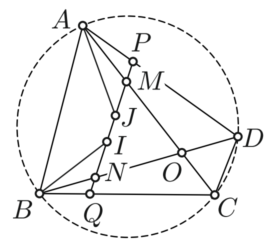
Călăraşi 2015 — Solutions

Problem 1. Let $ABCD$ be a cyclic quadrangle, let the diagonals AC and BD cross at O, and let I and J be the incentres of the triangles ABC and ABD, respectively. The line IJ crosses the segments OA and OB at M and N, respectively. Prove that the triangle OMN is isosceles.



Solution. We show that $\angle OMN \equiv \angle ONM$. To this end, let the line IJ cross the segments AD and BC at P and Q, respectively, and consider the position of J relative to the line AC, to write $\angle OMN \equiv \angle AJP \pm \angle JAM \equiv \angle AJP \pm (\pm\angle JAD \mp \angle CAD) \equiv \angle AJP + \angle JAD - \angle CAD \equiv \angle AJP + \angle BAJ - \angle CAD$. Similarly, $\angle ONM \equiv \angle BIQ + \angle ABI - \angle CBD$. Since the quadrangle $ABCD$ is cyclic, $\angle CAD \equiv \angle CBD$ and $\angle ACB \equiv \angle ADB$. The latter implies that $\angle AIB \equiv \angle AJB$, so the quadrangle $ABIJ$ is cyclic. Consequently, $\angle AJP \equiv \angle ABI$ and $\angle BAJ \equiv \angle BIQ$, and the conclusion follows.

Problem 2. Show that the edges of a connected finite simple graph can be oriented so that the number of edges leaving each vertex is even if and only if the total number of edges is even.

Solution 1. Given any orientation, the total number of edges equals the sum of all out-degrees. If the latter are all even, then so is the former.

To establish the converse, induct on the number of edges to show that any connected simple finite graph with an even number of edges splits into edge-disjoint paths of length 2. Orient the two edges of each of these paths away from the joint to obtain the required orientation.

Solution 2. The problem is a special case of the following general fact: Given a connected finite simple graph $G = (V, E)$ and an integral-valued function f on V, taken over all possible orientations of the edges of G, the minimum of the number of vertices at which outdeg and f have opposite parities is $|E| - \sum_{x\in V} f(x)$ reduced modulo 2.

Given any orientation, notice that the number of vertices at which the parities of outdeg and f disagree has the same parity as $|E| - \sum_{x\in V} f(x)$. Consider an orientation minimising the number of these vertices. If this number exceeds 1, choose two such vertices and use connectedness to join them by a path. Reverting orientations along the path changes the parity of out-degrees only at the end-points, so the outcome is an oriented graph with fewer vertices at which outdeg and f have opposite parities. This contradicts minimality and concludes the proof.

Problem 3. Determine all positive integers n such that all positive integers less than or equal to n and prime to n are pairwise coprime.

Solution. Let $p_1 = 2 < p_2 = 3 < p_3 = 5 < \cdots$ be the sequence of primes and let p_m be the first prime which does not divide n. A necessary and sufficient condition that n be of the required type is that $n < p_m^2$. Since each of the primes $p_1, \ldots, p_{m-1}$ divides n, so does their product, hence $p_1 \cdots p_{m-1} \le n < p_m^2$.

Notice that 5 is the first index k such that $p_1 \cdots p_{k-1} > p_k^2$. Now, if $p_1 \cdots p_{k-1} > p_k^2$ for some index $k \ge 5$, then $p_1 \cdots p_{k-1} p_k > p_k^3 > 4p_k^2 > p_{k+1}^2$, by the Bertrand-Tchebysheff theorem, so $p_1 \cdots p_{k-1} > p_k^2$ for all indices $k \ge 5$.

Consequently, $m \le 4$, so $n < p_4^2 = 49$. Examination of the possible cases quickly yields the required numbers: 1, 2, 3, 4, 6, 8, 12, 18, 24, 30.

Problem 4. Given an integer $n \ge 2$, determine the numbers that can be written in the form $\sum_{i=2}^k a_{i-1}a_i$, where k is an integer greater than or equal to 2, and $a_1, \ldots, a_k$ are positive integers that add up to n.

Solution. The required numbers are the integers in the range $n-1, \ldots, \lfloor n^2/4 \rfloor$.

Given an integer $k \ge 2$ and positive integers $a_1, \ldots, a_k$ that add up to n, we show that $n - 1 \le \sum_{i=2}^k a_{i-1}a_i \le \lfloor n^2/4 \rfloor$. To prove the first inequality, write

$$\sum_{i=2}^k a_{i-1}a_i = a_1a_2 + \sum_{i=3}^k a_{i-1}a_i \ge a_1 + a_2 - 1 + \sum_{i=3}^k a_{i-1}a_i \ge a_1 + a_2 - 1 + \sum_{i=3}^k a_i = n - 1.$$

To prove the second, write

$$\begin{aligned}\sum_{i=2}^k a_{i-1}a_i &\le \left(\sum_{i=1}^{\lfloor (k+1)/2 \rfloor} a_{2i-1}\right)\left(\sum_{i=1}^{\lfloor k/2 \rfloor} a_{2i}\right) = \left(\sum_{i=1}^{\lfloor (k+1)/2 \rfloor} a_{2i-1}\right)\left(\sum_{i=1}^k a_i - \sum_{i=1}^{\lfloor (k+1)/2 \rfloor} a_{2i-1}\right) \\ &\le \frac{1}{4}\left(\sum_{i=1}^k a_i\right)^2 = \frac{n^2}{4}.\end{aligned}$$

Since $\sum_{i=2}^k a_{i-1}a_i$ is integral, it cannot exceed $\lfloor n^2/4 \rfloor$.

Conversely, let s be an integer in the range $n-1, \ldots, \lfloor n^2/4 \rfloor$. If s is the product of two positive integers that add up to n, we are done; in particular, this is the case for $n-1 = 1 \cdot (n-1)$ and $\lfloor n^2/4 \rfloor = \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$. Otherwise, consider the largest integer $m < \lfloor n/2 \rfloor$ such that $m(n-m) < s$, to infer that $\ell = s - m(n-m)$ is a positive integer less than $n - 2m - 1$. Letting $a_1 = n - 2m - \ell - 1$, $a_2 = m$, $a_3 = \ell + m$, and $a_4 = 1$, it is readily checked that $a_1 + a_2 + a_3 + a_4 = n$, and $s = a_1a_2 + a_2a_3 + a_3a_4$.