The IMAR Competition, 2010 — Solutions

Problem 1. Show that a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of plus and minus ones is periodic with period a power of 2, if and only if $\epsilon_n = (-1)^{P(n)}$, $n \in \mathbb{N}$, where P is an integer-valued polynomial with rational coefficients.

(A polynomial P with complex coefficients is *integer-valued* if $P(m)$ is an integer number whenever m is an integer number.)

First Solution. Given an integer-valued polynomial P with rational coefficients, we show that the sequence $((-1)^{P(n)})_{n\in\mathbb{N}}$ is periodic with period a power of 2. Clearly, it is sufficient to show that, for some non-negative integer s, the integer numbers $P(n)$ and $P(n+2^s)$ both have the same parity, whatever $n \in \mathbb{N}$. To this end, consider a positive integer m such that $Q = mP$ is a polynomial with integral coefficients (e.g., let m be the least common multiple of the denominators of the coefficients of P when written in lowest terms), let 2^r be the highest power of 2 dividing m, and let s be an integer greater than r. Write $m = 2^r(2m'+1)$ and fix a positive integer n. Since 2^s divides the difference

$$Q(n+2^s) - Q(n) = m(P(n+2^s) - P(n)) = 2^r(2m'+1)(P(n+2^s) - P(n)),$$

and $s > r$, the conclusion follows.

Conversely, given a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of plus and minus ones which is periodic with period 2^r, let

$$a_k = \begin{cases} 0, & \text{if } \epsilon_k = 1, \\ 1, & \text{if } \epsilon_k = -1, \end{cases} \qquad k = 0, \cdots, 2^r - 1,$$

and consider the polynomial (Lagrange)

$$\begin{aligned} P &= \sum_{k=0}^{2^r-1} a_k \prod_{j\neq k} \frac{X-j}{k-j} \\ &= \sum_{k=0}^{2^r-1} (-1)^{k+1} a_k \left(\frac{1}{k!} \prod_{0\leq j<k} (X-j) \right) \left(\frac{1}{(2^r-k-1)!} \prod_{k<j\leq 2^r-1} (X-j) \right) \\ &= \sum_{k=0}^{2^r-1} (-1)^{k+1} a_k \binom{X}{k} \binom{X-k-1}{2^r-k-1}. \end{aligned}$$

Clearly, P has rational coefficients, is integer-valued, and $P(k) = a_k$, $k = 0, \cdots, 2^r - 1$, so $\epsilon_k = (-1)^{P(k)}$, $k = 0, \cdots, 2^r - 1$. To prove that the latter extends to all of $\mathbb{N}$, it is sufficient to show that $P(n)$ and $P(n + 2^r)$ both have the same parity, whatever $n \in \mathbb{N}$. This amounts to showing that if $j, m \in \mathbb{N}$ and $j < 2^r$, then $\binom{m}{j}$ and $\binom{m+2^r}{j}$ have the same parity. Recalling that $\binom{2^r}{i}$, $i = 1, \cdots, 2^r - 1$, are all even, this follows for instance from the identity

$$\binom{m + 2^r}{j} = \sum_{i=0}^{j} \binom{m - i}{j - i} \binom{2^r}{i};$$

or, what is actually the same, from the argument in the second solution below.

Second Solution. A polynomial P of degree at most k with complex coefficients is integer-valued if and only if

$$P = \sum_{j=0}^{k} a_j \binom{X}{j} = \sum_{j=0}^{k} \frac{a_j}{j!} X(X-1) \cdots (X - j + 1),$$

where the a_j are all integer numbers, so its coefficients are rational.

