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On 2-absorbing z-filters

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Abstract

In this work, 2-absorbing z-filters on a topological space X are defined and their general properties are examined. Moreover, the convergence of 2-absorbing z-filters is studied. A correspondence between 2-absorbing z-filters on X and 2-absorbing z-ideals of the ring $C(X)$ of all real-valued continuous functions on X is given.

Key Words: Ring of continuous functions, z-filters, 2-absorbing z-filters.
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1 Introduction

Let X be a topological space. We denote the ring of all continuous real-valued functions on X as $C(X)$. The ring $C(X)$ is a commutative ring with identity $1 : X \to \mathbb{R}$, defined as $1(x) = 1$ for all $x \in X$. Topological and algebraic properties of $C(X)$ have got a considerable attention in the last fifty years. For a detailed information on the ring $C(X)$ the reader may consult [4] and [8]. If X is a completely regular space, a wide range of properties of $C(X)$ can be characterized. Since for every topological space X, there exists a completely regular space Y such that $C(X)$ is isomorphic to $C(Y)$, as shown in [4, 3.9], the space X can be assumed to be completely regular.

An important concept related to $C(X)$ is z-filters. For an element f in $C(X)$, the zero-set $Z(f)$ of f is defined as $Z(f) = \{x \in X : f(x) = 0\}$. The collection of all zero-sets in X is denoted as $\mathcal{Z}(X)$. A nonempty subfamily $\mathscr{F}$ of $\mathcal{Z}(X)$ is called a *z-filter* on X provided that (i) $\varnothing \notin \mathscr{F}$, (ii) if $Z_1, Z_2 \in \mathscr{F}$, then $Z_1 \cap Z_2 \in \mathscr{F}$, (iii) if $Z \in \mathscr{F}$, $Z' \in \mathcal{Z}(X)$ and $Z \subseteq Z'$ then $Z \in \mathscr{F}$. The analoguous concept of z-filters is set theoretic filters. In a discrete space every set is a zero set, so z-filters and filters are the same in discrete spaces. A *prime z-filter* is a z-filter $\mathscr{F}$ with the following property: whenever the union of two zero-sets belongs to $\mathscr{F}$ at least one of them belongs to $\mathscr{F}$. By [4, 2E], a z-filter $\mathscr{F}$ is prime if and only if whenever the union of two zero-sets is all of X, at least one of them belongs to $\mathscr{F}$. Equivalently, a z-filter is prime if and only if it contains a prime z-filter. Prime z-filters are useful for studying prime ideal structure of the ring of continuous functions. Some of these researches can be found in [5], [6], and [7].

In this paper, we define 2-absorbing z-filters as a generalization of prime z-filters. A z-filter $\mathscr{F}$ is called a *2-absorbing z-filter* if whenever $Z_1 \cup Z_2 \cup Z_3 \in \mathscr{F}$ for $Z_1, Z_2, Z_3 \in \mathcal{Z}(X)$, one of the containments $Z_1 \cup Z_2 \in \mathscr{F}$, $Z_1 \cup Z_3 \in \mathscr{F}$ or $Z_2 \cup Z_3 \in \mathscr{F}$ holds.

Minimal prime z-filters are proved to be useful in the study of z-filters. A prime z-filter $\mathscr{P}$ is *minimal* over a z-filter $\mathscr{F}$ if it is a minimal element of the set of all prime z-filters containing $\mathscr{F}$. In Section 2, we examine some properties of minimal prime z-filters and obtain a characterization of them, Theorem 3. Further, we prove that every z-filter $\mathscr{F}$ is the intersection of all minimal prime z-filters containing $\mathscr{F}$, Proposition 1.

