Bull. Math. Soc. Sci. Math. Roumanie
Tome 61 (109) No. 2, 2018, 225–234

On sums of volume quotient functions

by
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Abstract

In this paper, we establish Brunn-Minkowski-type inequalities for the sums of volume quotient functions of star bodies, which in special case yield some of the recent results.

Key Words volume quotient function, volume sum function, radial Minkowski sum, radial Blaschke sum, harmonic Blaschke sum.
2010 Mathematics Subject Classification: Primary 52A30, Secondary 26D15

1 Introduction

Let K and L be star bodies in $\mathbb{R}^n$, then the classical dual Brunn-Minkowski inequality state that (see [3]).

$$V(K\tilde{+}L)^{1/n} \leq V(K)^{1/n} + V(L)^{1/n},$$

with equality holds if and only if K and L are dilates. Here, $\tilde{+}$ is radial Minkowski sum. If K and L are star bodies in $\mathbb{R}^n$, then The sum $K\tilde{+}L$ debotes the radial Minkowski sum, defined by

$$\rho(K\tilde{+}L,\cdot) = \rho(K,\cdot) + \rho(L,\cdot), \tag{1.1}$$

for star bodies K and L. Here, $\rho(K,u)$ denotes the radial function $\rho(K,\cdot) : S^{n-1} \to [0,\infty)$, defined for $u \in S^{n-1}$, by $\rho(K,u) = \max\{\lambda \geq 0 : \lambda u \in K\}$. If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Let $\mathcal{S}^n$ denote the set of star bodies in $\mathbb{R}^n$. Recently, the volume quotient functions has been introduced, and inequalities for it were established (see [7]):

Theorem A. *If $K, L \in \mathcal{S}^n$ and $i \leq n-1 \leq j \leq n$, then*

$$Q^{\frac{1}{j-i}}_{\widetilde{W}_{i,j}(K\tilde{+}L)} \leq Q^{\frac{1}{j-i}}_{\widetilde{W}_{i,j}(K)} + Q^{\frac{1}{j-i}}_{\widetilde{W}_{i,j}(L)}, \tag{1.2}$$

with equality if and only if K and L are dilates, where $\tilde{+}$ is the radial Minkowski sum.

Theorem B. *If $K, L \in \mathcal{S}^n$ and $i \leq 1 \leq j \leq n$, then*

$$Q^{\frac{n-1}{j-i}}_{\widetilde{W}_{i,j}(K\check{+}L)} \leq Q^{\frac{n-1}{j-i}}_{\widetilde{W}_{i,j}(K)} + Q^{\frac{n-1}{j-i}}_{\widetilde{W}_{i,j}(L)}, \tag{1.3}$$

with equality if and only if K and L are dilates, where $\check{+}$ is the radial Blaschke sum (see Section 2.)

Theorem C. *If $K, L \in \mathcal{S}^n$ and $i \leq -1 \leq j \leq n$ and, then*

$$\frac{Q^{\frac{1}{j-i}}_{\widetilde{W}_{i,j}(K\hat{+}L)}}{V(K\hat{+}L)} \leq \frac{Q^{\frac{1}{j-i}}_{\widetilde{W}_{i,j}(K)}}{V(K)} + \frac{Q^{\frac{1}{j-i}}_{\widetilde{W}_{i,j}(L)}}{V(L)}, \tag{1.4}$$

with equality if and only if K and L are dilates, where $\hat{+}$ is the harmonic Blaschke sum (see Section 2).

Similarly, we give a new definition of dual quermassintegral quotient function.

Definition 1.1 Let $K \in \mathcal{S}^n$, then dual quermassintegral quotient function of star body K, $Q_{\widetilde{W}_{i,j}(K)}$ $(i, j \in \mathbb{R})$, defined by

$$Q_{\widetilde{W}_{i,j}(K)} = \frac{\widetilde{W}_i(K)}{\widetilde{W}_j(K)}.$$

The aim of this paper is to establish the following Brunn-Minkowski-type inequalities for sums of volume quotient functions of star bodies.

