The 2017 Danube Competition in Mathematics, October 28^{th}

Problema 1. Să se găsească toate polinoamele P, cu coeficienţi întregi, care verifică relaţia

$$a^2 + b^2 - c^2 \mid P(a) + P(b) - P(c),$$

pentru orice numere întregi a, b, c.

Problema 2. Pe o tablă sunt scrise n numere reale, unde $n \geq 2$ este un număr natural. Printr-o *transformare*, alegem două dintre numere, le ştergem şi apoi le înlocuim (pe fiecare) cu produsul lor. Să se găsească numerele naturale n pentru care, oricare ar fi secvenţa de n numere scrise iniţial, este posibil ca, după un număr finit de *transformări*, să obţinem pe tablă n numere egale.

Problema 3. Fie H ortocentrul şi O centrul cercului circumscris triunghiului ABC. Fie F piciorul înălţimii din C şi M mijlocul segmentului CH. Fie N piciorul perpendicularei din C pe paralela din H la OM, fie $D \neq A$ pe dreapta AB astfel încât $CA = CD$, şi fie P intersecţia dreptelor BN şi CD. Dacă Q este intersecţia dreptelor HP şi AC arătaţi că QF este perpendiculară pe OF.

Problema 4. În fiecare dintre pătrăţelele unitate ale unei reţele infinite este scris un număr, astfel încât modulul sumei numerelor din orice pătrat ale cărui laturi sunt determinate de liniile reţelei este mai mic sau egal decât 1. Să se demonstreze că modulul sumei numerelor din orice dreptunghi cu laturile determinate de liniile reţelei este mai mic sau egal decât 4.

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Problem 1. Find all polynomials P, with integer cofficients, such that, for all integers a, b, c,

$$a^2 + b^2 - c^2 \mid P(a) + P(b) - P(c).$$

Solution. Take $a = 3k$, $b = 4k$, $c = 5k$, with integer k. Then 0 divides $P(3k) + P(4k) - P(5k)$, so $P(3k) + P(4k) - P(5k) = 0$ for infinitely many k, hence $P(3X) + P(4X) - P(5X) = 0$.

If P is not nil, say that P has degree n and leading coefficient a_n. Then $3^n a_n + 4^n a_n = 5^n a_n$, so, with a standard argument, $n = 2$.

Further $0 \mid P(0)$ implies $P = \alpha X^2 + \beta X$ and $a^2 \mid P(a)$ for all integer a implies $\beta = 0$, therefore $P = \alpha X^2$, with integer α.

Clearly, every such polynomial fulfills the condition.

Problem 2. There are written n real numbers on a blackboard, where $n \geq 2$ is a natural number. In a *transformation* we choose two numbers, erase them and replace each of them by their product. Find n such that, for any initial n-tuple, it is possible to obtain n equal numbers on the blackboard after a finite number of *transformations*.

Solution. We will show that every even n satisfies the condition.

We shall proceed by induction. The claim is trivial for $n = 2$. For $n = 4$ we apply the case $n = 2$ in four steps. Now we suppose that the claim is true for any even number $k \geq 4$ and prove it for $n = k + 2$. By the induction hypothesis, we first construct an n-tuple of the form

$$(a, a, \dots, a, b, b).$$

Now, for every $i = 3, 4, \dots, k$, we transform the pairs $(i, k+1)$, obtaining thus the n-tuple

$$(a, a, ab, a^2b, a^3b, \dots, a^{k-2}b, a^{k-2}b, b).$$

After selecting the last two numbers, we get

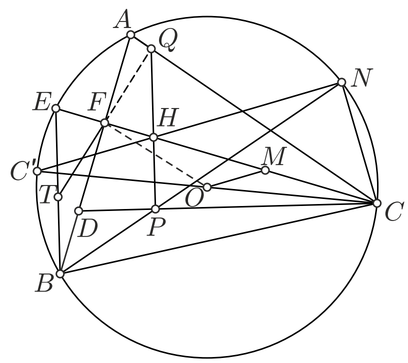
$$(a, a, ab, a^2b, \dots, a^{k-2}b, a^{k-2}b^2, a^{k-2}b^2).$$

Finally, we use the transformation step by step for the pairs at the positions $(i, n + 1 - i)$ for $i = 1, 2, \dots, \frac{n}{2}$, obtaining the sequence

$$(a^{k-1}b^2, a^{k-1}b^2, \dots, a^{k-1}b^2).$$

For odd $n \geq 3$ we consider the n-tuple $(3, 3, \dots, 2)$. Let m be the number of occurrences of the maximum elements (after each step) in the n-tuple. Initially we have $m = n - 1$. This number remains every time even, contradicting the fact that n is odd.

