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Non-integrated defect relations for the Gauss map of a complete minimal surface with finite total curvature in $\mathbb{R}^m$

by

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Dedicated to Professor Nguyen Dong Yen on his 60th birthday

Abstract

In this article, we give the non-integrated defect relations for the Gauss map of a complete minimal surface with finite total curvature in $\mathbb{R}^m$. This is a continuation of previous work of Ha-Trao (2015), which we extend here to targets of higher dimension.

Key Words: Minimal surface, Gauss map, Defect relation
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1 Introduction

In 1988, H. Fujimoto [6] proved Nirenberg's conjecture that if M is a complete non-flat minimal surface in $\mathbb{R}^3$, then its Gauss map can omit at most 4 points, and the bound is sharp. After that, he [8] also extended that result for minimal surfaces in $\mathbb{R}^m$. He proved that the Gauss map of a non-flat complete minimal surface in $\mathbb{R}^m$ can omit at most $m(m+1)/2$ hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ located in general position. He also gave an example to show that the number $m(m+1)/2$ is the best possible when m is odd. Beside that, many mathematicians also studied the value distribution of the Gauss map of a minimal surface with finite total curvature and got many good results (see Fang [4] and Ru [19] for examples). On the other hand, Mo-Osserman [18] (1990), Mo [17] (1994) and Ha-Phuong-Thoan [13] recently showed the relations between the value distribution of the Gauss map and the total curvature of a complete minimal surface. Related to the value distributrion of the Gauss map of a complete minimal surface and the value distribution of the Gauss map of a complete minimal surface with finite total curvature, many results were given (see [12], [2],[3], [16] and [15] for examples).

On the other hand, Fujimoto [7, 8, 9] improved the previous results on the value distribution theory of the Gauss map of a complete minimal surface by introducing the modified defect relations for the Gauss map of a complete minimal surface which have analogy to the defect relations given by R. Nevanlinna in his value distribution theory. The author and Trao [14] recently improved the Fujimoto's results in the case the Gauss map of a complete minimal surface with finite total curvature in $\mathbb{R}^3, \mathbb{R}^4$ by studying the non-integrated defect relations for the Gauss map. In this article, we would like to be continuous to study the non-integrated defect relations for the Gauss map of a complete minimal surface with finite total curvature in $\mathbb{R}^m$. These are the strict improvements of all previous results of Fujimoto on the modified defect relations for the Gauss map of a complete minimal surface with finite total curvature in $\mathbb{R}^m$. Thus, they also are the improvements of previous results on ramifications for the Gauss map of a complete minimal surface with finite total curvature in $\mathbb{R}^m$.

2 Statements of the main results

Let M be an open Riemann surface and f a nonconstant holomorphic map of M into $\mathbb{P}^k(\mathbb{C})$. Assume that f has reduced representation $f = (f_0 : \cdots : f_k)$. Set $||f|| = (|f_0|^2 + \cdots + |f_k|^2)^{1/2}$ and, for each a hyperplane $H : \overline{a}_0 w_0 + \cdots + \overline{a}_k w_k = 0$ in $\mathbb{P}^k(\mathbb{C})$ with $|a_0|^2 + \cdots + |a_k|^2 = 1$, we definition $f(H) := \overline{a}_0 f_0 + \cdots + \overline{a}_k f_k$.

Definition 1. *We definition the $S-$defect of H for f by*

$$\delta_f^S(H) := 1 - \inf\{\eta \geq 0; \eta \text{ satisfies condition } (*)_S\}.$$

Here, condition $(*)_S$ means that there exists a $[-\infty, \infty)-$valued continuous subharmonic function u ($\not\equiv -\infty$) on M satisfying the following conditions:
(C1) $e^u \leq ||f||^\eta$,
(C2) for each $\xi \in f^{-1}(H)$, there exists the limit

$$\lim_{z \to \xi} (u(z) - \min(\nu_{f(H)}(\xi), k) \log |z - \xi|) \in [-\infty, \infty),$$

where z is a holomorphic local coordinate around ξ and $\nu_{f(H)}$ is the divisor of $f(H)$.

Remark 1. *We always have that $\eta = 1$ satisfies condition $(*)_S$ with $u = \log |f(H)|$.*

Definition 2. *We definition the $H-$defect of H for f by*

$$\delta_f^H(H) := 1 - \inf\{\eta \geq 0; \eta \text{ satisfies condition } (*)_H\}.$$

Here, condition $(*)_H$ means that there exists a $[-\infty, \infty)-$valued continuous subharmonic function u on M which is harmonic on $M - f^{-1}(H)$ and satisfies the conditions (C1) and (C2).

Definition 3. *We definition the $O-$defect of H for f by*

$$\delta_f^O(H) := 1 - \inf\{\frac{1}{n}; \quad f(H) \text{ has no zero of order less than } n\}.$$

Remark 2. *We always have $0 \leq \delta_f^O(H) \leq \delta_f^H(H) \leq \delta_f^S(H) \leq 1$.*

Moreover, Fujimoto [5, page 672] also gave the reasons why he calls $\delta_f^S(H)$ the non-integrated defect by showing a relation between the non-integrated defect and the defect (as in Nevanlinna theory) of a nonconstant holomorphic map of Δ_R into $\mathbb{P}^k(\mathbb{C})$.

