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A refinement of Cauchy's bound for the moduli of zeros of a polynomial
by

V.K. Jain[(1)], V. Tewary[(2)]

Abstract

A refinement of Cauchy's bound

$$|z| < 1 + \max_{0 \le k \le n-1} |a_k|$$

for the moduli of all the zeros of the polynomial

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_0$$

has been obtained.

Key Words: refinement, zeros, polynomial, moduli, Cauchy's bound.
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1 Introduction and statement of results

Concerning an upper bound for the moduli of all the zeros of a polynomial, the following result due to Cauchy [3] is well known.

Theorem A. *All the zeros of $f(z) = a_0 + a_1 z + \ldots + a_n z^n, a_n \neq 0$, lie in the disc*

$$|z| < 1 + \max_{0 \le k \le n-1} |a_k / a_n|.$$

In the literature there exist many refinements ([9], [4], [2], [6], [10], [7], [8], [1]), of Theorem A. In this paper one more refinement, of Theorem A, with a simple bound and a very short proof has been obtained. More precisely we have proved

Theorem. *All the zeros of the polynomial*

$$f(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_0, (a_{-1} = 0),$$

lie in the disc

$$|z| < \max_{1 \le j \le \lfloor \frac{n+1}{2} \rfloor} \left\{ \frac{|a_{n-2j+1}| + \sqrt{|a_{n-2j+1}|^2 + 4 + 4|a_{n-2j}|}}{2} \right\}.$$

2 Proof of Theorem

Firsty let n be even. Then

$$\begin{aligned}
|f(z)| &\geq |z|^{n-2}\left\{|z|^2 - |a_{n-1}||z| - (1+|a_{n-2}|)\right\} + |z|^{n-4}\left\{|z|^2 - |a_{n-3}||z| - \right. \\
&\quad \left.(1+|a_{n-4}|)\right\} + \ldots + |z|^{n-2j}\left\{|z|^2 - |a_{n-2j+1}||z| - (1+|a_{n-2j}|)\right\} + \ldots + \\
&\quad \left\{|z|^2 - |a_1||z| - (1+|a_0|)\right\} + 1, \\
&> 0,
\end{aligned}$$

if

$$|z| \geq \max_{1\leq j\leq \lfloor \frac{n+1}{2} \rfloor}\left\{\frac{|a_{n-2j+1}| + \sqrt{|a_{n-2j+1}|^2 + 4 + 4|a_{n-2j}|}}{2}\right\}$$

and Theorem follows for the possibility under consideration. Now let n be odd. Then

$$\begin{aligned}
|f(z)| &\geq |z|^{n-2}\left\{|z|^2 - |a_{n-1}||z| - (1+|a_{n-2}|)\right\} + |z|^{n-4}\left\{|z|^2 - |a_{n-3}||z| - \right. \\
&\quad \left.(1+|a_{n-4}|)\right\} + \ldots + |z|^{n-2j}\left\{|z|^2 - |a_{n-2j+1}||z| - (1+|a_{n-2j}|)\right\} + \ldots + \\
&\quad |z|\left\{|z|^2 - |a_2||z| - (1+|a_1|)\right\} + \{|z| - |a_0|\}, \qquad n \geq 3
\end{aligned}$$

and

$$|f(z)| \geq \{|z| - |a_0|\} \qquad , \qquad n = 1.$$

Therefore

$$|f(z)| > 0,$$

if

$$|z| \geq \max_{1\leq j\leq \lfloor \frac{n+1}{2} \rfloor}\left\{\frac{|a_{n-2j+1}| + \sqrt{|a_{n-2j+1}|^2 + 4 + 4|a_{n-2j}|}}{2}\right\}$$

and Theorem follows for the possibility under consideration. This completes the proof of Theorem.

3 Additional justification for the bound of Theorem

For this purpose we will compare the bound of Theorem firstly with Cauchy's bound and its subsequent refinements and secondly with the bounds, not associated with Cauchy, namely those of Fujiwara [5] and Walsh [11]. Accordingly we proceed.

