome ($a_{i1} = a_{n-i+1,1}$, $i = 1, \ldots, n$), on the other.

To begin with, notice that the definition of the a_{ij} is equivalent to the Pascal-like relation in $\mathbb{Z}_2$:

$$a_{ij} + a_{i-1,j} + a_{i-1,j+1} = 0. \tag{1}$$

Henceforth, such a triangular array will be called a *Pascal binary* (or *dyadic*) *array*. Clearly, each binary string $\mathbf{a}$ in S yields a unique Pascal binary array $\hat{\mathbf{a}}$.

For more convenience, view a triangular array as a function on the standard lattice triangle

$$\Delta_n = \{(i, j)\colon i, j \geq 1 \text{ and } 2 \leq i + j \leq n + 1\},$$

situated in the first quadrant; thus, the first index runs horizontally and corresponds to columns, and the second index runs vertically and corresponds to rows.

Further, use the symbols $\to$, $\leftarrow$, $\uparrow$, $\downarrow$, $\searrow$ and $\nwarrow$ to denote the oriented sides of Δ_n; explicitly,

$$\begin{aligned}
\to &= \{(i,1)\colon i=1,\dots,n\}, & \leftarrow &= \{(n-i+1,1)\colon i=1.\dots,n\},\\
\uparrow &= \{(1,j)\colon j=1,\dots,n\}, & \downarrow &= \{(1,n-j+1)\colon j=1,\dots,n\},\\
\searrow &= \{(i,n-i+1)\colon i=1,\dots,n\}, & \nwarrow &= \{(n-j+1,j)\colon j=1,\dots,n\}.
\end{aligned}$$

With these notational conventions, a string $\mathbf{a}$ in S is systematically viewed as the $\hat{\mathbf{a}}_\uparrow$ of the unique Pascal dyadic array $\hat{\mathbf{a}}$ it generates, and $f\colon \mathbf{a}=\hat{\mathbf{a}}_\uparrow \mapsto \hat{\mathbf{a}}_\nwarrow$.

To establish a bijection between the fixed points of f and the binary palindromes of length n, consider the transformations ϱ and σ of Δ_n defined by

$$\varrho(i,j)=(n-i-j+2,i) \quad \text{and} \quad \sigma(i,j)=(n-i-j+2,j).$$

The former is a permutation of order 3 (ϱ^3 is the identity), and the latter is an involution (σ^2 is the identity). It is easily seen that

$$\varrho(\to)=\nwarrow, \quad \varrho(\uparrow)=\leftarrow, \quad \varrho(\nwarrow)=\downarrow \quad \text{and} \quad \sigma(\to)=\leftarrow, \quad \sigma(\uparrow)=\nwarrow, \quad \sigma(\nwarrow)=\uparrow,$$

so $\sigma\varrho(\to)=\uparrow$, $\sigma\varrho(\uparrow)=\to$, and $\sigma\varrho(\nwarrow)=\searrow$. It is also readily checked by (1) that, if $\hat{\mathbf{a}}=(a_{ij})$ is a Pascal binary array, then so are both

$$\varrho\hat{\mathbf{a}}=(a_{\varrho(i,j)}) \quad \text{and} \quad \sigma\hat{\mathbf{a}}=(a_{\sigma(i,j)}).$$

We are now in a position to prove the desired results.

Since $\sigma\varrho$ exchanges $\to$ and $\uparrow$, the assignment $\hat{\mathbf{a}}_\uparrow \mapsto \hat{\mathbf{a}}_\to$ is bijective.

Since σ exchanges $\uparrow$ and $\nwarrow$, and reverses orientation on the bottom row of Δ_n, if $\hat{\mathbf{a}}_\uparrow$ is fixed by f, then $\hat{\mathbf{a}}_\uparrow=f(\hat{\mathbf{a}}_\uparrow)=\hat{\mathbf{a}}_\nwarrow=(\sigma\hat{\mathbf{a}})_\uparrow$, so $\hat{\mathbf{a}}=\sigma\hat{\mathbf{a}}$ and consequently $\hat{\mathbf{a}}_\to=(\sigma\hat{\mathbf{a}})_\to=\hat{\mathbf{a}}_\leftarrow$; that is, the bottom row of $\hat{\mathbf{a}}$ is a palindrome.

Conversely, if the bottom row of $\hat{\mathbf{a}}$ is a palindrome, $\hat{\mathbf{a}}_\to=\hat{\mathbf{a}}_\leftarrow$, then $(\sigma\varrho\hat{\mathbf{a}})_\uparrow=(\varrho\hat{\mathbf{a}})_\uparrow$, so $\sigma\varrho\hat{\mathbf{a}}=\varrho\hat{\mathbf{a}}$ and consequently, $f(\hat{\mathbf{a}}_\uparrow)=\hat{\mathbf{a}}_\nwarrow=(\sigma\varrho\hat{\mathbf{a}})_\searrow=(\varrho\hat{\mathbf{a}})_\searrow=\hat{\mathbf{a}}_\uparrow$; that is, $\mathbf{a}=\hat{\mathbf{a}}_\uparrow$ is fixed by f. This ends the proof.