In Section 3, we define 2-absorbing z-filters and investigate their general properties. We show that a 2-absorbing z-filter has at most two minimal prime z-filters and acquire a characterization of 2-absorbing z-filters, Theorem 4. Let X and Y be two completely regular spaces and $\tau : X \to Y$ a continuous map. For any zero set $Z_Y(g) \in \mathcal{Z}(Y)$, denote the zero set $Z_X(g \circ \tau)$ in $\mathcal{Z}(X)$ as $\tau^{\leftarrow}[Z_Y(g)]$. Then, by [4, 4.12], for a z-filter $\mathscr{F}$ on X, the set $\tau^{\#}\mathscr{F} = \{Z \in \mathcal{Z}(Y) : \tau^{\leftarrow}[Z] \in \mathscr{F}\}$ is a z-filter on Y. We show that if $\mathscr{F}$ is a 2-absorbing z-filter so is $\tau^{\#}\mathscr{F}$, Proposition 3. Moreover, we deal with the convergence of 2-absorbing z-filters. Let X be a completely regular space. Following [4, 3.16], a point $p \in X$ is said to be a *cluster point* of a z-filter $\mathscr{F}$ if every neighbourhood of p meets every member of $\mathscr{F}$. A z-filter is said to *converge* to the *limit* p if every neighbourhood of p contains a member of $\mathscr{F}$. We prove that a 2-absorbing z-filter $\mathscr{F}$ converges to a point p if and only if all of the minimal prime z-filters over $\mathscr{F}$ converge to p, and a point p is a cluster point of a 2-absorbing z-filter $\mathscr{F}$ iff p is a cluster point of one of the minimal prime z-filters over $\mathscr{F}$, Theorem 3.

If I is an ideal in $C(X)$, then the family $Z[I] = \{Z(f) : f \in I\}$ is a z-filter on X, by [4, 2.3]. Analogously, if $\mathscr{F}$ is a z-filter on X, then the set $Z^{\leftarrow}[\mathscr{F}] = \{f : Z(f) \in \mathscr{F}\}$ is an ideal in $C(X)$. An ideal I in $C(X)$ is called a z-ideal if $Z(f) \in Z[I]$ implies $f \in I$, that is, if $I = Z^{\leftarrow}[Z[I]]$. In [2], 2-absorbing ideals of commutative rings with identity are defined. A nonzero proper ideal I of a commutative ring with identity R is called a 2-absorbing ideal whenever $abc \in I$ for $a, b, c \in R$, either $ab \in I$, or $ac \in I$, or $bc \in I$. A more general concept than 2-absorbing ideals is the concept of n-absorbing ideals for any $n \geq 1$, see [1]. For 2-absorbing commutative semigroups and their applications to rings, we recommend [3]. In Section 4, we prove that a z-ideal of $C(X)$ is a 2-absorbing ideal if and only if it contains a 2-absorbing ideal, Proposition 4. Moreover, we obtain a one-to-one correspondence between the set of 2-absorbing z-ideals and the set of 2-absorbing z-filters, Theorem 7.

2 Prime z-filters

In this section, we examine some properties of prime z-filters.

Definition 1. *A subset S of $\mathcal{Z}(X)$ is called a union closed subset if $\varnothing \in S$ and whenever two elements Z and Z' in S their union is also in S.*

The following theorem guarantees that every z-filter is contained in a prime z-filter.

Theorem 1. *Let $\mathscr{F}$ be a z-filter and $S \subseteq \mathcal{Z}(X)$ a union closed set such that $\mathscr{F} \cap S = \varnothing$. Then the set*

$$\Psi = \{\mathscr{G} \subseteq \mathcal{Z}(X) : \mathscr{G} \text{ is a } z\text{-filter, } \mathscr{F} \subseteq \mathscr{G} \text{ and } \mathscr{G} \cap S = \varnothing\}$$

has at least one maximal element and such a maximal element is a prime z-filter.

Proof: Since $\mathscr{F} \cap S = \varnothing$, the set Ψ is nonempty. Let $\{\mathscr{G}_i\}_{i \in I}$ be a chain in Ψ. The set $\mathscr{G} = \bigcup_{i \in I} \mathscr{G}_i$ is a z-filter containing $\mathscr{F}$ and $\mathscr{G} \cap S = \varnothing$ and, so $\mathscr{G}$ is an upper bound for the chain $\{\mathscr{G}_i\}_{i\in I}$ in Ψ. Then, by Zorn's Lemma, the set Ψ has a maximal element. Let $\mathscr{P}$ be such a maximal element of Ψ. Assume that $Z_1, Z_2 \notin \mathscr{P}$. The sets $\mathscr{P}_1 = \mathscr{P} \cup \{Z_1 \cup Z : Z \in \mathscr{P}\}$ and $\mathscr{P}_2 = \mathscr{P} \cup \{Z_2 \cup Z : Z \in \mathscr{P}\}$ are z-filters containing $\mathscr{F}$. By the maximality of $\mathscr{P}$, they should both meet S. Then there exist $Z, Z' \in \mathscr{P}$ such that $Z_1 \cup Z, Z_2 \cup Z' \in S$. Then we obtain $(Z_1 \cup Z_2) \cup Z \cup Z' \in S$. Thus, we must have $Z_1 \cup Z_2 \notin \mathscr{P}$. ∎

Theorem 2. *Let $\mathscr{F}$ be a z-filter. Then the set*

$$\Phi = \{\mathscr{G} \subseteq \mathcal{Z}(X) : \mathscr{G} \text{ is a prime } z\text{-filter, } \mathscr{F} \subseteq \mathscr{G}\}$$

has at least one minimal member with respect to inclusion.

