The 6th "STARS of MATHEMATICS" Competition – Juniors
December 8, 2012 ★★★ ICHB – Bucharest


Solutions

Problem 1. Let ℓ be a line in the plane, and a point $A \notin \ell$. Determine the locus of the points Q in the plane, for which there exists a point $P \in \ell$ so that $AQ = PQ$ and $\angle PAQ = 45°$.

DAN SCHWARZ

Solution. We claim the locus of the points Q is made by the $\boxed{\text{two main angle bisector lines through } O}$, the foot of the perpendicular dropped from A onto ℓ.

The angle $\angle AQP$ being right (since the triangle AQP is isosceles), the same as $\angle AOP$, it follows the points A, O, P, Q are concyclic, therefore we have $\angle QOA = \angle QPA = 45°$, or else $\angle QOA = 180° - \angle QPA = 135°$, so OQ is one of the two main angle bisectors through O. Of course, there appear two degenerate positions, when $P \equiv O$ and when $Q \equiv O$, but they are trivial. The fullness of the locus readily ensues from the fact a construction is possible in all cases (or else by a continuity argument).

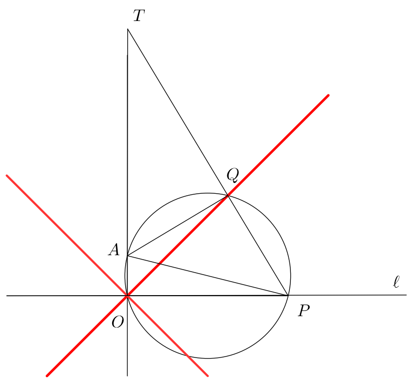

Figure courtesy of ANDREI ECKSTEIN.

An alternative proof, avoiding cyclic quadrilaterals, runs as follows. I will only present a case when prolonging PQ it meets OA at a point T on the same side of ℓ as A. Since AQ is an antiparallel in $\triangle TOP$, it follows triangles TQA and TOP are similar, therefore $\dfrac{TQ}{TO} = \dfrac{QA}{OP}$. But $QA = QP$, and so $\dfrac{TQ}{TO} = \dfrac{QP}{OP}$, meaning OQ is the angle bisector of $\angle TOP$. Any other case is treated in a similar manner.

The condition $A \notin \ell$ is not strictly necessary; that case is however trivial. The problem has been also, for any angle $\angle PAQ = \alpha \in (0, 90°)$, **asked to the Seniors**, arriving at similar conclusions. ■

Problem 2. Prove the value of the expression

$$\frac{\sqrt{n+\sqrt{0}}+\sqrt{n+\sqrt{1}}+\sqrt{n+\sqrt{2}}+\cdots+\sqrt{n+\sqrt{n^2-1}}+\sqrt{n+\sqrt{n^2}}}{\sqrt{n-\sqrt{0}}+\sqrt{n-\sqrt{1}}+\sqrt{n-\sqrt{2}}+\cdots+\sqrt{n-\sqrt{n^2-1}}+\sqrt{n-\sqrt{n^2}}}$$

is constant over all positive integers n.

FOLKLORE (ALSO PHILIPPINES OLYMPIAD)

Solution. (D. Schwarz) For all real numbers $0 \leq n, m \leq n^2$, we have

$$\sqrt{n+\sqrt{m}} = \sqrt{\frac{n+\sqrt{n^2-m}}{2}} + \sqrt{\frac{n-\sqrt{n^2-m}}{2}}.$$

Take now a positive integer n, and sum over all integers m between 0 and n^2

$$\sum_{m=0}^{n^2} \sqrt{n+\sqrt{m}} =$$
$$\sum_{m=0}^{n^2} \sqrt{\frac{n+\sqrt{n^2-m}}{2}} + \sum_{m=0}^{n^2} \sqrt{\frac{n-\sqrt{n^2-m}}{2}} =$$
$$\frac{1}{\sqrt{2}} \sum_{m=0}^{n^2} \sqrt{n+\sqrt{m}} + \frac{1}{\sqrt{2}} \sum_{m=0}^{n^2} \sqrt{n-\sqrt{m}}.$$

It follows

$$\sum_{m=0}^{n^2} \sqrt{n+\sqrt{m}} = \frac{1}{\sqrt{2}-1} \sum_{m=0}^{n^2} \sqrt{n-\sqrt{m}},$$

therefore

$$\left(\sum_{m=0}^{n^2} \sqrt{n+\sqrt{m}}\right) \Big/ \left(\sum_{m=0}^{n^2} \sqrt{n-\sqrt{m}}\right) = \boxed{1+\sqrt{2}}.$$

Such constant value is unexpected; a quick computation for $n = 1$ (and maybe also other small values of n) will expose it, and then remembering the formula for nested square roots finishes the proof by an easy double-counting argument. ■

Problem 3. For all triplets a, b, c of (pairwise) distinct real numbers, prove the inequality

$$\left|\frac{a+b}{a-b}\right| + \left|\frac{b+c}{b-c}\right| + \left|\frac{c+a}{c-a}\right| \geq 2$$

and determine all cases of equality.

