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The Italian identities for Hilbert series of hypergraphs

by
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Abstract

Given a graph with loops or a hypergraph, this paper demonstrates that the Hilbert series of the associated graph or hypergraph ring is determined by the Hilbert series of the graph or hypergraph rings associated to the contraction, deletion, and extraction of an edge of the original graph or hypergraph. This enables inductive procedures to compute the Hilbert series of graph and hypergraph rings.

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1 Introduction

Associated to every graph with n vertices is a monomial ideal of the polynomial ring in n indeterminates. This ideal, called the *edge ideal* of the graph, is defined by choosing as generators the monomials formed from the edges of the given graph. Any loops in a graph result in additional generators which are squares of the relevant indeterminate. This naturally generalizes to a corresponding definition for hypergraphs, so long as no edges contain another distinct edge. The quotient of the polynomial ring by the edge ideal for a given graph or hypergraph is referred to as its *graph ring* or *hypergraph ring*, respectively [9]. As a homogeneous quotient of a polynomial ring, these can be interpreted as a graded algebra, and thus Hilbert series can be computed. Additionally, the Hilbert-Serre Theorem [1, 5, 9] applies, leading to a relatively simple closed-form representation of these series.

Previous works by other authors have noted a relationship between the Hilbert series for graph and hypergraph rings and the edge-induced subgraph enumerating polynomial, defined below, first for simple graphs by Renteln [6] and later for hypergraphs by Goodarzi [4]. In this paper, we recall an identity for subgraph enumerating polynomials presented by Borzacchini and Pulito [2], translate it into an identity for Hilbert Series, and use an alternate proof method to generalize that identity to more broad circumstances.

For other recent work generalizing subgraph enumerating polynomials, applying them to Hilbert series, or related topics, we refer the reader to [3, 8, 10].

2 Definitions and Notation

Throughout this paper, we will assume that any graphs and hypergraphs are finite, do not contain multiple edges, and in the case of hypergraphs, no edges contain another distinct

Example:

G $\quad$ G_a $\quad$ $G_{[a]}$ $\quad$ $G_{\langle a\rangle}$

Graph $\quad$ Deletion $\quad$ Contraction $\quad$ Extraction

edge. Loops in graphs will be allowed unless otherwise noted. For simplicity, hypergraphs will be assumed to contain no edges of length one.

To begin, we collect some necessary definitions, some of which are standard and some of which are more specific to this topic. Most of these definitions and many of the naming conventions can also be found in [2].

Definition 1. *Let $G = (V, E)$ be a graph and let $a \in E$. Then*

(1) G_a is the graph formed by deleting the edge a.

(2) $G_{[a]}$ is the graph formed by contracting the edge a and ignoring any multiple edges.

(3) $G_{\langle a\rangle}$ is the hypergraph formed by deleting the vertices in the edge a and adding an isolated vertex, and is referred to as the extraction. Note that this also deletes any edges incident to a vertex in a.

We now extend the definition to hypergraphs, with a small addendum to prevent the introduction of edge inclusion.

Definition 2. *Let $H = (V, E)$ be a hypergraph and let $a \in E$. Then*

(1) H_a is the hypergraph formed by deleting the edge a.

(2) $H_{[a]}$ is the hypergraph formed by contracting the edge a and ignoring any multiple edges. In the case that this introduces edges strictly contained within one another, we only keep the edges minimal under inclusion.

(3) $H_{\langle a\rangle}$ is the hypergraph formed by deleting all of vertices in the edge a and adding an isolated vertex, and is referred to as the extraction. Note that this also deletes any edges incident to a vertex in a.

Note that no loops or multiple edges are created by this definition of contraction.

The following definition is presented for hypergraphs, but the definition for graphs is identical.

Definition 3. *Let H be a hypergraph. Then*

$$S_H(u, v) = \sum_{i,j} C_{ij} u^i v^j$$

is the edge induced subgraph enumerating polynomial *of H, where C_{ij} is the number of edge induced sub-hypergraphs of H with i vertices and j edges.*

These definitions allow the following theorem, stated and proven in [2].

