Liceul Internaţional de Informatică Bucureşti
Societatea de Ştiinţe Matematice

ALPHA BANK

Stelele Matematicii 2015, Seniori

Problema 1. Arătaţi că există un şir de numere naturale impare $m_1 < m_2 < \cdots$ şi un şir de numere naturale $n_1 < n_2 < \cdots$, astfel încât m_k să fie relativ prim cu n_k şi $m_k^4 - 2n_k^4$ să fie pătrat perfect, oricare ar fi indicele k.

Problema 2. Fie γ, γ_0, γ_1, γ_2 patru cercuri în plan, astfel încât γ_i să fie tangent interior lui γ în punctul A_i, iar γ_i şi γ_{i+1} să fie tangente exterior în punctul B_{i+2}, $i = 0, 1, 2$ (indicii sunt reduşi modulo 3). Tangenta în B_i, comună cercurilor γ_{i-1} and γ_{i+1}, intersectează cercul γ în punctul C_i, situat în semiplanul opus lui A_i în raport cu dreapta $A_{i-1}A_{i+1}$. Arătaţi că cele trei drepte A_iC_i sunt concurente.

Problema 3. Fie n un număr natural nenul şi fie $a_1, \ldots, a_n$ numere naturale nenule. Arătaţi că

$$\sum_{k=1}^{n} \frac{\sqrt{a_k}}{1 + a_1 + \cdots + a_k} < \sum_{k=1}^{n^2} \frac{1}{k}.$$

Problema 4. Fie S o mulţime finită de puncte în plan, situate în poziţie generală (oricare trei puncte din S nu sunt coliniare), şi fie

$$D(S, r) = \{\{x, y\} \colon x, y \in S,\ \operatorname{dist}(x, y) = r\},$$

unde r este un număr real strict pozitiv, iar $\operatorname{dist}(x, y)$ este distanţa euclidiană între punctele x şi y. Arătaţi că

$$\sum_{r>0} |D(S, r)|^2 \leq 3|S|^2(|S| - 1)/4.$$

Stars of Mathematics 2015, Senior Level — Solutions

Problem 1. Show that there are positive odd integers $m_1 < m_2 < \cdots$ and positive integers $n_1 < n_2 < \cdots$ such that m_k and n_k are relatively prime, and $m_k^4 - 2n_k^4$ is a perfect square for each index k.

Folklore

Solution. Let m and n be relatively prime positive integers such that m is odd and $m^4 - 2n^4$ is a perfect square, e.g., $m = 3$ and $n = 2$. Write $\ell^2 = m^4 - 2n^4$, so $\ell^4 = (m^4 - 2n^4)^2 = (m^4 + 2n^4)^2 - 8m^4n^4$, and $\ell^4 - 8m^4n^4 - (m^4 + 2n^4)^2 = -16m^4n^4 = -(2mn)^4$. Multiply the latter by $\ell^4 - 8m^4n^4 + (m^4 + 2n^4)^2 = 2\ell^4$ to get

$$\left(\ell^4 - 8m^4n^4 + (m^4 + 2n^4)^2\right)\left(\ell^4 - 8m^4n^4 - (m^4 + 2n^4)^2\right) = -2 \cdot (2\ell mn)^4;$$

that is, $(\ell^4 - 8m^4n^4)^2 - (m^4 + 2n^4)^4 = -2 \cdot (2\ell mn)^4$. Letting $m' = m^4 + 2n^4$ and $n' = 2\ell mn$, clearly $m' > m$, m' is odd, $n' > n$, the difference $m'^4 - 2n'^4$ is a perfect square, and it is readily checked that m' and n' are relatively prime. The conclusion follows.

Problem 2. Let γ, γ_0, γ_1, γ_2 be coplanar circles such that γ_i is internally tangent to γ at A_i, and γ_i and γ_{i+1} are externally tangent at B_{i+2}, $i = 0, 1, 2$ (indices are reduced modulo 3). The tangent at B_i, common to γ_{i-1} and γ_{i+1}, meets γ at C_i, located in the half-plane opposite A_i with respect to the line $A_{i-1}A_{i+1}$. Show that the three lines A_iC_i are concurrent.

Flavian Georgescu

Solution 1. Let γ_i cross the lines A_iC_{i+1} and A_iC_{i+2} again at X_{i+2} and Y_{i+1}, respectively.

Since the homothety centred at A_i, transforming γ_i into γ, sends X_{i+2} to C_{i+1} and Y_{i+1} to C_{i+2}, the lines $X_{i+2}Y_{i+1}$ and $C_{i+1}C_{i+2}$ are parallel, so $A_iC_{i+1}/A_iC_{i+2} = X_{i+2}C_{i+1}/Y_{i+1}C_{i+2}$.

