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Equilibrium existence for Bayesian generalized games in choice form and applications

by

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Abstract

In this paper, we define the Bayesian generalized game in choice form and its Bayesian equilibrium in choice. Our model generalizes the Bayesian abstract economy, introduced by the author, and the deterministic model, recently defined by Ferrara and Stefanescu. We apply the equilibrium results to prove the existence of solutions for the random quasi-variational inequalities.

Key Words: Bayesian game in choice form, Bayesian equilibrium in choice, Bayesian choice profile under restrictions, incomplete information.
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1 Introduction

We propose a new definition of a stochastic game, in the spirit of the competitive economy: the Bayesian generalized game in choice form. This game is characterized by constraint correspondences and a Bayesian choice profile under restrictions, expressing the choices of agents, depending on the set of nature states in the world. Our model generalizes, in a Bayesian setting, the ones introduced by Ferrara and Stefanescu in [7] and by the author in [10]. This work is also a continuation of the recent deterministic results obtained by the author.

The new stochastic model opens a new direction for obtaining results concerning possible types of exchange economies and their Walrasian equilibrium. Other applications could refer to random variational inequalities, equilibrium and optimization problems.

We exemplify these remarks by establishing random variational inequalities and fixed point-type theorems, as applications of the equilibrium results. The notion of a random variational inequality was introduced by Noor and Elsanousi [9]. We work with new assumptions. These new hypotheses are due to the Bayesian generalized game in choice form. Its particular form imposes original types of applications.

2 Definitions and notations

Let now $(\Omega, \mathcal{F}, \mu)$ be a complete, finite measure space, and Y be a topological space. Denote by 2^Y the set of all subsets of Y. The correspondence $T : \Omega \to 2^Y$ is said to be *lower* *measurable* if for every open subset V of Y, the set $\{\omega \in \Omega : T(\omega) \cap V \neq \emptyset\}$ is an element of $\mathcal{F}$. If T is closed valued and lower measurable, then it has a measurable graph.

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and Y be a Banach space. It is known (see [5], Theorem 2, p.45) that, if $x : \Omega \to Y$ is a μ-measurable function then x is Bochner integrable if only if $\int_{\Omega} \|x(\omega)\| d\mu(\omega) < \infty$. It is denoted by $L_1(\mu, Y)$ the space of equivalence classes of Y-valued Bochner integrable functions $x : \Omega \to Y$ normed by $\| x \| = \int_{\Omega} \|x(\omega)\| d\mu(\omega)$. Also it is known (see [5], p.50) that $L_1(\mu, Y)$ is a Banach space. We denote by S_T the set of all selections of the correspondence $T : \Omega \to 2^Y$ that belong to the space $L_1(\mu, Y)$, i.e. $S_T = \{x \in L_1(\mu, Y) : x(\omega) \in T(\omega)\ \mu\text{-a.e.}\}$. The *integral of correspondence* $T : \Omega \to 2^Y$ ([1]) is the set $\{\int_{\Omega} x(\omega) d\mu(\omega) : x \in S_T\}$. We will denote the above set by $\int T(\omega) d\mu(\omega)$ or simply $\int T$. The correspondence $T : \Omega \to 2^Y$ is said to be *integrably bounded* if there exists a map $h \in L_1(\mu, R)$, such that $\sup\{\| x \| : x \in T(\omega)\} \leq h(\omega)\ \mu - a.e.$

3 A new stochastic game model: the bayesian generalized game in choice form

Ferrara and Stefanescu defined in [7] the generalized game in choice form. This generalizes the well-known models of Nash [8], Debreu [4] and Shafer and Sonnenschein [11]. The main aim of this section is to define a stochastic version of the generalized game in choice form. In order to do this, we follow the ideas and the settings of Yannelis, who generalized the Debreu's deterministic model in [14] and worked in a Bayesian framework. Here, we introduce the model of the Bayesian generalized game in choice form, its Bayesian equilibrium in choice, and we make connections with the definitions in [10].

We work in the following setting. Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite measure space, where Ω denotes the set of states of nature of the world and the $\sigma-$algebra $\mathcal{F}$, denotes the set of events. Let Y denote the strategy or commodity space, where Y is a separable Banach space. Let I be the set of agents. For each $i \in I$, let $X_i : \Omega \to 2^Y$ and let us denote $L_{X_i} = \{\widetilde{x}_i \in S_{X_i} : \widetilde{x}_i$ is $\mathcal{F}_i$ -measurable$\}$. An element $\widetilde{x}_i$ of L_{X_i} is called a strategy for agent i. An element of L_{X_i} is denoted by $\widetilde{x}_i$ and that of $X_i(\omega)$ by $x_i(\omega)$. Let $L_X = \prod_{i \in I} L_{X_i}$. For each $i \in I$, let $L_{X_{-i}} = \prod_{j \neq i} L_{X_j}$. An element of $L_{X_{-i}}$ is denoted by $\widetilde{x}_{-i}$. Then, $\widetilde{x} = (\widetilde{x}_{-i}, \widetilde{x}_i) \in L_X$.

Definition 1. *A Bayesian generalized game in choice form is a family* $\Gamma = \{(\Omega, \mathcal{F}, \mu), (X_i, \mathcal{F}_i, A_i, C_i)_{i \in I}\}$, *where, for each* $i \in I$:

(a) $X_i : \Omega \to 2^Y$ is the *action correspondence* of agent i;

(b) $\mathcal{F}_i$ is a sub $\sigma-$algebra of $\mathcal{F}$ which denotes the *private information of agent* i;

(c) for each $\omega \in \Omega$, $A_i(\omega, \cdot) : L_{X_{-i}} \to 2^Y$ is the *random constraint correspondence of agent* i, where for all $(\omega, \widetilde{x}_{-i}) \in \Omega \times L_{X_{-i}}$, $A_i(\omega, \widetilde{x}_{-i}) \subset X_i(\omega)$;

(d) $A_i' : L_{X_{-i}} \to 2^{L_{X_i}}$ is defined by $A_i'(\widetilde{x}_{-i}) = \{\widetilde{y}_i \in L_{X_i} : \widetilde{y}_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})\ \mu - a.e.\}$, for each $\widetilde{x}_{-i} \in L_{X_{-i}}$ and $C_i \subseteq \mathrm{Gr}(A_i')$.

Definition 2. *The family $(C_i)_{i\in I}$ of nonempty sets, such that $C_i \subset Gr(A'_i)$, for each $i \in I$, is called a Bayesian choice profile under restrictions.*

If, for each $i \in I$, $A_i(\omega, \widetilde{x}_{-i}) = Y$, for each $\omega \in \Omega$ and $\widetilde{x}_{-i} \in L_{X_{-i}}$, then, we obtain a random variant of a game in choice form defined in [7].

We denote $\mathcal{C}_i(\widetilde{x}_{-i})$ *the upper section through* $(\widetilde{x}_{-i}, \widetilde{x}_i)$ *of the set* $\mathcal{C}_i$, i.e., $\mathcal{C}_i(\widetilde{x}_{-i}) = \{\widetilde{y}_i \in L_{X_i} : (\widetilde{x}_{-i}, \widetilde{y}_i) \in \mathcal{C}_i\}$ and $\mathcal{C}_i(\widetilde{x}_i)$ *the lower section through* $(\widetilde{x}_{-i}, \widetilde{x}_i)$ *of the set* $\mathcal{C}_i$, i.e., $\mathcal{C}_i(\widetilde{x}_i) = \{\widetilde{y}_{-i} \in L_{X_{-i}} : (\widetilde{y}_{-i}, \widetilde{x}_i) \in \mathcal{C}_i\}$.