Theorem 1.1 *If K, L and D are star bodies in $\mathbb{R}^n$ and D' is a dilated copy of D. If $i < j-1 \leq n-1 \leq j \leq n$, then for $\varepsilon > 0$*

$$\left(Q_{\widetilde{W}_{i,j}(K\tilde{+}\varepsilon L)} + Q_{\widetilde{W}_{i,j}(D\tilde{+}\varepsilon D')}\right)^{\frac{1}{j-i}}$$

$$\leq \left(Q_{\widetilde{W}_{i,j}(K)} + Q_{\widetilde{W}_{i,j}(D)}\right)^{\frac{1}{j-i}} + \varepsilon\left(Q_{\widetilde{W}_{i,j}(L)} + Q_{\widetilde{W}_{i,j}(D')}\right)^{\frac{1}{j-i}}, \tag{1.5}$$

with equality if and only if K and L are dilates and $(Q_{\widetilde{W}_{i,j}(K)}, Q_{\widetilde{W}_{i,j}(D)}) = \mu(Q_{\widetilde{W}_{i,j}(L)}, Q_{\widetilde{W}_{i,j}(D')})$, where μ is a constant.

Theorem 1.1 is special case of Theorems 4.1 established in Section 4.

Remark 1.1 Taking for $j = n$ and $\varepsilon = 1$ in (1.5), (1.5) changes to, for $i < n-1$

$$\left(\widetilde{W}_i(K\tilde{+}L) + \widetilde{W}_i(D\tilde{+}D')\right)^{1/(n-i)}$$

$$\leq \left(\widetilde{W}_i(K)^{1/(n-i)} + \widetilde{W}_i(D)\right)^{1/(n-i)} + \left(\widetilde{W}_i(L) + \widetilde{W}_i(D')\right)^{1/(n-i)}, \tag{1.6}$$

with equality if and only if K and L are dilates and $(\widetilde{W}_i(K), \widetilde{W}_i(L)) = \mu(\widetilde{W}_i(D), \widetilde{W}_i(D'))$, where μ is a constant.

Theorem 1.2 *If K, L and D are star bodies in $\mathbb{R}^n$ and D' is a dilated copy of D. If $n \geq j \geq 1 \geq j-n+1 > i$, then for $\varepsilon > 0$*

$$\left(Q_{\widetilde{W}_{i,j}(K\check{+}\varepsilon L)} + Q_{\widetilde{W}_{i,j}(D\check{+}\varepsilon D')}\right)^{\frac{n-1}{j-i}}$$

$$\leq \left(Q_{\widetilde{W}_{i,j}(K)} + Q_{\widetilde{W}_{i,j}(D)}\right)^{\frac{n-1}{j-i}} + \varepsilon\left(Q_{\widetilde{W}_{i,j}(L)} + Q_{\widetilde{W}_{i,j}(D')}\right)^{\frac{n-1}{j-i}}, \tag{1.7}$$

with equality if and only if K and L are dilates and $(Q_{\widetilde{W}_{i,j}(K)}, Q_{\widetilde{W}_{i,j}(D)}) = \mu(Q_{\widetilde{W}_{i,j}(L)}, Q_{\widetilde{W}_{i,j}(D')})$*, where μ is a constant.*

Theorem 1.2 is special case of Theorems 4.2 established in Section 4.

Remark 1.2 Taking for $j = n$ and $\varepsilon = 1$ in (1.7), (1.7) changes to, for $i < n$

$$\left(\widetilde{W}_i(K\check{+}L) + \widetilde{W}_i(D\check{+}D')\right)^{(n-1)/(n-i)}$$

$$\leq \left(\widetilde{W}_i(K) + \widetilde{W}_i(D)\right)^{(n-1)/(n-i)} + \left(\widetilde{W}_i(L) + \widetilde{W}_i(D')\right)^{(n-1)/(n-i)}, \tag{1.8}$$

with equality if and only if K and L are dilates and $(\widetilde{W}_i(K), \widetilde{W}_i(L)) = \mu(\widetilde{W}_i(D), \widetilde{W}_i(D'))$, where μ is a constant.

Theorem 1.3 *If K, L and D are star bodies in $\mathbb{R}^n$ and D' is a dilated copy of D. If $i < j - n - 1 \leq -1 \leq j \leq n$, then for $\varepsilon > 0$*

$$\left(\frac{Q_{\widetilde{W}_{i,j}(K\hat{+}\varepsilon L)}}{V(K\hat{+}\varepsilon L)^{\frac{j-i}{n+1}}} + \frac{Q_{\widetilde{W}_{i,j}(D\hat{+}\varepsilon D')}}{V(D\hat{+}\varepsilon D')^{\frac{j-i}{n+1}}}\right)^{\frac{n+1}{j-i}}$$