Problem 3. Denote H the orthocentre, O the circumcentre, F the foot of the altitude from C and M the midpoint of the segment CH in a triangle ABC. Let N be the orthogonal projection of C onto the parallel from H to OM, let $D \neq A$ be on AB such that $CA = CD$ and let P be the common point of BN and CD. If HP intersects AC in Q, prove that $QF \perp OF$.



Solution. In order to get the conclusion, we prove that F is the midpoint of the chord of $\mathcal{C}(ABC)$ through Q and F.

To this end, denote C' the diametrally opposed point of C in $\mathcal{C}(ABC)$. Then OM is a midline in triangle CHC', so $HC' \parallel OM$, hence $N \in HC'$.

We show that the quadrilateral $CPHN$ is cyclic (1). Indeed,

$$\begin{aligned}\angle CPN &= \angle BCP + \angle CBP = \angle ADC - \angle ABC + \angle CBN \\ &= \angle BAC - \angle ABC - \angle CBN\end{aligned}$$

$$\begin{aligned}\angle CHN &= \angle HCC' + \angle HC'C = \angle FCB - \angle BCC' + \angle CC'N \\ &= 90^\circ - \angle FBC - 90^\circ + \angle BC'C + \angle CC'N \\ &= \angle BAC - \angle ABC + \angle CBN\end{aligned}$$

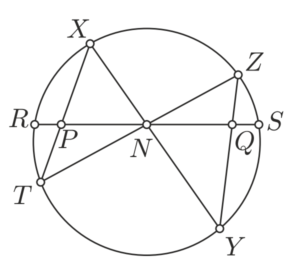
therefore $\angle CPN = \angle CHN$, which proves (1).

This yields $\angle PHC = \angle PNC = \angle BAC$.

Denote $E = CH \cap \mathcal{C}(ABC)$ and $T = QF \cap BE$. Then $\Delta HFQ \equiv \Delta EFT$ (ASA), because $HF = FE$, $\angle HFQ = \angle EFT$ and $\angle FHQ = \angle PHC = \angle PNC = \angle BNC = \angle BAC = \angle FET$. This shows that $QF = FT$. (2)

We will use now the following result, which is a converse of *The Butterfly Theorem* (BT).

Converse of BT. If the chords XY and ZT of a circle $\mathcal{C}$ meet at N and a line through N meets the chords at P and Q such that $NP = NQ$ and meets $\mathcal{C}$ at R and S, then $NR = NS$.

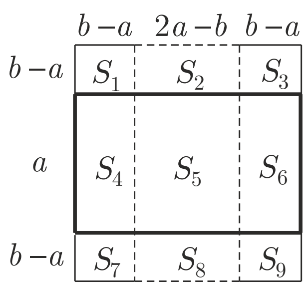


Proof of the lemma. If N is the center of the circle, everything is clear. Otherwise, consider the chord of $\mathcal{C}$ whose midpoint is N and its intersections P', Q' with XY and ZT. Then BT shows that $NP' = NQ'$. So Q' belongs to the reflection of XT across the perpendicular bisector of this chord. Since this reflection is on a line different from YZ, the point Q' is uniquely determined, so it must coincide with Q.

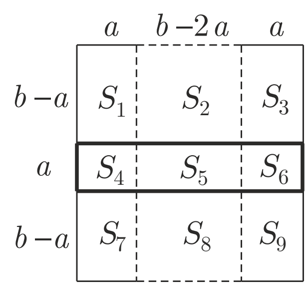
Now the Lemma and (2) shows that F is the midpoint of the chord of $\mathcal{C}$ through Q, F and T, whence the conclusion.

Problem 4. In each unit square of an infinite plane grid we write a real number such that the absolute value of the sum of the numbers in any square whose sides are determined by the lines of the grid is less than or equal to 1. Prove that the absolute value of the sum of the numbers in any rectangle whose sides are determined by the lines of the grid is less than or equal to 4.