On the other hand, since C_i has equal powers relative to γ_{i+1} and γ_{i+2}, $C_iX_{i+1} \cdot C_iA_{i+2} = C_iB_i^2 = C_iY_{i+2} \cdot C_iA_{i+1}$, it follows that $C_iX_{i+1} = C_iB_i^2/C_iA_{i+2}$ and $C_iY_{i+2} = C_iB_i^2/C_iA_{i+1}$.

Hence $A_iC_{i+1}/A_iC_{i+2} = (B_{i+1}C_{i+1}/B_{i+2}C_{i+2})^2(A_iC_{i+2}/A_iC_{i+1})$, so $A_iC_{i+1}/A_iC_{i+2} = B_{i+1}C_{i+1}/B_{i+2}C_{i+2}$, and consequently $\prod_{i=0}^{2}(A_iC_{i+1}/A_iC_{i+2}) = \prod_{i=0}^{2}(B_{i+1}C_{i+1}/B_{i+2}C_{i+2}) = 1$. Since the sines of the angles $C_iA_iC_j$ are proportional to the lengths of the corresponding chords A_iC_j, the conclusion follows by Ceva's theorem in trigonometric form in the triangle $C_0C_1C_2$.

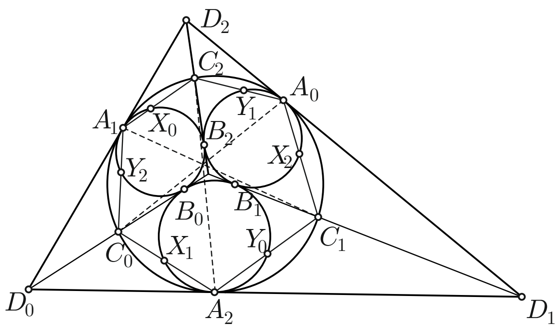

Fig. 1

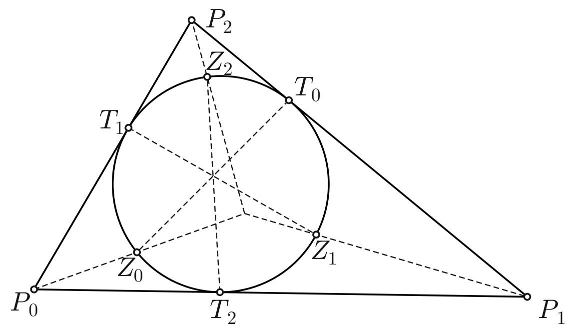

Fig. 2

Solution 2. Let the tangents to γ at A_i and A_{i+1} meet at D_{i+2} (fig 1), and notice that the latter has equal powers relative to γ_i and γ_{i+1} to deduce that it lies on their radical axis,

$B_{i+2}C_{i+2}$. Consequently, the lines C_iD_i are concurrent at the radical centre of the γ_i, and the conclusion follows by the lemma below (see Figure 2).

Lemma. *Let $P_0P_1P_2$ be a triangle, and let T_i be the touchpoint of the side $P_{i+1}P_{i+2}$ and the incircle γ of the triangle $P_0P_1P_2$. Let further Z_i be a point on the arc $T_{i+1}T_{i+2}$ of γ not containing T_i. Then the lines T_iZ_i are concurrent if and only if the lines P_iZ_i are concurrent.*

The lemma can be proved either projectively, by sending the point of intersection of T_0Z_0 and T_1Z_1 to the incentre, or by a trigonometric version of Ceva's theorem.

Problem 3. Let n be a positive integer and let $a_1, \ldots, a_n$ be n positive integers. Show that

$$\sum_{k=1}^{n} \frac{\sqrt{a_k}}{1 + a_1 + \cdots + a_k} < \sum_{k=1}^{n^2} \frac{1}{k}.$$

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Solution. Set $b_0 = 1$ and $b_k = 1 + a_1 + \cdots + a_k$, $k = 1, \ldots, n$, to obtain a strictly increasing string of positive integers $1 = b_0 < b_1 < \cdots < b_n$, and write the sum in the left-hand member in the form $\sum_{k=1}^{n}(b_k - b_{k-1})^{1/2}/b_k$.

Next, let $m = \min\{k\colon b_k > n^2\} \geq 1$ — if there is no $b_k > n^2$, let $m = n + 1$ —, to split the above sum into

$$\sum_{k=1}^{m-1} \frac{\sqrt{b_k - b_{k-1}}}{b_k} + \sum_{k=m}^{n} \frac{\sqrt{b_k - b_{k-1}}}{b_k}, \tag{$*$}$$

where empty sums are zero. We show that the first sum does not exceed $\sum_{k=2}^{n^2} 1/k$, and the second is always less than 1.