$$\leq \left(\frac{Q_{\widetilde{W}_{i,j}(K)}}{V(K)^{\frac{j-i}{n+1}}} + \varepsilon\frac{Q_{\widetilde{W}_{i,j}(D)}}{V(D)^{\frac{j-i}{n+1}}}\right)^{\frac{n+1}{j-i}} + \left(\frac{Q_{\widetilde{W}_{i,j}(L)}}{V(L)^{\frac{j-i}{n+1}}} + \frac{Q_{\widetilde{W}_{i,j}(D')}}{V(D')^{\frac{j-i}{n+1}}}\right)^{\frac{n+1}{j-i}}, \tag{1.9}$$

with equality if and only if K and L are dilates and

$$\frac{V(L)^{\frac{n+1}{j-i}} Q_{\widetilde{W}_{i,j}(K)}}{V(K)^{\frac{n+1}{j-i}} Q_{\widetilde{W}_{i,j}(L)}} = \frac{V(D')^{\frac{n+1}{j-i}} Q_{\widetilde{W}_{i,j}(D)}}{V(D)^{\frac{n+1}{j-i}} Q_{\widetilde{W}_{i,j}(D')}}.$$

Theorem 1.3 is special case of Theorems 4.3 established in Section 4.

Remark 1.3 Let D and D' be single points, and taking for $j = n$ and $\varepsilon = 1$ in (1.9), we have for $i < 1$

$$\frac{\widetilde{W}_i(K\hat{+}L)^{\frac{n+1}{n-i}}}{V(K\hat{+}L)} \leq \frac{\widetilde{W}_i(K)^{\frac{n+1}{n-i}}}{V(K)} + \frac{\widetilde{W}_i(L)^{\frac{n+1}{n-i}}}{V(L)},$$

with equality if and only if K and L are dilates.

2 Notations and preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n (n > 2)$. We reserve the letter u for unit vectors, and the letter B for the unit ball centered at the origin. The surface of B is S^{n-1}. For $K_1, \ldots, K_r \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, the volume of the radial Minkowski linear combination $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$ is a homogeneous nth polynomial in the λ_i (see e.g. [2])

$$V(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r) = \sum \tilde{V}_{i_1,\ldots,i_n} \lambda_{i_1} \cdots \lambda_{i_n}, \tag{2.1}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ whose entries are positive integers not exceeding r. If we require the coefficients of the polynomial in (2.1) to be symmetric in their argument, then they are uniquely determined. The coefficient $\tilde{V}_{i_1,\ldots,i_n}$ is nonnegative and depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$. Here we denote $\tilde{V}_{i_1,\ldots,i_n}$ to $\tilde{V}(K_{i_1}, \ldots, K_{i_n})$ and is called the dual mixed volume of $K_{i_1}, \ldots, K_{i_n}$. The radial Minkowski linear combination, $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$, defined by (see [4])

$$\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r : x_i \in K_i\}, \tag{2.2}$$

for $K_1, \ldots, K_r \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$. If $K_1, \ldots, K_n \in \mathcal{S}^n$, the dual mixed volume $\tilde{V}(K_1, \ldots, K_n)$ defined by (see [3])

$$\tilde{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u) dS(u). \tag{2.3}$$

If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, the dual mixed volume is written as $\tilde{V}_i(K, L)$. If $L = B$, the dual mixed volume $\tilde{V}_i(K, L)$ is written as $\widetilde{W}_i(K)$ and call it quermassintegral of K, defined by (see [3])

$$\widetilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} dS(u), \tag{2.4}$$

where $i \in \mathbb{R}$. When $i = 0$, $\widetilde{W}_i(K)$ becomes the usual volume $V(K)$. If $K, L \in \mathcal{S}^n$, then from (2.1), it follows immediately that

$$\lim_{\varepsilon \to 0} \frac{V(K \tilde{+} \varepsilon \cdot L) - V(K)}{\varepsilon} = n \tilde{V}_1(K, L). \tag{2.5}$$

If K and L are star bodies in $\mathbb{R}^n$ and $\lambda, \mu \geq 0$, then $\lambda \cdot K \breve{+} \mu \cdot L$, is the star body, call it radial Blaschke linear combination, whose radial function is given by (see [4])

$$\rho(\lambda \cdot K \breve{+} \mu \cdot L, \cdot)^{n-1} = \lambda \rho(K, \cdot)^{n-1} + \mu \rho(L, \cdot)^{n-1}. \tag{2.6}$$

The harmonic Blaschke linear combinations, denoted by $\lambda K \hat{+} \mu L$, defined by (see [5])

$$V(\lambda K \hat{+} \mu L)^{-1} \rho(\lambda K \hat{+} \mu L, \cdot)^{n+1} = \lambda V(K)^{-1} \rho(K, \cdot)^{n+1} + \mu V(L)^{-1} \rho(L, \cdot)^{n+1}, \tag{2.7}$$

where $K, L \in \mathcal{S}^n$ and $\lambda, \mu \geq 0$ (not both zero).