Definition 4. *One says that f is ramified over a hyperplane H in $\mathbb{P}^k(\mathbb{C})$ with multiplicity at least e if all the zeros of the function $f(H)$ have orders at least e. If the image of f omits H, one will say that f is ramified over H with multiplicity ∞.*

Remark 3. *If f is ramified over a hyperplane H in $\mathbb{P}^k(\mathbb{C})$ with multiplicity at least n, then $\delta_f^S(H) \geq \delta_f^H(H) \geq \delta_f^O(H) \geq 1 - \dfrac{1}{n}$. In particular, if $f^{-1}(H) = \emptyset$, then $\delta_f^O(H) = 1$.*

In this article, we would like to study the $S-$defect relations for the Gauss maps of minimal surfaces with finite total curvature in $\mathbb{R}^m$ which generalize the previous results of Ha-Trao in [14] to targets of higher dimension. In particular, we prove the following.

Main theorem. *Let M be a non-flat complete minimal surface with finite total curvature in $\mathbb{R}^m$ and its Gauss map G. Let $H_1, ..., H_q$ be hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ located in N-subgeneral position $(q > 2N-k+1, N \geq m-1)$. Assume that G is $k-$non-degenerate (that is $G(M)$ is contained in a $k-$dimensional linear subspace in $\mathbb{P}^{m-1}(\mathbb{C})$, but none of lower dimension), $1 \leq k \leq m-1$, then*

$$\sum_{j=1}^{q} \delta_G^S(H_j) \leq (k+1)(N - \frac{k}{2}) + (N+1).$$

Remark 4. *For the case of the Gauss maps of minimal surfaces with finite total curvature, we can show that the Main theorem improved strictly Theorem 1.2 in [5](by reducing the number m^2 to the number $m(m+1)/2$) and Theorem 2.8 in [8](by changing the $H-$ defect relations to the $S-$ defect relations).*

Remark 5. *It is well known that the image of the (generalized) Gauss map $g: M \to \mathbb{P}^{m-1}(\mathbb{C})$ is contained in the hyperquadric $Q_{m-2}(\mathbb{C}) \subset \mathbb{P}^{m-1}(\mathbb{C})$, and that $Q_1(\mathbb{C})$ is biholomorphic to $\mathbb{P}^1(\mathbb{C})$ and that $Q_2(\mathbb{C})$ is biholomorphic to $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. So the results Ha-Trao in ([14]) which only treat the cases $m = 3$ and $m = 4$ are better than a result which holds for any $m \geq 3$ can be if restricted to the special cases $m = 3, 4$. The easiest way to see the difference is to observe that 6 lines in $\mathbb{P}^2(\mathbb{C})$ in general position may have only 4 points of intersection with the quadric $Q_1(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C})$.*

3 Preliminaries and auxiliary lemmas

In this section, we recall some auxiliary lemmas in [9, 10, 11].
Let M be an open Riemann surface and ds^2 a pseudo-metric on M, namely, a metric on M with isolated singularities which is locally written as $ds^2 = \lambda^2 |dz|^2$ in terms of a nonnegative real-value function λ with mild singularities and a holomorphic local coordinate z. We definition the divisor of ds^2 by $\nu_{ds} := \nu_\lambda$ for each local expression $ds^2 = \lambda^2 |dz|^2$, which is globally well-definitiond on M. We say that ds^2 is a continuous pseudo-metric if $\nu_{ds} \geq 0$ everywhere.

Definition 5. *(see [9]) We definition the Ricci form of ds^2 by*

$$Ric_{ds^2} := -dd^c \log \lambda^2$$

for each local expression $ds^2 = \lambda^2 |dz|^2$.

In some cases, a $(1,1)-$form Ω on M is regarded as a current on M by defining $\Omega(\varphi) := \int_M \varphi\Omega$ for each $\varphi \in \mathcal{D}$, where $\mathcal{D}$ denotes the space of all C^∞ differentiable functions on M with compact supports.

Definition 6. *(see [9]) We say that a continuous pseudo-metric ds^2 has strictly negative curvature on M if there is a positive constant C such that*

$$-Ric_{ds^2} \geq C \cdot \Omega_{ds^2},$$

where Ω_{ds^2} denotes the area form for ds^2, namely,

$$\Omega_{ds^2} := \lambda^2(\sqrt{-1}/2)dz \wedge d\bar{z}$$

for each local expression $ds^2 = \lambda^2 |dz|^2$.

As is well-known, if the universal covering surface of M is biholomorphic with the unit disc in $\mathbb{C}$, then M has the complete conformal metric with constant curvature -1 which is called the Poincaré metric of M and denoted by $d\sigma_M^2$.

