IMAR 2015

Problema 1. Determinaţi numerele naturale N, care au următoarea proprietate: oricare ar fi numărul natural $n \geq 3$, există n numere naturale nenule $a_1, a_2, \ldots, a_n$, distincte două câte două, astfel încât

$$N = \frac{(a_1 + 1)(a_2 + 1)\cdots(a_n + 1) - 1}{a_1 a_2 \cdots a_n}.$$

Problema 2. Fie n un număr natural nenul şi fie $\mathcal{G}_n$ mulţimea grafurilor simple care au exact n vârfuri. Pentru fiecare vârf v al unui graf din $\mathcal{G}_n$, fie $k(v)$ cardinalul unei mulţimi independente maximale de vecini ai lui v. Determinaţi $\max_{G \in \mathcal{G}_n} \sum_{v \in V(G)} k(v)$ şi grafurile din $\mathcal{G}_n$ care realizează acest maximum.

Un graf este *simplu*, dacă nu are nici bucle, nici muchii multiple. O mulţime de vârfuri ale unui graf este *independentă*, dacă oricare două vârfuri din această mulţime nu sunt legate printr-o muchie.

Problema 3. Fie ABC un triunghi, fie A_1, B_1, respectiv C_1, punctele diametral opuse vârfurilor A, B, respectiv C, în cercul ABC şi fie X un punct în planul ABC, nesituat pe niciuna dintre dreptele BC, CA, AB. Perpendiculara în B, pe dreapta XB, intersectează perpendiculara în C, pe dreapta XC, în punctul A_2; punctele B_2 şi C_2 sunt definite în mod analog. Arătaţi că dreptele A_1A_2, B_1B_2 şi C_1C_2 sunt concurente.

Problema 4. **(a)** Arătaţi că, dacă $I \subset \mathbb{R}$ este un interval închis şi mărginit şi $f\colon I \to \mathbb{R}$ este o funcţie polinomială monică neconstantă, astfel încât $\max_{x \in I} |f(x)| < 2$, atunci există o funcţie polinomială monică neconstantă $g\colon I \to \mathbb{R}$, astfel încât $\max_{x \in I} |g(x)| < 1$.

(b) Arătaţi că există un interval închis şi mărginit $I \subset \mathbb{R}$, astfel încât $\max_{x \in I} |f(x)| \geq 2$, oricare ar fi funcţia polinomială monică neconstantă $f\colon I \to \mathbb{R}$.

IMAR 2015 — Solutions

Problem 1. Determine all positive integers expressible, for every integer $n \geq 3$, in the form $((a_1+1)(a_2+1)\cdots(a_n+1)-1)/(a_1a_2\cdots a_n)$, where $a_1, a_2, \ldots, a_n$ are pairwise distinct positive integers.

Solution. The required numbers are 2 and 3. Clearly, a positive integer satisfying the condition in the statement is at least 2.

The integers greater than 3 are ruled out by noticing that if a_1, a_2, a_3 are pairwise distinct positive integers, then

$$\begin{aligned}\frac{(a_1+1)(a_2+1)(a_3+1)-1}{a_1a_2a_3} &= 1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_1a_2}+\frac{1}{a_1a_3}+\frac{1}{a_2a_3}\\ &\leq 1+1+\frac{1}{2}+\frac{1}{3}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 3}+\frac{1}{2\cdot 3}=3+\frac{5}{6}.\end{aligned}$$

To complete the proof, fix an integer $n \geq 3$, consider an integer $a \geq 3$, and let $a_1 = a-2$, let $a_k = a^{2^{k-2}}$, $k = 2, \ldots, n-1$, and let $a_n = a^{2^{n-2}} - 2$. Then $1 \leq a_1 < a_2 < \cdots < a_n$, and

$$\begin{aligned}\frac{(a_1+1)(a_2+1)\cdots(a_n+1)-1}{a_1a_2\cdots a_n} &= \frac{\left((a-1)\prod_{k=2}^{n-1}\left(a^{2^{k-2}}+1\right)\right)\left(a^{2^{n-2}}-1\right)-1}{(a-2)\,a^{1+2+\cdots+2^{n-3}}\left(a^{2^{n-2}}-2\right)}\\ &= \frac{\left(a^{2^{n-2}}-1\right)^2-1}{(a-2)\,a^{2^{n-2}-1}\left(a^{2^{n-2}}-2\right)} = \frac{a^{2^{n-2}}\left(a^{2^{n-2}}-2\right)}{(a-2)\,a^{2^{n-2}-1}\left(a^{2^{n-2}}-2\right)}\\ &= \frac{a}{a-2} = 1+\frac{2}{a-2},\end{aligned}$$

which is integral if and only if $a = 3$ or $a = 4$. The former shows that 3 satisfies the required condition, and the latter shows that so does 2.

Problem 2. Let n be a positive integer and let $\mathcal{G}_n$ be the set of all simple graphs on n vertices. For each vertex v of a graph in $\mathcal{G}_n$, let $k(v)$ be the maximal cardinality of an independent set of neighbours of v. Determine $\max_{G\in\mathcal{G}_n}\sum_{v\in V(G)} k(v)$ and the graphs in $\mathcal{G}_n$ that achieve this value.

Solution. The required maximum is $\lfloor n^2/2 \rfloor$ and is achieved by the complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ alone. To prove this, let G be a simple graph on n vertices, and let S be a maximal independent set of vertices of G. If v is a member of S, then $k(v) \leq n - |S|$, since v has at most $n-|S|$ neighbours. If a vertex v is not in S, then $k(v) \leq |S|$, since $k(v)$ is the size of an independent set. Consequently,

$$\sum_{v\in V(G)} k(v) \leq |S|(n-|S|) + (n-|S|)|S| \leq \lfloor n^2/2 \rfloor.$$

The complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ clearly achieves the upper bound. Achieving the upper bound requires that all of S be adjacent to all of $V(G)\setminus S$ and that $V(G)\setminus S$ be an independent set, so the extremal graph is unique.