3 An improvement of Beckenbach-Dresher's inequality

Lemma 3.1 [1] (Beckenbach-Dresher's inequality) *If* $p \geq 1 \geq r \geq 0$, $f, g \geq 0$, *then*

$$\left(\frac{\int_{\mathbb{E}} (f+g)^p d\phi}{\int_{\mathbb{E}} (f+g)^r d\phi}\right)^{\frac{1}{p-r}} \leq \left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}\right)^{\frac{1}{p-r}} + \left(\frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}\right)^{\frac{1}{p-r}}, \tag{3.1}$$

with equality if and only if the functions f and g are positively proportional. Here $\mathbb{E}$ is a bounded measurable subset in $\mathbb{R}^n$.

Lemma 3.2 *If $p > r+1 \geq 1 \geq r \geq 0$, $f, g, u, v \geq 0$ and the functions u and v are positively proportional, then*

$$\left(\frac{\int_{\mathbb{E}}(f+g)^p d\phi}{\int_{\mathbb{E}}(f+g)^r d\phi}+\frac{\int_{\mathbb{E}}(u+v)^p d\phi}{\int_{\mathbb{E}}(u+v)^r d\phi}\right)^{\frac{1}{p-r}}$$

$$\leq \left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}+\frac{\int_{\mathbb{E}} u^p d\phi}{\int_{\mathbb{E}} v^r d\phi}\right)^{\frac{1}{p-r}}+\left(\frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}+\frac{\int_{\mathbb{E}} v^p d\phi}{\int_{\mathbb{E}} v^r d\phi}\right)^{\frac{1}{p-r}}, \tag{3.2}$$

with equality if and only if the functions f and g are positively proportional and $\left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}, \frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}\right) = \mu\left(\frac{\int_{\mathbb{E}} u^p d\phi}{\int_{\mathbb{E}} u^r d\phi}, \frac{\int_{\mathbb{E}} v^p d\phi}{\int_{\mathbb{E}} v^r d\phi}\right)$, *where μ is a constant.*

Proof: By Lemma 3.1, for $0 \leq r \leq 1 \leq p$, we have

$$\left(\frac{\int_{\mathbb{E}}(f+g)^p d\phi}{\int_{\mathbb{E}}(f+g)^r d\phi}\right)^{\frac{1}{p-r}} \leq \left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}\right)^{\frac{1}{p-r}}+\left(\frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}\right)^{\frac{1}{p-r}}, \tag{3.3}$$

with equality if and only if the functions f and g are positively proportional, and

$$\left(\frac{\int_{\mathbb{E}}(u+v)^p d\phi}{\int_{\mathbb{E}}(u+v)^r d\phi}\right)^{\frac{1}{p-r}} = \left(\frac{\int_{\mathbb{E}} u^p d\phi}{\int_{\mathbb{E}} u^r d\phi}\right)^{\frac{1}{p-r}}+\left(\frac{\int_{\mathbb{E}} v^p d\phi}{\int_{\mathbb{E}} v^r d\phi}\right)^{\frac{1}{p-r}}, \tag{3.4}$$

From (3.3), (3.4), and by using Minkowski inequality, we obtain for $p > r+1$

$$\frac{\int_{\mathbb{E}}(f+g)^p d\phi}{\int_{\mathbb{E}}(f+g)^r d\phi}+\frac{\int_{\mathbb{E}}(u+v)^p d\phi}{\int_{\mathbb{E}}(u+v)^r d\phi}$$

$$\leq \left[\left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}\right)^{\frac{1}{p-r}}+\left(\frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}\right)^{\frac{1}{p-r}}\right]^{p-r}+\left[\left(\frac{\int_{\mathbb{E}} u^p d\phi}{\int_{\mathbb{E}} u^r d\phi}\right)^{\frac{1}{p-r}}+\left(\frac{\int_{\mathbb{E}} v^p d\phi}{\int_{\mathbb{E}} v^r d\phi}\right)^{\frac{1}{p-r}}\right]^{p-r}$$