Proof: Applying Theorem 1 with $S = \{\varnothing\}$, we obtain a prime z-filter containing $\mathscr{F}$. Hence Φ is nonempty. The set Φ is partially ordered by the relation

$$\mathscr{G} \preccurlyeq \mathscr{H} \text{ if and only if } \mathscr{H} \subseteq \mathscr{G}, \text{ where } \mathscr{G}, \mathscr{H} \in \Phi.$$

Let $\{\mathscr{G}_i\}_{i\in I}$ be a chain in Φ. The set $\mathscr{G} = \bigcap_{i\in I} \mathscr{G}_i$ is a prime z-filter containing $\mathscr{F}$. By Zorn's Lemma, we conclude that Φ has a minimal element. ∎

Definition 2. *A prime z-filter $\mathscr{P}$ is minimal over a z-filter $\mathscr{F}$ if it is a minimal element of the set of all prime z-filters containing $\mathscr{F}$.*

The following theorem gives a characterization of elements of minimal prime z-filter over a z-filter.

Theorem 3. *Let $\mathscr{P}$ be a prime z-filter containing $\mathscr{F}$. The following are equivalent:*

(i) $\mathscr{P}$ is minimal over $\mathscr{F}$.

(ii) $\mathcal{Z}(X)\backslash\mathscr{P}$ is a union closed set that is maximal with resprect to missing $\mathscr{F}$.

(iii) For each $Z \in \mathscr{P}$ there exists a $Z' \in \mathcal{Z}(X)\backslash\mathscr{P}$ such that $Z \cup Z' \in \mathscr{F}$.

Proof: (i)⇒(ii): Clearly, $\mathcal{Z}(X)\backslash\mathscr{P}$ is a union closed set. Let S be a union closed set containing $\mathcal{Z}(X)\backslash\mathscr{P}$ and $\mathscr{F} \cap S = \varnothing$. Let $\mathscr{Q}$ be a z-filter containing $\mathscr{F}$ that is maximal with respect to being disjoint from S. Then, by Theorem 1, the z-filter $\mathscr{Q}$ is prime. Since $\mathscr{Q} \cap \mathcal{Z}(X)\backslash\mathscr{P} = \varnothing$, we get $\mathscr{Q} \subseteq \mathscr{P}$. Since $\mathscr{P}$ is minimal over $\mathscr{F}$, we have $\mathscr{Q} = \mathscr{P}$. Thus $S = \mathcal{Z}(X)\backslash\mathscr{P}$.

(ii)⇒(iii): If $Z = X$ the statement trivially holds. Let $Z \in \mathscr{P}\backslash\{X\}$. Set

$$S = \{Z \cup Z' : Z' \in \mathcal{Z}(X)\backslash\mathscr{P}\} \cup \mathcal{Z}(X)\backslash\mathscr{P}.$$

Then S is a union closed set and properly contains $\mathcal{Z}(X)\backslash\mathscr{P}$. Then $\mathscr{F} \cap S \neq \varnothing$. Hence there exists a $Z' \in \mathcal{Z}(X)\backslash\mathscr{P}$ such that $Z \cup Z' \in \mathscr{P}$.

(iii)⇒(i): Assume that $\mathscr{F} \subseteq \mathscr{Q} \subseteq \mathscr{P}$ where $\mathscr{Q}$ is a prime z-filter. If there exists $Z \in \mathscr{P}\backslash\mathscr{Q}$, then there is a $Z' \in \mathcal{Z}(X)\backslash\mathscr{P}$ such that $Z \cup Z' \in \mathscr{F} \subseteq \mathscr{Q}$. Since $Z \notin \mathscr{Q}$, we have the contradiction $Z' \in \mathscr{P}$. ∎

Proposition 1. *Let $\mathscr{F}$ be a z-filter. Then $\mathscr{F}$ is the intersection of all minimal prime z-filters containing $\mathscr{F}$.*