Definition 3. *a) A Bayesian equilibrium in choice of the generalized game in the choice form* Γ *is any strategy profile* $\widetilde{x}^* \in L_X$, *with the property:*

$\forall i \in I$, $(\widetilde{x}^*_{-i}, A_i(\widetilde{x}^*_{-i})) \cap C_i \neq \emptyset \Rightarrow \widetilde{x}^* \in C_i$.

This means that $\widetilde{x}^*$ is a Bayesian equilibrium in choice if $\widetilde{x^*}_{-i} \in \mathcal{C}_i(\widetilde{x}^*_{-i})$, for each $i \in I$ for which $A_i(\widetilde{x}^*_{-i}) \cap \mathcal{C}_i(\widetilde{x}^*_{-i}) \neq \emptyset$.

b) A *strong Bayesian equilibrium in choice for* Γ is a strategy profile $\widetilde{x}^* \in L_X$, such that $\widetilde{x}^* \in \bigcap_{i\in I} C_i$.

A particular case of a Bayesian generalized game in choice form is the general Bayesian abstract economy defined in [10].

A *general Bayesian abstract economy* [10] is a set $G = \{(\Omega, \mathcal{F}, \mu), (X_i, \mathcal{F}_i, A_i, P_i)_{i\in I}\}$, where:

(a) $X_i : \Omega \to 2^Y$ is the *action correspondence* of agent i;

(b) $\mathcal{F}_i$ is a sub $\sigma-$algebra of $\mathcal{F}$ which denotes the *private information of agent* i;

(c) for each $\omega \in \Omega$, $A_i(\omega,\cdot) : L_{X_{-i}} \to 2^Y$ is the *random constraint correspondence of agent* i, where for all $(\omega, \widetilde{x}_{-i}) \in \Omega \times L_{X_{-i}}$, $A_i(\omega, \widetilde{x}_{-i}) \subset X_i(\omega)$;

(d) for each $\omega \in \Omega$, $P_i(\omega,\cdot) : L_X \to 2^Y$ is the *random preference correspondences of agent* i, where for all $(\omega, \widetilde{x}) \in \Omega \times L_X$, $P_i(\omega, \widetilde{x}) \subset X_i(\omega)$.

A *strong Bayesian equilibrium for* G is a strategy profile $\widetilde{x}^* \in L_X$, such that, for all $i \in I$: $\widetilde{x}^*_i(\omega) \in A_i(\omega, \widetilde{x}^*_{-i})$ $\mu - a.e.$ and $A_i(\omega, \widetilde{x}^*_{-i}) \cap P_i(\omega, \widetilde{x}^*) = \emptyset$ $\mu - a.e.$

If, for each $i \in I$, X_i is a compact convex nonempty subset of Y and, for each $\omega \in \Omega$, $X_i(\omega) = X_i$, we obtain a version of the deterministic classical model of Yannelis-Prabhakar in [12] for an abstract economy with any set of players.

We note that we form the Bayesian choice profile under restrictions by setting, for each $i \in I$: $C_i = \{\widetilde{x} \in L_X : A_i(\omega, \widetilde{x}_{-i}) \cap P_i(\omega, \widetilde{x}) = \emptyset \ \mu - a.e\} \cap \mathrm{Gr}(A'_i)$. Then, $\widetilde{x}^*$ is a strong Bayesian equilibrium for the general Bayesian abstract economy $\{(\Omega, \mathcal{F}, \mu),$ $(X_i, \mathcal{F}_i, A_i, P_i)_{i\in I}\}$ if it is a strong equilibrium for the Bayesian generalized game in choice form $\{(\Omega, \mathcal{F}, \mu), (X_i, \mathcal{F}_i, A_i, C_i)_{i\in I}\}$.

4 Bayesian equilibrium theorems

This section is devoted to establishing theorems which state the existence of the Bayesian equilibrium for both Bayesian generalized games in choice form and general Bayesian abstract economies. The new approach for the Bayesian equilibrium in choice leads to new hypotheses which ensure the existence of the Bayesian equilibrium for general Bayesian abstract economies, which are very different from the classical ones. In Theorem 1, the constraint correspondences verify the assumptions of having measurable graphs and weakly open lower sections.

Theorem 1. *Let I be an index set. Let $\Gamma = \{(\Omega, \mathcal{F}, \mu), (X_i, \mathcal{F}_i, A_i, C_i)_{i\in I}\}$ be a Bayesian generalized game in choice form. Suppose that for each $i \in I$:*

A.1) (a) $X_i : \Omega \to 2^Y$ is a nonempty, convex, weakly compact-valued and integrably bounded correspondence;

(b) $X_i : \Omega \to 2^Y$ is $\mathcal{F}_i-$lower measurable, i.e., for every open subset V of Y, the set $\{\omega \in \Omega : X_i(\Omega) \cap V \neq \emptyset\}$ belongs to $\mathcal{F}_i$;

A.2) (a) For each $(\omega, \widetilde{x}_{-i}) \in \Omega \times L_{X_{-i}}$, $A_i(\omega, \widetilde{x}_{-i})$ has a non-empty interior in the relative norm topology of $X_i(\omega)$;

(b) A_i has a measurable graph, i.e. $\{(\omega, \widetilde{x}_{-i}, y) \in \Omega \times L_{X_{-i}} \times Y : y \in A_i(\omega, \widetilde{x}_{-i})\} \in \mathcal{F} \otimes \beta_w(L_{X_{-i}}) \otimes \beta(Y)$, where $\beta_w(L_{X_{-i}})$ is the Borel $\sigma-$algebra for the weak topology on $L_{X_{-i}}$ and $\beta(Y)$ is the Borel $\sigma-$algebra for the norm topology on Y;

A.3)(a) C_i is nonempty, where $C_i \subset Gr(A_i')$, and $A_i^{'} : L_{X_{-i}} \to 2^{L_{X_i}}$ is defined by $A_i^{'}(\widetilde{x}_{-i}) = \{\widetilde{y}_i \in L_{X_i} : \widetilde{y}_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})\ \mu - a.e.\}$, for each $\widetilde{x}_{-i} \in L_{X_{-i}}$;

(b) $C_i(\widetilde{x}_{-i})$ is nonempty and convex, for each $\widetilde{x}_{-i} \in L_{X_{-i}}$;

(c) $C_i(\widetilde{x}_i)$ is open in $L_{X_{-i}}$ with respect to the product topology, for each $\widetilde{x}_i \in L_{X_i}$.

Then, there exists a strong Bayesian equilibrium in choice for Γ.