(A) Comparison of bound of Theorem with Cauchy's bound and its subsequent refinements

(A_1) Comparison of bound of Theorem with Cauchy's bound

Using the

$$\begin{aligned}\text{bound of Theorem} &= \text{B (, say)},\\ &= \max_{1\le j\le \lfloor \frac{n+1}{2}\rfloor}\left\{\frac{|a_{n-2j+1}| + \sqrt{|a_{n-2j+1}|^2 + 4 + 4|a_{n-2j}|}}{2}\right\},\end{aligned} \tag{3.1}$$

we can say that all the zeros of

$$f_1(z) = z^5 + a_1z + a_0, |a_1| = |a_0| = 1 \tag{3.2}$$

lie in

$$|z| < 1.618 \tag{3.3}$$

and all the zeros of

$$f_2(z) = z^5 + a_4z^4 + a_3z^3 + a_2z^2 + a_1z + a_0, |a_4| = 2, |a_3| = 3, |a_2| = 4, |a_1| = 5, |a_0| = 6 \tag{3.4}$$

lie in

$$|z| < 6.162. \tag{3.5}$$

Further using

$$\text{Cauchy's bound [3]} = B_1(\text{, say}) = 1 + \max_{0\le k\le n-1} |a_k| = 1 + M, (M = \max_{0\le k\le n-1} |a_k|), \tag{3.6}$$

we can say that all the zeros of $f_1(z)$ lie in

$$|z| < 2$$

and all the zeros of $f_2(z)$ lie in

$$|z| < 7,$$

thereby implying by (3.3) and (3.5) that the result obtained by bound of Theorem is <u>better</u> than that obtained by Cauchy's bound for both $f_1(z)$ and $f_2(z)$.

Now

$$\frac{B}{B_1} = \frac{\max_{1\le j\le \lfloor \frac{n+1}{2}\rfloor}\left\{\frac{|a_{n-2j+1}|+\sqrt{|a_{n-2j+1}|^2+4+4|a_{n-2j}|}}{2}\right\}}{1 + \max_{0\le k\le n-1}|a_k|}. \tag{3.7}$$

Using (3.7) or otherwise we will get certain information about B in terms of B_1 or vice versa, by imposing certain restrictions on size of coefficients (i.e. moduli of coefficients) of polynomial and sometimes imposing restrictions on degree of polynomial also:

(a) All coefficients are equal in size (i.e $|a_j| = t, 0 \leq j \leq n-1$). Then (3.8)

$B = B_1, (n \geq 2),$ (by usual method).

(b) (i) The first two leading coefficients (i.e. a_{n-1} and a_{n-2}) dominate the remaining coefficients in size and themselves are equal in size (i.e. $|a_{n-1}| = |a_{n-2}| = t$ and $|a_j| < t, 0 \leq j \leq n-3$). Then (3.9)

$B = B_1, (n \geq 3),$ (by usual method).

(ii) (a′) The first two leading coefficients (i.e. a_{n-1} and a_{n-2}) dominate the remaining coefficients in size and themselves are unequal in size with

$$|a_{n-1}|, (= t) > |a_{n-2}|, (= t_1), \tag{3.10}$$

(i.e. $|a_j| < t_1, 0 \leq j \leq n-3$). Then (3.11)

$B < B_1, (n \geq 3),$ (by usual method).

(b′) The first two leading coefficents (i.e. a_{n-1} and a_{n-2}) dominate the remaining coefficients in size and themselves are unequal in size with

$$|a_{n-1}|, (= t) < |a_{n-2}|, (= t_1), \tag{3.12}$$

(i.e. $|a_j| < t, 0 \leq j \leq n-3$). Then (3.13)

$B < B_1, (n \geq 3),$ (by usual method).