Let f be a linearly non-degenerate holomorphic map of M into $\mathbb{P}^k(\mathbb{C})$. Take a reduced representation $f = (f_0 : \cdots : f_k)$. Then $F := (f_0, \cdots, f_k) : M \to \mathbb{C}^{k+1} \backslash \{0\}$ is a holomorphic map with $\mathbb{P}(F) = f$. Consider the holomorphic map

$$F_p = (F_p)_z := F^{(0)} \wedge F^{(1)} \wedge \cdots \wedge F^{(p)} : M \longrightarrow \wedge^{p+1}\mathbb{C}^{k+1}$$

for $0 \leq p \leq k$, where $F^{(0)} := F = (f_0, \cdots, f_k)$ and $F^{(l)} = (F^{(l)})_z := (f_0^{(l)}, \cdots, f_k^{(l)})$ for each $l = 0, 1, \cdots, k$, and where the l-th derivatives $f_i^{(l)} = (f_i^{(l)})_z$, $i = 0, ..., k$, are taken with respect to z. (Here and for the rest of this paper the index $|_z$ means that the corresponding term is definitiond by using differentiation with respect to the variable z, and in order to keep notations simple, we usually drop this index if no confusion is possible). The norm of F_p is given by

$$|F_p| := \left(\sum_{0 \leq i_0 < \cdots < i_p \leq k} \left| W(f_{i_0}, \cdots, f_{i_p}) \right|^2 \right)^{\frac{1}{2}},$$

where $W(f_{i_0}, \cdots, f_{i_p}) = W_z(f_{i_0}, \cdots, f_{i_p})$ denotes the Wronskian of $f_{i_0}, \cdots, f_{i_p}$ with respect to z.

Proposition 1. *([11, Proposition 2.1.6]).*
For two holomorphic local coordinates z and ξ and a holomorphic function $h : M \to \mathbb{C}$, the following holds :
a) $W_\xi(f_0, \cdots, f_p) = W_z(f_0, \cdots, f_p) \cdot (\frac{dz}{d\xi})^{p(p+1)/2}$.
b) $W_z(hf_0, \cdots, hf_p) = W_z(f_0, \cdots, f_p) \cdot (h)^{p+1}$.

Proposition 2. *([11, Proposition 2.1.7]).*
For holomorphic functions $f_0, \cdots, f_p : M \to \mathbb{C}$ the following conditions are equivalent:
(i) $f_0, \cdots, f_p$ are linearly dependent over $\mathbb{C}$.
(ii) $W_z(f_0, \cdots, f_p) \equiv 0$ for some (or all) holomorphic local coordinate z.

We now take a hyperplane H in $\mathbb{P}^k(\mathbb{C})$ given by

$$H : \bar{c}_0\omega_0 + \cdots + \bar{c}_k\omega_k = 0\,,$$

with $\sum_{i=0}^{k} |c_i|^2 = 1$. We set

$$F_0(H) := F(H) := \bar{c}_0 f_0 + \cdots + \bar{c}_k f_k$$

and

$$|F_p(H)| = |(F_p)_z(H)| := \Bigg(\sum_{0 \le i_1 < \cdots < i_p \le k} \Bigg| \sum_{l \ne i_1, \ldots, i_p} \overline{c}_l W(f_l, f_{i_1}, \cdots, f_{i_p}) \Bigg|^2 \Bigg)^{\frac{1}{2}},$$

for $1 \le p \le k$. We note that by using Proposition 1, $|(F_p)_z(H)|$ is multiplied by a factor $|\frac{dz}{d\xi}|^{p(p+1)/2}$ if we choose another holomorphic local coordinate ξ, and it is multiplied by $|h|^{p+1}$ if we choose another reduced representation $f = (hf_0 : \cdots : hf_k)$ with a nowhere zero holomorphic function h. Finally, for $0 \le p \le k$, set the p-th contact function of f for H to be $\phi_p(H) := \dfrac{|F_p(H)|^2}{|F_p|^2} = \dfrac{|(F_p)_z(H)|^2}{|(F_p)_z|^2}$.

We next consider q hyperplanes $H_1, \cdots, H_q$ in $\mathbb{P}^k(\mathbb{C})$ given by

$$H_j : \langle \omega, A_j \rangle \equiv \overline{c}_{j0}\omega_0 + \cdots + \overline{c}_{jk}\omega_k \quad (1 \le j \le q)$$

where $A_j := (c_{j0}, \cdots, c_{jk})$ with $\sum_{i=0}^k |c_{ji}|^2 = 1$.

Assume now $N \ge k$ and $q \ge N + 1$. For $R \subseteq Q := \{1, 2, \cdots, q\}$, denote by $d(R)$ the dimension of the vector subspace of $\mathbb{C}^{k+1}$ generated by $\{A_j; j \in R\}$.

The hyperplanes $H_1, \cdots, H_q$ are said to be in N-subgeneral position if $d(R) = k+1$ for all $R \subseteq Q$ with $\sharp(R) \ge N+1$, where $\sharp(A)$ means the number of elements of a set A. In the particular case $N = k$, these are said to be in general position.

