IMAR 2016

Problema 1. Fie n un număr natural mai mare sau egal cu 3 şi fie $a_0 = n$. Există o permutare $a_1, a_2, \ldots, a_{n-1}$ a primelor $n-1$ numere naturale nenule, astfel încât $\sum_{j=0}^{k-1} a_j$ să fie divizibil cu a_k, pentru toţi indicii $k < n$?

Problema 2. Fie n un număr natural nenul. Există un poligon planar şi un punct în planul său, astfel încât orice dreaptă care trece prin acest punct să intersecteze frontiera poligonului în exact $2n$ puncte?

Problema 3. Fie n un număr întreg mai mare sau egal cu 2, fie Q_n graful format din vârfurile şi muchiile unui n-cub şi fie T un subgraf conex, fără cicluri, care conţine toate vârfurile lui Q_n. Arătaţi că există o muchie în Q_n, care, adăugată lui T, produce un ciclu de lungime mai mare sau egală cu $2n$, ale cărui vârfuri şi muchii sunt distincte două câte două.

Vârfurile n-cubului sunt cele 2^n n-tuplete binare posibile; două vârfuri sunt legate printr-o muchie, dacă şi numai dacă n-tupletele corespunzătoare diferă într-o singură poziţie.

Problema 4. Un număr natural nenul m se numeşte *perfect*, dacă suma divizorilor săi naturali, inclusiv 1 şi m, este egală cu $2m$. Determinaţi numerele naturale nenule n, pentru care numărul $n^n + 1$ este perfect.

IMAR 2016 — Solutions

Problem 1. Fix an integer $n \geq 3$ and let $a_0 = n$. Does there exist a permutation a_1, a_2, …, a_{n-1} of the first $n-1$ positive integers such that $\sum_{j=0}^{k-1} a_j$ is divisible by a_k for all indices $k < n$?

Solution. The answer is in the affirmative. If n is odd, set $a_1 = 1$ and $a_2 = 2$, and if $n > 3$ define the other a_k recursively by $a_k a_{k-1} = a_0 + a_1 + \cdots + a_{k-1}$, $k = 3, \ldots, n$. It is easily seen that $a_{k+1} = a_{k-1} + 1$, $k = 3, \ldots, n-1$, so $a_{2k} = k+1$ and $a_{2k+1} = k + (n+1)/2$, $k = 1, \ldots, (n-1)/2$. By construction, the integers a_k satisfy the divisibility condition (even at $k = n$). The a_{2k} and a_{2k+1}, $k = 1, \ldots, (n-1)/2$, form strictly increasing sequences of integers greater than 1, and no a_{2i} equals an a_{2j+1}, $i, j = 1, \ldots, (n-1)/2$, so the a_k, $k = 1, \ldots, n-1$, form an injective sequence of positive integers. Since they all lie in the range 1, 2, …, $n-1$, they form indeed a permutation of the latter.

Similarly, if n is even, set $a_1 = 2$ and define a_k, $k = 2, \ldots, n-2$, by the same recursive relation, to get $a_{2k-1} = k+1$ and $a_{2k} = k + n/2$, $k = 1, \ldots, n/2 - 1$. Setting $a_{n-1} = 1$ settles the case as above and completes the proof.

Remark. For an odd $n > 3$, we may equally well start by setting $a_1 = 1$ and $a_2 = (n+1)/2$, and define the remaining a_k by the recursive relation in the solution, to obtain another sequence of positive integers satisfying the divisibility condition. Explicitly, this time $a_{2k} = k + (n-1)/2$, $k = 1, \ldots, (n-1)/2$, and $a_{2k+1} = k+2$, $k = 1, \ldots, (n-3)/2$. The trouble with this sequence is that it is not injective: $a_2 = a_{n-2} = (n+1)/2$; in fact, $(2, n-2)$ is the only pair of indices where injectivity fails. Since a_1, a_2, …, a_{n-1} all lie in the range 1, 2, …, $n-1$, they miss some integer in this range: indeed, 2 is never hit. Forcing $a_{n-2} = 2$ without changing the other a_k makes a_1, a_2, …, a_{n-1} into a permutation of 1, 2, …, $n-1$, but divisibility fails at the last step: $a_{n-1} = n-1$ does not divide $\sum_{k=0}^{n-2} a_k = n(n-1)/2 + 1$. Forcing $a_2 = 2$ and keeping the recursive definition of the a_k produces the permutation in the solution.

A similar phenomenon occurs if n is even, $n \geq 6$; and if $n \geq 8$, there are even two pairs of indices where injectivity fails.

Problem 2. Given a positive integer n, does there exist a planar polygon and a point in its plane such that every line through that point meets the boundary of the polygon at exactly $2n$ points?