(A_2) <u>Comparison of bound of Theorem with Joyal et al.'s bound</u>

Using

$$\text{Joyal et al.'s bound [9]} = B_2 \text{ (, say)} = \frac{1}{2}\left\{1 + |a_{n-1}| + \sqrt{(1-|a_{n-1}|)^2 + 4M_1}\right\}, \quad (M_1 = \max_{0 \leq l \leq n-2} |a_l|), \tag{3.14}$$

we can say that all the zeros of $f_1(z)$ lie in

$$|z| \leq 1.618$$

and all the zeros of $f_2(z)$ lie in

$$|z| \leq 4,$$

thereby implying by (3.3) and (3.5) that the result obtained by bound of Theorem is <u>better</u> than that obtained by Joyal et al.'s bound for $f_1(z)$ but the result obtained by bound of Theorem is <u>worse</u> than that obtained by Joyal et al.'s bound for $f_2(z)$.

Now

$$\frac{B}{B_2} = \frac{\max_{1 \leq j \leq \lfloor \frac{n+1}{2} \rfloor} \left\{|a_{n-2j+1}| + \sqrt{|a_{n-2j+1}|^2 + 4 + 4|a_{n-2j}|}\right\}}{1 + |a_{n-1}| + \sqrt{(1-|a_{n-1}|)^2 + 4(\max_{0 \leq l \leq n-2} |a_l|)}}. \tag{3.15}$$

Further what we did in (A_1) after getting $\frac{B}{B_1}$, we will now do the same thing here also with

B_1 replaced by B_2 and we will be using subsections of (A_1) only, with same meaning.

(a) $B = B_2, (n \geq 2),$ (by usual method).
(b) (i) $B = B_2, (n \geq 3),$ (by usual method).
(ii) (a')$B > B_2, (n \geq 3),$ (by usual method).
(b') $B < B_2, (n \geq 3),$ (by usual method).
(A_3) Comparison of bound of Theorem with Datt and Govil's bound
Using

$$\text{Datt and Govil's bound [4]} = B_3\ (\text{,say}) = 1 + \lambda_0 M, (M \text{ is as in (3.6) and } \lambda_0 = \text{ unique root of } (1 + Mx)^n(x-1) + 1 = 0, \text{ in } (0,1)), \tag{3.16}$$

we can say that all the zeros of $f_1(z)$ lie in

$$|z| \leq R, R > 1.75$$

and all the zeros of $f_2(z)$ lie in

$$|z| \leq R', R' > 6.25$$

thereby implying by (3.3) and (3.5) that the result obtainted by bound of Theorem is better than that obtained by Datt and Govil's bound for both $f_1(z)$ an $f_2(z)$.

Now

$$\frac{B}{B_3} = \frac{\max_{1\leq j\leq \lfloor \frac{n+1}{2} \rfloor} \left\{ |a_{n-2j+1}| + \sqrt{|a_{n-2j+1}|^2 + 4 + 4|a_{n-2j}|} \right\} /2}{1 + \lambda_0 (\max_{0\leq k\leq n-1} |a_k|)}. \tag{3.17}$$

Further what we did in (A_1) after getting $\frac{B}{B_1}$, we will now do the same thing here also with B_1 replaced by B_3 and we will be using subsections of (A_1) only, with same meaning.

(a) $B > B_3, (n \geq 2),$ (by usual method).
(b) (i) $B > B_3, (n \geq 3),$ (by usual method).
(ii) (a') $B > B_3,$ for values of $t_1 (< t)$, sufficiently close to $t, (n \geq 2)$, (by usual method).
(b') $B > B_3,$ for values of $t (< t_1)$, sufficiently close to $t_1, (n \geq 2)$, (by usual method).
(A_4) Comparison of bound of Theorem with Boese and Luther's bound
Using

Boese and Luther's bound [2] = B_4 (, say) =

$$\begin{cases} \{M(1-nM)/[1-(nM)^{1/n}]\}^{1/n}, & M < 1/n, \\ \min\{(1+M)(1-M/[(1+M)^{n+1}-nM]), 1 + 2(nM-1)/(n+1)\}, & M \geq 1/n, \end{cases} \quad (M \text{ is as in (3.6)}), \tag{3.18}$$

we can say that all the zeros of $f_1(z)$ lie in

$$|z| < 1.966$$

and all the zeros of $f_2(z)$ lie in

$$|z| < 6.999$$

thereby implying by (3.3) and (3.5) that the result obtained by bound of Theorem is better than that obtained by Boese and Luther's bound for both $f_1(z)$ and $f_2(z)$.