$$\leq \left\{\left[\left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}\right)^{\frac{1}{p-r}}+\left(\frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}\right)^{\frac{1}{p-r}}\right]^{p-r}+\left[\left(\frac{\int_{\mathbb{E}} u^p d\phi}{\int_{\mathbb{E}} u^r d\phi}\right)^{\frac{1}{p-r}}+\left(\frac{\int_{\mathbb{E}} v^p d\phi}{\int_{\mathbb{E}} v^r d\phi}\right)^{\frac{1}{p-r}}\right]^{p-r}\right\}^{\frac{1}{p-r}}$$

$$\leq \left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}+\frac{\int_{\mathbb{E}} u^p d\phi}{\int_{\mathbb{E}} v^r d\phi}\right)^{\frac{1}{p-r}}+\left(\frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}+\frac{\int_{\mathbb{E}} v^p d\phi}{\int_{\mathbb{E}} v^r d\phi}\right)^{\frac{1}{p-r}}.$$

From the equality conditions of (3.3) and Minkowski inequality, it follows the equality in (3.2) holds if and only if f and g are positively proportional and $\left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}, \frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}\right) = \mu\left(\frac{\int_{\mathbb{E}} u^p d\phi}{\int_{\mathbb{E}} u^r d\phi}, \frac{\int_{\mathbb{E}} v^p d\phi}{\int_{\mathbb{E}} v^r d\phi}\right)$, where μ is a constant.

4 Inequalities for sums of volume quotient functions

Theorem 4.1 *If $K, L, D \in \mathcal{S}^n$ and D' is a dilated copy of D. If $0 \le r \le 1 \le r+1 < p$, then for $\varepsilon > 0$*

$$\left(Q_{\widetilde{W}_{n-p,n-r}(K\tilde{+}\varepsilon L)} + Q_{\widetilde{W}_{n-p,n-r}(D\tilde{+}\varepsilon D')}\right)^{\frac{1}{p-r}} \le \left(Q_{\widetilde{W}_{n-p,n-r}(K)} + Q_{\widetilde{W}_{n-p,n-r}(D)}\right)^{\frac{1}{p-r}} + \varepsilon\left(Q_{\widetilde{W}_{n-p,n-r}(L)} + Q_{\widetilde{W}_{n-p,n-r}(D')}\right)^{\frac{1}{p-r}}, \tag{4.1}$$

with equality if and only if K and L are dilates and $\frac{Q_{\widetilde{W}_{n-p,n-r}(K)}}{Q_{\widetilde{W}_{n-p,n-r}(L)}} = \frac{Q_{\widetilde{W}_{n-p,n-r}(D)}}{Q_{\widetilde{W}_{n-p,n-r}(D')}}$.

Proof: From (1.1) and (2.4), we have

$$\widetilde{W}_{n-p}(K\tilde{+}\varepsilon L) = \frac{1}{n}\int_{S^{n-1}} \rho(K\tilde{+}\varepsilon L, u)^p dS(u) = \frac{1}{n}\int_{S^{n-1}} (\rho(K,u) + \varepsilon\rho(L,u))^p dS(u). \tag{4.2}$$

Similarly

$$\widetilde{W}_{n-r}(K\tilde{+}\varepsilon L) = \frac{1}{n}\int_{S^{n-1}} (\rho(K,u) + \varepsilon\rho(L,u))^r dS(u), \tag{4.3}$$

Hence, from (4.2) and (4.3) and using Lemma 3.2, we have for $0 \le r \le 1 \le r+1 < p$