Theorem 1 ([2]). *For any graph $G = (V, E)$, and for every $a \in E$, we have:*

(1) if a is not a loop:

$$S_G = S_{G_a} + uvS_{G_{[a]}} + uv(u-1)S_{G_{\langle a \rangle}}$$

(2) if a is a loop:

$$S_G = (1+v)S_{G_a} + v(u-1)S_{G_{\langle a \rangle}}$$

These polynomials have been connected via the following theorem to the Hilbert series of graph and hypergraph rings, first by Renteln [6] for graphs, then by Goodarzi [4]. For generality, Goodarzi's version is presented here, though their proof techniques sufficiently vary that both are of independent interest.

Theorem 2 ([4]). *Let $I \subset R = k[x_1, ..., x_n]$ be a squarefree monomial ideal and H its associated hypergraph. Then*

$$HS(R/I) = \frac{S_H(t,-1)}{(1-t)^n},$$

where $HS(R/I)$ is the Hilbert Series of the ring R/I.

3 Graphs

We first aim to prove the following translation of the first half of Theorem 1 to Hilbert series:

Theorem 3. *Let G be a graph without multiple edges with an edge $a = \{\alpha, \beta\}$ which is not a loop. Let $R, R_a, R_{[a]}$, and $R_{\langle a \rangle}$ be the graph rings corresponding to $G, G_a, G_{[a]}$, and $G_{\langle a \rangle}$, respectively. Then,*

(1) If there is a loop at at most one of α and β :

$$HS(R) = HS(R_a) - t\left(\frac{HS(R_{[a]})}{1-t} - HS(R_{\langle a \rangle})\right).$$

(2) If there are loops at both α and β :

$$HS(R) = HS(R_a) - t\left(HS(R_{[a]}) - (1-t)HS(R_{\langle a \rangle})\right).$$

The proof when the graph contains no loops is immediate from Theorems 1 and 2. However, as the latter theorem requires a squarefree monomial ideal, it does not apply to graphs with loops.

Our proof requires a lemma:

Lemma 1. *Let G be a graph without multiple edges with an edge $a = \{\alpha, \beta\}$ which is not a loop, and assume that there is a loop at at most one of α and β. Let R_a and $R_{[a]}$ be the graph rings corresponding to G_a and $G_{[a]}$, respectively. Define γ to be the vertex created by the identification of α and β in the contraction, and let δ be an indeterminate with no relations in R. Then, the ring homomorphism*

$$\frac{R_{[a]}[\delta]}{(0:\gamma)} \xrightarrow[\delta\mapsto\beta]{\gamma\mapsto\alpha} \frac{R_a}{(0:\alpha\beta)}$$

is a (grade preserving) isomorphism.

Alternatively, if there are loops at both α and β,

$$\frac{R_{[a]}}{(0:\gamma)} \xrightarrow{\gamma\mapsto\alpha} \frac{R_a}{(0:\alpha\beta)}$$

is a (grade preserving) isomorphism.

Proof. The reader may wish to note that, while δ solely serves as an extra isolated vertex within this lemma, it will later refer to the isolated vertex resulting from the extraction of the edge a.

First, we note that, given the structure of the edge ideal, the colon ideal $(0 : \gamma)$ is generated by the neighbors of γ in $G_{[a]}$. As γ can be understood as the identification of α and β in G_a, a vertex x is adjacent to γ in $G_{[a]}$ if and only if x is adjacent to α or β in G_a. If there is a loop at a vertex, the vertex is considered to be adjacent to itself.

Since the generators of the edge ideal of a graph are of degree two, and α and β are not neighbors in the edge-deleted graph G_a, the colon ideal $(0 : \alpha\beta) = (0 : \alpha) + (0 : \beta)$ is generated by the neighbors of α and β.