Now here we will have three different expressions for B_4, one for $M < 1/n$ and other two for $M \geq 1/n$, (depending on which one of the two expressions is minimum one for $M \geq 1/n$), thereby giving three different expressions for $\frac{B}{B_4}$.

(1) $M < 1/n$.

$$\frac{B}{B_4} = \frac{\max_{1\leq j\leq \lfloor \frac{n+1}{2} \rfloor} \left\{|a_{n-2j+1}| + \sqrt{|a_{n-2j+1}|^2 + 4 + 4|a_{n-2j}|}\right\}/2}{\{M(1-nM)/[1-(nM)^{1/n}]\}^{1/n}}. \tag{3.19}$$

Further what we did in (A_1) after getting $\frac{B}{B_1}$, we will now do the same thing here also, (as well as after getting remaining two expressions of $\frac{B}{B_4}$ also), with B_1 replaced by B_4 and we will be using subsections of (A_1) only, with same meaning.

(a) $B > B_4$ for values of $t(< 1/n)$, sufficiently close to $1/n, (n \geq 2)$, (by usual method).

(b) (i) $B > B_4$ for values of $t(< 1/n)$, sufficiently close to $1/n, (n \geq 3)$, (by usual method).

(ii) (a') $B > B_4$ for values of $t_1(< t < 1/n)$, sufficiently close to $1/n, (n \geq 3)$, (by usual method).

(b') $B > B_4$ for values of $t(< t_1 < 1/n)$, sufficiently close to $1/n, (n \geq 3)$, (by usual method).

(2) $M \geq 1/n$. (d_1).

$$\frac{B}{B_4} = \frac{\max_{1\leq j\leq \lfloor \frac{n+1}{2} \rfloor} \left\{|a_{n-2j+1}| + \sqrt{|a_{n-2j+1}|^2 + 4 + 4|a_{n-2j}|}\right\}/2}{(1+M)\left(1 - \frac{M}{[(1+M)^{n+1} - nM]}\right)}. \tag{3.20}$$

(a) $B > B_4, (n \geq 2)$, (by usual method).

(b) (i) $B > B_4, (n \geq 3)$, (by usual method).

(ii) (a') $B > B_4$ for values of $t_1(< t)$, sufficiently close to $t, (n \geq 3)$, (by usual method).

(b') $B > B_4$ for values of $t(< t_1)$, sufficiently close to $t_1, (n \geq 3)$, (by usual method).

(d_2).

$$\frac{B}{B_4} = \frac{\max_{1\leq j\leq \lfloor \frac{n+1}{2} \rfloor} \left\{|a_{n-2j+1}| + \sqrt{|a_{n-2j+1}|^2 + 4 + 4|a_{n-2j}|}\right\}/2}{1 + \frac{2(nM-1)}{n+1}}. \tag{3.21}$$

(a) $B > B_4$ for $t < 2/(n-1)$ and $B < B_4$ for $t > 2/(n-1), (n \geq 2)$, (by usual method).

(b) (i) $B > B_4$ for $t < 2/(n-1)$ and $B < B_4$ for $t > 2/(n-1), (n \geq 3)$, (by usual method).

(ii) (a′) $B > B_4$ for values of $t_1(< t < 2/(n-1))$, sufficiently close to t and $B < B_4$ for values of $t_1(< t, t > 2/(n-1))$, sufficiently close to $t, (n \geq 3)$, (by usual method).

(b′) $B > B_4$ for values of $t(< t_1 < 2/(n-1))$, sufficiently close to t_1 and $B < B_4$ for values of $t(< t_1, t_1 > 2/(n-1))$, sufficiently close to $t_1, (n \geq 3)$, (by usual method).

