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Algorithms in the classical Néron Desingularization

by

Asma Khalid[(1)], Adrian Popescu[(2)], Dorin Popescu[(3)]

Abstract

We give algorithms to construct the Néron Desingularization and the easy case from [3] of the General Néron Desingularization.

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Introduction

Let $A \subset A'$ be an unramified extension of discrete valuation rings (shortly DVR), that is the generator x of the maximal ideal of A generates also the maximal ideal of A'. Suppose that the induced field extensions $A/(x) \to A'/xA'$, $\mathrm{Fr}(A) \to \mathrm{Fr}(A')$ are separable, in other words the inclusion $A \to A'$ is regular. Néron [6] (see here Theorem 1) proved that every sub-A-algebra $B \subset A'$ of finite type can be embedded in a regular subring $D \subset A'$, essentially of finite type over A. Moreover, D could be chosen contained in $\mathrm{Fr}(B)$, that is D could be seen as a desingularization of B made with respect of the inclusion $B \to A'$.

By Jacobian Criterion we see that D is a localization of a smooth sub-A-algebra $C \subset A'$ containing B. Suppose that $B = A[Y]/(f)$, $Y = (Y_1, \dots, Y_m)$, $f = (f_1, \dots, f_p)$ and the inclusion $v : B \to A'$ is given by $Y \to y \in A'^m$. Replacing B by C means in fact to substitute the system of polynomial equations f by a system of polynomials in more variables where it is possible to apply the Implicit Function Theorem. If A' is Henselian then this could be very helpful. One example is the applications of Néron Desingularization to the so-called the Artin approximation property of algebraic power series rings with coefficients in an excellent Henselian DVR (see [1]). Actually for this aim it is not necessary the injectivity of v and $C \subset A'$. Ploski [9] proved that if $A = \mathbb{C}\{x\}$, $x = (x_1, \dots, x_n)$, $B = \mathbb{C}\{x, Y\}/(f)$ for some complex convergent power series f in x, Y then a A-morphism $v : B \to \mathbb{C}[[x]]$ factors through an A-algebra of type $\mathbb{C}\{x, Z\}$ for some new variables Z.

Ploski's result gave the idea of the so-called the General Néron Desingularization, (see [12], [13], [16]) which says that for special (that is regular) morphisms $A \to A'$ of Noetherian rings any A-morphism $v : B \to A'$ with B a A-algebra of finite type, factors through a smooth A-algebra C, that is v is a composite A-morphism $B \to C \to A'$. This desingularization is constructive when $\dim A = \dim A' = 1$ (see [10], [8], [4]).

In [3] an easy proof of the General Néron Desingularization is given in the case when $\dim A = 0$, $\dim A' = 1$ and A, A' have the same residue field. Here it is our goal to provide some algorithms in this direction. We start presenting an algorithm to construct the Néron

Desingularization in the case when A, A' are DVR. In Section 1 we recall Néron's proof in the idea presented by Artin [1, Theorem 4.5] in a special case (see also [11, Theorem 5.1, page 683], [7, pages 171-172]). The present proof is taken from [14] and we give it here for the sake of completeness. In Sections 2, 3 we present shortly a constructive proof and give an algorithm to construct Néron Desingularization in the easier case when A, A' have the same residue field as in [15, Theorem 10]. We do not implement our algorithm. The implementation done in [10] gives a hard General Néron Desingularization because it is constructed in a general case, that is when A, A' are Noetherian local domains of one dimension and with possible different residue fields and so involving some extra equations and new variables. In the last sections we present an algorithm to construct the easy case from [3] of the General Néron Desingularization.

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1 Preliminaries on Néron Desingularization

Let $A \subset A'$ be the unramified extension of DVR as above and B a finite type sub-A-algebra of A', let us say $B = A[y]$ for some elements $y = (y_1, \dots, y_N)$ of A'. Let $\sigma : A[Y] \to A'$ be the A-morphism $Y \to y$, $I = \text{Ker } \sigma$ and $f = (f_1, \dots, f_r)$ be a regular system of parameters of the regular local ring $A[Y]_I$. Thus $r = \text{height } I$. By separability of Fr $A \subset$ Fr B we see that the Jacobian matrix $(\partial f/\partial Y)$ has a $r \times r$-minor M such that $M(y) \neq 0$.