Solution. The answer is in the affirmative. To describe the configuration, fix a coordinate frame and let a_0, a_1, …, a_{4n-1} be real numbers such that $a_0 > 0 > a_2 > a_{4n-2} > a_4 > a_{4n-4} > \cdots > a_{2n-2} > a_{2n+2} > a_{2n}$, and $a_{2n+1} < a_{2n+3} < \cdots < a_{4n-1} < 0 < a_1 < a_3 < \cdots < a_{2n-1}$. Setting $A_{2k} = 0 \times a_{2k}$ and $A_{2k+1} = a_{2k+1} \times 0$, $k = 0, \ldots, 2n-1$, the polygon $A_0 A_1 \ldots A_{4n-1}$ and the origin satisfy the condition in the statement: the x-axis (respectively, y-axis) contains all vertices of odd (respectively, even) rank and no other points on the boundary; and every line through the first and third (respectively, second and fourth) quadrants crosses the sides $A_0 A_1$ and $A_{2n+k-1} A_{2n+k}$ (respectively, $A_{4n-1} A_0$ and $A_k A_{k+1}$), $k = 1, \ldots, 2n-1$, and no other side, since the remaining sides all lie in the other two quadrants.

Problem 3. Fix an integer $n \geq 2$, let Q_n be the graph consisting of all vertices and all edges of an n-cube, and let T be a spanning tree in Q_n. Show that Q_n has an edge whose adjunction to T produces a simple cycle of length at least $2n$.

Solution. For every vertex v of Q_n, let v' be the antipodal (opposite) vertex, consider the unique path in T from v to v' and orient its first edge away from v. Since T has fewer edges than vertices, some edge, say xy, has been assigned two orientations. The (combinatorial) distance between two antipodes of Q_n is n in Q_n, so it is at least n in T, and the unique path in T, $x' \ldots yx \ldots y'$, from x' to y' has length at least $2n-1$. Finally, since xy is an edge in Q_n, so is $x'y'$; adjunction of the latter to the unique path in T from x' to y' yields the required cycle.

Remark. The result is best possible, since adding any further edge to any breadth-first search tree in Q_n yields a simple cycle of length $2n$ or less. The argument also shows that the diameter of any spanning tree in Q_n is at least $2n-1$.

Problem 4. A positive integer m is *perfect* if the sum of all its positive divisors, 1 and m inclusive, is equal to $2m$. Determine the positive integers n such that $n^n + 1$ is a perfect number.

Solution. There is only one such integer, namely, $n = 3$; it is readily checked that that $3^3 + 1 = 28$ is perfect.

If n is odd, then $n^n + 1$ is even, so it is of the form $2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are both prime (Euler's theorem on the structure of perfect even integers). Rule out the trivial case $n = 1$, to assume $n > 1$, and write $n^n + 1 = (n+1)(n^{n-1} - n^{n-2} + \cdots - n + 1)$. Since n is odd, the first factor is even and the second is odd; and since $n > 1$, the latter is greater than 1 (simply rewrite it in the form $1 + n(n-1)(1 + n^2 + \cdots + n^{n-3})$). It follows that $n + 1 = 2^{p-1}$, so $2^p - 1 = 2n + 1$, and $n^n + 1 = (n+1)(2n+1) = 2n^2 + 3n + 1$ which forces $n = 3$.

We now rule out the other parity of n; recall that the existence of perfect odd numbers is still an open question.

To begin, we show that n is divisible by 3. Suppose, if possible, this is not the case, so $N = n^n + 1 \equiv 2 \pmod 3$, since n is even. It follows that N is not square, and $d + N/d$ is divisible by 3 for each divisor d of N, so $\sigma(N) = \sum_{d \mid N} d = \sum_{d \mid N,\, d < \sqrt{N}} (d + N/d)$ is divisible by 3, while $2N$ is certainly not. Consequently, n is divisible by 3, and since it is even, n is divisible by 6.

Next, consider the positive even integer $k = n^{n/6}$ to write $N = k^6 + 1 = (k^2+1)(k^4 - k^2 + 1)$. Since $k^4 - k^2 + 1 = (k^2 - 2)(k^2 + 1) + 3$, and $k^2 + 1$ is never divisible by 3, the integers $k^2 + 1$ and $k^4 - k^2 + 1$ are coprime, so $2N = \sigma(N) = \sigma(k^2 + 1)\sigma(k^4 - k^2 + 1)$. Since N is odd, it follows that (exactly) one of $\sigma(k^2 + 1)$ and $\sigma(k^4 - k^2 + 1)$ is odd, and since $k^2 + 1$ and $k^4 - k^2 + 1$ are both odd, one of them is square. However, $k^2 < k^2 + 1 < (k+1)^2$ and $(k^2 - 1)^2 < k^4 - k^2 + 1 < (k^2)^2$, and we reach a contradiction. This ends the proof.