We show that for such a P, the sequence $((-1)^{P(n)})_{n \in \mathbb{N}}$ is periodic with period 2^r, where $r = \min\{s : 2^s > k\}$. To this end, it suffices to show that if $j < 2^s$ and m is integer, then $\binom{m}{j} \equiv \binom{m+2^s}{j} \bmod 2$. These are the coefficients of X^j in the expansions of $(1 + X)^m$ and $(1 + X)^{m+2^s}$, respectively. The congruence $(1 + X)^{2^s} \equiv 1 + X^{2^s} \bmod 2$ follows easily by induction on s. Hence $(1 + X)^{m+2^s} = (1 + X)^m \left(1 + X^{2^s}\right) \bmod 2$. Since j is less than 2^s, it is immediate that the coefficients of X^j in $(1 + X)^m$ and $(1 + X)^{m+2^s}$ have the same parity.

Conversely, let $\alpha_0, \cdots, \alpha_{2^r-1}$ be arbitrary integers, and let $\beta_0, \cdots, \beta_{2^r-1}$ be the solution to the lower triangular system of linear equations

$$\sum_{i=0}^{2^r-1} \beta_i \binom{j}{i} = \alpha_j, \quad j = 0, \cdots, 2^r - 1.$$

Since the coefficient of β_j in the j-th equation is 1, the β_i are all integers. The polynomial

$$\sum_{i=0}^{2^r-1} \beta_i \binom{X}{i} = \sum_{i=0}^{2^r-1} \frac{\beta_i}{i!} X(X-1) \cdots (X - i + 1)$$

realizes the sequence $(-1)^{\alpha_j}$, $j = 0, \cdots, 2^r - 1$, and its extension with period 2^r.

Problem 2. Given a triangle ABC, let D be the point where the incircle of the triangle ABC touches the side BC. A circle through the vertices B and C is tangent at point E to the incircle of the triangle ABC. Show that the line DE passes through the excentre of the triangle ABC corresponding to the vertex A.

Solution. Let I be the incentre of the triangle ABC, let I_A be the excentre corresponding to the vertex A, and notice that the vertices B and C both lie on the circle of diametre II_A. The line DI_A meets again the latter circle at point K, and the lines BC and IK meet at point L (unless $AB = AC$ in which case the conclusion is obvious). Notice that the line BC is the radical axis of the circles BEC and BIC to deduce that $LB \cdot LC = LI \cdot LK$. On the other hand, $LI \cdot LK = LD^2$, for K is the perpendicular foot dropped from the right-angled vertex D of the triangle DIL. Consequently, the point L is the radical centre of the following three circles: the incircle of the triangle ABC, the circle BEC, and the circle BIC. Since the common tangent at E of the first two circles is their radical axis, it must pass through L. It follows that E is the reflection of D across the line IL, so the lines DE and IL are perpendicular and we are done.

Problem 3. Given an integer number $n \geq 2$, a positive real number A, and $n + 1$ distinct points in the plane, X_0, X_1, $\cdots$, X_n, show that the number of triangles $X_0X_iX_j$ of area A does not exceed $4n\sqrt{n}$.

Solution. Suppose that for some integer $n \geq 2$, there exist $n + 1$ distinct points in the plane, X_0, X_1, $\cdots$, X_n, such that the number of triangles $X_0X_iX_j$ of area A be greater than $4n\sqrt{n}$. Choose the minimal such n, notice that $n \geq 4$, and let G be the graph whose vertices are X_1, $\cdots$, X_n and whose edges are the X_iX_j such that $\operatorname{area} X_0X_iX_j = A$. Then every vertex X_i of G is adjacent to at least $\lfloor 4\sqrt{n} \rfloor$ other vertices, since otherwise, removing X_i would reduce the number of triangles by at most $4\sqrt{n}$, and we would be left with a configuration of n distinct points such that the number of triangles $X_0X_jX_k$ of area A is at least $4n\sqrt{n} - 4\sqrt{n} > 4(n-1)\sqrt{n-1}$, contradicting the minimal choice of n. Consequently, for each X_i, there are at least $\lfloor 4\sqrt{n} \rfloor$ points X_j such that the triangle $X_0X_iX_j$ has area A. These points lie on two parallel lines to the line X_0X_i. One of these linear sets of points, say S_i, contains at least $\frac{1}{2}\lfloor 4\sqrt{n} \rfloor$ points. Consider the points X_j on the first $\lfloor\sqrt{n}\rfloor$ lines S_i, $i = 1, \cdots, \lfloor\sqrt{n}\rfloor$, to get