Second, the graphs $G_{[a]}$ and G_a differ only in whether or not the identification of α and β has been made and the related differences in edges incident to α, β, or γ. The quotients that occur in the domain and codomain of the maps are graph rings for graphs obtained from $G_{[a]}$ and G_a respectively. These graphs are derived from $G_{[a]}$ and G_a by deletion of all neighbors of γ, or respectively α, and β. This deletes all edges incident to γ, or respectively α, and β. The vertices γ, or respectively α, and β. are themselves either deleted or just isolated depending on whether there is a loop at the vertex. They are deleted if there is a loop at the corresponding vertex as the vertex is its own neighbor; otherwise the vertex is isolated.

The graphs associated to the quotients in the domain and codomain are therefore isomorphic up to a potential isolated vertex. This isolated vertex occurs exactly when there is a loop at at most one of α and β. This is accounted for by adjoining the indeterminate δ to $R_{[a]}$ in that case.

Since the maps are the identity up to the relabeling of isolated vertices, in either case they are grade-preserving isomorphisms.

□

With this lemma in hand, we proceed to the proof of Theorem 3.

Proof of Theorem 3. The bulk of the proof consists of noting that the following diagrams connect two short exact sequences via the appropriate isomorphism from the above lemma.

In the case where there is a loop at at most one of α and β:

$$\begin{tikzcd} 0 \arrow[r] & \frac{R_{[a]}[\delta]}{(0:\gamma)}(-2) \arrow[r, hook, "\cdot\gamma"] \arrow[d, "\cong"] & R_{[a]}[\delta](-1) \arrow[r, two heads, "\gamma\,\mapsto\,0"] & R_{\langle a\rangle}(-1) \arrow[r] & 0 \\ 0 \arrow[r] & \frac{R_a}{(0:\alpha\beta)}(-2) \arrow[r, hook, "\cdot\alpha\beta"] & R_a \arrow[r, two heads, "\alpha\beta\,\mapsto\,0"] & R \arrow[r] & 0 \end{tikzcd}$$

In the case where there is a loop at both α and β,

$$\begin{tikzcd} 0 \arrow[r] & \frac{R_{[a]}[\delta]}{(0:\gamma)}(-2) \arrow[r, hook, "\cdot\gamma"] \arrow[d, "\cong"] & R_{[a]}[\delta](-1) \arrow[r, two heads, "\gamma\,\mapsto\,0"] & R_{\langle a\rangle}(-1) \arrow[r] & 0 \\ 0 \arrow[r] & \frac{R_a[\delta]}{(0:\alpha\beta)}(-2) \arrow[r, hook, "\cdot\alpha\beta"] & R_a[\delta] \arrow[r, two heads, "\alpha\beta\,\mapsto\,0"] & R[\delta] \arrow[r] & 0 \end{tikzcd}$$

A shift is made in the first sequence in order to make the isomorphism a degree zero homomorphism.

The only difficulty in the construction of these sequences is the map from $R_{[a]}[\delta]$ to $R_{\langle a\rangle}$. Graphically, the map deletes the vertex γ created from the contraction, leaving the remainder of the graph and an extra isolated vertex δ. As this is exactly how extraction was defined in this paper, the surjectivity is clear.

First, assume that there is a loop at at most one of α and β, including the case where there is a loop at neither.

Appealing to the fact that Hilbert series satisfy an additive relation on exact complexes, we can collect the following equations:

$$\begin{aligned} t^2 HS\left(\frac{R_{[a]}[\delta]}{(0:\gamma)}\right) &= tHS\left(R_{[a]}[\delta]\right) - tHS\left(R_{\langle a\rangle}\right), \text{ and} \\ t^2 HS\left(\frac{R_a}{(0:\alpha\beta)}\right) &= HS\left(R_a\right) - HS\left(R\right). \end{aligned}$$

Invoking the grade preserving isomorphism,

$$\begin{aligned} HS\left(R\right) &= HS\left(R_a\right) - t\left(HS\left(R_{[a]}[\delta]\right) - HS\left(R_{\langle a\rangle}\right)\right) \\ &= HS\left(R_a\right) - t\left(\frac{HS\left(R_{[a]}\right)}{1-t} - HS\left(R_{\langle a\rangle}\right)\right). \end{aligned}$$

The proof follows in a similar manner in the case where there are loops at both α and β.