The minimum valuation $l(B)$ of the values of the $r \times r$-minors of the Jacobian matrix $(\partial f_i/\partial Y_j)$ in y is an invariant of B. Indeed, tensorizing with A' the module of differentials $\Omega_{B/A}$ we get the following exact sequence

$$A' \otimes_B I/I^2 \xrightarrow{\phi} \sum_{i \in [N]} A' dY_i \to A' \otimes_B \Omega_{B/A} \to 0,$$

where $[N] = \{1, \dots, N\}$ and ϕ is given by $g \to \sum_i (\partial g/\partial Y_i)(y) dY_i$. Applying the Invariant Factor Theorem we see that the finite type A'-module $A' \otimes_B \Omega_{B/A}$ is isomorphic with $(\oplus_{i=1}^t A'/(x^{a_i})) \oplus A'^k$ for some $t, k, a_i \in \mathbb{N}$. Thus we may suppose that Im ϕ has the diagonal form, where the only non zero elements are (x^{a_i}), that is $\sum_{i \in [t]} a_i$ is the minimum valuation of the values of the $t \times t$-minors of the Jacobian matrix $(\partial f_i/\partial Y_j)$ in y. It follows that $t = r$ and $l(B) = \sum_{i \in [r]} a_i$.

If $l(B) = 0$ then we may choose $f_1, \dots, f_r$ such that the Jacobian matrix $(\partial f/\partial Y)$ has an $r \times r$-minor M such that $M(y)$ is invertible in A', that is $M \notin q = \sigma^{-1}(xA')$. By Jacobian Criterion [5, Theorem 30.4] we see that $B_{xA' \cap B} \cong (A[Y]/I)_q$ is regular.

Theorem 1 (Néron)**.** *If $A \subset A'$ induces separable field extensions on fraction and residue fields then there exists a sub-A-algebra of finite type C of A' containing B such that $C_{xA' \cap C}$ is a regular local ring.*

Proof. If $B_{xA' \cap B}$ is regular then we may take for C a localization of B. Suppose that $B_{xA' \cap B}$ is not regular. Then $f_1, \dots, f_r$ cannot be part of a regular system of parameters of $A[Y]_q$, and so $f_1, \dots, f_r$ induce a linearly dependent system of elements $\bar{f}$ in $qA[Y]_q/q^2A[Y]_q$. Also note that $l(B) > 0$. Assume that $\bar{f}_1, \dots, \bar{f}_e$, $e < r$, induce modulo q^2 a maximal linearly independent subsystem of $\bar{f}$ in $qA[Y]_q/q^2A[Y]_q$. Then we may complete $\bar{f}_1, \dots, \bar{f}_e$

with $\bar{h}_1, \ldots, \bar{h}_s$ up to a regular system of parameters of the regular ring $A/(x)[Y]_q$, thus $e+s = \text{height } q > \text{height } I = r$. Since the field extension $A/(x) \to Q(A[Y]/q)$ is separable we see that the Jacobian matrix $((\partial f_i/\partial Y_j)|(\partial h_k/\partial Y_j))_{i\in[e],k\in[s],j\in[N]}$ has rank $e+s$ modulo q. Note that $\bar{f}_i$ is linearly dependent on $\bar{f}_1, \ldots, \bar{f}_e$ modulo q^2 for $e < i \leq r$ and there exist $E_i, L_{ic} \in A[Y]$, $E_i \notin q$ such that

$$E_i f_i - \sum_{c=1}^{e} L_{ic} f_c \in q^2.$$

Moreover we may choose E_i, L_{ic} such that

$$E_i f_i - \sum_{c=1}^{e} L_{ic} f_c \in (x, h_1, \ldots, h_s)^2,$$

for some $h_k \in A[Y]$ lifting $\bar{h}_k$. Set $g_k = xT_k - h_k \in A[Y,T]$, $k \in [s]$, $T = (T_1, \ldots, T_s)$. Since $(x, h_1, \ldots, h_s) = (x, g_1, \ldots, g_s)$ we get

$$E_i f_i - \sum_{c=1}^{e} L_{ic} f_c = \sum_{k,k'=1}^{s} S_{ikk'} g_k g_{k'} + x \sum_{k=1}^{s} F_{ik} g_k + x^2 R_i,$$

for some $R_i, S_{ikk'}, F_{ik} \in A[Y,T]$. By construction, $h(y) \equiv 0$ modulo x, that is there exists $t \in A'^s$ such that $h(y) = xt$ and so $g_k(y,t) = 0$. It follows that $R_i(y,t) = 0$. Taking derivations above we get

$$E_i(y)(\frac{\partial f_i}{\partial Y_j})(y) - \sum_{c=1}^{e} L_{ic}(y)(\frac{\partial f_c}{\partial Y_j})(y) = x \sum_{k=1}^{s} F_{ik}(y,t)(\frac{\partial g_k}{\partial Y_j})(y,t) + x^2(\frac{\partial R_i}{\partial Y_j})(y,t),$$

$$0 = x \sum_{k=1}^{s} F_{ik}(y,t)(\frac{\partial g_k}{\partial T_{k'}})(y,t) + x^2(\frac{\partial R_i}{\partial T_{k'}})(y,t),$$

for $e < i \leq r$. Note that the Jacobian matrix J associated to the $(r+s)$-system $(f_1, \ldots, f_e, E_{e+1}f_{e+1}, \ldots, E_r f_r, g_1, \ldots, g_s)$ has a minor M whose value in (y,t) has the valuation $\leq l(B) + s$, because $E_i(y) \notin xR'$ and the Jacobian matrix $(\partial g_k/\partial T_{k'})$ is $x\mathrm{Id}_s$, Id_s being the identity matrix.