Theorem 1. *([11, Theorem 2.4.11]) For given hyperplanes $H_1, \cdots, H_q$ $(q > 2N - k + 1)$ in $\mathbb{P}^k(\mathbb{C})$ located in N-subgeneral position, there are some rational numbers $\omega(1), \cdots, \omega(q)$ and θ satisfying the following conditions:*
(i) $0 < \omega(j) \le \theta \le 1 \quad (1 \le j \le q)$,
(ii) $\sum_{j=1}^q \omega(j) = k + 1 + \theta(q - 2N + k - 1)$,
(iii) $\frac{k+1}{2N-k+1} \le \theta \le \frac{k+1}{N+1}$,
(iv) If $R \subset Q$ and $0 < \sharp(R) \le n+1$, then $\sum_{j \in R} \omega(j) \le d(R)$.

Constants $\omega(j)$ $(1 \le j \le q)$ and θ with the properties of Theorem 1 are called Nochka weights and a Nochka constant for $H_1, \cdots, H_q$ respectively.

We need the three following results of Fujimoto combining the previously introduced concept of contact functions with Nochka weights:

Theorem 2. *([11, Theorem 2.5.3]) Let $H_1, \cdots, H_q$ be hyperplanes in $\mathbb{P}^k(\mathbb{C})$ located in $N-$subgeneral position and let $\omega(j)$ $(1 \le j \le q)$ and θ be Nochka weights and a Nochka constant for these hyperplanes. For every $\epsilon > 0$ there exist some positive numbers $\delta(> 1)$ and C, depending only on ϵ and H_j, $1 \le j \le q$, such that*

$$\begin{aligned} &dd^c \log \frac{\Pi_{p=0}^{k-1} |F_p|^{2\epsilon}}{\Pi_{1 \le j \le q, 0 \le p \le k-1} \log^{2\omega(j)}(\delta/\phi_p(H_j))} \\ &\ge C \Bigg(\frac{|F_0|^{2\theta(q-2N+k-1)} |F_k|^2}{\Pi_{j=1}^q (|F(H_j)|^2 \Pi_{p=0}^{k-1} \log^2(\delta/\phi_p(H_j)))^{\omega(j)}} \Bigg)^{\frac{2}{k(k+1)}} dd^c |z|^2. \end{aligned}$$

Proposition 3. *([11, Proposition 2.5.7]) Set $\sigma_p = p(p+1)/2$ for $0 \le p \le k$ and $\tau_k = \sum_{p=0}^{k} \sigma_p$. Then,*

$$dd^c \log(|F_0|^2|F_1|^2 \cdots |F_{k-1}|^2) \ge \frac{\tau_k}{\sigma_k}\left(\frac{|F_0|^2|F_1|^2 \cdots |F_k|^2}{|F_0|^{2\sigma_{k+1}}}\right)^{1/\tau_k} dd^c|z|^2.$$

Proposition 4. *([11, Lemma 3.2.13]) Let f be a non-degenerate holomorphic map of a domain in $\mathbb{C}$ into $\mathbb{P}^k(\mathbb{C})$ with reduced representation $f = (f_0 : \cdots : f_k)$ and let $H_1, \cdots, H_q$ be hyperplanes located in N-subgeneral position $(q > 2N - k + 1)$ with Nochka weights $\omega(1), \cdots, \omega(q)$ respectively. Then,*

$$\nu_\phi + \sum_{j=1}^{q} \omega(j) \cdot \min(\nu_{(f,H_j)}, k) \ge 0,$$

where $\phi = \dfrac{|F_k|}{\Pi_{j=1}^{q} \mid F(H_j) \mid^{\omega(j)}}$.

Lemma 1. *(Generalized Schwarz's Lemma [1]) Let v be a non-negative real-valued continuous subharmonic function on Δ_R. If v satisfies the inequality $\Delta \log v \ge v^2$ in the sense of distribution, then*

$$v(z) \le \frac{2R}{R^2 - |z|^2}.$$

4 Proof of the Main Theorem

Proof. For the convenience of the reader, we first recall some notations on the Gauss map of minimal surfaces in $\mathbb{R}^m$. Let M be a complete immersed minimal surface in $\mathbb{R}^m$. Take an immersion $x = (x_0, ..., x_{m-1}) : M \to \mathbb{R}^m$. Then M has the structure of a Riemann surface and any local isothermal coordinate (x, y) of M gives a local holomorphic coordinate $z = x + \sqrt{-1}y$. The generalized Gauss map of x is definitiond to be

$$g : M \to \mathbb{P}^{m-1}(\mathbb{C}), g = \mathbb{P}(\frac{\partial x}{\partial z}) = (\frac{\partial x_0}{\partial z} : \cdots : \frac{\partial x_{m-1}}{\partial z}).$$

Since $x : M \to \mathbb{R}^m$ is immersed,

$$G = G_z := (g_0, ..., g_{m-1}) = ((g_0)_z, ..., (g_{m-1})_z) = (\frac{\partial x_0}{\partial z}, \cdots, \frac{\partial x_{m-1}}{\partial z})$$

is a (local) reduced representation of g, and since for another local holomorphic coordinate ξ on M we have $G_\xi = G_z \cdot (\frac{dz}{d\xi})$, g is well definitiond (independently of the (local) holomorphic coordinate). Moreover, if ds^2 is the metric on M induced by the standard metric on $\mathbb{R}^m$, we have

$$ds^2 = 2|G_z|^2|dz|^2. \tag{4.1}$$

Finally since M is minimal, g is a holomorphic map.