(A_5) Comparison of bound of Theorem with Jain's first old bound

Using

Jain's first old bound [6] $= B_5$ (, say) $= 1 + d_0 r, (r = \frac{1}{2}\{-b + \sqrt{b^2 + 4M_1}\}, b = 1 - |a_{n-1}|,$

$$\begin{array}{l} M_1 \text{ is as in (3.14) and } d_0 \text{ (= greatest root of} \\ (1+xr)^{n-1}(x^2r^2 + xrb - M_1) + M_1 = 0 \\ \text{in } [0,1]) \text{ is always} < 1, \text{ except when} \\ M_1 = 0, \text{ in which case it is } 1), \end{array} \tag{3.22}$$

we can say that all the zeros of $f_1(z)$ lie in

$$|z| \leq R_1,\ R_1 < 1.556$$

and all the zeros of $f_2(z)$ lie in

$$|z| \leq R_1^{'},\ R_1^{'} < 4,$$

thereby implying by (3.3) and (3.5) that the result obtained by bound of Theorem is worse than that obtained by Jain's first old bound for both $f_1(z)$ and $f_2(z)$.

Now

$$\frac{B}{B_5} = \frac{\max_{1\leq j\leq\lfloor\frac{n+1}{2}\rfloor}\left\{|a_{n-2j+1}| + \sqrt{|a_{n-2j+1}|^2 + 4 + 4|a_{n-2j}|}\right\}/2}{1 + \frac{d_0}{2}\left\{-1 + |a_{n-1}| + \sqrt{(1-|a_{n-1}|)^2 + 4M_1}\right\}}. \tag{3.23}$$

Further what we did in (A_1) after getting $\frac{B}{B_1}$, we will now do the same thing here also with B_1 replaced by B_5 and we will be using subsections of (A_1) only, with same meaning.

(a) $B > B_5, (n \geq 2)$, (by usual method).

(b) (i) $B > B_5, (n \geq 3)$, (by usual method).

(ii) (a′) $B > B_5, (n \geq 3)$, (by usual method).

(b′) Here we are unable to say anything.

(A_6) Comparison of bound of Theorem with Sun and Hsieh's bound

Using

Sun and Hsieh's bound [10] = B_6 (, say) = $1 + \delta_1$, ($\delta_1 =$ unique positive root of

$$\begin{aligned}&x^3 + (2 - |a_{n-1}|)x^2 + (1 - |a_{n-1}| - |a_{n-2}|)x - M = 0\\ &\text{and } M \text{ is as in (3.6)}),\end{aligned} \tag{3.24}$$

we can say that all the zeros of $f_1(z)$ lie in

$$|z| \le R_2,\ R_2 < 1.5$$

and all the zeros of $f_2(z)$ lie in

$$|z| \le R_2',\ R_2' < 4,$$

thereby implying by (3.3) and (3.5) that the result obtained by bound of Theorem is worse than that obtained by Sun and Hsieh's bound for both $f_1(z)$ and $f_2(z)$.

Now

$$\frac{B}{B_6} = \frac{\max_{1\le j\le \lfloor \frac{n+1}{2}\rfloor}\left\{|a_{n-2j+1}| + \sqrt{|a_{n-2j+1}|^2 + 4 + 4|a_{n-2j}|}\right\}/2}{1+\delta_1}. \tag{3.25}$$

Further what we did in (A_1) after getting $\frac{B}{B_1}$, we will now do the same thing here also with B_1 replaced by B_6 and we will be using subsections of (A_1) only, with same meaning.

(a) $B = B_6, (n \ge 2)$, (by usual method).

(b) (i) $B = B_6, (n \ge 3)$, (by usual method).

(ii) (a') $B < B_6, (n \ge 3)$, (by usual method).

(b') $B < B_6, (n \ge 3)$, (by usual method).