Proof: Clearly $\mathscr{F}$ is contained in the intersection of all minimal prime z-filters over $\mathscr{F}$. For the reverse inclusion, assume that there exists an element Z contained in the intersection of all minimal prime z-filters over $\mathscr{F}$ but not contained in $\mathscr{F}$. The set $S = \{Z' \in \mathcal{Z}(X) : Z' \subseteq Z\}$ is a union closed set disjoint from $\mathscr{F}$. Then, by Theorem 1, there exists a prime z-filter $\mathscr{P}$ containing $\mathscr{F}$ and disjoint from S. However, since $Z \in \mathscr{P}$, we have the contradiction $Z \in \mathscr{P} \cap S$. □

3 2-absorbing z-filters

Inspiring from prime z-filters we can define 2-absorbing z-filters as follows:

Definition 3. *A z-filter $\mathscr{F}$ is called as a 2-absorbing z-filter if whenever $Z_1 \cup Z_2 \cup Z_3 \in \mathscr{F}$ for $Z_1, Z_2, Z_3 \in \mathcal{Z}(X)$, one of the containments $Z_1 \cup Z_2 \in \mathscr{F}$, $Z_1 \cup Z_3 \in \mathscr{F}$ or $Z_2 \cup Z_3 \in \mathscr{F}$ holds.*

We note that the set of 2-absorbing z-filters properly contains the set of prime z-filters.

Example 1. *Let $\mathscr{F} = \{Z \in \mathcal{Z}(\mathbb{R}) : \{0, 1\} \subseteq Z\}$. It is easy to show that $\mathscr{F}$ is a 2-absorbing z-filter. However, it is not a prime z-filter since $Z(sinx) \cup Z(sin(x-1)) \in \mathscr{F}$ and neither $Z(sinx)$ nor $Z(sin(x-1))$ is in $\mathscr{F}$.*

Theorem 4. *Let $\mathscr{F}$ be a 2-absorbing z-filter. Then there are at most two minimal prime z-filters over $\mathscr{F}$.*

Proof: Suppose that $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3$ be distinct minimal prime z-filters over $\mathscr{F}$. Then there exist $Z_1 \in \mathscr{P}_1 \backslash \mathscr{P}_2$ and $Z_2 \in \mathscr{P}_2 \backslash \mathscr{P}_1$. Since $\mathscr{P}_1$ and $\mathscr{P}_2$ are minimal, by Theorem 3, there exist elements $Z \in \mathcal{Z}(X) \backslash \mathscr{P}_1$ and $Z' \in \mathcal{Z}(X) \backslash \mathscr{P}_2$ such that $Z_1 \cup Z, Z_2 \cup Z' \in \mathscr{F}$. We have

$$(Z_1 \cap Z_2) \cup (Z_1 \cap Z') \cup (Z \cap Z_2) \cup (Z \cap Z') = (Z_1 \cup Z) \cap (Z_2 \cup Z') \in \mathscr{F}.$$

Then we obtain $Z_1 \cap Z' \in \mathscr{P}_1$ and $Z_2 \cap Z \in \mathscr{P}_2$, and hence $Z' \in \mathscr{P}_1$ and $Z \in \mathscr{P}_2$. As

$$(Z \cap Z') \cup Z_1 \cup Z_2 = (Z \cup Z_1 \cup Z_2) \cap (Z' \cup Z_1 \cup Z_2) \in \mathscr{F}$$

and $\mathscr{F}$ is a 2-absorbing z-filter, we get $Z_1 \cup Z_2 \in \mathscr{F}$. Now, since $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3$ are all distinct z-filters, there exist elements $Y_1 \in \mathscr{P}_1 \backslash (\mathscr{P}_2 \cup \mathscr{P}_3)$ and $Y_2 \in \mathscr{P}_2 \backslash (\mathscr{P}_1 \cup \mathscr{P}_3)$. Then, by the above argument we have $Y_1 \cup Y_2 \in \mathscr{F}$. As $\mathscr{F} \subseteq \mathscr{P}_1 \cap \mathscr{P}_2 \cap \mathscr{P}_3$, we get one of the contradicting arguments $Y_1 \in \mathscr{P}_3$ or $Y_2 \in \mathscr{P}_3$. Hence, there are at most two minimal prime z-filters over $\mathscr{F}$. □

Corollary 1. *Let $\mathscr{F}$ be a 2-absorbing z-filter. Then $\mathscr{F}$ is a prime z-filter or $\mathscr{F} = \mathscr{P}_1 \cap \mathscr{P}_2$ where $\mathscr{P}_1$ and $\mathscr{P}_2$ are the only minimal prime z-filters over $\mathscr{F}$.*

Now, we are to give a characterization of 2-absorbing z-filters. First, we need the following lemma.