Since by hypothesis of the Main theorem, g is k-non-degenerate $(1 \leq k \leq m-1)$ without loss of generality, we may assume that $g(M) \subset \mathbb{P}^k(\mathbb{C})$; then

$$g : M \to \mathbb{P}^k(\mathbb{C}), g = \mathbb{P}(\frac{\partial x}{\partial z}) = (\frac{\partial x_0}{\partial z} : \cdots : \frac{\partial x_k}{\partial z})$$

is linearly non-degenerate in $\mathbb{P}^k(\mathbb{C})$ (so in particular g is not constant) and the other facts mentioned above still hold.

Let $H_j (j = 1, ..., q)$ be $q (\geq N + 1)$ hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ in N-subgeneral position $(N \geq m-1 \geq k)$. Then $H_j \cap \mathbb{P}^k(\mathbb{C}) (j = 1, ..., q)$ are q hyperplanes in $\mathbb{P}^k(\mathbb{C})$ in N-subgeneral position. Let each $H_j \cap \mathbb{P}^k(\mathbb{C})$ be represented as

$$H_j \cap \mathbb{P}^k(\mathbb{C}) : \overline{c}_{j0}\omega_0 + \cdots + \overline{c}_{jk}\omega_k = 0$$

with $\sum_{i=0}^{k} |c_{ji}|^2 = 1$.
Set

$$G(H_j) = G_z(H_j) := \overline{c}_{j0} g_0 + \cdots + \overline{c}_{jk} g_k.$$

We will now, for each contact function $\phi_p(H_j)$ for each of our hyperplanes H_j, choose one of the components of the numerator $|((G_z)_p)_z(H_j)|$ which is not identically zero: More precisely, for each j, p $(1 \leq j \leq q, 1 \leq p \leq k)$, we can choose $i_1, \cdots, i_p$ with $0 \leq i_1 < \cdots < i_p \leq k$ such that

$$\psi(G)_{jp} = (\psi(G_z)_{jp})_z := \sum_{l \neq i_1,..,i_p} \overline{c}_{jl} W_z(g_l, g_{i_1}, \cdots, g_{i_p}) \not\equiv 0,$$

(indeed, otherwise, we have $\sum_{l \neq i_1,...,i_p} \overline{c}_{jl} W(g_l, g_{i_1}, \cdots, g_{i_p}) \equiv 0$ for all $i_1, ..., i_p$, so

$$W(\sum_{l \neq i_1,..,i_p} \overline{c}_{jl} g_l, g_{i_1}, \cdots, g_{i_p}) \equiv 0, \quad \text{for all} \quad i_1, ..., i_p\,,$$

which contradicts the non-degeneracy of g in $\mathbb{P}^k(\mathbb{C})$. Alternatively we simply can observe that in our situation none of the contact functions vanishes identically).

Now we prove the Main theorem by reduction to absurd in four steps:
Step 1: Suppose the Main theorem is not true. This means that

$$\sum_{j=1}^{q} \delta_g^S(H_j) > (k+1)(N - \frac{k}{2}) + (N+1). \tag{4.2}$$

By definition, there exist constants $\eta_j \geq 0 (1 \leq j \leq q)$ such that $q - \sum_{j=1}^{q} \eta_j > (k+1)(N - \frac{k}{2}) + (N+1)$ and continuous subharmonic functions $u_j (1 \leq j \leq q)$ on M satisfying conditions (C1) and (C2). Thus

$$\sum_{j=1}^{q} (1 - \eta_j) - 2N + k - 1 > \frac{(2N - k + 1)k}{2} > 0, \tag{4.3}$$

and this implies in particular

$$q > 2N - k + 1 \geq N + 1 \geq k + 1. \tag{4.4}$$

By Theorem 1, we have

$$(q - 2N + k - 1)\theta = \sum_{j=1}^{q} \omega(j) - k - 1,\ \theta \geq \omega(j) > 0 \text{ and } \theta \geq \frac{k+1}{2N-k+1},$$

so

$$\begin{aligned}
2\bigg(\sum_{j=1}^{q}\omega(j)(1-\eta_j) - k - 1\bigg) &= 2(\sum_{j=1}^{q}\omega(j) - k - 1) - 2\sum_{j=1}^{q}\omega(j)\eta_j \\
&= 2(q - 2N + k - 1)\theta - 2\sum_{j=1}^{q}\omega(j)\eta_j \\
&\geq 2(q - 2N + k - 1)\theta - 2\sum_{j=1}^{q}\theta\eta_j \\
&= 2\theta\bigg(\sum_{j=1}^{q}(1-\eta_j) - 2N + k - 1\bigg) \\
&\geq 2\frac{(k+1)\Big(\sum_{j=1}^{q}(1-\eta_j) - 2N + k - 1\Big)}{2N - k + 1}.
\end{aligned}$$

