IMAR Mathematical Competition
Bucharest, November 7$^{\text{th}}$, 2009

Problem 1. Given a and b distinct positive integers, show that the system of equations

$$xy + zw = a$$
$$xz + yw = b$$

has only finitely many solutions in integers x, y, z, w.

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Solution. By adding and subtracting the equations we get $(x+w)(y+z) = a+b$ and $(x-w)(y-z) = a-b$, hence by multiplying, $0 < |(x^2-w^2)(y^2-z^2)| = |a^2-b^2|$. Therefore $0 < |x^2-w^2| \leq |a^2-b^2|$ and $0 < |y^2-z^2| \leq |a^2-b^2|$. But an equation $0 < |A^2-B^2| \leq |C|$ has only finitely many integer solutions A, B for fixed C. ■

Remark. Notice that if we allow $a = b$, then we can take $y = z = 1$, arbitrary x, and $w = a - x$.

Problem 2. Of the vertices of a cube, 7 of them have assigned the value 0, and the eighth the value 1. A *move* is selecting an edge and increasing the numbers at its ends by an integer value $k > 0$. Prove that after any finite number of moves, the g.c.d. of the 8 numbers at vertices is equal to 1.

Russia MO

Solution. Let us alternately color the vertices in black and white. After any move, the difference between the sums of the numbers at the black and the white vertices remains 1, therefore the g.c.d. of the 8 numbers is equal to 1 (as it is dividing the difference of the sums mentioned above). ■

Remark. The problem has a short and sweet solution, but it is tricky, since a much stricter truth yields the answer. Question: is it true that after any number of moves, there exist two vertices having co-prime numbers assigned to them?

Problem 3. Consider a convex quadrilateral $ABCD$ with

$$AB = CB \quad \text{and} \quad \angle ABC + 2\angle CDA = \pi,$$

and let E be the midpoint of AC. Show that $\angle CDE = \angle BDA$.

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Solution. Let point X be lying on line BE such that $\angle CXE = \angle CDE$ (X is the (other than C) meeting point of the circumcircle of ΔCDE and the line BE).

Therefore the quadrilateral $DECX$ is cyclic, so $\angle DXE = \angle DCE = \pi - \angle CDA - \angle CAD$. But $\angle CDA = \frac{1}{2}(\pi - \angle ABC) = \angle CAB$, so $\angle DCE = \pi - \angle CAB - \angle CAD = \pi - \angle BAD$, hence the quadrilateral $ABXD$ is cyclic.

It follows $\angle BDA = \angle BXA = \angle CXE = \angle CDE$. ■

Alternative Solution. (D. Schwarz) We are asked to prove that DE and DB are isogonal conjugate, and so, since DE is median in ΔCDA, that DB is symmedian in that triangle.

Let γ be the circumcircle of ΔABC, and Γ be the circumcircle of ΔCDA, of center Ω. We have $\angle C\Omega A = 2\angle CDA = \pi - \angle ABC$, hence $\Omega \in \gamma$, and $C\Omega = A\Omega$, hence $\Omega \in BE$; therefore $\Omega = \gamma \cap BE$. Since $B\Omega$ is a diameter for circle γ, it follows $\angle BA\Omega = \angle BC\Omega = \pi/2$, therefore BA and BC are tangent to circle Γ.

A well-known Lemma states that a symmedian in a triangle (ΔCDA) connects the vertex (D) it originates at with the intersection (B) of the tangents (BA and BC) to the circumcircle (Γ) of the triangle at the other two vertices (A and C); for us that yields $\boxed{DB \text{ symmedian}}$ in ΔCDA. ■

Remark. In what concerns the Lemma, see for example `http://web.mit.edu/yufeiz/www/geolemmas.pdf`, a Yufei Zhao article concerning the construction of the symmedians (and others). The neatest proof uses methods of projective geometry.

Another interesting mention of this result (and others) is at `http://www.cut-the-knot.org/triangle/` (the symmedians page), Alexander Bogomolny's Cut-The-Knot site.

Problem 4. Given any n positive integers, and a sequence of 2^n integers (with terms among them), prove there exists a subsequence made of consecutive terms, such that the product of its terms is a perfect square.

Also show that we cannot replace 2^n with any lower value (therefore 2^n is the threshold value for this property).