□

This theorem translates half of Theorem 1. It is a natural question to ask whether the other half also translates well. The answer is no, or at least not in the same way. Since the edge being examined in that half is a loop, again Theorem 2 is unavailable, and, in fact, would predict an incorrect result. However, our proof technique, applied in a manner analogous to the one just used, produces an identity for the case where a is a loop:

Theorem 4. *Let G be a graph without multiple edges with a loop a at the vertex α. Let R, R_a, and $R_{\langle a\rangle}$ be the graph rings corresponding to G, G_a, and $G_{\langle a\rangle}$, respectively. Then,*

$$HS(R) = (1-t)HS(R_a) + t(1-t)HS(R_{\langle a\rangle})$$

Proof. Note that, for a loop, contraction and deletion are identical by our definitions, and thus induce the same graph rings.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{R_{[a]}[\delta]}{(0:\alpha)}(-2) & \xhookrightarrow{\cdot\gamma} & R_{[a]}[\delta](-1) & \xrightarrow{\gamma\ \mapsto\ 0} & R_{\langle a\rangle}(-1) & \longrightarrow & 0 \\ & & \downarrow\cong & & & & & & \\ 0 & \longrightarrow & \frac{R_a[\delta]}{(0:\alpha)}(-2) & \xhookrightarrow{\cdot\alpha^2} & R_a[\delta] & \xrightarrow{\alpha^2\ \mapsto\ 0} & R[\delta] & \longrightarrow & 0 \end{array}$$

The rest follows in a manner similar to the previous proof. ∎

4 Hypergraphs

We extend the previous result to the case of hypergraphs. Again, we first need to verify an isomorphism. The intuition of the isomorphism is that if vertex contraction (see Schrijver [7]) is applied to each vertex in a specified edge, then the resulting hypergraph remains the same even if the edge were deleted or contracted first. However, a more algebraic approach is presented.

Lemma 2. *Let H be a hypergraph with an edge $a = \{\delta_1, \delta_2, \ldots, \delta_n\}$. Let R_a and $R_{[a]}$ be the hypergraph rings corresponding to H_a and $H_{[a]}$, respectively. Let the vertex created by the contraction of the edge a retain the name δ_1. Then, the ring homomorphism*

$$\frac{R_{[a]}[\delta_2, \ldots, \delta_n]}{(0:\delta_1)} \xrightarrow{\quad i \quad} \frac{R_a}{(0:\delta_1\ldots\delta_n)}$$

is a (grade preserving) isomorphism.

Proof. The isomorphism is understood to be the identity.

Let S be a polynomial ring over a field with an indeterminate for each vertex of H, and let I_a and $I_{[a]}$ be the edge ideals for R_a and $R_{[a]}$. Then

$$\frac{R_{[a]}[\delta_2, \ldots, \delta_n]}{(0:\delta_1)} \cong \frac{\frac{S}{I_{[a]}}}{(0:_{R_{[a]}}\delta_1)} \cong \frac{\frac{S}{I_{[a]}}}{\frac{(I_{[a]}:\delta_1)}{I_{[a]}}} \cong \frac{S}{(I_{[a]}:\delta_1)}$$

and

$$\frac{R_a}{(0:\delta_1\ldots\delta_n)} \cong \frac{\frac{S}{I_a}}{(0:_{R_a}\delta_1\ldots\delta_n)} \cong \frac{\frac{S}{I_a}}{\frac{(I_a:\delta_1\ldots\delta_n)}{I_a}} \cong \frac{S}{(I_a:\delta_1\ldots\delta_n)}$$

.

Thus, it suffices to show that $(I_{[a]}:\delta_1) = (I_a:\delta_1\ldots\delta_n)$.

We will proceed by constructing generating sets for both ideal quotients, and noting that they are the same generating set.