Thus, we now can conclude with (4.3) that

$$\begin{aligned}
&2\bigg(\sum_{j=1}^{q}\omega(j)(1-\eta_j) - k - 1\bigg) > k(k+1) \\
&\Rightarrow \sum_{j=1}^{q}\omega(j)(1-\eta_j) - k - 1 - \frac{k(k+1)}{2} > 0.
\end{aligned} \tag{4.5}$$

We set $\gamma := \sum_{j=1}^{q}\omega(j)(1-\eta_j) - k - 1$.
Then, by (4.5), we get $\gamma > \sigma_k \Rightarrow \dfrac{\gamma}{\sigma_k} > 1$. So we can choose a positive real number ϵ such that $\dfrac{\gamma - \epsilon\sigma_{k+1}}{\sigma_k + \epsilon\tau_k} > 1$, where σ_k, τ_k were definitiond in Proposition 3.
Step 2: We set

$$\lambda_z := \bigg(\frac{|G_z|^{\gamma - \epsilon\sigma_{k+1}} e^{\sum_{j=1}^{q}\omega(j)u_j}.|(G_k)_z|.\prod_{p=0}^{k}|(G_p)_z|^{\epsilon}}{\prod_{j=1}^{q}(|G(H_j)|\Pi_{p=0}^{k-1}\log(\delta/\phi_p(H_j)))^{\omega(j)}}\bigg)^{\frac{1}{\sigma_k + \epsilon\tau_k}},$$

and definition the pseudometric $d\tau_z^2 := \lambda_z^2|dz|^2$. Using Proposition 1, we can see that

$$\begin{aligned}
d\tau_\xi &:= \Big(\frac{|G_\xi|^{\gamma-\epsilon\sigma_{k+1}}e^{\sum_{j=1}^q \omega(j)u_j}.|(G_k)_\xi|.\prod_{p=0}^k |(G_p)_\xi|^\epsilon}{\prod_{j=1}^q(|G(H_j)|\Pi_{p=0}^{k-1}\log(\delta/\phi_p(H_j)))^{\omega(j)}}\Big)^{\frac{1}{\sigma_k+\epsilon\tau_k}}|d\xi| \\
&= \Big(\frac{|G_z|^{\gamma-\epsilon\sigma_{k+1}}e^{\sum_{j=1}^q \omega(j)u_j}.|(G_k)_z||\frac{dz}{d\xi}|^{\sigma_k}.\prod_{p=0}^k |(G_p)_z|^\epsilon.|\frac{dz}{d\xi}|^{\sum_{p=0}^k \epsilon\frac{p(p+1)}{2}}}{\prod_{j=1}^q(|G(H_j)|\Pi_{p=0}^{k-1}\log(\delta/\phi_p(H_j)))^{\omega(j)}}\Big)^{\frac{1}{\sigma_k+\epsilon\tau_k}}|\frac{d\xi}{dz}|.|dz| \\
&= \Big(\frac{|G_z|^{\gamma-\epsilon\sigma_{k+1}}e^{\sum_{j=1}^q \omega(j)u_j}.|(G_k)_z|.\prod_{p=0}^k |(G_p)_z|^\epsilon.|\frac{dz}{d\xi}|^{\sigma_k+\epsilon\tau_k}}{\prod_{j=1}^q(|G(H_j)|\Pi_{p=0}^{k-1}\log(\delta/\phi_p(H_j)))^{\omega(j)}}\Big)^{\frac{1}{\sigma_k+\epsilon\tau_k}}|\frac{d\xi}{dz}|.|dz| \\
&= d\tau_z.
\end{aligned}$$

Thus $d\tau_z^2$ is independent of the choice of the local coordinate z. We will denote $d\tau_z^2$ by $d\tau^2$ for convenience. So $d\tau^2$ is well-definitiond on M and

$$d\tau^2 = \Big(\frac{|G|^{\gamma-\epsilon\sigma_{k+1}}e^{\sum_{j=1}^q \omega(j)u_j}.|G_k|.\prod_{p=0}^k |G_p|^\epsilon}{\prod_{j=1}^q(|G(H_j)|\Pi_{p=0}^{k-1}\log(\delta/\phi_p(H_j)))^{\omega(j)}}\Big)^{\frac{2}{\sigma_k+\epsilon\tau_k}}|dz|^2 := \lambda^2|dz|^2.$$

Step 3: We will show that $d\tau^2$ is continuous and has strictly negative curvature on M in this step.