Moreover the valuation of $M(y,t)$ can be $\leq l(B)+r-e$. Indeed, by elementary transformations on the first r-lines and the first N-columns of $J(y,t)$ we get a matrix G with a diagonal $r \times N$-block on the above lines and rows. Since $(\partial(f_i,h_k)/\partial Y_j)$, $i \in [e], k \in [s]$, $j \in [N]$ has rank $e+s$ modulo q, we may choose an invertible $(s-r+e) \times (s-r+e)$-minor U of the block P of $(\partial h_k/\partial Y_j)$ given on the columns $r+1, \ldots, e+s$ after renumbering Y. Assume that U is given on the lines $r+j_1, \ldots, r+j_{s-r+e}$ and let $\{u_1, \ldots, u_{r-e}\} = [s] \setminus \{j_1, \ldots, j_{s-r+e}\}$. G has the following form

$$\begin{pmatrix} \mathrm{Id}_e & 0 & 0 & 0 & 0 \\ 0 & D & 0 & 0 & 0 \\ \square & \square & P & \square & x\mathrm{Id}_s \end{pmatrix}$$

where D is a diagonal matrix with $x^{a_{e+1}}, \ldots, x^{a_r}$ on the diagonal and $l(B) = \sum_{i=1}^{r-e} a_i$.

Then the $(r+s)\times(r+s)$-minor of G given on the columns $1,\dots,r,r+1,\dots,e+s,N+u_1,\dots,N+u_{r-s}$ has the form $x^{l(B)+r-e}\det U$, which has the valuation $l(B)+r-e$.

On the other hand, adding to the block of J given by the rows $e+1,\dots,r$ the first e rows multiplied on left in order with $-L_{e+1,c}(y,t),\dots,-L_{r,c}(y,t)$ and the last s rows multiplied on left in order with $-xF_{e+1,k}(y,t),\dots,-xF_{r,k}(y,t)$ we get the Jacobian matrix associated to $(f_1,\dots,f_e,x^2R_{e+1},\dots,x^2R_r,g_1,\dots,g_s)$. Thus we found a system of polynomials $(f_1,\dots,f_e,R_{e+1},\dots,R_r,g_1,\dots,g_s)$ in the kernel Q of the A-morphism $A[Y,T]\to A'$ such that its Jacobian matrix has in (y,t) a maximal minor of valuation $l(B)-r+e<l(B)$. Note that $C=B[t]$ has the same dimension with B having the same fraction field and so height $Q=N+s-\dim C=N+s-\dim B=r+s$. It follows that $l(C)<l(B)$. Using this construction by recurrence we arrive finally to a C with $l(C)=0$ which is enough. ◻

Remark 1. The above proof is not constructive since it is based on the induction on $l(B)$. This is the reason to recall a constructive proof in the next section.

2 A Constructive Néron Desingularization

A ring morphism $u: A\to A'$ of Noetherian rings has *regular fibers* if for all prime ideals $P\in\operatorname{Spec} A$ the ring A'/PA' is a regular ring, i.e. its localizations are regular local rings. It has *geometrically regular fibers* if for all prime ideals $P\in\operatorname{Spec} A$ and all finite field extensions K of the fraction field of A/P the ring $K\otimes_{A/P}A'/PA'$ is regular. If $A\supset\mathbb{Q}$ then regular fibers of u are geometrically regular. We call u *regular* if it is flat and its fibers are geometrically regular. A regular morphism is *smooth* if it is finitely presented and it is *essentially smooth* if it is a localization of a finitely presented morphism.

The Néron Desingularization presented in the first section is a kind of *smoothification* with respect to the inclusion $B\to A'$. Now we want to find a smoothification with respect to an A-morphism $v: B\to A'$ in the case when $A=k[x]_{(x)}$, $A'=k[[x]]$, k being a field and x a variable. Since we cannot give to a computer the whole informations concerning the coefficients of the formal power series $v(Y_j)$, it is better to work from the beginning with A-morphisms $v: B\to A'/x^mA'$ for some $m\gg 0$. This is done in [15, Theorem 10]. Next we recall in sketch this construction since we need it for the algorithm. In [10], [8] such algorithms are presented and even implemented for the so-called the General Néron Desingularization but in our case the things are simpler.