As S is an integral domain, note that

$$(I_{[a]}:\delta_1) = \frac{1}{\delta_1}\left(I_{[a]}\cap(\delta_1)\right), \text{ and}$$
$$(I_a:\delta_1\ldots\delta_n) = \frac{1}{\delta_1\ldots\delta_n}\left(I_{[a]}\cap(\delta_1\ldots\delta_n)\right)$$

An intersection of monomial ideals of a polynomial ring over a field is generated by the least common multiples of the generators of the ideals being intersected. In the case of $I_{[a]}$ and (δ_1), this is straightforward: take the typical list of generators of $I_{[a]}$ and multiply any generators not yet in (δ_1) by δ_1. We then find a generating set for $(I_{[a]}:\delta_1)$ by dividing each of these by δ_1. The combination of these processes results in taking the edges in $I_{[a]}$ and removing δ_1 from them if necessary. Finally, remove any monomials which are products of a smaller degree monomial, and thus unnecessary.

Similarly, to find a generating set for $(I_a:\delta_1\ldots\delta_n)$, we apply the same technique and arrive at a generating set of monomials constructed by removing any copies of $\delta_1,\delta_2,\ldots,\delta_n$ present in the edges of I_a. As before, remove any unnecessary monomials.

Since the edges of the contraction $H_{[a]}$ are simply the edges of the edge deletion H_a after identifying the vertices in the specified edge a, removing the vertices incident to a (or contracted to via a) removes any differences in the edge lists, up to removing edges that strictly contain another edge. Therefore, we have constructed two identical generating sets, and the ideal quotients in question are indeed isomorphic.

□

This allows for the following theorem, which generalizes the earlier result for graphs in the case that the graph is simple.

Theorem 5. *Let H be a hypergraph with an edge $a=\{\delta_1,\delta_2,\ldots,\delta_n\}$. Let $R,R_a,R_{[a]}$, and $R_{\langle a\rangle}$ be the hypergraph rings corresponding to $H,H_a,H_{[a]}$, and $H_{\langle a\rangle}$, respectively. Let the vertex created by either the contraction or extraction of the edge a retain the name δ_1. Then,*

$$HS(R) = HS(R_a) - \frac{t^{n-1}}{(1-t)^{n-2}}\left(\frac{HS(R_{[a]})}{1-t} - HS(R_{\langle a\rangle})\right).$$

Proof.

$$\begin{tikzcd}
0 \arrow[r] & \frac{R_{[a]}[\delta_2,\ldots,\delta_n]}{(0:\delta_1)}(-n) \arrow[r, hook, "\cdot\delta_1"] \arrow[d, "\cong"] & R_{[a]}[\delta_2,\ldots,\delta_n](1-n) \arrow[r, two heads, "\delta_1 \mapsto 0", "\delta_n \mapsto \delta_1"'] & R_{\langle a\rangle}[\delta_2,\ldots,\delta_{n-1}](1-n) \arrow[r] & 0 \\
0 \arrow[r] & \frac{R_a}{(0:\delta_1\ldots\delta_n)}(-n) \arrow[r, hook, "\cdot\delta_1\ldots\delta_n"] & R_a \arrow[r, two heads, "\delta_1\ldots\delta_n \mapsto 0"] & R \arrow[r] & 0
\end{tikzcd}$$

Again, the Hilbert series result follows after some simple computation. ∎

5 An Example

In [7], Schrijver notes an example, denoted J_n, of a minimally non-ideal hypergraph, minimal under vertex contraction and vertex deletion. In this section, we will use the result of the previous section to compute the Hilbert series for this example. This is done purely to illustrate the technique presented in this paper. The Hilbert series of the example can be determined in a simpler way by other means.

