Fourth Selection Test — Solutions

Problem 1. Fix a point O in the plane and an integer $n \geq 3$. Consider a finite set $\mathcal{D}$ of closed unit discs in the plane such that:

(a) No disc in $\mathcal{D}$ contains the point O; and
(b) For each positive integer $k < n$, the closed disc of radius $k+1$ centred at O contains the centres of at least k discs in $\mathcal{D}$.

Show that some line through O stabs at least $\frac{2}{\pi} \log \frac{n+1}{2}$ discs in $\mathcal{D}$.

Solution. For each disc D in $\mathcal{D}$, let ω_D denote the centre of D, and let α_D be the arc-length of the image of D under radial projection from O onto the unit circle centred at O. Clearly, $\alpha_D/2 > \sin(\alpha_D/2) = 1/O\omega_D$.

Now, for each positive integer $k < n$, let $\mathcal{D}_k$ be the set of those discs in $\mathcal{D}$ whose centres lie in the closed disc of radius $k+1$ centred at O. Since $\mathcal{D}_i \subseteq \mathcal{D}_j$ if $i \leq j$, and each $\mathcal{D}_k$ contains at least k elements, we may recursively choose (or apply Hall's marriage theorem to produce) a system of distinct representatives, $D_1, \ldots, D_{n-1}$, for the collection $\mathcal{D}_1, \ldots, \mathcal{D}_{n-1}$, to obtain

$$\sum_{D \in \mathcal{D}_{n-1}} \alpha_D > 2 \sum_{D \in \mathcal{D}_{n-1}} 1/O\omega_D \geq 2 \sum_{k=1}^{n-1} 1/O\omega_{D_k} \geq 2 \sum_{k=1}^{n-1} 1/(k+1) > 2 \log \frac{n+1}{2}.$$

Finally, if N_{n-1} is the maximal number of discs in $\mathcal{D}_{n-1}$ stabbed by a line through O as it makes a half-turn about O, then $\pi N_{n-1} \geq \sum_{D \in \mathcal{D}_{n-1}} \alpha_D$ and the conclusion follows.

Problem 2. Let n be an integer greater than 1 and let $\mathcal{S}$ be the set of n-element subsets of the set $\{1, 2, \ldots, 2n\}$. Determine

$$\max_{S \in \mathcal{S}} \min_{x,y \in S,\, x \neq y} [x, y],$$

where $[x, y]$ denotes the least common multiple of the integers x and y.

Solution. The required value is $6(\lfloor n/2 \rfloor + 1)$, unless $n = 4$ in which case it is 24.

Let S be a member of $\mathcal{S}$. We first show that

$$\min_{x,y \in S,\, x \neq y} [x, y] \leq 6(\lfloor n/2 \rfloor + 1), \qquad (*)$$

unless $n = 4$. To this end, for each x in S, choose a positive integer m_x such that $n < m_x x \leq 2n$ and consider the set $S' = \{m_x x \colon x \in S\}$.

If $|S'| < n$, then $m_x x = m_y y$ for some distinct elements x and y in S, so $[x, y] \leq 2n$.

If $|S'| = n$, then $S' = \{n+1, n+2, \ldots, 2n\}$. The first even number in S' is $2(\lfloor n/2 \rfloor + 1)$, and the number $3(\lfloor n/2 \rfloor + 1)$ is also in S' if $n = 3$ or $n \geq 5$. Consequently, $(*)$ holds for $n = 3$ or $n \geq 5$, and it clearly holds for $n = 2$.

If $n = 4$, then

$$\min \{[x, y] \colon x, y \in \{5, 6, 7, 8\}, x \neq y\} = 24,$$

which is the required value by the preceding.

Finally, we show that, if $1 \leq i < j \leq n$, then $[n+i, n+j] \geq 6(\lfloor n/2 \rfloor + 1)$. Suppose, if possible, that $[n+i, n+j] < 6(\lfloor n/2 \rfloor + 1)$. Since $[n+1, n+2] = (n+1)(n+2) \geq 6(\lfloor n/2 \rfloor + 1)$, it follows that $j \geq 3$, so $n+j \geq 2(\lfloor n/2 \rfloor + 1)$. Hence $[n+i, n+j] = 2(n+j) = m(n+i)$, where m is an integer greater than 2. If $m = 3$, then $n+i$ must be an even number less than

$2(\lfloor n/2 \rfloor + 1)$ which is impossible. If $m \geq 4$, then $n+i < 3(\lfloor n/2 \rfloor + 1)/2 \leq n+1$ which is again impossible. This ends the proof.