Proof. Let $i \in I$. We prove first that L_{X_i} is a non-empty, convex, weakly compact subset in $L_1(\mu, Y)$. Since $(\Omega,\ \mathcal{F}, \mu)$ is a complete finite measure space, Y is a separable Banach space and $X_i : \Omega \to 2^Y$ has a measurable graph, according to Aumann's measurable selection theorem (see [14], page 64), it follows that there exists a $\mathcal{F}_i$-measurable function $f_i : \Omega \to Y$, such that $f_i(\omega) \in X_i(\omega)\ \mu-a.e.$ Since X_i is integrably bounded, we have that $f_i \in L_1(\mu, Y)$. Hence, L_{X_i} is non-empty. Obviously, L_{X_i} is convex. Since $X_i : \Omega \to 2^Y$ is integrably bounded and has convex weakly compact values, according to Diestel's Theorem (Theorem 3.1 in [13]), it follows that L_{X_i} is a weakly compact subset of $L_1(\mu, Y)$. $L_1(\mu, Y)$, equipped with the weak topology, is a locally convex topological vector space and $\prod\limits_{i\in I} L_1(\mu, Y)$ is also a locally convex space. $L_X = \prod\limits_{i\in I} L_{X_i}$ is non-empty and convex. Tychonoff's Theorem implies that L_X is compact with respect to the product topology.

A_i is nonempty valued and, for each $\widetilde{x}_{-i} \in L_{X_{-i}}$, $A_i(\cdot, \widetilde{x}_{-i})$ has a measurable graph. Hence, according to the Aumann measurable selection theorem (see [14], page 64), for each fixed $\widetilde{x} \in L_X$, there exists an $\mathcal{F}_i-$measurable function $y_i : \Omega \to Y$, such that $y_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})\ \mu - a.e.$ Since, for each $(\omega, \widetilde{x}_{-i}) \in \Omega \times L_{X_{-i}}$, $A_i(\omega, \widetilde{x}_{-i})$ is contained in $X_i(\omega)$, where $X_i(\cdot)$ is an integrably bounded correspondence, then $y_i \in L_{X_i}$ and we conclude that $y_i \in A_i^{'}(\widetilde{x}_{-i})$. Thus, $A_i^{'}$ is non-empty valued.

Let $\Phi_i : L_{X_{-i}} \to 2^{X_i}$, be defined by $\Phi_i(\widetilde{x}_{-i}) = C_i(\widetilde{x}_{-i}) \subset A_i^{'}(\widetilde{x}_{-i})$, for each $\widetilde{x}_{-i} \in L_{X_{-i}}$. According to Assumptions A3) (a) (b), (c), the correspondence Φ_i is nonempty and convex valued, and $(\Phi_i(\widetilde{x}_i))^{-1}$ is an open set in $L_{X_{-i}}$, for each $\widetilde{x}_i \in L_{X_i}$. We apply the Yannelis and Prabhakar's Theorem (Theorem 3.1 in [12]). Then, Φ_i has a continuous selection $f_i : L_{X_{-i}} \to L_{X_i}$. Let $f : L_X \to L_X$ be defined by $f(\widetilde{x}) := \prod_{i\in I} f_i(\widetilde{x}_{-i})$, for each $\widetilde{x} \in L_X$. The function f is continuous with respect to the product topology of L_X, and, according to the Brouwer-Schauder fixed point Theorem, there exists $\widetilde{x}^* \in L_X$, such that $f(\widetilde{x}^*) = \widetilde{x}^*$.

Hence, $\widetilde{x}^* \in \prod_{i\in I} T_i(\widetilde{x}^*_{-i})$ and obviously, $\widetilde{x}^*_i \in T_i(\widetilde{x}^*_{-i})$, for each $i \in I$. Consequently, $\widetilde{x}^* \in C_i$, for each $i \in I$. □

Example 1. *We can construct an example by directly defining:*

Let $(\Omega, \mathcal{F}, \mu)$ the measure space, where $\Omega = [0,1]$, $\mathcal{F} = \beta([0,1])$ is the sigma algebra of the Borel measurable subsets in $[0,1]$ and μ is the Lebesgue measure.

Let $Y = \mathbb{R}$ and $I = \{1, 2, ..., n\}$.

For each $i \in I$, let us define the following:

$\mathcal{F}_i = \mathcal{F}$.

The correspondence $X_i : [0,1] \to 2^{\mathbb{R}}$ is defined by $X_i(\omega) = [0,1]$, for each $\omega \in [0,1]$.

It is a non-empty convex compact valued and integrably bounded correspondence. It is also $\mathcal{F}_i-$lower measurable.

Let $L_{X_i} = \{\widetilde{x}_i \in S_{X_i(\cdot)} : \widetilde{x}_i$ is $\mathcal{F}_i$ -measurable $\}$ and $L_X = \prod\limits_{i\in I} L_{X_i}$.

Then, the correspondence $A_i : [0,1] \times L_X \to 2^{[0,1]}$ is defined by

$A_i(\omega, \widetilde{x}) = (0, \frac{9}{10})$, $(\omega, \widetilde{x}) \in [0,1] \times L_X$.

A_i has a measurable graph.

For each $(\omega, \widetilde{x}) \in [0,1] \times L_X$, $A_i(\omega, \widetilde{x})$ is convex and with non-empty interior in $[0,1]$.

For each $i \in I$, let us define $D_i = \prod\limits_{j\neq i} L_{X_j} \times \{\widetilde{x} : [0,1] \to [0,1],\ \widetilde{x}(\omega) = k_{\widetilde{x}}\omega^i,\ \omega \in [0,1],$ $k_{\widetilde{x}} \in [0,1]\}$. D_i is weakly closed in L_X.

Let also define $C_i = \{\widetilde{x} \in L_X : \widetilde{x}_i \in D_i,\ \widetilde{x}_i(\omega) \in (0, \frac{9}{10})\ \mu - a.e,\ \frac{\widetilde{x}_i(\omega)+4}{5} \notin (0, \frac{9}{10})$ $\mu - a.e\} \subset \mathrm{Gr} A_i^{\prime}$.

$C_i(\widetilde{x}_{-i})$ is nonempty and convex, for each $\widetilde{x}_{-i} \in L_{X_{-i}}$ and $C_i(\widetilde{x}_i)$ is open in $L_{X_{-i}}$ with respect to the product topology, for each $\widetilde{x}_i \in L_{X_i}$.

All the assumptions of Theorem 1 are fulfilled, then an equilibrium exists.

For example, $\widetilde{x^*} \in L_X$, such that for each $i \in I$, $\widetilde{x_i^*}(\omega) = \frac{3}{4}\omega^i$, $\omega \in [0,1]$ is an equilibrium for the abstract economy, that is, for each $i \in I$, $\widetilde{x^*} \in C_i$.

Now, we establish a result,- concerning the existence of the Bayesian equilibrium for general Bayesian abstract economies. Theorem 2 generalizes the Yannelis and Prabhakar's deterministic theorem in [12].

