IMAR 2014 — Solutions

Problem 1. Let ABC be a triangle and let M be the midpoint of the side BC. The circle of radius MA centred at M meets the lines AB and AC again at B' and C', respectively, and the tangents to this circle at B' and C' meet at D. Show that the perpendicular bisector of the segment BC bisects the segment AD.

Solution. Let A' be the antipode of A in the circle $AB'C'$, and let A'' be the point where this circle meets again the line through A parallel to BC (the points A and A'' may coincide). Since M is the midpoint of the side BC, the lines AA'', AB', AA', AC' form a harmonic pencil. Consequently, so do the lines XA'', XB', XA', XC' for any point X on the circle $AB'C'$. Now let $X = B'$ and $X = C'$ to infer that the pencils $B'A''$, $B'D$, $B'A'$, $B'C'$ and $C'A''$, $C'B'$, $C'A'$, $C'D$ are both harmonic. Since the two pencils share the line $B'C'$, the points A'', A' D lie on a line which is clearly perpendicular to BC and the conclusion follows.

Problem 2. Let ϵ be a positive real number. A positive integer will be called ϵ-*squarish* if it is the product of two integers a and b such that $1 < a < b < (1+\epsilon)a$. Prove that there are infinitely many occurences of six consecutive ϵ-squarish integers.

Solution. If N is a large enough positive integer, then $N^2 - 1 = (N-1)(N+1)$ and $N^2 - 4 = (N-2)(N+2)$ are both ϵ-squarish. Next, if k is a large enough positive integer and $N = (k-1)(k+2) = k^2 + k - 2$, then $N^2 = (k-1)^2(k+2)^2$, $N^2 - 2 = (k^2-2)(k^2+2k-1)$ and $N^2 - 5 = (k^2-k-1)(k^2+3k+1)$ are all three ϵ-squarish. Finally, if n is a large enough positive integer and $N = 2n^2 - 2$, then $N^2 - 3 = (2n^2 - 2n - 1)(2n^2 + 2n - 1)$ is ϵ-squarish.

Consequently, $N^2 - 5$, $N^2 - 4$, $\ldots$, N^2 are six consecutive ϵ-squarish integers, provided that $N = k^2 + k - 2 = 2n^2 - 2$, where k and n are sufficiently large integers. To conclude, write $m = 2k + 1$ to turn the condition into a Pell equation, $m^2 - 8n^2 = 1$, which has arbitrarily large solutions,

$$\begin{pmatrix} m_r \\ n_r \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 1 & 3 \end{pmatrix}^r \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad r \in \mathbb{N}.$$

Problem 3. Let f be a primitive polynomial with integral coefficients (their highest common factor is 1) such that f is irreducible in $\mathbb{Q}[X]$, and $f(X^2)$ is reducible in $\mathbb{Q}[X]$. Show that $f = \pm(u^2 - Xv^2)$ for some polynomials u and v with integral coefficients.

For instance, if a and b are coprime integers and a is odd, then $f = a^4X^2 + 4b^4$ is a primitive polynomial in $\mathbb{Z}[X]$, irreducible in $\mathbb{R}[X] \supset \mathbb{Q}[X]$, $f(X^2) = a^4X^4 + 4b^4 = (a^2X^2 - 2abX + 2b^2)(a^2X^2 + 2abX + 2b^2)$ is reducible in $\mathbb{Z}[X] \subset \mathbb{Q}[X]$, and $f = (a^2X + 2b^2)^2 - X \cdot (2ab)^2$.

Solution. Unless otherwise stated, we work in $\mathbb{Q}[X]$. Since the case $\deg f = 1$ is easily dealt with, let $\deg f \geq 2$ and write $f(X^2) = gh$, where g and h both have a positive degree, and g is irreducible. Next, write $g = a(X^2) + Xb(X^2)$ and $h = c(X^2) + Xd(X^2)$ to infer (from $f(X^2) = gh$ by an obvious argument on the parity of degrees) that

$$ad + bc = 0, \tag{1}$$

so $f = ac + Xbd$, whence

$$af = (a^2 - Xb^2)c. \tag{2}$$

We now show that a and b are coprime. Alternatively, but equivalently, $\delta = \gcd(a, b)$ is a constant. To this end, write $a = a_1\delta$ and $b = b_1\delta$, and refer to the irreducibility of g to deduce that $\delta(X^2)$ is either associated with g, a case to be ruled out in the sequel, or a constant, in which case we are through.