Lemma 1. *Intersection of two prime z-filters is a 2-absorbing z-filter.*

Proof: Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two prime z-filters and $\mathscr{F} = \mathscr{P}_1 \cap \mathscr{P}_2$. assume that $Z_1 \cup Z_2 \cup Z_3 \in \mathscr{F}$ where $Z_1, Z_2, Z_3 \in \mathcal{Z}(X)$. Since $\mathscr{P}_1$ is prime, either Z_1 or $Z_2 \cup Z_3$ is in $\mathscr{P}_1$. Similarly, either Z_1 or $Z_2 \cup Z_3$ is in $\mathscr{P}_2$. If $Z_1 \in \mathscr{P}_1$ and $Z_1 \in \mathscr{P}_2$, then $Z_1 \cup Z_2 \in \mathscr{P}_1 \cap \mathscr{P}_2 = \mathscr{F}$. If $Z_1 \in \mathscr{P}_1$ and $Z_2 \cup Z_3 \in \mathscr{P}_2$, then since Z_2 or Z_3 is in $\mathscr{P}_2$, we have $Z_1 \cup Z_2$ or $Z_1 \cup Z_3$ is in $\mathscr{P}_1 \cap \mathscr{P}_2 = \mathscr{F}$. The exactly same argument is applied for the other cases. □

Theorem 5. *A z-filter $\mathscr{F}$ is 2-absorbing if and only if it contains a 2-absorbing z-filter.*

Proof: The necessity part is clear.

For the sufficiency, let $\mathscr{G}$ be a 2-absorbing z-filter contained in $\mathscr{F}$. Then $\mathscr{G} = \mathscr{P}$ or $\mathscr{G} = \mathscr{P}_1 \cap \mathscr{P}_2$ where $\mathscr{P}, \mathscr{P}_1, \mathscr{P}_2$ are minimal prime z-filters over $\mathscr{G}$. In the former case, $\mathscr{P} = \mathscr{G} \subseteq \mathscr{F}$. Since $\mathscr{F}$ contains a prime z-filter, it is a prime, and hence a 2-absorbing z-filter. Now assume that $\mathscr{G} = \mathscr{P}_1 \cap \mathscr{P}_2$. Then $\mathscr{P}_1 \cap \mathscr{P}_2 \subseteq \mathscr{F}$. We can write $\mathscr{F} = \mathscr{A} \cap \mathscr{B}$ where $\mathscr{A}$ is the intersection of prime z-filters over $\mathscr{F}$ containing $\mathscr{P}_1$ and $\mathscr{B}$ is the intersection of prime z-filters over $\mathscr{F}$ containing $\mathscr{P}_2$. As $\mathscr{A}$ and $\mathscr{B}$ are intersection of z-filters, they are z-filters, too. Besides, since they contain prime z-filters $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively, they are prime z-filters. Thus, we conclude that, being the intersection of two prime z-filters, the z-filter $\mathscr{F}$ is 2-absorbing.

□

Theorem 6. *A z-filter $\mathscr{F}$ is 2-absorbing if and only if whenever $Z_1 \cup Z_2 \cup Z_3 = X$ with $Z_1, Z_2, Z_3 \in \mathcal{Z}(X)$, either $Z_1 \cup Z_2 \in \mathscr{F}$ or $Z_1 \cup Z_3 \in \mathscr{F}$ or $Z_2 \cup Z_3 \in \mathscr{F}$.*

Proof: Since $X \in \mathscr{F}$, the necessity part is clear. For the sufficiency part, let $\mathscr{G}$ be the set of all $Z_1 \cup Z_2 \in \mathscr{F}$ with $Z_1, Z_2 \in \mathcal{Z}(X) \backslash \{X\}$ and such that there exists a $Z_3 \in \mathcal{Z}(X) \backslash \{X\}$ satisfying $Z_1 \cup Z_2 \cup Z_3 = X$. Since $\mathscr{G}$ is contained in $\mathscr{F}$, it does not contain the empty set. Let $Z_1 \cup Z_2, Y_1 \cup Y_2 \in \mathscr{G}$. Then there exists $Z_3, Y_3 \in \mathcal{Z}(X)$ such that $Z_1 \cup Z_2 \cup Z_3 = X$ and $Y_1 \cup Y_2 \cup Y_3 = X$. Then, we have

$$[(Z_1 \cup Z_2) \cap (Y_1 \cup Y_2)] \cup [((Z_1 \cup Z_2) \cap Y_3) \cup (Z_3 \cap (Y_1 \cup Y_2)) \cup (Z_3 \cap Y_3)] = X.$$