Indeed, it is easy to see that $d\tau$ is continuous at every point z_0 with $\Pi_{j=1}^q G(H_j)(z_0) \neq 0$. Now we take a point z_0 such that $\Pi_{j=1}^q G(H_j)(z_0) = 0$. Hence, it follows from (C2) that we get

$$\lim_{z\to z_0} e^{u_j(z)}|z-z_0|^{-\min(\nu_{G(H_j)}(z_0),k)} = \lim_{z\to z_0} e^{u_j(z)-\min(\nu_{G(H_j)}(z_0),k)\log|z-z_0|} < \infty, \forall\ 1 \le j \le q.$$

Thus, combining this with Proposition 4, we get

$$\begin{aligned}
\nu_{d\tau}(z_0) \geq \frac{1}{\sigma_k+\epsilon\tau_k}\Big(&\nu_{G_k}(z_0) - \sum_{j=1}^q \omega(j)\nu_{G(H_j)}(z_0) + \sum_{j=1}^q \omega(j)\min\{\nu_{G(H_j)}(z_0), k\} \\
&+ \sum_{j=1}^q \omega(j)\nu_{e^{u_j(z)}|z-z_0|^{-\min(\nu_{G(H_j)}(z_0),k)}}(z_0)\Big) \\
&\geq 0.
\end{aligned}$$

This concludes the proof that $d\tau$ is continuous on M.

On the other hand, by using Proposition 3, Theorem 2 and noting that $dd^c\log|G_k| =$

$0, dd^c \log e^{u_j} \geq 0 (1 \leq j \leq q)$, we have

$$
\begin{aligned}
dd^c \log \lambda \geq & \frac{\gamma - \epsilon\sigma_{k+1}}{\sigma_k + \epsilon\tau_k} dd^c \log |G| + \frac{\epsilon}{4(\sigma_k + \epsilon\tau_k)} dd^c \log(|G_0|^2 \cdots |G_{k-1}|^2) \\
& + \frac{1}{2(\sigma_k + \epsilon\tau_k)} dd^c \log \frac{\prod_{p=0}^{k-1} |G_p|^{2(\frac{\epsilon}{2})}}{\prod_{j=1}^{q} \Pi_{p=0}^{k-1} \log^{2\omega(j)}(\delta/\phi_p(H_j))} \\
& \geq \frac{\epsilon}{4(\sigma_k + \epsilon\tau_k)} \frac{\tau_k}{\sigma_k} \left(\frac{|G_0|^2 |G_1|^2 \cdots |G_k|^2}{|G_0|^{2\sigma_{k+1}}} \right)^{1/\tau_k} dd^c |z|^2 \\
& + C_0 \left(\frac{|G_0|^{2\theta(q-2N+k-1)} |G_k|^2}{\Pi_{j=1}^{q} (|G(H_j)|^2 \Pi_{p=0}^{k-1} \log^2(\delta/\phi_p(H_j)))^{\omega(j)}} \right)^{\frac{2}{k(k+1)}} dd^c |z|^2 \\
& \geq \min\{ \frac{1}{4\sigma_k(\sigma_k + \epsilon\tau_k)}, \frac{C_0}{\sigma_k} \} \left(\epsilon\tau_k \left(\frac{|G_0|^2 |G_1|^2 \cdots |G_k|^2}{|G_0|^{2\sigma_{k+1}}} \right)^{1/\tau_k} \right. \\
& \left. + \sigma_k \left(\frac{|G_0|^{2\theta(q-2N+k-1)} |G_k|^2}{\Pi_{j=1}^{q} (|G(H_j)|^2 \Pi_{p=0}^{k-1} \log^2(\delta/\phi_p(H_j)))^{\omega(j)}} \right)^{\frac{1}{\sigma_k}} \right) dd^c |z|^2
\end{aligned}
$$

where C_0 is the positive constant. So, by using the basic inequality

$$
\alpha A + \beta B \geq (\alpha + \beta) A^{\frac{\alpha}{\alpha+\beta}} B^{\frac{\beta}{\alpha+\beta}} \text{ for all } \alpha, \beta, A, B > 0,
$$

we can find a positive constant C_1 satisfing the following

$$
\begin{aligned}
dd^c \log \lambda & \geq C_1 \left(\frac{|G|^{\theta(q-2N+k-1) - \epsilon\sigma_{k+1}} . |G_k| . \prod_{p=0}^{k} |G_p|^{\epsilon}}{\prod_{j=1}^{q} (|G(H_j)| \cdot \Pi_{p=0}^{k-1} \log(\delta/\phi_p(H_j)))^{\omega(j)}} \right)^{\frac{2}{\sigma_k + \epsilon\tau_k}} dd^c |z|^2 \\
& = C_1 \left(\frac{|G|^{\sum_{j=1}^{q} \omega(j) - k - 1 - \epsilon\sigma_{k+1}} . |G_k| . \prod_{p=0}^{k} |G_p|^{\epsilon}}{\prod_{j=1}^{q} (|G(H_j)| \cdot \Pi_{p=0}^{k-1} \log(\delta/\phi_p(H_j)))^{\omega(j)}} \right)^{\frac{2}{\sigma_k + \epsilon\tau_k}} dd^c |z|^2 \quad \text{(by Theorem 1)} \\
& = C_1 \left(\frac{|G|^{\gamma - \epsilon\sigma_{k+1}} . e^{\sum_{j=1}^{q} \omega(j) u_j} . |G_k| . \prod_{p=0}^{k} |G_p|^{\epsilon} . \prod_{j=1}^{q} \left(\frac{|G|^{\eta_j}}{e^{u_j}} \right)^{\omega(j)}}{\prod_{j=1}^{q} (|G(H_j)| \cdot \Pi_{p=0}^{k-1} \log(\delta/\phi_p(H_j)))^{\omega(j)}} \right)^{\frac{2}{\sigma_k + \epsilon\tau_k}} dd^c |z|^2.
\end{aligned}
$$