Theorem 2. *Let I be an index set. Let $G = \{(\Omega, \mathcal{F}, \mu), (X_i, \mathcal{F}_i, A_i, P_i)_{i\in I}\}$ be a general Bayesian abstract economy. Suppose that the following conditions are satisfied, for each $i \in I$:*

A.1) (a) $X_i : \Omega \to 2^Y$ is a nonempty, convex, weakly compact-valued and integrably bounded correspondence;

(b) $X_i : \Omega \to 2^Y$ is $\mathcal{F}_i-$lower measurable, i.e., for every open subset V of Y, the set $\{\omega \in \Omega : X_i(\Omega) \cap V \neq \emptyset\}$ belongs to $\mathcal{F}_i$;

A.2) (a) For each $(\omega, \widetilde{x}_{-i}) \in \Omega \times L_{X_{-i}}$, $A_i(\omega, \widetilde{x}_{-i})$ has a non-empty interior in the relative norm topology of $X_i(\omega)$;

(b) A_i has a measurable graph, i.e. $\{(\omega, \widetilde{x}_{-i}, y) \in \Omega \times L_{X_{-i}} \times Y : y \in A_i(\omega, \widetilde{x}_{-i})\} \in \mathcal{F} \otimes \beta_w(L_{X_{-i}}) \otimes \beta(Y)$, *where* $\beta_w(L_{X_{-i}})$ *is the Borel* $\sigma-$*algebra for the weak topology on* $L_{X_{-i}}$ *and* $\beta(Y)$ *is the Borel* $\sigma-$*algebra for the norm topology on* Y;

(c) for each $\widetilde{y}_i \in L_{X_i}$, *the set* $\{\widetilde{x}_{-i} \in L_{X_{-i}} : \widetilde{y}_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})\ \mu - a.e.\}$ *is open in* $L_{X_{-i}}$ *with respect to the product topology of* $L_{X_{-i}}$;

A.3) (a) $P_i : \Omega \times L_X \to 2^Y$ *has nonempty values, such that* $P_i(\omega, \widetilde{x}) \subset X(\omega)$, *for each* $(\omega, \widetilde{x}) \in \Omega \times L_X$;

(b) For each $\widetilde{x}_{-i} \in L_{X_{-i}}$, $\{\widetilde{x}_i \in L_{X_i} : A_i(\omega, \widetilde{x}_{-i}) \cap P_i(\omega, \widetilde{x}) = \emptyset\ \mu - a.e\} \cap \{\widetilde{x}_i \in L_{X_i} : \widetilde{x}_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})\ \mu - a.e.\}$ *is nonempty;*

(c) For each $\widetilde{x}_{-i} \in L_{X_{-i}}$, $\{\widetilde{x}_i \in L_{X_i} : A_i(\omega, \widetilde{x}_{-i}) \cap P_i(\omega, \widetilde{x}) = \emptyset\ \mu - a.e\}$ *is convex;*

(d) For each $\widetilde{x}_i \in L_{X_i}$, $\{\widetilde{x}_{-i} \in L_{X_{-i}} : A_i(\omega, \widetilde{x}_{-i}) \cap P_i(\omega, \widetilde{x}) = \emptyset\ \mu - a.e\}$ *is open in* $L_{X_{-i}}$ *with respect to the product topology of* $L_{X_{-i}}$.

Then, G *has a Bayesian equilibrium.*

Proof. Let $i \in I$. As in the previous proof, L_{X_i} is a non-empty, convex, weakly compact subset in $L_1(\mu, Y)$. $A_i^{'}$ is non-empty valued, as in the previous proof. $A_i^{'}$ is convex valued, since A_i is so. For each $\widetilde{x}_i \in L_{X_i}$, $(A_i^{'}(\widetilde{x}_i))^{-1} = \{\widetilde{x}_{-i} \in L_{X_{-i}} : \widetilde{x}_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})\ \mu - a.e.\}$ is open, according to assumption A2 (c). Let us define $C_i = \{\widetilde{x} \in L_X : A_i(\omega, \widetilde{x}_{-i}) \cap P_i(\omega, \widetilde{x}) = \emptyset\ \mu - a.e\} \cap \mathrm{Gr}(A_i')$.

$C_i(\widetilde{x}_{-i}) = \{\widetilde{x}_i \in A_i'(\widetilde{x}_{-i}) : A_i(\omega, \widetilde{x}_{-i}) \cap P_i(\omega, \widetilde{x}) = \emptyset\ \mu - a.e\}$ is nonempty and convex, for each $\widetilde{x}_{-i} \in L_{X_{-i}}$, since Assumptions A3 (b) and (c) hold. According to assumptions A2 (c), A3 (d), $C_i(\widetilde{x}_i)$ is open in $L_{X_{-i}}$, for each $\widetilde{x}_i \in L_{X_i}$, where $C_i(\widetilde{x}_i) = \{\widetilde{x}_{-i} \in L_{X_{-i}} : A_i(\omega, \widetilde{x}_{-i}) \cap P_i(\omega, \widetilde{x}) = \emptyset\ \mu - a.e\} \cap (A_i^{'}(\widetilde{x}_i))^{-1}$. We apply Theorem 1 and we find that there exists $\widetilde{x}^* \in C_i$, for each $i \in I$. Consequently, for each $i \in I$, $\widetilde{x}_i^*(\omega) \in A_i(\omega, \widetilde{x}_{-i}^*)\ \mu - a.e.$ and $A_i(\omega, \widetilde{x}_{-i}^*) \cap P_i(\omega, \widetilde{x}^*) = \emptyset\ \mu - a.e.$ □

In the second theorem, concerning the existence of the Bayesian equilibrium for Bayesian generalized games in choice form, the constraint correspondences verify the assumption of the measurable graph. In addition, we suppose the upper semicontinuouity of the correspondences, formed with the upper sections of $(C_i)_{i \in I}$.

We recall that if X, Y are topological spaces, then, the correspondence $T : X \to 2^Y$ is said to be *upper semicontinuous* if, for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X, such that $T(x) \subset V$, for each $y \in U$.

Theorem 3. *Let* I *be a countable index set. Let* $\Gamma = \{(\Omega, \mathcal{F}, \mu), (X_i, \mathcal{F}_i, A_i, C_i)_{i \in I}\}$ *be a Bayesian generalized game in choice form. Suppose that the following conditions are satisfied, for each* $i \in I$*:*

A.1) (a) $X_i : \Omega \to 2^Y$ *is a nonempty, convex, weakly compact-valued and integrably bounded correspondence;*

(b) $X_i : \Omega \to 2^Y$ *is* $\mathcal{F}_i-$*lower measurable, i.e., for every open subset* V *of* Y, *the set* $\{\omega \in \Omega : X_i(\Omega) \cap V \neq \emptyset\}$ *belongs to* $\mathcal{F}_i$;

A.2) (a) For each $(\omega, \widetilde{x}_{-i}) \in \Omega \times L_{X_{-i}}$, $A_i(\omega, \widetilde{x}_{-i})$ *has a non-empty interior in the relative norm topology of* $X_i(\omega)$;

(b) A_i *has a measurable graph, i.e.* $\{(\omega,\widetilde{x}_{-i},y)\in\Omega\times L_{X_{-i}}\times Y : y\in A_i(\omega,\widetilde{x}_{-i})\}\in\mathcal{F}\otimes\beta_w(L_{X_{-i}})\otimes\beta(Y)$, *where* $\beta_w(L_{X_{-i}})$ *is the Borel* $\sigma-$*algebra for the weak topology on* $L_{X_{-i}}$ *and* $\beta(Y)$ *is the Borel* $\sigma-$*algebra for the norm topology on* Y*;*

A.3) (a) C_i *is nonempty, where* $C_i\subset\mathrm{Gr}(A_i')$, *and* $A_i^{'} : L_{X_{-i}}\to 2^{L_{X_i}}$ *is defined by* $A_i^{'}(\widetilde{x}_{-i})=\{\widetilde{y}_i\in L_{X_i} : \widetilde{y}_i(\omega)\in A_i(\omega,\widetilde{x}_{-i})\ \mu-a.e.\}$, *for each* $\widetilde{x}_{-i}\in L_{X_{-i}}$*;*

(b) $C_i(\widetilde{x}_{-i})$ *is nonempty, convex and weakly closed in* L_{X_i}, *for each* $\widetilde{x}_{-i}\in L_{X_{-i}}$*;*

(c) $\Phi_i : L_{X_{-i}}\to 2^{L_{X_i}}$, *defined by* $\Phi_i(\widetilde{x}_{-i})=C_i(\widetilde{x}_{-i})$, *for each* $\widetilde{x}_{-i}\in L_{X_{-i}}$, *is weakly upper semicontinuous, in the sense that the set* $\{\widetilde{x}_{-i}\in L_{X_{-i}} : \Phi_i(\widetilde{x}_{-i})\subset V\}$ *is weakly open in* $L_{X_{-i}}$, *for every weakly open subset* V *of* L_{X_i}.