$$\begin{aligned}
\left(Q_{\widetilde{W}_{n-p,n-r}(K\tilde{+}\varepsilon L)} + Q_{\widetilde{W}_{n-p,n-r}(D\tilde{+}\varepsilon D')}\right)^{\frac{1}{p-r}} &= \left(\frac{\widetilde{W}_{n-p}(K\tilde{+}\varepsilon L)}{\widetilde{W}_{n-r}(K\tilde{+}\varepsilon L)} + \frac{\widetilde{W}_{n-p}(D\tilde{+}\varepsilon D')}{\widetilde{W}_{n-r}(D\tilde{+}\varepsilon D')}\right)^{\frac{1}{p-r}} \\
&= \left(\frac{\int_{S^{n-1}}(\rho(K,u)+\varepsilon\rho(L,u))^p dS(u)}{\int_{S^{n-1}}(\rho(K,u)+\varepsilon\rho(L,u))^r dS(u)} + \frac{\int_{S^{n-1}}(\rho(D,u)+\varepsilon\rho(D',u))^p dS(u)}{\int_{S^{n-1}}(\rho(D,u)+\varepsilon\rho(D',u))^r dS(u)}\right)^{\frac{1}{p-r}} \\
&\le \left(\frac{\int_{S^{n-1}}\rho(K,u)^p dS(u)}{\int_{S^{n-1}}\rho(K,u)^r dS(u)} + \frac{\int_{S^{n-1}}\rho(D,u)^p dS(u)}{\int_{S^{n-1}}(\rho(D,u)^r dS(u)}\right)^{\frac{1}{p-r}} \\
&\quad +\varepsilon\left(\frac{\int_{S^{n-1}}\rho(L,u)^p dS(u)}{\int_{S^{n-1}}\rho(L,u)^r dS(u)} + \frac{\int_{S^{n-1}}\rho(D',u)^p dS(u)}{\int_{S^{n-1}}(\rho(D',u)^r dS(u)}\right)^{\frac{1}{p-r}} \\
&= \left(\frac{\widetilde{W}_{n-p}(K)}{\widetilde{W}_{n-r}(K)} + \frac{\widetilde{W}_{n-p}(D)}{\widetilde{W}_{n-r}(D)}\right)^{\frac{1}{p-r}} + \varepsilon\left(\frac{\widetilde{W}_{n-p}(L)}{\widetilde{W}_{n-r}(L)} + \varepsilon\frac{\widetilde{W}_{n-p}(D')}{\widetilde{W}_{n-r}(D')}\right)^{\frac{1}{p-r}}. \\
&= \left(Q_{\widetilde{W}_{n-p,n-r}(K)} + Q_{\widetilde{W}_{n-p,n-r}(D)}\right)^{\frac{1}{p-r}} + \varepsilon\left(Q_{\widetilde{W}_{n-p,n-r}(L)} + Q_{\widetilde{W}_{n-p,n-r}(D')}\right)^{\frac{1}{p-r}}.
\end{aligned} \tag{4.4}$$

From the equality conditions of Lemma 3.2, the equality in (4.4) holds if and only if K and L are dilates and $\left(Q_{\widetilde{W}_{n-p,n-r}(K)}, Q_{\widetilde{W}_{n-p,n-r}(D)}\right) = \mu\left(Q_{\widetilde{W}_{n-p,n-r}(L)}, Q_{\widetilde{W}_{n-p,n-r}(D')}\right)$, where μ is a constant.

Let $n-p = i$ and $n-r = j$, in view of $0 \le r \le 1 \le r+1 < p$, then

$$0 \le r \le 1 \le r+1 < p \Rightarrow i < j-1 \le n-1 \le j \le n. \tag{4.5}$$

Taking for $n-p=i$ and $n-r=j$ in (4.1) and notice (4.5), (4.1) changes to the inequality in Theorem 1.1 stated in the introduction.

Theorem 4.2 *If $K, L \in \mathcal{S}^n$ and let $0 \le r \le n-1 \le r+n-1 < p$, then for $\varepsilon > 0$*

$$\left(Q_{\widetilde{W}_{n-p,n-r}(K\check{+}\varepsilon L)} + Q_{\widetilde{W}_{n-p,n-r}(D\check{+}\varepsilon D')}\right)^{\frac{n-1}{p-r}} \le \left(Q_{\widetilde{W}_{n-p,n-r}(K)} + Q_{\widetilde{W}_{n-p,n-r}(D)}\right)^{\frac{n-1}{p-r}} + +\varepsilon\left(Q_{\widetilde{W}_{n-p,n-r}(L)} + Q_{\widetilde{W}_{n-p,n-r}(D')}\right)^{\frac{n-1}{p-r}}, \tag{4.6}$$

with equality if and only if K and L are dilates and $\dfrac{Q_{\widetilde{W}_{n-p,n-r}(K)}}{Q_{\widetilde{W}_{n-p,n-r}(L)}} = \dfrac{Q_{\widetilde{W}_{n-p,n-r}(D)}}{Q_{\widetilde{W}_{n-p,n-r}(D')}}$.