Problem 3. Let ABC be a triangle, let A_1, B_1, C_1 be the antipodes of the vertices A, B, C, respectively, in the circle ABC, and let X be a point in the plane ABC, collinear with no two vertices of the triangle ABC. The line through B, perpendicular to the line XB, and the line through C, perpendicular to the line XC, meet at A_2; the points B_2 and C_2 are defined similarly. Show that the lines A_1A_2, B_1B_2 and C_1C_2 are concurrent.

Solution. Let the lines B_2C_2, C_2A_2, A_2B_2 meet the circle ABC again at A_3, B_3, C_3, respectively. The angle AA_3A_1 is a right angle, so A_1A_3 is perpendicular to B_2C_2 and hence parallel to XA, and the parallel lines XA and A_1A_3 are equidistant from the circumcentre O of the triangle ABC. Hence A_1A_3 passes through the reflection X' of X across O. Similarly, the lines B_1B_3 and C_1C_3 pass through X', and therefore $X'A_1 \cdot X'A_3 = X'B_1 \cdot X'B_3 = X'C_1 \cdot X'C_3$, a quantity denoted r^2.

It follows that A_1, B_1, C_1 are the poles of the lines B_2C_2, C_2A_2, A_2B_2, respectively, relative to the circle of radius r centred at X', so the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are in perspective, by the well-known lemma below.

Lemma. *If XYZ is a triangle, and X', Y', Z' are the poles of the lines YZ, ZX, XY, respectively, relative to a conic γ, then the triangles XYZ and $X'Y'Z'$ are in perspective.*

For completeness, we include a proof of the lemma. By projectivity, we may and will assume that γ is a circle (which is precisely the case in the problem). Let the lines XX' and YY' (projectively) meet at O. If necessary, apply a projective transformation that transforms γ into another circle and sends O to the centre of this circle, to assume further that γ is centred at O. Then the lines OX' and YZ are perpendicular, and the points O, X, X' are collinear, so the lines OX and YZ are perpendicular. Similarly, the lines OY and ZX are perpendicular, so O is the orthocentre of the triangle XYZ. Consequently, the lines OZ and XY are perpendicular, and since so are the lines OZ' and XY, the points O, Z, Z' are indeed collinear.

Problem 4. **(a)** Show that, if $I \subset \mathbb{R}$ is a closed bounded interval, and $f\colon I \to \mathbb{R}$ is a non-constant monic polynomial function such that $\max_{x\in I} |f(x)| < 2$, then there exists a non-constant monic polynomial function $g\colon I \to \mathbb{R}$ such that $\max_{x\in I} |g(x)| < 1$.

(b) Show that there exists a closed bounded interval $I \subset \mathbb{R}$ such that $\max_{x\in I} |f(x)| \geq 2$ for every non-constant monic polynomial function $f\colon I \to \mathbb{R}$.

Solution. **(a)** Let $I \subset \mathbb{R}$ be a closed bounded interval, let $\mathcal{P}(I)$ be the set of all polynomial functions $f\colon I \to \mathbb{R}$, and let $\|f\| = \max_{x\in I} |f(x)|$. Define $A\colon \mathcal{P}(I) \to \mathcal{P}(I)$ by $Af(x) = (f(x))^2 - \frac{1}{2}\|f\|^2$, $x \in I$. If f is monic (respectively, non-constant), then so is Af. Further, $\|Af\| = \frac{1}{2}\|f\|^2$, so

$$\|A^n f\| = 2\left(\frac{1}{2}\|f\|\right)^{2^n},$$

where $A^n = A \circ \cdots \circ A$, n times, is the n-th iterate of A. Consequently, if $\|f\| < 2$ for some f in $\mathcal{P}(I)$, then $\|A^n f\| < 1$ for n large enough.

(b) In the notation above, let $I = [-2, 2]$, and define $B\colon \mathcal{P}(I) \to \mathcal{P}(I)$ by

$$Bf(x) = \frac{1}{2}f(\sqrt{x+2}) + \frac{1}{2}f(-\sqrt{x+2}), \quad x \in I.$$

If f is monic, so is Bf. Further, $\|Bf\| \leq \|f\|$, and $\deg Bf = \lfloor \frac{1}{2}\deg f \rfloor$. Consequently, if $2^n \leq \deg f < 2^{n+1}$, where n is a non-negative integer, then $\deg B^n f = 1$. To conclude the proof, notice that every monic polynomial function of degree 1 on I has norm at least 2.

Remark. Tchebysheff's polynomials and their properties provide an alternative solution. If n is a non-negative integer, and $-1 \le x \le 1$, the Tchebysheff polynomial (real-valued function) of degree n is defined by

$$T_n(x) = \cos(n \arccos x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} x^{n-2k}(1-x^2)^k = 2^{n-1}x^n + \cdots \ ;$$

$\|T_n\| = \max_{|x|\le 1} |T_n(x)| = 1$. It is well-known that if $f\colon [-1,1] \to \mathbb{R}$ is a monic polynomial function of degree $n \ge 1$, then $\|f\| = \max_{|x|\le 1} |f(x)| \ge 2^{1-n}$, and equality holds if and only if $f = 2^{1-n}T_n$. Normalising accordingly, if $f\colon [-2,2] \to \mathbb{R}$ is a monic polynomial function of degree $n \ge 1$, then $\|f\| = \max_{|x|\le 2} |f(x)| \ge 2$, and equality holds if and only if $f(x) = 2T_n(x/2)$.