(A_7) Comparison of bound of Theorem with Jain's second old bound

Using

Jain's second old bound [7] = B_7 (, say) = $1 + \delta_0$, (δ_0 = unique positive root of

$$\begin{aligned}&x^4 + (3 - |a_{n-1}|)x^3 + (3 - 2|a_{n-1}| - |a_{n-2}|)x^2\\ &+(1 - |a_{n-1}| - |a_{n-2}| - |a_{n-3}|)x - M = 0\\ &\text{and } M \text{ is as in (3.6)}),\end{aligned} \tag{3.26}$$

we can say that all the zeros of $f_1(z)$ lie in

$$|z| < R_3, R_3 < 1.4$$

and all the zeros of $f_2(z)$ lie in

$$|z| < R_3', R_3' < 3.5,$$

thereby implying by (3.3) and (3.5) that the result obtained by bound of Theorem is worse than that obtained by Jain's second old bound for both $f_1(z)$ and $f_2(z)$.

Now

$$\frac{B}{B_7} = \frac{\max_{1\le j\le \lfloor \frac{n+1}{2} \rfloor} \left\{ |a_{n-2j+1}| + \sqrt{|a_{n-2j+1}|^2 + 4 + 4|a_{n-2j}|} \right\} /2}{1+\delta_0}. \tag{3.27}$$

Further what we did in (A_1) after getting $\frac{B}{B_1}$, we will now do the same thing here also with B_1 replaced by B_7 and we will be using subsections of (A_1) only, with same meaning.

(a) $B = B_7, (n \ge 3)$, (by usual method).

(b) (i) $B > B_7, (n \ge 3)$, (by usual method).

(ii) (a′) $B >$ or $< B_7 \Leftrightarrow (t-1-t_2)(\sqrt{t^2+4+4t_1} + t - 2) + 2t_1 >$ or $< 0, (t_2 = |a_{n-3}|), (n \ge 3)$, (by usual method).

(b′) $B >$ or $< B_7 \Leftrightarrow (t-1-t_2)(\sqrt{t^2+4+4t_1} + t) + 2(1+t_2) >$ or $< 0, (t_2 = |a_{n-3}|), (n \ge 3)$, (by usual method).

(A_8) Comparison of bound of Theorem with Jain's third old bound

Using

Jain's third old bound [8] = B_8 (, say) = $\frac{1}{2}\left\{\alpha_1 + \beta_1 + \sqrt{(\alpha_1-\beta_1)^2 + 4\sigma_1}\right\}, (\beta_1 = |a_{n-1}|;$

$\beta_2 \ge \beta_3 \ge \ldots \ge \beta_n$, being the ordered non-negative numbers $|a_j|, j = 0, 1, \ldots, n-2$, with $a_j \ne 0$ for at least one $j, 0 \le j \le n-2$; $\alpha_1 = \max_{2\le k\le n-1}(\beta_{k+1}/\beta_k)$, (maximum being taken over all k such that $\beta_k \ne 0$); $\delta_k = \alpha_1\beta_k - \beta_{k+1}, k = 2, 3, 4, \ldots, n; \beta_{n+1} = 0$; $t_1^{'} = \frac{1}{2}\left\{\alpha_1 + \beta_1 + \sqrt{(\alpha_1-\beta_1)^2 + 4\beta_2}\right\} - \alpha_1$, $\sigma_1 = \beta_2 - (\delta_2/(\alpha_1 + t_1^{'})) - (\delta_3/(\alpha_1 + t_1^{'})^2) - \ldots - (\delta_n/(\alpha_1 + t_1^{'})^{n-1}))$, (3.28)

we can say that all the zeros of $f_1(z)$ lie in

$$|z| \le 1.432$$

and all the zeros of $f_2(z)$ lie in

$$|z| \le 3.968,$$

thereby implying by (3.3) and (3.5) that the result obtained by bound of Theorem is worse than that obtained by Jain's third old bound for both $f_1(z)$ and $f_2(z)$.