Prove that if we also impose $a, b, c \geq 0$, then

$$\left|\frac{a+b}{a-b}\right| + \left|\frac{b+c}{b-c}\right| + \left|\frac{c+a}{c-a}\right| > 3,$$

with the value 3 being the best constant possible.

DAN SCHWARZ

Solution. Denote $x = \dfrac{a+b}{a-b}$, $y = \dfrac{b+c}{b-c}$, $z = \dfrac{c+a}{c-a}$; then it is easily seen that

$$\prod(x-1) = \frac{8bca}{\prod(a-b)} = \frac{8abc}{\prod(a-b)} = \prod(x+1),$$

whence $xy + yz + zx = -1$.

But since $(xy)(yz)(zx) = (xyz)^2 \geq 0$ we must have at least one of the factors being non-negative, say $xy \geq 0$. Then

$$|x| + |y| + |z| \geq |x+y| + |z| \geq 2\sqrt{|zx+yz|} = 2\sqrt{1+xy} \geq 2.$$

Equality occurs for $xy = 0$, say $x = 0$, and also $|y| = 1$, with $z = -y$, when $b = -a$ and $c = 0$. Thus equality cases are $\boxed{\{a,b,c\} = \{t,-t,0\}}$, for $t \neq 0$. However, since clearly the value of the expression for a, b, c is the same as that for ta, tb, tc, with $t \neq 0$, the only essential solution is $\boxed{\{1, -1, 0\}}$.

If we also impose $a, b, c \geq 0$ then, since the expression is symmetric, we may assume $0 \leq a < b < c$. Consequently

$$\left|\frac{a+b}{a-b}\right| + \left|\frac{b+c}{b-c}\right| + \left|\frac{c+a}{c-a}\right| = 3 + \frac{2a}{b-a} + \frac{2b}{c-b} + \frac{2a}{c-a} > 3.$$

The fact we can approach 3 as close as wanted is simply argued by taking $a = 0$, $c = (2n+1)b$ for some $n \in \mathbb{N}^*$, when

$$\frac{2a}{b-a} + \frac{2b}{c-b} + \frac{2a}{c-a} = \frac{1}{n}$$

is taking as small a value as wanted for large enough n. ■

Remarks. Consider $\sigma = a + b + c$; the inequality becomes

$$\left|\frac{a-\sigma}{b-c}\right| + \left|\frac{b-\sigma}{c-a}\right| + \left|\frac{c-\sigma}{a-b}\right| \geq 2.$$

Taking $a' = a - \sigma$, $b' = b - \sigma$, $c' = c - \sigma$, it writes again

$$\left|\frac{a'}{b'-c'}\right| + \left|\frac{b'}{c'-a'}\right| + \left|\frac{c'}{a'-b'}\right| \geq 2,$$

a seemingly more difficult inequality, **asked to the Seniors**. Its equality cases are totally similar. However, imposing there $a', b', c' \geq 0$ just rules out the equality cases, and 2 stays best constant possible.

Problem 4. Consider a set X with $|X| = n \geq 1$ elements. A family $\mathscr{F}$ of distinct subsets of X is said to have property $\mathscr{P}$ if there exist $A, B \in \mathscr{F}$ so that $A \subset B$ and $|B \setminus A| = 1$.

i) Determine the least value m, so that any family $\mathscr{F}$ with $|\mathscr{F}| > m$ has property $\mathscr{P}$.

ii) Describe all families $\mathscr{F}$ with $|\mathscr{F}| = m$, and not having property $\mathscr{P}$.

Dan Schwarz

Solution. i) We claim that $\boxed{m = 2^{n-1}}$. Denote by $\mathscr{P}(X)$ the set of all parts (subsets) of X (inclusive the empty set $\emptyset$, and X). It is well-known (and easy to prove) that $\mathscr{P}(X)$ contains $2^{|X|} = 2^n$ elements. Take an arbitrary $x \in X$, and consider the 2^{n-1} doubletons $\{S, S \cup \{x\}\}$, with $S \in \mathscr{P}(X \setminus \{x\})$. These doubletons make up a partition of $\mathscr{P}(X)$

$$\mathscr{P}(X) = \bigcup_{S \in \mathscr{P}(X \setminus \{x\})} \{S, S \cup \{x\}\}.$$

By the pigeonhole principle, if $|\mathscr{F}| > 2^{n-1}$, then $\mathscr{F}$ will have to contain one full doubleton $\{S_0, S_0 \cup \{x\}\}$, and then, taking $A = S_0$ and $B = S_0 \cup \{x\}$, we will have $A \subset B$ and $|B \setminus A| = 1$, therefore the family $\mathscr{F}$ will have property $\mathscr{P}$. On the other hand, a family $\mathscr{F}$ with $|\mathscr{F}| = 2^{n-1}$, not having property $\mathscr{P}$, must contain exactly one member each from each of such doubletons, by the same pigeonhole argument; we will see this argument used at point ii).