Then, Γ *has a strong Bayesian equilibrium in choice.*

Proof. Let $i\in I$. L_{X_i} is a non-empty, convex, weakly compact subset in $L_1(\mu,Y)$. By applying Theorem 3 in Dunford-Scwartz ([3], pag 434), we conclude that L_{X_i} is metrizable. $L_{X_{-i}}$ is also metrizable (since I is a countable set). In addition, $L_{X_{-i}}$ is weakly compact. $A_i^{'}$ is non-empty valued, as we showed in the proof of Theorem 1. According to Assumptions A3) (b) and (c), the correspondence Φ_i is nonempty, closed and convex valued, and weakly upper semicontinuous. Let $\Phi : L_X\to 2^X$, be defined by $\Phi(\widetilde{x})=\prod_{i\in I}\Phi_i(\widetilde{x}_{-i})$, for each $\widetilde{x}\in L_X$. Φ is also nonempty, closed, convex valued, and weakly upper semicontinuous. L_X is nonempty, convex and weakly compact. We apply the Ky Fan fixed point Theorem [6]. Then, there exists $\widetilde{x}^*\in L_X$, such that $\widetilde{x}^*\in\Phi(\widetilde{x}^*)$. Obviously, $\widetilde{x}_i^*\in\Phi_i(\widetilde{x}_{-i}^*)$, for each $i\in I$. Consequently, $\widetilde{x}^*\in C_i$, for each $i\in I$. □

Now, we establish a Bayesian equilibrium existence theorem for general Bayesian abstract economies with upper semi-continuous correspondences formed by using the constraints and the preference correspondences. We emphasize the new assumptions of our theorem.

Theorem 4. *Let* I *be a countable index set. Let* $G=\{(\Omega,\mathcal{F},\mu),(X_i,\mathcal{F}_i,A_i,P_i)_{i\in I}\}$ *be a general Bayesian abstract economy. Suppose that the following conditions are satisfied, for each* $i\in I$*:*

A.1) (a) $X_i : \Omega\to 2^Y$ *is a nonempty, convex, weakly compact-valued and integrably bounded correspondence;*

(b) $X_i : \Omega\to 2^Y$ *is* $\mathcal{F}_i-$*lower measurable, i.e., for every open subset* V *of* Y, *the set* $\{\omega\in\Omega : X_i(\Omega)\cap V\neq\emptyset\}$ *belongs to* $\mathcal{F}_i$*;*

A.2) (a) For each $(\omega,\widetilde{x}_{-i})\in\Omega\times L_{X_{-i}}$, $A_i(\omega,\widetilde{x}_{-i})$ *has a non-empty interior in the relative norm topology of* $X_i(\omega)$*;*

(b) A_i *has a measurable graph, i.e.* $\{(\omega,\widetilde{x}_{-i},y)\in\Omega\times L_{X_{-i}}\times Y : y\in A_i(\omega,\widetilde{x}_{-i})\}\in\mathcal{F}\otimes\beta_w(L_{X_{-i}})\otimes\beta(Y)$, *where* $\beta_w(L_{X_{-i}})$ *is the Borel* $\sigma-$*algebra for the weak topology on* $L_{X_{-i}}$ *and* $\beta(Y)$ *is the Borel* $\sigma-$*algebra for the norm topology on* Y*;*

(c) for each $\omega\in\Omega$, $A_i(\omega,\cdot) : L_{X_{-i}}\to 2^Y$ *is upper semicontinuous, in the sense that the set* $\{\widetilde{x}_{-i}\in L_{X_{-i}} : A_i(\omega,\widetilde{x}_{-i})\}\subset V\}$ *is weakly open in* $L_{X_{-i}}$, *for every norm open subset* V *of* Y*;*

A.3) (a) For each $\widetilde{x}_{-i}\in L_{X_{-i}}$, $\{\widetilde{x}_i\in L_{X_i} : A_i(\omega,\widetilde{x}_{-i})\cap P_i(\omega,\widetilde{x})=\emptyset\ \mu-a.e\}\cap\{\widetilde{x}_i\in L_{X_i} : \widetilde{x}_i(\omega)\in A_i(\omega,\widetilde{x}_{-i})\ \mu-a.e.\}$ *is nonempty;*

(b) For each $\widetilde{x}_{-i} \in L_{X_{-i}}$, $\{\widetilde{x}_i \in L_{X_i} : A_i(\omega, \widetilde{x}_{-i}) \cap P_i(\omega, \widetilde{x}) = \emptyset \ \ \mu - a.e\}$ *is convex and weakly closed in* L_{X_i}*;*

(c) The correspondence $T_i : L_{X_{-i}} \to 2^{L_{X_i}}$, *defined by* $T_i(\widetilde{x}_{-i}) = \{\widetilde{x}_i \in L_{X_i} : A_i(\omega, \widetilde{x}_{-i}) \cap P_i(\omega, \widetilde{x}) = \emptyset \ \ \mu - a.e\}$, *is weakly upper semicontinuous, in the sense that the set* $\{\widetilde{x}_{-i} \in L_{X_{-i}} : T_i(\widetilde{x}_{-i}) \subset V\}$ *is weakly open in* $L_{X_{-i}}$, *for every weakly open subset* V *of* L_{X_i}.