If $k = 1, \ldots, m-1$, write $(b_k - b_{k-1})^{1/2}/b_k \leq (b_k - b_{k-1})/b_k = 1/b_k + \cdots + 1/b_k \leq \sum_{j=b_{k-1}+1}^{b_k} 1/j$, to deduce that the first sum in $(*)$ does not exceed $\sum_{k=1}^{m-1}\sum_{j=b_{k-1}+1}^{b_k} 1/j = \sum_{k=2}^{b_{m-1}} 1/k \leq \sum_{k=2}^{n^2} 1/k$.

Finally, if $k = m, \ldots, n$, write $(b_k - b_{k-1})^{1/2}/b_k < b_k^{-1/2} < 1/n$, to deduce that the second sum in $(*)$, when non-empty, is less than $(n - m + 1)/n \leq 1$. The conclusion follows.

Remarks. (1) The upper bound H_{n^2} can be lowered to $(H_n + H_{n^2})/2$, where $H_k = \sum_{j=1}^{k} 1/j$. This can be done by first applying the Cauchy-Schwarz inequality to the first sum in $(*)$, then using the fact that $b_k \geq k + 1$ for all k, along with the inequalities established above:

$$\sum_{k=1}^{m-1} \frac{\sqrt{b_k - b_{k-1}}}{b_k} \leq \left(\sum_{k=1}^{m-1} \frac{1}{b_k}\right)^{1/2} \left(\sum_{k=1}^{m-1} \frac{b_k - b_{k-1}}{b_k}\right)^{1/2} \leq \left(\sum_{k=1}^{m-1} \frac{1}{k+1}\right)^{1/2} \left(\sum_{k=2}^{n^2} \frac{1}{k}\right)^{1/2}$$
$$= (H_m - 1)^{1/2}(H_{n^2} - 1)^{1/2} \leq (H_m + H_{n^2})/2 - 1.$$

Since $H_{n+1} < H_n + 1$, this settles the case $m = n + 1$. Otherwise, $m \leq n$, so $H_m \leq H_n$, and the conclusion follows by recalling that the second sum in $(*)$ is less than 1.

(2) If the a_k are arbitrary real numbers greater than or equal to 1 — alternatively, but equivalently, the b_k are real numbers, $b_0 = 1$, and $b_k \geq 1 + b_{k-1}$, $k = 1, \ldots, n$ —, using calculus we obtain, along the lines in the previous remark, upper bounds of the form $(H_n + H_{n^2-1} + 1)/2$ or $(H_n + H_{n^2})/2 + (1 - \gamma)/2$, where γ is Euler's constant, $1/2 < \gamma < 3/5$.

Problem 4. Let S be a finite planar set no three points of which are collinear, and let $D(S,r) = \{\{x,y\} : x,y \in S,\ \operatorname{dist}(x,y) = r\}$, where r is a positive real number, and $\operatorname{dist}(x,y)$ is the Euclidean distance between the points x and y. Show that

$$\sum_{r>0} |D(S,r)|^2 \le 3|S|^2(|S|-1)/4.$$

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Solution. Given a point x in S and a real number r, let $S(x,r) = \{y : y \in S,\ \operatorname{dist}(x,y) = r\}$, and notice that the $S(x,r)$, $r \ge 0$, partition S.

The number of non-degenerate isosceles triangles with vertices in S and apex at x is $\sum_{r>0} \binom{|S(x,r)|}{2}$, so the total number of non-degenerate isosceles triangles with vertices in S is $N = \sum_{x\in S}\sum_{r>0}\binom{|S(x,r)|}{2}$, equilateral triangles with vertices in S being counted three times each. Now,

$$\begin{aligned} N &= \sum_{x\in S}\sum_{r>0}\binom{|S(x,r)|}{2} = \sum_{r>0}\sum_{x\in S}\binom{|S(x,r)|}{2} \\ &\ge \sum_{r>0}|S|\binom{\frac{1}{|S|}\sum_{x\in S}|S(x,r)|}{2} = \sum_{r>0}|S|\binom{\frac{2|D(S,r)|}{|S|}}{2} \\ &= \frac{2}{|S|}\sum_{r>0}|D(S,r)|^2 - \sum_{r>0}|D(S,r)| = \frac{2}{|S|}\sum_{r>0}|D(S,r)|^2 - \binom{|S|}{2}, \end{aligned}$$

by Jensen's inequality applied to the convex function $t \mapsto \binom{t}{2} = t(t-1)/2$, $t \in \mathbb{R}$.

On the other hand, given two distinct points x and y in S, there are at most two non-degenerate isosceles triangles with vertices in S, base xy, and apex at a third point in $S\setminus\{x,y\}$: the apex must lie on the perpendicular bisector of the segment xy, and since no three points in S are collinear, there are at most two such. Hence $N \le 2\binom{|S|}{2}$.

Combining the lower and upper bounds for N and rearranging terms yields the required inequality.