Definition 4. *Let J_n refer to the hypergraph with vertex set $V = \{\delta_0, \delta_1, \ldots, \delta_n\}$ and edge set $E = \{\{\delta_1, \delta_2, \ldots, \delta_n\}, \{\delta_0, \delta_1\}, \{\delta_0, \delta_2\}, \ldots, \{\delta_0, \delta_n\}\}$.*

Note, then, that if we pick the edge $a = \{\delta_1, \delta_2, \ldots, \delta_n\}$,

- $(J_n)_a = S_n = K_{1,n}$
- $(J_n)_{[a]} = K_2 = K_{1,1}$
- $(J_n)_{\langle a\rangle} = \overline{K}_2$

For the sake of notational brevity, we will define the Hilbert series of a hypergraph to be the Hilbert series of its associated hypergraph ring.

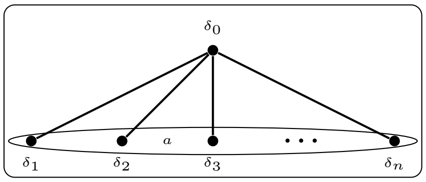
The Hilbert series for $K_{1,1}$ and $\overline{K}_n$ are easily understood from the definition of the Hilbert series, and thus their formal calculations are omitted. In particular, $HS(K_{1,1}) = \frac{1+t}{1-t}$ and $HS(\overline{K}_n) = \frac{1}{(1-t)^n}$.

We begin by inductively building the Hilbert series of $K_{1,n}$.

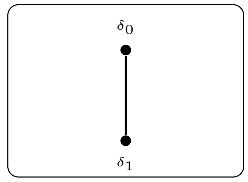
Lemma 3. *Let $n \geq 1$. The Hilbert series of the hypergraph ring associated to $K_{1,n}$, the star with n edges, is*

$$HS(K_{1,n}) = \frac{1 + t(1-t)^{n-1}}{(1-t)^n}.$$

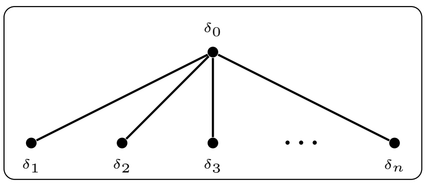
J_n



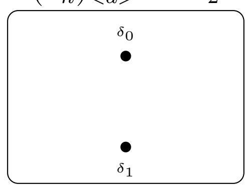
$(J_n)_{[a]} = K_{1,1}$



$(J_n)_a = K_{1,n}$



$(J_n)_{<a>} = \overline{K}_2$



Proof. The case when $n = 1$ is trivial, as mentioned above. Assume the lemma holds at $n = k$. By Theorem 5,

$$\begin{aligned}
HS(K_{1,k+1}) &= \frac{HS(K_{1,k})}{1-t} - t\left(\frac{HS(K_{1,k})}{1-t} - HS(\overline{K}_{k+1})\right) \\
&= \frac{1+t(1-t)^{k-1}}{(1-t)^{k+1}} - t\left(\frac{1+t(1-t)^{k-1}}{(1-t)^{k+1}} - \frac{1}{(1-t)^{k+1}}\right) \\
HS(K_{1,k+1}) &= \frac{1+t(1-t)^{k}}{(1-t)^{k+1}}, \text{ as desired.}
\end{aligned}$$

Thus, the lemma holds by induction. ∎

Now that we know the Hilbert series of the contraction, deletion, and extraction of a specified edge of J_n, we can easily find the Hilbert series of J_n itself:

Example 1. *Let $n \geq 1$. The Hilbert series of the hypergraph ring associated to J_n is*

$$HS(J_n) = \frac{1 - t^n + t(1-t)^{n-1}}{(1-t)^n}$$

Proof.

$$\begin{aligned}
HS(J_n) &= HS(K_{1,n}) - \frac{t^{n-1}}{(1-t)^{n-2}}\left(\frac{HS(K_{1,1})}{1-t} - HS(\overline{K}_2)\right) \\
&= \frac{1+t(1-t)^{n-1}}{(1-t)^n} - \frac{t^{n-1}}{(1-t)^{n-2}}\left(\frac{1+t-1}{(1-t)^2}\right) \\
&= \frac{1-t^n+t(1-t)^{n-1}}{(1-t)^n}
\end{aligned}$$

∎

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