Since

$$(Z_1 \cup Z_2) \cap (Y_1 \cup Y_2) = (Z_1 \cap (Y_1 \cup Y_2)) \cup (Z_2 \cap (Y_1 \cup Y_2)),$$

$(Z_1 \cup Z_2) \cap (Y_1 \cup Y_2)$ is an element of $\mathscr{G}$. For $Z \in \mathcal{Z}(X)$ with $Z_1 \cup Z_2 \subseteq Z$, we have $Z \cup (Z_1 \cup Z_2) \cup Z_3 = X$. Hence $Z \in \mathscr{G}$. Therefore, $\mathscr{G}$ is a z-filter. Let $Z_1 \cup Z_2 \cup Z_3 \in \mathscr{G}$. assume first that one of the unions $Z_1 \cup Z_2$, $Z_1 \cup Z_3$ and $Z_2 \cup Z_3$ is equal to X, then it is trivially in $\in \mathscr{G}$. Now, suppose none of them is equal to X. Since there exists $Z \in \mathcal{Z}(X)$ satsifying $Z_1 \cup Z_2 \cup Z_3 \cup Z = X$, by assumption, either $Z_1 \cup Z_2 \in \mathscr{F}$ or $Z_1 \cup Z_3 \cup Z \in \mathscr{F}$ or $Z_2 \cup Z_3 \cup Z \in \mathscr{F}$. Then, we have $Z_1 \cup Z_2 \in \mathscr{G}$ or $Z_1 \cup Z_3 \in \mathscr{G}$ or $Z_2 \cup Z_3 \in \mathscr{G}$. Thus, $\mathscr{G}$ is a 2-absorbing z-filter. By Theorem 5, we conclude that $\mathscr{F}$ is a 2-absorbing z-filter. □

Let X and Y be two completely regular spaces and $\tau : X \to Y$ a continuous mapping. For any zero set $Z_Y(g) \in \mathcal{Z}(Y)$, denote the zero set $Z_X(g \circ \tau)$ in $\mathcal{Z}(X)$ as $\tau^{\leftarrow}[Z_Y(g)]$. Then, by [4, 4.12], for a z-filter $\mathscr{F}$ on X, the set

$$\tau^{\#}\mathscr{F} = \{Z \in \mathcal{Z}(Y) : \tau^{\leftarrow}[Z] \in \mathscr{F}\}$$

is a z-filter on Y. Moreover, if $\mathscr{F}$ is prime, so is $\tau^{\#}\mathscr{F}$. We have a further result:

Proposition 2. *With the above notation, if $\mathscr{F}$ is a 2-absorbing z-filter, then $\tau^{\#}\mathscr{F}$ is a 2-absorbing z-filter.*

Proof: Let $Z_1 \cup Z_2 \cup Z_3 \in \tau^{\#}\mathscr{F}$ where $Z_1 = Z_Y(g_1), Z_2 = Z_Y(g_2), Z_3 = Z_Y(g_3)$ for some $g_1, g_2, g_3 \in C(Y)$. Then we have

$$\begin{aligned} Z_X[g_1 \circ \tau] \cup Z_X[g_2 \circ \tau] \cup Z_X[g_3 \circ \tau] &= Z_X[(g_1 \circ \tau)(g_2 \circ \tau)(g_3 \circ \tau)] \\ &= Z_X[(g_1 g_2 g_3) \circ \tau] \\ &= \tau^{\leftarrow}[Z_Y(g_1 g_2 g_3)] \\ &= \tau^{\leftarrow}[Z_Y(g_1) \cup Z_Y(g_2) \cup Z_Y(g_3)] \\ &= \tau^{\leftarrow}[Z_1 \cup Z_2 \cup Z_3] \\ &\in \mathscr{F}. \end{aligned}$$

Since $\mathscr{F}$ is a 2-absorbing z-filter, for some $i \neq j$, we have

$$Z_X[g_i \circ \tau] \cup Z_X[g_j \circ \tau] \in \mathscr{F}.$$

This implies

$$\tau^{\leftarrow}[Z_Y(g_i) \cup Z_Y(g_j)] = \tau^{\leftarrow}[Z_Y(g_i g_j)] = Z_X[g_i g_j \circ \tau] \in \mathscr{F}.$$