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Solution. (D. Schwarz) Since the integers could be distinct primes, this is equivalent to proving the stricter problem of, given a finite alphabet A of n letters $a_1, a_2, \ldots, a_n$, and a word w of length 2^n on this alphabet, to show it contains a nonempty contiguous subword x ($w = \overline{uxv}$, where u, v could be the empty word), in which each letter appears at an even number of times.

Let us, further on, identify the letter a_k with the element $\mathbf{e}_k \in \mathbb{Z}_2^n$ given by $\mathbf{e}_k = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is at the k^{th} position. Now the requirement is to find a subword such that the sum of its elements is $\mathbf{0} = (0, 0, \ldots, 0)$.

We will apply a classical idea of Erdös. Let $w = \overline{x_1 x_2 \ldots x_{2^n}}$ be the word, with $x_i \in \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for all $i = 1, 2, \ldots, 2^n$. Define $\sigma_k = \sum_{i=1}^{k} x_i$ for all $k = 1, 2, \ldots, 2^n$. If any of $\sigma_k = \mathbf{0}$, we are done, since $\overline{x_1 \ldots x_k}$ can be taken as the subword; otherwise there must exist $1 \leq p < q \leq 2^n$ such that $\sigma_p = \sigma_q$, so then $\mathbf{0} = \sigma_q - \sigma_p = \sum_{i=1}^{q-p} x_{p+i}$, and we can take the subword $\overline{x_{p+1} \ldots x_q}$.

Let us notice that the result is tight: there exist words of length $\boxed{2^n - 1}$ without this property. We can build them inductively. Take $w_1 = \overline{a_1}$, and build $w_{n+1} = \overline{w_n a_{n+1} w_n}$ for all $n \geq 1$. ■

Remark. The problem fits nicely into the larger, classical topic of combinatorics on words, of squarefree and abelian squarefree words, with quite a large literature on it.

Problem 5. Determine the least real number c, such that for any integer $n \geq 1$ and any positive real numbers $a_1, a_2, \ldots, a_n$, the following holds

$$\sum_{k=1}^{n} \frac{k}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_k}} < c \sum_{k=1}^{n} a_k.$$

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Solution. (C. Popescu) We claim $\boxed{c_{\min}=2}$.

Taking $a_j = \frac{1}{j}$ for $j = 1, 2, \ldots, n$, we have

$\sum_{k=1}^{n} \frac{k}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_k}} = \sum_{k=1}^{n} \frac{k}{1+2+\cdots+k} = 2\sum_{k=1}^{n} \frac{1}{k+1}$,

while $c\sum_{k=1}^{n} a_k = c\sum_{k=1}^{n} \frac{1}{k}$, hence for the inequality to hold we need $(c-2)\sum_{k=1}^{n} \frac{1}{k} > -2 + \frac{2}{n+1} > -2$, therefore $c \geq 2$ (since the sum $\sum_{k=1}^{n} \frac{1}{k}$ grows (with n) as large as wanted).

We will now prove that $c = 2$ is suitable.

We have, from the Cauchy-Schwartz inequality,

$\frac{k^2(k+1)^2}{4} = \left(\sum_{j=1}^{k} j\right)^2 \leq \left(\sum_{j=1}^{k} j^2 a_j\right)\left(\sum_{j=1}^{k} \frac{1}{a_j}\right)$, hence

$\frac{k}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_k}} \leq \frac{4}{k(k+1)^2}\sum_{j=1}^{k} j^2 a_j.$

Therefore

$\sum_{k=1}^{n} \frac{k}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_k}} \leq \sum_{k=1}^{n}\left(\frac{4}{k(k+1)^2}\sum_{j=1}^{k} j^2 a_j\right) =$

$= \sum_{j=1}^{n}\left(j^2 a_j \sum_{k=j}^{n} \frac{4}{k(k+1)^2}\right) = 2\sum_{j=1}^{n}\left(j^2 a_j \sum_{k=j}^{n} \frac{2k}{k^2(k+1)^2}\right) <$

$< 2\sum_{j=1}^{n}\left(j^2 a_j \sum_{k=j}^{n} \frac{2k+1}{k^2(k+1)^2}\right)$. But

$\sum_{k=j}^{n} \frac{2k+1}{k^2(k+1)^2} = \sum_{k=j}^{n}\left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right) = \frac{1}{j^2} - \frac{1}{(n+1)^2} < \frac{1}{j^2}$,

hence $\sum_{k=1}^{n} \frac{k}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_k}} < 2\sum_{k=1}^{n} a_k$. ■

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