A model for a family $\mathscr{F}$ having 2^{n-1} elements, but not having property $\mathscr{P}$, is the family $\mathscr{F}_e$ of all even cardinality subsets of X. This ensures that for any $A, B \in \mathscr{F}_e$ with $A \subset B$ we have $|B \setminus A| \geq 2$. It remains to prove that $|\mathscr{F}_e| = 2^{n-1}$.

For those of you who know about binomial coefficients, the proof is classical. We have

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n.$$

Taking $x = 1$ yields $(1+x)^n = (1+1)^n = 2^n$; on the other hand, taking $x = -1$ yields $(1+x)^n = (1-1)^n = 0$. This shows that

$$\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots = 2^{n-1},$$

and since $\binom{n}{k}$ counts the subsets of X having k elements, it means the LHS is the cardinality of $\mathscr{F}_e$, while the RHS is the cardinality of $\mathscr{F}_o$, the family of all odd cardinality subsets of X, which thus also works as a second model.

In the absence of such knowledge, a simple argument shows that the number of subsets of even cardinality of a set X with n elements is equal to the number of subsets of odd cardinality. For $n = 1$ this trivially checks. Now consider a set X with $n > 1$ elements and an arbitrary fixed element x of it. A one-to-one correspondence between the subsets of even cardinality of X and the 2^{n-1} subsets of $X \setminus \{x\}$ is now readily established; if such subset S of even cardinality does not contain x, it is mapped to itself, while if it contains x, it is mapped to $S \setminus \{x\}$. Thus the total number of subsets of even cardinality of X is 2^{n-1}, just the same as the total number of subsets of odd cardinality of X.

ii) Let $0 \leq k < n$. If such a family $\mathscr{F}$ contains all subsets of X having cardinality k, then no subset of X of cardinality $k+1$ may belong to $\mathscr{F}$. This is so, because for any subset B with $|B| = k+1$, taking an arbitrary $x \in B$ and $A = B \setminus \{x\}$, we will have $A \subset B$ and $|B \setminus A| = 1$. But $|A| = k$, so $A \in \mathscr{F}$, therefore $B \notin \mathscr{F}$, since $\mathscr{F}$ does not have property $\mathscr{P}$.

Also, if such a family $\mathscr{F}$ contains no subset of X having cardinality k, then all subsets of X having cardinality $k+1$ must belong to $\mathscr{F}$. This is so, because for any subset B with $|B| = k+1$, taking an arbitrary $x \in B$ and $A = B \setminus \{x\}$, we will have $A \subset B$ and $|B \setminus A| = 1$. But $|A| = k$, so $A \notin \mathscr{F}$, therefore $B \in \mathscr{F}$, since $\mathscr{F}$ does not have property $\mathscr{P}$, so by the pigeonhole argument previously announced, exactly one member of the doubleton $\{A, B\}$ must belong to $\mathscr{F}$.

Now, either $\emptyset$ belongs to $\mathscr{F}$, or it does not. In the former case, it follows by what has been said, that all subsets with $0, 2, \ldots$ elements need belong to $\mathscr{F}$, while those with $1, 3, \ldots$ elements must not, hence $\mathscr{F} = \mathscr{F}_e$. In the latter case, again by what has been said, all subsets with $1, 3, \ldots$ elements need belong to $\mathscr{F}$, while those with $0, 2, \ldots$ elements must not, hence $\mathscr{F} = \mathscr{F}_o$. Thus only the two families $\boxed{\mathscr{F}_e, \mathscr{F}_o}$ answer the condition.[1] ■

END

[1] The issue of the threshold value m, renouncing the condition $|B \setminus A| = 1$, is settled by Sperner's theorem, stating that the size of the largest antichain of the poset of the parts of a set with n elements (ordered by inclusion) is $m = \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}$. So for $|\mathscr{F}| > m$ there will exist $A, B \in \mathscr{F}$ with $A \subset B$. The model(s) is/are given by the set of parts having $\lfloor n/2 \rfloor$, respectively $\lceil n/2 \rceil$ elements. For even n having $\lfloor n/2 \rfloor = \lceil n/2 \rceil = n/2$, there is only one model; while for odd n there are two. One may explore other variations on this theme.