Now

$$\frac{B}{B_8} = \frac{\max_{1\le j\le \lfloor \frac{n+1}{2} \rfloor} \left\{ |a_{n-2j+1}| + \sqrt{|a_{n-2j+1}|^2 + 4 + 4|a_{n-2j}|} \right\}}{\alpha_1 + \beta_1 + \sqrt{(\alpha_1-\beta_1)^2 + 4\sigma_1}}. \tag{3.29}$$

Further what we did in (A_1) after getting $\frac{B}{B_1}$, we will now do the same thing here also with B_1 replaced by B_8 and we will be using subsections of (A_1) only, with same meaning.

(a) $B > B_8, (n \geq 2)$, (by usual method).
(b) (i) $B > B_8, (n \geq 3)$, (by usual method).
(ii) (a') $B > B_8, (n \geq 3)$, (by usual method).
(b') Here we are unable to say anything.

(A_9) Comparison of bound of Theorem with Affane-Aji et al.'s bound

We will not compare the bound of Theorem with Affane-Aji et al.'s bound as Affane-Aji et al.'s result is the generalization of the results due to Sun and Hsieh [10] and Jain [7] and we have already compared the bound of Theorem with Sun and Hsieh's bound in (A_6) and with Jain's bound in (A_7).

(A') Comparison of bound of Theorem with the bounds, not associated with Cauchy

(A_1') Comparison of bound of Theorem with Fujiwara's bound

Using

$$\text{Fujiwara's bound [5]} = B_1' \text{ (, say)} = \left(1 + \sum_{j=0}^{n-1} |a_j|^2\right)^{1/2}, \tag{3.30}$$

we can say that all the zeros of $f_1(z)$ lie in

$$|z| < 1.732$$

and all the zeros of $f_2(z)$ lie in

$$|z| < 9.539,$$

thereby implying by (3.3) and (3.5) that the result obtained by bound of Theorem is better than that obtained by Fujiwara's bound for both $f_1(z)$ and $f_2(z)$.

Now

$$\frac{B}{B_1'} = \frac{\max_{1 \leq j \leq \lfloor \frac{n+1}{2} \rfloor} \left\{ |a_{n-2j+1}| + \sqrt{|a_{n-2j+1}|^2 + 4 + 4|a_{n-2j}|} \right\} /2}{(1 + \sum_{j=0}^{n-1} |a_j|^2)^{1/2}}. \tag{3.31}$$

Further what we did in (A_1) after getting $\frac{B}{B_1}$, we will now do the same thing here also with

B_1 replaced by $B_1^{'}$ and we will be using subsections of (A_1) only, with same meaning.

(a) $B >$ or $< B_1^{'} \Leftrightarrow 2/(n-1) >$ or $< t, (n \geq 2)$, (by usual method).

(b) (i) $B < B_1^{'}, (s > 1)$ and $B >$ or $< B_1^{'} \Leftrightarrow \sqrt{1-s} >$ or $< |t-1|, (s \leq 1), (s = \sum_{j=0}^{n-3} |a_j|^2)$,

$(n \geq 3)$, (by usual method).

(ii) (a′) For $t^2 + 2t_1(t_1 - 1) + 2s < 0$,

$B > B_1^{'}$

and for $t^2 + 2t_1(t_1 - 1) + 2s \geq 0$,

$B < B_1^{'}$, if $2 - s - (t_1 - 1)^2 \leq 0$,

$B >$ or $< B_1^{'} \Leftrightarrow |t| >$ or $< D$, if $2 - s - (t_1 - 1)^2 > 0, \left(D = \dfrac{|s + t_1(t_1 - 1)|}{\sqrt{2 - s - (t_1 - 1)^2}}\right)$,

$\left(s = \sum_{j=0}^{n-3} |a_j|^2\right), (n \geq 3)$, (by usual method).

(b′) Same as in (a′).