Then, G *has a Bayesian equilibrium.*

Proof. As in the proof of Theorem 1, it results that for each $i \in I$, L_{X_i} is non-empty and convex. $L_X = \prod\limits_{i \in I} L_{X_i}$ is also non-empty and convex. For each $i \in I$, define $A_i^{'} : L_{X_{-i}} \to 2^{L_{X_i}}$, by $A_i^{'}(\widetilde{x}_{-i}) = \{\widetilde{y}_i \in L_{X_i} : \widetilde{y}_i(\omega) \in A_i(\omega, \widetilde{x}_{-i}) \ \mu - a.e.\}$, for each $\widetilde{x}_{-i} \in L_{X_{-i}}$. Thus, $A_i^{'}$ is non-empty valued, as we showed in the proof of Theorem 1. According to the projection Theorem (see [14], page 64), for each $\widetilde{x}_{-i} \in L_{X_{-i}}$, $A_i(\cdot, \widetilde{x}_{-i})$ has a measurable graph. In addition, for each $\omega \in \Omega$, $A_i(\omega, \cdot) : L_{X_{-i}} \to 2^Y$ is upper semicontinuous and $A_i(\omega, \widetilde{x}_{-i}) \subset X_i(\omega)$, for each $(\omega, \widetilde{x}_{-i}) \in \Omega \times L_{X_{-i}}$. By applying Theorem 3 in Dunford-Scwartz ([3], pag 434), we conclude that L_{X_i} is metrizable. $L_{X_{-i}}$ is also metrizable (since I is a countable set). In addition, $L_{X_{-i}}$ is weakly compact. Then, according to the u. s. c. Lifting Theorem (Lemma 2.1 in [14]), it follows that $A_i^{'}$ is weakly upper semicontinuous in the sense that the set $\{\widetilde{x}_{-i} \in L_{X_{-i}} : A_i^{'}(\widetilde{x}_{-i}) \subset V\}$ is weakly open in $L_{X_{-i}}$ for every weakly open subset V of L_{X_i}. $A_i^{'}$ is closed and convex valued, since A_i is so. Let us define $\Phi_i : L_{X_{-i}} \to 2^{X_i}$, be defined by $\Phi_i(\widetilde{x}_{-i}) = T_i(\widetilde{x}_{-i}) \cap A_i^{'}(\widetilde{x}_{-i})$, for each $\widetilde{x}_{-i} \in L_{X_{-i}}$. According to Assumptions A 3 (a), (b) and (c), the correspondence Φ_i is nonempty, closed, convex valued, and weakly upper semicontinuous. Let $\Phi : L_X \to 2^X$, be defined by $\Phi(\widetilde{x}) = \prod_{i \in I} \Phi_i(\widetilde{x}_{-i})$, for each $\widetilde{x} \in L_X$. Φ_i is also nonempty, closed, convex valued, and weakly upper semicontinuous. L_X is nonempty, convex and weakly compact. We apply the Ky Fan fixed point Theorem [6]. Then, there exists $\widetilde{x}^* \in L_X$, such that $\widetilde{x}^* \in \Phi(\widetilde{x}^*)$. Obviously, $\widetilde{x}_i^* \in \Phi_i(\widetilde{x}_{-i}^*)$, for each $i \in I$. Consequently, for each $i \in I$, $\widetilde{x}_i^*(\omega) \in A_i(\omega, \widetilde{x}_{-i}^*)$ $\mu - a.e.$ and $A_i(\omega, \widetilde{x}_{-i}^*) \cap P_i(\omega, \widetilde{x}^*) = \emptyset$ $\mu - a.e.$ □

5 New random quasi-variational inequalities

In this section, we apply the equilibrium results established in the previous section, in order to prove the existence of solutions for systems of random quasi-variational inequalities, under new settings and new hypotheses. We also report a new random fixed-point-type theorem.

In this section, we consider a complete finite separable measure space $(\Omega, \mathcal{F}, \mu)$ and a separable Banach space Y.

Theorem 5. *Let* I *be an index set. Suppose that the following conditions are satisfied, for each* $i \in I$*:*

A.1) (a) $X_i : \Omega \to 2^Y$ *is a nonempty, convex, weakly compact-valued and integrably bounded correspondence;*

(b) $X_i : \Omega \to 2^Y$ is $\mathcal{F}_i-$lower measurable, i.e., for every open subset V of Y, the set $\{\omega \in \Omega : X_i(\Omega) \cap V \neq \emptyset\}$ belongs to $\mathcal{F}_i$;

A.2) (a) For each $(\omega, \widetilde{x}_{-i}) \in \Omega \times L_{X_{-i}}$, $A_i(\omega, \widetilde{x}_{-i})$ has a non-empty interior in the relative norm topology of $X_i(\omega)$;

(b) A_i has a measurable graph, i.e. $\{(\omega, \widetilde{x}_{-i}, y) \in \Omega \times L_{X_{-i}} \times Y : y \in A_i(\omega, \widetilde{x}_{-i})\} \in \mathcal{F} \otimes \beta_w(L_{X_{-i}}) \otimes \beta(Y)$, where $\beta_w(L_{X_{-i}})$ is the Borel $\sigma-$algebra for the weak topology on $L_{X_{-i}}$ and $\beta(Y)$ is the Borel $\sigma-$algebra for the norm topology on Y;

A.3) $\psi_i : \Omega \times L_X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ is such that:

(a) For each $\widetilde{x}_{-i} \in L_{X_{-i}}$, $\{\widetilde{x}_i \in L_{X_i} : \widetilde{x}_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})$ $\mu-$a.e. and $\psi_i(\omega, \widetilde{x}, y_i(\omega)) \leq 0$ $\mu-a.e.$, for each $y_i : \Omega \to Y$, such that $y_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})$ $\mu - a.e\}$ is nonempty;

(b) For each $\widetilde{x}_{-i} \in L_{X_{-i}}$, $\{\widetilde{x}_i \in L_{X_i} : \psi_i(\omega, \widetilde{x}, y_i(\omega)) \leq 0$ $\mu-$a.e., for each $y_i : \Omega \to Y$, such that $y_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})$ $\mu - a.e\}$ is convex;

(c) For each $\widetilde{x}_i \in L_{X_i}$, $\{\widetilde{x}_{-i} \in L_{X_{-i}} : \psi_i(\omega, \widetilde{x}, y_i(\omega)) \leq 0$ $\mu-$a.e., for each $y_i : \Omega \to Y$, such that $y_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})$ $\mu - a.e\}$ is open in $L_{X_{-i}}$, with respect to the product topology.

Then, there exists $\widetilde{x}^* \in L_X$, such that, for every $i \in I$ and $\mu - a.e.$:

$\widetilde{x}_i^*(\omega) \in A_i(\omega, \widetilde{x}_{-i}^*)$ and $sup_{y \in A_i(\omega, \widetilde{x}_{-i}^*)} \psi_i(\omega, \widetilde{x}^*, y) \leq 0$.

Proof. For each $i \in I$, let us define $P_i : \Omega \times L_X \to 2^Y$ by

$P_i(\omega, \widetilde{x}) = \{y \in X_i(\omega) : \psi_i(\omega, \widetilde{x}, y) > 0\}$, for each $(\omega, \widetilde{x}) \in \Omega \times L_X$. We shall show that the general Bayesian abstract economy $G = \{(\Omega, \mathcal{F}, \mu), (X_i, \mathcal{F}_i, A_i, P_i)_{i \in I}\}$ satisfies all hypotheses of Theorem 2. For each $\widetilde{x}_{-i} \in L_{X_{-i}}$, $\{\widetilde{x}_i \in L_{X_i} : A_i(\omega, \widetilde{x}_{-i}) \cap P_i(\omega, \widetilde{x}) = \emptyset$ $\mu - a.e\} \cap \{\widetilde{x}_i \in L_{X_i} : \widetilde{x}_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})$ $\mu - a.e.\} = \{\widetilde{x}_i \in L_{X_i} : \widetilde{x}_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})$ $\mu - a.e.$ and $\psi_i(\omega, \widetilde{x}, y_i(\omega)) \leq 0$ $\mu - a.e.$, for each $y_i : \Omega \to Y$, such that $y_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})$ $\mu - a.e\}$ is nonempty, according to A3 (a).