Remarks. **(1)** Since σ is an involution exchanging $\uparrow$ and $\nwarrow$, it follows that f is also an involution:

$$f(\hat{\mathbf{a}}_\uparrow)=\hat{\mathbf{a}}_\nwarrow=(\sigma\hat{\mathbf{a}})_\uparrow \quad \text{and} \quad f^2(\hat{\mathbf{a}}_\uparrow)=f((\sigma\hat{\mathbf{a}})_\uparrow)=(\sigma^2\hat{\mathbf{a}})_\uparrow=\hat{\mathbf{a}}_\uparrow.$$

(2) The problem is a skillful application of the action of the dihedral group

$$D_3=\langle \varrho,\sigma \mid \varrho^3=1, \sigma^2=1, \sigma\varrho=\varrho^2\sigma\rangle$$

on the set of Pascal binary arrays. This may also be seen by transforming Δ_n into an equilateral triangle via

$$\begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix},$$

or simply viewing Δ_n as one such, and recalling that the dihedral group D_3 is precisely the full planar symmetry group of the equilateral triangle. In this setting, ϱ is the counterclockwise rotation through $2\pi/3$ about the centre, and σ is the reflection in one of the symmetry axes.

Solution 2. (*Ilya Bogdanov*) For convenience, we denote $b_k=a_{1,n-k}$ and $c_k=a_{k+1,n-k}$ for every $k=0,1,\dots,n-1$. Our aim is to find the set of relations for (b_k) which are equivalent to the relation $(b_k)=(c_k)$. All the calculations will be made in $\mathbb{Z}_2$.

The definition of a_{ij} is equivalent to $a_{ij} = a_{i-1,j} + a_{i-1,j+1}$. A straightforward check shows then that

$$a_{ij} = \sum_{\ell=0}^{i-1} \binom{i-1}{\ell} a_{1,\ell+j}.$$

For two nonnegative integers k and ℓ, we will write $\ell \preceq k$ if the binary representation of ℓ can be obtained from that of k by replacing some ones by zeroes (the leading zeroes are allowed; thus $0 \preceq k$ and $k \preceq k$ for every k). We write $\ell \prec k$ if $\ell < k$ and $\ell \preceq k$. Recall that by Lucas' theorem, $\binom{k}{\ell}$ is odd if and only if $\ell \preceq k$. Thus,

$$c_k = a_{k+1,n-k} = \sum_{\ell=0}^{k} \binom{k}{\ell} a_{1,\ell+n-k} = \sum_{\ell=0}^{k} \binom{k}{\ell} b_{k-\ell} = \sum_{\ell=0}^{k} \binom{k}{\ell} b_\ell = \sum_{\ell \preceq k} b_\ell.$$

Now, the conditions $(b_k) = (c_k)$ rewrite as the set of equations

$$0 = \sum_{\ell \prec k} b_\ell \qquad (*_k)$$

for all $k = 0, 1, \ldots, n-1$.

Denote by T the set of all strings (b_k) such that $(*_k)$ are satisfied for all *odd* $k \leq n-1$. Each string in this set is determined uniquely by the values of b_{2i-1} $(2i < n)$ and b_{n-1}: the values of b_{2i} (for $2i < n-1$) are found inductively from $(*_{2i+1})$. Thus $|T| = 2^{\lceil n/2 \rceil} = 2^{\lfloor (n+1)/2 \rfloor}$. Now we claim that T is exactly the desired set of fixed points; in fact, we will prove that all the relations $(*_d)$ for even d follow from the relations $(*_k)$ for odd k.

Consider any even $d \leq n-1$. To establish $(*_d)$, we add up all the relations $(*_{k+1})$, where $k \prec d$, obtaining a sum

$$\begin{aligned} 0 = \sum_{k \prec d} \sum_{\ell \prec k+1} b_\ell &= \sum_{k \prec d} \left(\sum_{\ell \preceq k} b_\ell + \sum_{\ell \prec k} b_{\ell+1} \right) \\ &= \sum_{\ell \prec d} b_\ell \cdot |\{k \colon \ell \preceq k \prec d\}| + \sum_{\ell \prec d} b_{\ell+1} \cdot |\{k \colon \ell \prec k \prec d\}|. \end{aligned}$$

But one can easily check that $|\{k \colon \ell \preceq k \prec d\}|$ is odd, and $|\{k \colon \ell \prec k \prec d\}|$ is even for all $\ell \prec d$. Thus our equality rewrites exactly as $(*_d)$.