Proof: From (2.4) and (2.6), we have

$$\widetilde{W}_{n-p}(K\check{+}\varepsilon L) = \frac{1}{n}\int_{S^{n-1}} \rho(K\check{+}\varepsilon L, u)^p dS(u) = \frac{1}{n}\int_{S^{n-1}} (\rho(K,u)^{n-1} + \varepsilon\rho(L,u)^{n-1})^{\frac{p}{n-1}} dS(u). \tag{4.7}$$

Similarly

$$\widetilde{W}_{n-r}(K\check{+}\varepsilon L) = \frac{1}{n}\int_{S^{n-1}} (\rho(K,u)^{n-1} + \varepsilon\rho(L,u)^{n-1})^{\frac{r}{n-1}} dS(u), \tag{4.8}$$

From (4.7) and (4.8) and in view of Lemma 3.2, the same as the proof of Theorem 4.1 and with appropriate transformation, we may get the inequality in Theorem 4.2. Here, we omit the details.

Let $n-p=i$ and $n-r=j$, in view of $0 \le r \le n-1 \le r+n-1 < p$, then

$$n \ge j \ge 1 \ge j-n+1 > i. \tag{4.9}$$

Taking for $n-p=i$ and $n-r=j$ in (4.6) and notice (4.9), (4.6) changes to the inequality in Theorem 1.2 stated in the introduction.

Theorem 4.3 *If $K, L, D \in \mathcal{S}^n$ and D' is a dilated copy of D. If $0 \le r \le n+1 \le r+n+1 < p$, then for $\varepsilon > 0$*

$$\left(\frac{Q_{\widetilde{W}_{n-p,n-r}(K\hat{+}\varepsilon L)}}{V(K\hat{+}\varepsilon L)^{\frac{p-r}{n+1}}} + \frac{Q_{\widetilde{W}_{n-p,n-r}(D\hat{+}\varepsilon D')}}{V(D\hat{+}\varepsilon D')^{\frac{p-r}{n+1}}}\right)^{\frac{n+1}{p-r}} \le \left(\frac{Q_{\widetilde{W}_{n-p,n-r}(K)}}{V(K)^{\frac{p-r}{n+1}}} + \frac{Q_{\widetilde{W}_{n-p,n-r}(D)}}{V(D)^{\frac{p-r}{n+1}}}\right)^{\frac{n+1}{p-r}} + +\varepsilon\left(\frac{Q_{\widetilde{W}_{n-p,n-r}(L)}}{V(L)^{\frac{p-r}{n+1}}} + \frac{Q_{\widetilde{W}_{n-p,n-r}(D')}}{V(D')^{\frac{p-r}{n+1}}}\right)^{\frac{n+1}{p-r}}, \tag{4.10}$$

with equality if and only if K and L are dilates and

$$\frac{V(L)^{\frac{n+1}{p-r}} Q_{\widetilde{W}_{n-p,n-r}(K)}}{V(K)^{\frac{n+1}{p-r}} Q_{\widetilde{W}_{n-p,n-r}(L)}} = \frac{V(D')^{\frac{n+1}{p-r}} Q_{\widetilde{W}_{n-p,n-r}(D)}}{V(D)^{\frac{n+1}{p-r}} Q_{\widetilde{W}_{n-p,n-r}(D')}}.$$

Proof: From (2.4), (2.7) and in view of $\xi = V(K\hat{+}L)$, we have

$$\widetilde{W}_{n-p}(K\hat{+}\varepsilon L) = \frac{1}{n}\int_{S^{n-1}} \rho(K\hat{+}\varepsilon L, u)^p dS(u)$$

$$= \frac{1}{n}\int_{S^{n-1}}(\xi V^{-1}(K)\rho(K,u)^{n+1}+\varepsilon\xi V^{-1}(L)\rho(L,u)^{n+1})^{\frac{p}{n+1}}dS(u). \tag{4.11}$$

Similarly

$$\widetilde{W}_{n-r}(K\hat{+}\varepsilon L) = \frac{1}{n}\int_{S^{n-1}}(\xi V^{-1}(K)\rho(K,u)^{n+1}+\varepsilon\xi V^{-1}(L)\rho(L,u)^{n+1})^{\frac{r}{n+1}}dS(u), \tag{4.12}$$

From (4.11) and (4.12) and in view of Lemma 3.2, the same as the proof of Theorem 4.1 and with appropriate transformation, we may get the inequality in Theorem 4.3. Here, we omit the details.

Let $n-p=i$ and $n-r=j$, in view of $0\leq r\leq n+1\leq r+n+1<p$, then

$$i<j-n-1\leq -1\leq j\leq n. \tag{4.13}$$

Taking for $n-p=i$ and $n-r=j$ in (4.10) and notice (4.13), (4.10) changes to the inequality in Theorem 1.3 stated in the introduction.