For each $\widetilde{x}_{-i} \in L_{X_{-i}}$, $\{\widetilde{x}_i \in L_{X_i} : A_i(\omega, \widetilde{x}_{-i}) \cap P_i(\omega, \widetilde{x}) = \emptyset$ $\mu - a.e\} =$

$\{\widetilde{x}_i \in L_{X_i} : \psi_i(\omega, \widetilde{x}, y_i(\omega)) \leq 0$ $\mu-a.e.$, for each $y_i : \Omega \to Y$, such that $y_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})$ $\mu - a.e\}$ is convex, according to A3 (b). For each $\widetilde{x}_i \in L_{X_i}$, $\{\widetilde{x}_{-i} \in L_{X_{-i}} : A_i(\omega, \widetilde{x}_{-i}) \cap P_i(\omega, \widetilde{x}) = \emptyset$ $\mu - a.e\} = \{\widetilde{x}_{-i} \in L_{X_{-i}} : \psi_i(\omega, \widetilde{x}, y_i(\omega)) \leq 0$ $\mu - a.e.$, for each $y_i : \Omega \to Y$, such that $y_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})$ $\mu - a.e\}$ is open, according to A3 (c).

Thus, the general Bayesian abstract economy G satisfies all hypotheses of Theorem 2 and there exists an equilibrium for G, which is a solution for the random variational inequality. □

As a consequence of Theorem 5, we prove the following Tan and Yuan-type [15] random quasi-variational inequality. We recall that, if X, Y are topological spaces, then the correspondence $T : X \to 2^Y$ is said to be *lower semicontinuous* if, for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x in X, such that $T(y) \cap V \neq \emptyset$, for each $y \in U$.

Corollary 1. *Let I be an index set. Suppose that the following conditions are satisfied, for each $i \in I$:*

A.1) (a) $X_i : \Omega \to 2^Y$ is a nonempty, convex, weakly compact-valued and integrably bounded correspondence;

(b) $X_i : \Omega \to 2^Y$ *is* $\mathcal{F}_i-$*lower measurable, i.e., for every open subset* V *of* Y, *the set* $\{\omega \in \Omega : X_i(\Omega) \cap V \neq \emptyset\}$ *belongs to* $\mathcal{F}_i$;

A.2) (a) For each $(\omega, \widetilde{x}_{-i}) \in \Omega \times L_{X_{-i}}$, $A_i(\omega, \widetilde{x}_{-i})$ *has a non-empty interior in the relative norm topology of* $X_i(\omega)$;

(b) A_i *has a measurable graph, i.e.* $\{(\omega, \widetilde{x}_{-i}, y) \in \Omega \times L_{X_{-i}} \times Y : y \in A_i(\omega, \widetilde{x}_{-i})\} \in \mathcal{F} \otimes \beta_w(L_{X_{-i}}) \otimes \beta(Y)$, *where* $\beta_w(L_{X_{-i}})$ *is the Borel* $\sigma-$*algebra for the weak topology on* $L_{X_{-i}}$ *and* $\beta(Y)$ *is the Borel* $\sigma-$*algebra for the norm topology on* Y;

A.3) $G_i : \Omega \times Y \to 2^{Y'}$ *is such that:*

(a) for each $\omega \in \Omega$, $y \to G_i(\omega, y) : Y \to 2^{Y'}$ *is monotone (that is* $Re\langle u - v, y - x\rangle \geq 0$, *for all* $u \in G_i(\omega, y)$, $v \in G_i(\omega, x)$ *and* $x, y \in Y$), *with non-empty values*;

(b) for each $\omega \in \Omega$, $y \to G_i(\omega, y) : L \cap Y \to 2^{Y'}$ *is lower semicontinuous from the relative topology of* Y *into the weak*$^*-$*topology* $\sigma(Y', Y)$ *of* Y', *for each one-dimensional flat* $L \subset Y$;

(c) For each $\widetilde{x}_{-i} \in L_{X_{-i}}$, $\{\widetilde{x}_i \in L_{X_i} : \widetilde{x}_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})$ $\mu - a.e.$ *and* $\sup_{u \in G_i(\omega, y_i(\omega))} Re\ \langle u, \widetilde{x}_i(\omega) - y_i(\omega)\rangle \leq 0$, *for each* $y_i : \Omega \to Y$, *such that* $y_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})$ $\mu - a.e\}$ *is nonempty;*

(d) For each $\widetilde{x}_{-i} \in L_{X_{-i}}$, $\{\widetilde{x}_i \in L_{X_i} : \sup_{u \in G_i(\omega, y_i(\omega))} Re\langle u, \widetilde{x}_i(\omega) - y_i(\omega)\rangle \leq 0$ $\mu - a.e.$, *for each* $y_i : \Omega \to Y$, *such that* $y_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})$ $\mu - a.e\}$ *is convex;*

(e) For each $\widetilde{x}_i \in L_{X_i}$, $\{\widetilde{x}_{-i} \in L_{X_{-i}} : \sup_{u \in G_i(\omega, y_i(\omega))} Re\langle u, \widetilde{x}_i(\omega) - y_i(\omega)\rangle \leq 0$ $\mu -$ *a.e., for each* $y_i : \Omega \to Y$, *such that* $y_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})$ $\mu - a.e\}$ *is open.*

Then, there exists $\widetilde{x}^* \in L_X$, *such that, for every* $i \in I$ *and* $\mu - a.e.$: $\widetilde{x}^*_i(\omega) \in \mathrm{cl}A_i(\omega, \widetilde{x}^*_{-i})$ *and* $sup_{u \in G_i(\omega, \widetilde{x}^*(\omega))} \mathrm{Re}\langle u, \widetilde{x}^*_i(\omega) - y\rangle \leq 0$, *for all* $y \in A_i(\omega, \widetilde{x}^*_{-i})$.

Proof. Let us define $\psi_i : \Omega \times L_X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ by $\psi_i(\omega, \widetilde{x}, y) = \sup_{u \in G_i(\omega, y)} Re\langle u, \widetilde{x}_i(\omega) - y\rangle$, for each $(\omega, \widetilde{x}, y) \in \Omega \times L_X \times Y$.

All hypotheses of Theorem 5 are satisfied. According to Theorem 5, there exists $\widetilde{x}^* \in L_X$, such that $\widetilde{x}^*_i(\omega) \in A_i(\omega, \widetilde{x}^*_{-i})$, $\mu - a.e.$, for every $i \in I$ and

(1) $\sup_{y \in A_i(\omega, \widetilde{x}^*_{-i})} \sup_{u \in G_i(\omega, y)} [Re\langle u, \widetilde{x}^*_i(\omega) - y\rangle] \leq 0$, $\mu - a.e.$, for every $i \in I$. We claim that $\sup_{y \in A_i(\omega, \widetilde{x}^*_{-i})} \sup_{u \in G_i(\omega, \widetilde{x}^*(\omega))} [Re\langle u, \widetilde{x}^*_i(\omega) - y\rangle] \leq 0$, $\mu - a.e.$, for every $i \in I$. Indeed, let us consider $i \in I$ and the fixed point $\omega \in \Omega$. Let $y \in A_i(\omega, \widetilde{x}^*_{-i})$, $\lambda \in [0, 1]$ and $z^i_\lambda(\omega) := \lambda y + (1 - \lambda)\widetilde{x}^*_i(\omega)$. According to assumption A2 (a), $z^i_\lambda(\omega) \in A_i(\omega, \widetilde{x}^*_{-i})$. According to (1), $\sup_{u \in G_i(\omega, z^i_\lambda(\omega))} [Re\langle u, \widetilde{x}^*_i(\omega) - z_\lambda(\omega)\rangle] \leq 0$, for each $\lambda \in [0, 1]$.