On the other hand,

$$
\left(\frac{|G|^{\eta_j}}{e^{u_j}} \right)^{\omega(j)} \geq 1 \text{ for all } j = 1, ..., q,
$$

so we get

$$
\begin{aligned}
dd^c \log \lambda & \geq C_1 \left(\frac{|G|^{\gamma - \epsilon\sigma_{k+1}} . e^{\sum_{j=1}^{q} \omega(j) u_j} . |G_k| . \prod_{p=0}^{k} |G_p|^{\epsilon}}{\prod_{j=1}^{q} (|G(H_j)| \cdot \Pi_{p=0}^{k-1} \log(\delta/\phi_p(H_j)))^{\omega(j)}} \right)^{\frac{2}{\sigma_k + \epsilon\tau_k}} dd^c |z|^2 \\
& = C_1 \lambda^2 dd^c |z|^2.
\end{aligned}
$$

This concludes the proof that $d\tau^2$ has strictly negative curvature on M.
Step 4: Set $\Omega_g = dd^c \log ||G||^2$.

Now, by using the classical Nevanlinna theory [11, Theorem 3.3.15] and remarks of Fujimoto [5, Proposition 4.7] we get $\sum_{j=1}^{q} \delta_g^S(H_j) \leq 2N - k + 1$ if the universal covering surface of M is biholomorphic to $\mathbb{C}$. This is a contradiction with (4.2). Thus, we only need consider the case that the universal covering surface of M is biholomorphic to the unit disc. By Lemma 1, there exists a positive constant C_0 such that

$$d\tau^2 \leq C_0 d\sigma_M^2,$$

where $d\sigma_M^2$ denotes the Poincaré metric on M.
Now, it follows from the assumption M has finite total curvature in $\mathbb{R}^m$ that M is biholomorphic with a compact Riemann surface $\overline{M}$ with finitely many points a_l's removed. For each a_l, we take a neighborhood U_l of a_l which is biholomorphic to $\Delta^* = \{z; 0 < |z| < 1\}$, where $z(a_l) = 0$. The Poincaré metric on domain Δ^* is given by

$$d\sigma_{\Delta^*}^2 = \frac{4|dz|^2}{|z|^2 \log^2 |z|^2}.$$

By using the distance decreasing property of the Poincaré metric, we have

$$d\tau^2 \leq C_l \frac{|dz|^2}{|z|^2 \log^2 |z|^2}$$

with some $C_l > 0$. This implies that, for a neighborhood U_l^* of a_l which is relatively compact in U_l, we have

$$\int_{U_l^*} \Omega_{d\tau^2} < +\infty.$$

Since $\overline{M}$ is compact, we have

$$\int_M \Omega_{d\tau^2} \leq \int_{\overline{M} - \cup_l U_l^*} \Omega_{d\tau^2} + \sum_l \int_{U_l^*} \Omega_{d\tau^2} < +\infty. \tag{4.6}$$

On the other hand, we have

$$\begin{aligned} dd^c \log \lambda &\geq (\frac{\gamma - \epsilon\sigma_{k+1}}{\sigma_k + \epsilon\tau_k}) dd^c \log |G| + \frac{1}{(\sigma_k + \epsilon\tau_k)} dd^c \log \frac{\prod_{p=0}^{k-1} |G_p|^\epsilon}{\prod_{j=1}^{q} \Pi_{p=0}^{k-1} \log^{2\omega(j)}(\delta/\phi_p(H_j))} \\ &\geq (\frac{\gamma - \epsilon\sigma_{k+1}}{\sigma_k + \epsilon\tau_k}) dd^c \log |G|, \text{ by Theorem 2.} \end{aligned}$$

This implies

$$dd^c \log \lambda^2 \geq (\frac{\gamma - \epsilon\sigma_{k+1}}{\sigma_k + \epsilon\tau_k}) \Omega_g.$$

Thus, we now can find a subharmonic function v such that

$$\begin{aligned} \lambda^2 |dz|^2 &= e^v ||G||^{2(\frac{\gamma - \epsilon\sigma_{k+1}}{\sigma_k + \epsilon\tau_k})} |dz|^2 \\ &= e^{v + (\frac{\gamma - \epsilon\sigma_{k+1}}{\sigma_k + \epsilon\tau_k} - 1) \log ||G||^2} ||G||^2 |dz|^2 \\ &= e^w ds^2. \end{aligned}$$

So $d\tau^2 = e^w ds^2$, where w is a subharmonic function. Here, we can apply the result of Yau in [20] to see

$$\int_M e^w \Omega_{ds^2} = +\infty,$$

because of the completeness of M with respect to the metric ds^2. This contradicts the assertion (4.6). The proof of the Main theorem is completed. □

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