Solution. Suppose that there is a rectangle (the bolded rectangle in the figures) with sides a and b ($a < b$) such that the absolute value of the sum of its numbers is equal to $4 + \epsilon$, ($\epsilon > 0$). In the exterior of this rectangle we construct four squares with sides $b - a$ (see the figures) to obtain a new rectangle (the dotted rectangle in the figures) with sides $2b - a$ and $|2a - b|$. It is clear that the case $b = 2a$ is not possible.



Case 1



Case 2

Denote by S_i, $i = \overline{1,9}$ the corresponding sums of the numbers in the nine rectangles. Suppose that the sum of the numbers in the initial rectangle (i.e. $S_4 + S_5 + S_6$) is $4 + \epsilon$, otherwise we change the signs of all numbers. We consider two cases:

Case 1. $b < 2a$. Then

$$S_5 = (S_4 + S_5) + (S_5 + S_6) - (S_4 + S_5 + S_6) \leq 1 + 1 - 4 - \epsilon = -2 - \epsilon,$$

$$S_2 + S_8 = (S_1 + S_2 + S_3 + S_4 + S_5 + S_6) + (S_4 + S_5 + S_6 + S_7 + S_8 + S_9) -$$

$$-S_1 - S_3 - S_7 - S_9 - 2(S_4 + S_5 + S_6) \leq 1 + 1 + 1 + 1 + 1 + 1 - 2(4 + \epsilon) = -2 - 2\epsilon,$$

and thus $S_2 + S_5 + S_8 \leq -4 - 3\epsilon$.

Case 2. $b > 2a$. Then

$$S_5 = (S_4 + S_5 + S_6) - S_4 - S_6 \geq 4 + \epsilon - 1 - 1 = 2 + \epsilon,$$

$$S_2+S_8=(S_1+S_2)+(S_2+S_3)+(S_7+S_8)+(S_8+S_9)-$$
$$-(S_1+S_2+S_3+S_4+S_5+S_6)-(S_4+S_5+S_6+S_7+S_8+S_9)+$$
$$+2(S_4+S_5+S_6)\geq 2+2\epsilon,$$

and thus $S_2+S_5+S_8\geq 4+3\epsilon$.

In both cases, starting from a rectangle $(a_1=a, b_1=b)$, we obtain a new rectangle (a_2,b_2) with sides $a_2=2a-b$, $b_2=2b-a$ (in the first case) and $a_2=b-2a$, $b_2=2b-a$ (in the second case), such that the absolute value of the sum of its numbers is greater than or equal to $4+3\epsilon$.

We can continue the procedure to get a sequence (a_n,b_n) of rectangles such that the absolute value of the sum of their numbers is greater than or equal to $4+3^{n-1}\epsilon$.

We shall prove now that all rectangles (a_n,b_n) in this sequence (for any $n\geq n_0$, where n_0 is large enough) are of the second type, that is $b_n>2a_n$. First of all it easy to check that if $\frac{a_n}{b_n}<\frac{1}{2}$ then $\frac{a_{n+1}}{b_{n+1}}=\frac{b_n-2a_n}{2b_n-a_n}<\frac{1}{2}$. On the other hand, if $\frac{a_n}{b_n}>\frac{1}{2}$, then

$$\frac{1-\frac{a_{n+1}}{b_{n+1}}}{1-\frac{a_n}{b_n}}=\frac{1-\frac{2a_n-b_n}{2b_n-a_n}}{1-\frac{a_n}{b_n}}=\frac{3b_n}{2b_n-a_n}>2,$$

and thus after a finite number of steps we get $1-\frac{a_n}{b_n}>\frac{1}{2}$, that is $\frac{a_n}{b_n}<\frac{1}{2}$.

For simplicity we suppose now that $\frac{a_1}{b_1}<\frac{1}{2}$. Then

$$a_3=b_2-2a_2=(2b_1-a_1)-2(b_1-2a_1)=3a_1$$

$$b_3=2b_2-a_2=2(2b_1-a_1)-(b_1-2a_1)=3b_1.$$

In general we obtain $a_{2k+1}=3^k a_1$ and $b_{2k+1}=3^k b_1$. The absolute value of the sum of the numbers in the rectangle $(3^k a_1, 3^k b_1)$ is greater than $4+9^k\epsilon$. This value, for a large enough k, could be as large as we wish.

The contradiction is obtained now because any rectangle (ma_1, mb_1), with a_1, b_1, m natural numbers, can be decomposed in a_1b_1 squares (with side m) and thus, the absolute value of the sum of the numbers in this rectangle is less than or equal to a_1b_1.