$$\begin{aligned} n \geq \left| S_1 \cup \cdots \cup S_{\lfloor\sqrt{n}\rfloor} \right| &\geq \sum_{i=1}^{\lfloor\sqrt{n}\rfloor} |S_i| \; - \sum_{1\leq i<j\leq\lfloor\sqrt{n}\rfloor} |S_i \cap S_j| \\ &\geq \frac{1}{2} \lfloor\sqrt{n}\rfloor \lfloor 4\sqrt{n}\rfloor - \binom{\lfloor\sqrt{n}\rfloor}{2}, \end{aligned}$$

which is false for $n \geq 4$.

Problem 4. Let r be a positive integer and let N_r be the smallest positive integer such that the numbers

$$\frac{N_r}{n+r}\binom{2n}{n}, \quad n = 0, 1, 2, \cdots,$$

are all integer. Show that

$$N_r = \frac{r}{2}\binom{2r}{r}.$$

Solution. We first show that $N_r \leq \frac{r}{2}\binom{2r}{r}$ by proving that

$$K(n, r) = \frac{r}{2(n+r)}\binom{2n}{n}\binom{2r}{r}$$

is an integer for all $n \geq 0$ and $r \geq 1$. Notice that

$$K(0, r) = \binom{2r-1}{r} \quad \text{and} \quad K(n, 1) = \frac{1}{n+1}\binom{2n}{n},$$

the latter being a Catalan number, so $K(0, r)$ and $K(n, 1)$ are integers for all n and r. Next, the recurrence relation (which will be proved at the end of the solution)

$$K(n, r+1) - K(n+1, r) = 2K(n, 1)K(0, r)$$

shows by induction on r that $K(n, r)$ is indeed an integer for all n and r.

Suppose now that, for some r, $N_r < M_r = \frac{r}{2}\binom{2r}{r}$. The integer N_r is the least common multiple, over all $n \geq 0$, of the denominators of the numbers $\frac{1}{n+r}\binom{2n}{n}$ when written in lowest terms. By the argument above, M_r is a multiple of all these denominators. Hence N_r divides M_r. By the definition of N_r, any prime p that divides M_r/N_r also divides $K(n, r)$ for each $n \geq 0$.

Since $K(n, s+1) = K(n+1, s) + 2K(n,1)K(0,s)$, induction on m shows that p divides $K(n,m)$ for all $n \geq 0$ and $m \geq r$.

Now choose k such that $p^k \geq r$. Since p divides $\binom{p^k}{j}$, $j = 1, 2, \cdots, p^k - 1$, the identity $\binom{2n}{n} = \sum_{j=1}^{n} \binom{n}{j}^2$ yields $\binom{2p^k}{p^k} \equiv 2 \pmod{p}$. Therefore p does not divide $\frac{1}{2}\binom{2p^k}{p^k} = K(0, p^k)$. This contradicts the preceding paragraph, so $N_r = \frac{r}{2}\binom{2r}{r}$ for all r.

To prove the recurrence relation, notice that

$$\binom{2m+2}{m+1} = 2\binom{2m+1}{m} = 2\,\frac{2m+1}{m+1}\binom{2m}{m},$$

to get

$$\begin{aligned}
&K(n, r+1) - K(n+1, r) = \\
&\frac{r+1}{2(n+r+1)}\binom{2n}{n}\binom{2r+2}{r+1} - \frac{r}{2(n+r+1)}\binom{2n+2}{n+1}\binom{2r}{r} = \\
&\frac{1}{n+r+1}\binom{2n}{n}\binom{2r}{r}\left(2r+1-r\frac{2n+1}{n+1}\right) = \\
&\frac{1}{n+1}\binom{2n}{n}\binom{2r}{r} = 2K(n,1)K(0,r).
\end{aligned}$$