Therefore, for each $\lambda \in [0, 1]$, $t\{\sup_{u \in G_i(\omega, z^i_\lambda(\omega))} [Re\langle u, \widetilde{x}^*_i(\omega) - y\rangle]\} =$ $\sup_{u \in G_i(\omega, z^i_\lambda(\omega))} t[Re\langle u, \widetilde{x}^*_i(\omega_i) - y)\rangle] = \sup_{u \in G_i(\omega, z^i_\lambda(\omega))} [Re\langle u, \widetilde{x}^*_i(\omega) - z^i_\lambda(\omega)\rangle] \leq 0$. It follows that, for each $\lambda \in [0, 1]$, (2) $\sup_{u \in G_i(\omega, z^i_\lambda(\omega))} [Re\langle u, \widetilde{x}^*_i(\omega) - y\rangle] \leq 0$.

Now, we show the conclusion, by using the lower semicontinuity of $G_i(\omega, \cdot) : L \cap Y \to 2^{Y'}$. For each $z_0 \in G_i(\omega, \widetilde{x}^*_i(\omega))$ and $e > 0$, let us consider $U^i_{z_0}$, the neighborhood of z_0 in the topology $\sigma(Y', Y)$, defined by $U^i_{z_0} := \{z \in Y' : |\Re\mathrm{e}\langle z_0 - z, \widetilde{x}^*_i(\omega) - y\rangle| < e\}$. The correspondence $G_i(\omega, \cdot) : L \cap Y \to 2^{Y'}$ is lower semicontinuous, where $L = \{z^i_\lambda(\omega) : \lambda \in [0, 1]\}$ and $U^i_{z_0} \cap G_i(\omega, \widetilde{x}^*_i(\omega)) \neq \emptyset$. Then, there exists a non-empty neighborhood $N(\widetilde{x}^*_i(\omega))$ of $\widetilde{x}^*_i(\omega)$ in L, such that $U^i_{z_0} \cap G_i(\omega, z) \neq \emptyset$, for each $z \in N(\widetilde{x}^*_i(\omega))$. Hence, there exists $\delta \in (0, 1]$, $t \in (0, \delta)$ and $u \in G_i(\omega, z^i_\lambda(\omega)) \cap U^i_{z_0} \neq \emptyset$, such that $Re\langle z_0 - u, \widetilde{x}^*_i(\omega) - y\rangle < e$. Therefore, $Re\langle z_0, \widetilde{x}^*_i(\omega) - y\rangle < Re\langle u_i, \widetilde{x}^*_i(\omega) - y\rangle + e$. It follows that $Re\langle z_0, \widetilde{x}^*_i(\omega) - y\rangle < Re\langle u, \widetilde{x}^*_i(\omega) - y\rangle + e < e$. The last inequality comes from (2). Since $e > 0$ and $z_0 \in$

$G_i(\omega, \widetilde{x}_i^*(\omega))$ have been chosen arbitrarily, $Re\langle z_0, \widetilde{x}_i^*(\omega) - y\rangle < 0$. Hence, for each $i \in I$, we have that $\sup_{u\in G_i(\omega,\widetilde{x}^*(\omega))}[Re\langle z_0, \widetilde{x}_i^*(\omega) - y\rangle] \leq 0$, for every $y \in A_i(\omega, \widetilde{x}_{-i}^*)$. □

As a corollary, we obtain the following random fixed point-type theorem. It is a generalization of Browder fixed-point Theorem *[2]*.

Corollary 2. *Let I be an index set. Assumptions A.1) and A2) of Theorem 5 hold. Then, there exists $\widetilde{x}^* \in L_X$, such that, for every $i \in I$ and $\mu - a.e.$, $\widetilde{x}_i^*(\omega) \in A_i(\omega, \widetilde{x}_{-i}^*)$.*

Theorem 6. *Theorem 5 holds if I is a countable index set and Conditions A.3) (b), (c) are replaced by:*

A.3) (b') For each $\widetilde{x}_{-i} \in L_{X_{-i}}$, $\{\widetilde{x}_i \in L_{X_i} : \psi_i(\omega, \widetilde{x}, y_i(\omega)) \leq 0\ \mu - a.e$, for each $y_i : \Omega \to Y$, such that $y_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})\ \mu - a.e\}$ is convex and weakly closed;

(c') The correspondence $T_i : L_{X_{-i}} \to 2^{L_{X_i}}$, defined by $T_i(\widetilde{x}_{-i}) = \{\widetilde{x}_i \in L_{X_i} : \psi_i(\omega, \widetilde{x}, y_i(\omega)) \leq 0\ \mu - a.e.$, for each $y_i : \Omega \to Y$, such that $y_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})\ \mu - a.e\}$ is weakly upper semicontinuous.

Proof. The proof follows similar lines as the proof of Theorem 5. It is obtained by applying Theorem 4. □

In the case of the upper semicontinuity of the correspondences, we can obtain corollaries which are similar to Corollaries 1 and 2.

Corollary 3. *Let I be a countable index set. Assumptions A.1), A2) and A3) (a), (b) of Corrolary 1 are fulfilled and, in addition:*

A.4) (a) For each $\widetilde{x}_{-i} \in L_{X_{-i}}$, $\{\widetilde{x}_i \in L_{X_i} : \widetilde{x}_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})\ \mu - a.e.$ and $\sup_{u\in G_i(\omega,y_i(\omega))} Re\langle u, \widetilde{x}_i(\omega) - y_i(\omega)\rangle \leq 0\ \mu - a.e.$, for each $y_i : \Omega \to Y$, such that $y_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})\ \mu - a.e\}$ is nonempty;

(b) For each $\widetilde{x}_{-i} \in L_{X_{-i}}$, $\{\widetilde{x}_i \in L_{X_i} : \sup_{u\in G_i(\omega,y_i(\omega))} Re\langle u, \widetilde{x}_i(\omega) - y_i(\omega)\rangle \leq 0\ \mu - a.e.$, for each $y_i : \Omega \to Y$, such that $y_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})\ \mu - a.e\}$ is convex and weakly closed;

(c) The correspondence $T_i : L_{X_{-i}} \to 2^{L_{X_i}}$, defined by

$T_i(\widetilde{x}_{-i}) = \{\widetilde{x}_i \in L_{X_i} : \sup_{u\in G_i(\omega,y_i(\omega))} Re\langle u, \widetilde{x}_i(\omega) - y_i(\omega)\rangle \leq 0\ \mu - a.e$, for each $y_i : \Omega \to Y$, such that $y_i(\omega) \in A_i(\omega, \widetilde{x}_{-i})\ \mu - a.e\}$, for each $\widetilde{x}_{-i} \in L_{X_{-i}}$ is weakly upper semicontinuous.

Then, the conclusion of Corollary 1 holds.

Corollary 4. *Assume that I is a countable index set and Conditions A.1), A2) and A3) (c) of Theorem 4 are fulfilled. Then, there exists $\widetilde{x}^* \in L_X$, such that, for every $i \in I$ and $\mu - a.e.$, $\widetilde{x}_i^*(\omega) \in A_i(\omega, \widetilde{x}_{-i}^*)$.*

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