We finally remark that inequalities for quotient function were given in [8-9], volume sum or difference functions were given in [10-12].

5 Applications

As applications of our results, we prove the Minkowski inequality for volume sum and further obtained the classical dual Minkowski inequality.

Theorem 5.1 (Dual Minkowski inequality) *If $K,L\in\mathcal{S}^n$, then*

$$\widetilde{V}_1(K,L)^n\leq V(K)^{n-1}V(K)^n, \tag{5.1}$$

with equality if and only if K and L are dilates.

Theorem 5.2 (Dual Minkowski inequality for volumes sum) *If $K,L,D\in\mathcal{S}^n$ and D' is a dilated copy of D, then*

$$\left(\widetilde{V}_1(K,L)+\widetilde{V}_1(D,D')\right)^n\leq (V(K)+V(D))^{n-1}(V(L)+V(D')), \tag{5.2}$$

with equality if and only if K and L are dilates and $(V(K),V(D))=\lambda(V(L),V(D'))$.

Proof: From (2.5), we have

$$n(\widetilde{V}_1(K,L)+\widetilde{V}_1(D,D'))=\lim_{\varepsilon\to 0}\frac{[V(K\tilde{+}\varepsilon L)+V(D\tilde{+}\varepsilon D')]-(V(K)+V(D))}{\varepsilon}.$$

By using the case $j=n$ and $i=0$ of (1.5), we obtain

$$n(\widetilde{V}_1(K,L)+\widetilde{V}_1(D,D'))\leq\lim_{\varepsilon\to 0}\frac{[(V(K)+V(D))^{1/n}+\varepsilon(V(L)+V(D'))^{1/n}]^n-(V(K)+V(D))}{\varepsilon}, \tag{5.3}$$

with equality if and only if K and L are dilates and $(V(K), V(D)) = \lambda(V(L), V(D'))$.

On the other hand, from (5.3) and in view of L'Hôpital's rule, we have

$$\widetilde{V}_1(K,L)+\widetilde{V}_1(D,D') \leq \lim_{\varepsilon\to 0}[(V(K)+V(D))^{1/n}+\varepsilon(V(L)+V(D'))^{1/n}]^{n-1}\cdot(V(L)+V(D'))^{1/n}$$

$$= (V(K)+V(D))^{(n-1)/n}(V(L)+V(D'))^{1/n},$$

with equality if and only if K and L are dilates and $(V(K), V(D)) = \lambda(V(L), V(D'))$.

Remark 5.3 Let D and D' be single points in (5.2), (5.2) changes to the classical dual Minkowski inequality (5.1).

References

[1] E. F. Bechenbach, R. Bellman, *Inequalities*, Springer-Verlag, Berlin-Göttingen, Heidelberg, (1961).

[2] R. J. Gardner, *Geometric Tomography*, Cambridge Univ. Press, New York, (1996).

[3] E. Lutwak, Dual mixed volumes, *Pacific J. Math.*, **58**, 531-538 (1975).

[4] E. Lutwak, Intersection bodies and dual mixed volumes, *Adv. Math.*, **71**, 232-261 (1988).

[5] E. Lutwak, Centroid bodies and dual mixed volumes, *Proc. London Math. Soc.*, **60**, 365-391 (1990).

[6] R. Schneider, Convex Bodies: The Brunn-Minkowski Theory, Cambridge Univ. Press, 1993).

[7] C.-J. Zhao, On volume quotient functions, *Indagations Mathematicae.*, **24**, 57-67 (2013).

[8] C.-J. Zhao, Quotient functions of dual quermassintegrals, *Quaestiones Math.*, **38**, 271-283 (2015).

[9] C.-J. Zhao, On radial and polar Blaschke-Minkowski homomorphisms, *Proc. Amer. Math. Soc.*, **141**, 667-675 (2013).

[10] C.-J. Zhao, Volume sums of polar Blaschke-Minkowski homomorphisms, *Proc. Indian Acad. Sci.*, **125**, 209-219 (2015).

[11] C.-J. Zhao, On Blaschke-Minkowski Homomorphisms and Radial Blaschke-Minkowski Homomorphisms , *J. Geom. Anal.*, **26**, 1523-1538 (2016).

[12] C.-J. Zhao, The dual difference Aleksandrov–Fenchel inequality, *Indagationes Math.*, **28**, 362-371 (2017).

Received: 02.03.2018
Revised: 02.04.2018
Accepted: 15.04.2018

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