Problem 3. Given an integer $n \geq 2$, determine all non-constant polynomials f with complex coefficients satisfying the condition

$$1 + f(X^n + 1) = \big(f(X)\big)^n.$$

Solution. If n is even, there are no such polynomials. If n is odd, the required polynomials are precisely those recursively defined by $f_0(X) = -X$, and $f_{k+1}(X) = f_k(X^n + 1)$, $k \geq 0$.

It is readily checked that the polynomials in the above sequence all satisfy the condition in the statement.

Conversely, let f be a polynomial with complex coefficients satisfying the condition

$$1 + f(X^n + 1) = \big(f(X)\big)^n. \tag{1}$$

To begin, we show that, if $f(0) = 0$, then $f = -X$ and n must be odd. To prove this, consider the sequence defined by $x_0 = 0$ and $x_{k+1} = x_k^n + 1$, $k \geq 0$. Clearly, $f(x_{k+1}) = \big(f(x_k)\big)^n - 1$, $k \geq 0$, and $f(x_1) = -1$.

If n is even, then $f(x_2) = 0$, so $f(x_{2k}) = 0$ (and $f(x_{2k+1}) = -1$), $k \geq 0$. Since the x_k form a strictly increasing sequence, we reach a contradiction.

If n is odd, induct on k to prove that $f(x_k) = -x_k$, $k \geq 0$. This is clearly true if $k = 0, 1, 2$. For the induction step, use (1) to get $f(x_{k+1}) = (-x_k)^n - 1 = -(x_k^n + 1) = -x_{k+1}$. With reference again to the monotonicity of the x_k, we conclude that $f = -X$.

Finally, consider the case $f(0) \neq 0$. Let ω be a primitive n-th root of unity and use (1) to deduce that $\big(f(X)\big)^n = \big(f(\omega X)\big)^n$, so $f(X) = \omega^m f(\omega X)$ for some non-negative integer $m < n$. Since $f(0) \neq 0$, identification of the constant terms yields $\omega^m = 1$, so $m = 0$, for ω is primitive. Hence $f(X) = f(\omega X)$ and identification of coefficients shows that $f(X)$ is a polynomial in X^n with complex coefficients. Alternatively, but equivalently, $f(X) = g(X^n + 1)$ for some polynomial g with complex coefficients. Since g also satisfies (1), the conclusion now follows recursively.

Alternative solution — case $f(0) = 0$. Use (1) repeatedly to obtain $f(1) = -1$, $f(2) = (-1)^n - 1$, $f(2^n + 1) = \big((-1)^n - 1\big)^n - 1$, and deduce thereby that

$$|f(2^n + 1)| \leq 2^n + 1. \tag{2}$$

We now take time out to show that the roots of f all lie in the disc $|z| < 2$ in the complex plane. To this end, let α_0 be a root of f of maximal absolute value. Since the absolute value of the leading coefficient of f is 1, (1) yields

$$\prod_{\alpha \text{ is a root of } f} |\alpha_0^n + 1 - \alpha| = 1. \tag{3}$$

Suppose, if possible, that $|\alpha_0| \geq 2$. If α is a root of f, then

$$|\alpha_0^n + 1 - \alpha| \geq |\alpha_0|^n - 1 - |\alpha| \geq 2|\alpha_0| - 1 - |\alpha| = \big(|\alpha_0| - 1\big) + \big(|\alpha_0| - |\alpha|\big) \geq |\alpha_0| - 1 \geq 1.$$

Since $f(0) = 0$, at least one of the factors of the product in (3) is $|\alpha_0^n + 1| \geq |\alpha_0|^n - 1 \geq 2^n - 1 \geq 3$, so the product is at least 3 — in contradiction with (3).

Back to the problem, write (2) in the form

$$\prod_{\alpha \text{ is a root of } f} |2^n + 1 - \alpha| \leq 2^n + 1. \tag{2'}$$

By the preceding, if α is a non-zero root of f, then $|2^n + 1 - \alpha| \geq 2^n - 1 - |\alpha| > 2^n - 3 \geq 1$, so, if the multiplicity of 0 exceeds 1 or f has a non-zero root, then the product in $(2')$ exceeds $2^n + 1$ and we reach a contradiction. Consequently, $f = aX$, where a is a complex number of absolute value 1, and (1) forces $a = -1$ and n odd.