Thus, we obtain

$$Z_i \cup Z_j = Z_Y(g_i) \cup Z_Y(g_j) \in \tau^{\#}\mathscr{F}.$$

Hence, we conclude that $\tau^{\#}\mathscr{F}$ is a 2-absorbing z-filter. □

Let X be a completely regular space. Following [4, 3.16], a point $p \in X$ is said to be a cluster point of a z-filter $\mathscr{F}$ if every neighbourhood of p meets every member of $\mathscr{F}$. A z-filter is said to converge to the limit p if every neighbourhood of p contains a member of $\mathscr{F}$. Note that every limit point is a cluster point. We are to investigate convergence of 2-absorbing z-filters.

Proposition 3. *Let $\mathscr{F}$ be a 2-absorbing z-filter on a completely regular space X with minimal prime z-filters $\mathscr{P}_1$ and $\mathscr{P}_2$. Let $p \in X$.*

(i) $\mathscr{F}$ converges to p if and only if $\mathscr{P}_1$ and $\mathscr{P}_2$ both converge to p.

(ii) p is a cluster point of $\mathscr{F}$ if and only if p is a cluster point of $\mathscr{P}_1$ or $\mathscr{P}_2$.

Proof: (i) Assume that $\mathscr{F}$ converges to p. Let $\mathscr{G}$ be a z-filter containing $\mathscr{F}$. Every neighborhood of p contains an element of $\mathscr{F}$, and hence an element of $\mathscr{G}$. Then $\mathscr{G}$ converges to p. This fact apply to minimal prime z-filters of $\mathscr{F}$, as well.

Conversely, assume that $\mathscr{P}_1$ and $\mathscr{P}_2$ both converge to p. Let $\mathscr{N}$ be a neighborhood of p. Then there exist $Z_1 \in \mathscr{P}_1$ and $Z_2 \in \mathscr{P}_1$, both contained in $\mathscr{N}$. Since $Z_1 \cup Z_2 \in \mathscr{P}_1 \cup \mathscr{P}_2 = \mathscr{F}$ and $Z_1 \cup Z_2 \subseteq \mathscr{N}$, we conclude that $\mathscr{F}$ converges to p.

(ii) Suppose that p is not a cluster point of both $\mathscr{P}_1$ and $\mathscr{P}_2$. Then, there exist neighborhoods $\mathscr{N}_1$ and $\mathscr{N}_2$ of p and elements $Z_1 \in \mathscr{P}_1$ and $Z_2 \in \mathscr{P}_2$ such that $\mathscr{N}_1 \cap Z_1 = \varnothing$ and $\mathscr{N}_2 \cap Z_2 = \varnothing$. We have $Z_1 \cup Z_2 \in \mathscr{F}$ and $\mathscr{N}_1 \cap \mathscr{N}_2$ a neighborhood of p. Moreover, we observe that $(\mathscr{N}_1 \cap \mathscr{N}_2) \cap (Z_1 \cup Z_2) = \varnothing$. Therefore p is not a cluster point of $\mathscr{F}$.

Conversely, assume that p is a cluster point of some z-filter $\mathscr{G}$ containing $\mathscr{F}$. Let $\mathscr{N}$ be a neighborhood of p. For any $z \in \mathscr{F}$, since $Z \in \mathscr{G}$, we have $\mathscr{N} \cap Z \neq \varnothing$. Therefore p is a cluster point if $\mathscr{F}$. □

Note that, a prime z-filter on a completely regular space has at most one cluster point, [4, 3.17]. We conclude, by Proposition 3, that a 2-absorbing z-filter on a completely regular space can have at most two cluster points.

4 2-absorbing z-ideals

In this section, we give a correspondence between 2-absorbing z-filters on a topological space X and 2-absorbing z-ideals of $C(X)$. If I is an ideal in $C(X)$, then the family $Z[I] = \{Z(f) : f \in I\}$ is a z-filter on X. Analogously, if $\mathscr{F}$ is a z-filter on X, then the family $Z^{\leftarrow}[\mathscr{F}] = \{f : Z(f) \in \mathscr{F}\}$ is an ideal in $C(X)$. An ideal I in $C(X)$ is called a z-ideal if $Z(f) \in Z[I]$ implies $f \in I$, that is, if $I = Z^{\leftarrow}[Z[I]]$. If $\mathscr{F}$ is a z-filter then the ideal $Z^{\leftarrow}[\mathscr{F}]$ is a z-ideal. Hence, if J is an ideal in $C(X)$, then $I = Z^{\leftarrow}[Z[J]]$ is the smallest z-ideal containing J. Note that, every maximal ideal is z-ideal, and the intersection of an arbitrary family of z-ideals is a z-ideal. Moreover, it is proved in [4, 14.7] that every minimal prime ideal over an ideal I is a z-ideal.