In the former case, $a_1(X^2)+Xb_1(X^2)$ is a constant, so $b_1 = 0$, whence $b = 0$ and $g = a(X^2)$, and (1) forces one of a and d to be 0. The fact that g is not constant rules out the case $a = 0$, so $d = 0$, $f = ac$ and $h = c(X^2)$. Since f is irreducible, one of a and c must be a constant, hence so must be one of g and h — a contradiction, since both have a positive degree. Incidentally, notice that we have just proved that $b \neq 0$.

Notice further that a and X are also coprime: otherwise, $a(0) = 0$, so $g(0) = 0$, hence $f(0) = 0$, contradicting the fact that f is irreducible and $\deg f \geq 2$.

Consequently, a and $a^2 - Xb^2$ are coprime, so a divides c by (2), and $f = (a^2 - Xb^2)c_1$ for some c_1 in $\mathbb{Q}[X]$. Since f is irreducible, one of $a^2 - Xb^2$ and c_1 must be a constant. Since $b \neq 0$ by the remark at the end of the last but one paragraph, $a^2 - Xb^2$ cannot be constant, so c_1 is a constant.

From now on we work in $\mathbb{Z}[X]$. By the preceding, $nf = m(a^2 - Xb^2)$ for some integers m and n, and some u and v in $\mathbb{Z}[X]$. Fix a prime integer p and write $m = p^\mu m_1$, $n = p^\nu n_1$, $u = p^\alpha u_1$, $v = p^\beta v_1$, where α, β, μ, ν are non-negative integers, and none of m_1, n_1, u_1, v_1 is divisible by p. To make a choice, let $\alpha \leq \beta$; the case $\alpha > \beta$ is dealt with similarly. Since f is primitive, the relation

$$p^\nu n_1 f = p^{\mu+2\alpha} m_1 \left(u_1^2 - Xp^{2(\beta-\alpha)}v_1^2\right)$$

implies that $\nu \geq \mu + 2\alpha$. If we show that $\nu = \mu + 2\alpha$, we are through.

Suppose, if possible, that $\nu > \mu + 2\alpha$, to deduce that p divides $u_1^2 - Xp^{2(\beta-\alpha)}v_1^2$, so it also divides

$$u_1^2(X^2) - X^2p^{2(\beta-\alpha)}v_1^2(X^2) = \left(u_1(X^2) - Xp^{\beta-\alpha}v_1(X^2)\right)\left(u_1(X^2) + Xp^{\beta-\alpha}v_1(X^2)\right).$$

Since p is prime, it must divide one of $u_1(X^2) \pm Xp^{\beta-\alpha}v_1(X^2)$, and an obvious argument on the parity of degrees shows that $u_1(X^2)$ must be divisible by p, and hence so must be u_1 — a contradiction which concludes the proof.

Problem 4. Let n be a positive integer. A *Steiner tree* associated with a finite set S of points in the Euclidean n-space is a finite collection T of straight-line segments in that space such that any two points in S are joined by a unique path in T; its length is the sum of the segment lengths. Show that there exists a Steiner tree of length $1 + (2^{n-1} - 1)\sqrt{3}$ associated with the vertex set of a unit n-cube.

Solution. We describe a recursive procedure for constructing the desired Steiner tree. The case $n = 1$ is handled by a single line segment.

Assume a Steiner tree of length $1 + (2^{n-1} - 1)\sqrt{3}$ associated with the vertex set of a unit n-cube has been constructed such that each vertex of the n-cube is the endpoint of just one segment, and the length of this segment is greater than $\sqrt{3}/6$. Now consider a unit $(n+1)$-cube. Select a pair of opposite n-faces and consider the n-cube whose vertices are the midpoints of the edges joining the corresponding vertices of these faces. Start with the assumed Steiner tree associated with the vertex set of this n-cube. For each vertex v of this n-cube, delete a segment of length $\sqrt{3}/6$ from the segment terminating at v, then add the segments joining the new endpoint to the two vertices of the $(n+1)$-cube that are endpoints of the edge containing v. By the Pythagorean theorem, these segments each have length $1/\sqrt{3}$. The net effect of these changes is to add $2^n(2/\sqrt{3} - \sqrt{3}/6) = 2^{n-1}\sqrt{3}$ to the length of the tree. Thus the resulting tree has length $1 + (2^n - 1)\sqrt{3}$. It is clearly a Steiner tree associated with the vertex set of the $(n + 1)$-cube and has the additional properties required for the induction.