$(A_2^{'})$ <u>Comparison of bound of Theorem with Walsh's bound</u>

Using

$$\text{Walsh's bound [11]} = B_2^{'} \text{ (, say)} = \sum_{j=1}^{n} |a_{n-j}|^{1/j}, \tag{3.32}$$

we can say that all the zeros of $f_1(z)$ lie in

$$|z| \leq 2$$

and all the zeros of $f_2(z)$ lie in

$$|z| \leq 8.244$$

thereby implying by (3.3) and (3.5) that the result obtained by bound of Theorem is <u>better</u> than that obtained by Walsh's bound for both $f_1(z)$ and $f_2(z)$.

Now

$$\frac{B}{B_2^{'}} = \frac{\max_{1 \leq j \leq \lfloor \frac{n+1}{2} \rfloor} \left\{ |a_{n-2j+1}| + \sqrt{|a_{n-2j+1}|^2 + 4 + 4|a_{n-2j}|} \right\} /2}{\sum_{j=1}^{n} |a_{n-j}|^{1/j}}. \tag{3.33}$$

Further what we did in (A_1) after getting $\frac{B}{B_1}$, we will now do the same thing here also with B_1 replaced by $B_2^{'}$ and we will be using subsections of (A_1) only, with same meaning.

(a) $B >$ or $< B_2^{'} \Leftrightarrow 1 >$ or $< \sum_{j=2}^{n} t^{1/j}, (n \geq 2),$ (by usual method).

(b) (i) $B >$ or $< B_2^{'} \Leftrightarrow 1 >$ or $< t^{1/2} + \sum_{j=3}^{n} |a_{n-j}|^{1/j}, (n \geq 3),$ (by usual method).

(ii) (a′) For $1 - s_1^2 - ts_1 \leq 0$,

$$B < B_2^{'}$$

and for $1 - s_1^2 - ts_1 > 0$,

$$B > \text{ or } < B_2^{'} \Leftrightarrow \frac{1 - s_1^2 - ts_1}{t + 2s_1} > \text{ or } < \sqrt{t_1}, \left(s_1 = \sum_{j=3}^{n} |a_{n-j}|^{1/j} \right), (n \geq 3),$$

(by usual method).

(b′) Same as in (a′).

References

[1] C. Affane-Aji, N. Agarwal, N. K. Govil, Location of zeros of polynomials, *Math. Comput. Modelling* **50**, 306–313 (2009).

[2] F. G. Boese, W. J. Luther, A note on classical bound for the moduli of all zeros of polynomial, *IEEE Trans. Automat. Contr.* **34**, 998–1001 (1989).

[3] A. L. Cauchy, *Exercises de Mathématiques*, IV Annee de Bure Fiéres, Paris (1829).

[4] B. Datt, N. K. Govil, On the location of zeros of polynomial, *J. Approx. Theory* **24**, 78–82 (1978).

[5] M. Fujiwara, Ueber die Wurzeln der algebraischen Gleichungen, *Tôhoku Math. J.* **8**, 78–85 (1915).

[6] V. K. Jain, On Cauchy's bound for zeros of a polynomial, *Approx. Theory Appl.* **6**, 18–24 (1990).

[7] V. K. Jain, On Cauchy's bound for zeros of a polynomial, *Turk. J. Math.* **30**, 95–100 (2006).

[8] V. K. Jain, On Cauchy's bound for zeros of a polynomial, *Bull. Math. Soc. Sci. Math. Roumanie* **50 (98)**, 273–279 (2007).

[9] A. Joyal, G. Labelle, Q. I. Rahman, On the location of zeros of polynomials, *Canad. Math. Bull.* **10**, 53–63 (1967).

[10] Y. J. Sun, J. G. Hsieh, A note on circular bound of polynomial zeros, *IEEE Trans. Circuits Syst. I* **43**, 476–478 (1996).

[11] J. L. WALSH, An inequality for the roots of an algebraic equation, *Ann. of Math.* **25**, 285–286 (1924).

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Mathematics Department, I.I.T., Kharagpur, India - 721302
(1)E-mail: `vinayjain.kgp@gmail.com`
(2)E-mail: `vivektewary@gmail.com`