The following proposition is the z-ideal counterpart of Theorem 5.

Proposition 4. *A z-ideal I of $C(X)$ is a 2-absorbing ideal if and only if it contains a 2-absorbing ideal.*

Proof: The necessity part is clear.

Let I be a z-ideal of $C(X)$ containing a 2-absorbing ideal J. Note that, since I is a z-ideal, by [4, 2.8], it is a radical ideal. Then $\mathrm{rad}J \subseteq I$. As J is a 2-absorbing ideal it has at most two minimal prime ideals, that is $\mathrm{rad}J = P$ or $\mathrm{rad}J = P_1 \cap P_2$ where P, P_1, P_2 are minimal prime ideals of J. In the former case, since $P \subseteq I$, by [4, 2.9], we have I as a prime, and hence a 2-absorbing ideal. Now, assume that $\mathrm{rad}J = P_1 \cap P_2$. Since $P_1 \cap P_2 \subseteq I$ and I is a radical ideal, it can be written as $I = A \cap B$ where A is the intersection of minimal prime ideals of I containing P_1 and B is the intersection of minimal prime ideals of I containing P_2. Since every minimal prime ideal is a z-ideal, being the intersection of any arbitrary nonempty family of z-ideals, both A and B are z-ideals. Besides, containing prime ideals P_1 and P_2 respectively, they are prime ideals. Therefore, we conclude that $I = A \cap B$, being an intersection of two prime ideals, is a 2-absorbing ideal. $\square$

The following theorem gives a correspondence between 2-absorbing ideals of $C(X)$ and 2-absorbing z-filters.

Theorem 7. *Let I be a 2-absorbing ideal in $C(X)$. Then $Z[I]$ is a 2-absorbing z-filter. Conversely, if $\mathscr{F}$ is a 2-absorbing z-filter, the ideal $Z^{\leftarrow}[F]$ is a 2-absorbing z-ideal. Thus, there is a one-to-one correspondence between the set of 2-absorbing ideals of $C(X)$ and the set of 2-absorbing z-filters.*

Proof: Let I be a 2-absorbing ideal in $C(X)$ and $J = Z^{\leftarrow}[Z[I]]$. Then we have $I \subseteq Z^{\leftarrow}[Z[I]] = J$. As a z-ideal containing a 2-absrobing ideal, by Proposition 4, the ideal J is a 2-absorbing ideal. Assume that $Z(f) \cup Z(g) \cup Z(h) \in Z[I]$ where $f, g, h \in C(X)$. Then we have $Z(fgh) \in Z[I] = Z[J]$. Since J is a z-ideal, we obtain $fgh \in J$. As J is a 2-absorbing ideal, either $fg \in J$ or $fh \in J$ or $gh \in J$. Thus we get $Z(f) \cup Z(g) = Z(fg) \in Z[J] = Z[I]$ or $Z(f) \cup Z(h) = Z(fh) \in Z[J] = Z[I]$ or $Z(g) \cup Z(h) = Z(gh) \in Z[J] = Z[I]$. Therefore $Z[I]$ is a 2-absorbing z-filter.

Conversely, let $\mathscr{F}$ be a 2-absorbing z-filter. Note that $Z^{\leftarrow}[\mathscr{F}]$ is a z-ideal. Let $fgh \in Z^{\leftarrow}[\mathscr{F}]$. Then

$$Z(f) \cup Z(g) \cup Z(h) = Z(fgh) \in \mathscr{F}.$$

Since $\mathscr{F}$ is a 2-absorbing z-filter, either $Z(fg) = Z(f) \cup Z(g) \in \mathscr{F}$ or $Z(fh) = Z(f) \cup Z(h) \in \mathscr{F}$ or $Z(gh) = Z(g) \cup Z(h) \in \mathscr{F}$. This implies $fg \in Z^{\leftarrow}[\mathscr{F}]$ or $fh \in Z^{\leftarrow}[\mathscr{F}]$ or $gh \in Z^{\leftarrow}[\mathscr{F}]$. $\square$

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