If $f=(f_1,\dots,f_r)$, $r\le n$ is a system of polynomials from I then we consider an $r\times r$-minor M of the Jacobian matrix $(\partial f_i/\partial Y_j)$. Let $c\in\mathbb{N}$. Suppose that there exist an A-morphism $v: B\to A'/(x^{2c+1})$ and $L\in((f):I)$ such that $A'v(LM)=(x)^c/(x)^{2c}$, where for simplicity we write $v(LM)$ instead $v(LM+I)$. We may assume that $M=\det((\partial f_i/\partial Y_j)_{i,j\in[r]})$.

Theorem 2 (Popescu [15])**.** *There exists a B-algebra C which is smooth over A such that every A-morphism $v': B\to A'$ with $v'\equiv v$ modulo x^{2c+1} (that is $v'(Y)\equiv v(Y)$ modulo x^{2c+1}) factors through C.*

Proof. Since $A/(x^{2c+1})\cong A'/(x^{2c+1})$, $y'\in A^n$ can be choose such that $v(Y)=y'+(x^{2c+1})$. Set $P=LM$ and $d=P(y')$. We have $dA=x^cA$.

Let H be the $n \times n$-matrix obtained adding down to $(\partial f/\partial Y)$ as a border the block $(0|\mathrm{Id}_{n-r})$. Let G' be the adjoint matrix of H and $G = LG'$. We have $GH = HG = LM\mathrm{Id}_n = P\mathrm{Id}_n$ and so $d\mathrm{Id}_n = P(y')\mathrm{Id}_n = G(y')H(y')$.

Let $h = Y - y' - dG(y')T$, where $T = (T_1, \ldots, T_n)$ are new variables. Since $Y - y' \equiv dG(y')T$ modulo h and $f(Y) - f(y') \equiv \sum_j \partial f/\partial Y_j(y')(Y_j - y'_j)$ modulo higher order terms in $Y_j - y'_j$, by Taylor's formula we see that we have

$$f(Y) - f(y') \equiv \sum_j d\partial f/\partial Y_j(y')G_j(y')T_j + d^2Q = dP(y')T + d^2Q = d^2(T + Q)$$

modulo h, where $Q \in T^2A[T]^r$. This is because $(\partial f/\partial Y)G = (P\mathrm{Id}_r|0)$. We have $f(y') = d^2a$ for some $a \in xA^r$. Set $g_i = a_i + T_i + Q_i$, $i \in [r]$ and $E = A[Y,T]/(I, g, h)$. Then there exists s, s' in $1 + (T)$ as it is proved in [15, Theorem 10] such that $C := E_{ss'} \cong (A[T]/(g))_{ss'}$ is smooth . Moreover, v' factors through C. □

3 Néron Desingularization Algorithm

Neron-Desingularization_Dim1

Input: $N \in \mathbb{Z}_{>0}$ a bound. $A = k[x]_{(x)}$, k being a field, $A' = k[[x]]$, $B = A[Y]/I$, $I = (f_1, \ldots, f_q), f_i \in k[x, Y], Y = (Y_1, \ldots, Y_n)$, $v : B \to A'$ an A-morphism and $y' \in k[x]^n$ approximations mod $(x)^N$ of $v(Y)$.

Output: (C, π) given by the following data: $C = (A[Z]/(L))_{hM}$ standard smooth, $Z = (Z_1, \ldots, Z_p)$, $L = (b_1, \ldots, b_{q'}) \subset k[x, Z]$, $h \in k[x, Z]$, M a $q \times q$-minor of $(\partial b_i/\partial Z_j)$, $\pi : B \to C$ an A-morphism given by $\pi(Y_1), \ldots, \pi(Y_n)$ factorizing v, or the message "y', N are not well chosen".

1. Compute $f := (f_1, \ldots, f_r)$ in I such that $v(((f) : I)\Delta_f) \not\subset (x)^N$
2. Reorder the variables Y such that for $M := \det(\partial f_i/\partial Y_j)_{i,j \in [r]})$ and a suitable $L \in (f) : I$ we have $v(LM) \notin (x)^N$
3. $P := LM$; $d := P(y')$; $c := ord(d)$
4. If $2c + 1 > N$, return "y', N are not well chosen"
5. Complete $(\partial f_i/\partial Y_j)_{i \le r}$ with $(0|(\mathrm{Id}_{n-r}))$ in order to obtain a square matrix H
6. Compute G' the adjoint matrix of H and $G := LG'$
7. $h = Y - y' - dG(y')T,\ T = (T_1, \ldots, T_n)$
8. Compute $Q \in T^2A[T]^r$ such that
   $f(Y) - f(y') = \sum_j d(\partial f/\partial Y_j)(y')G_j(y')T + d^2Q$
9. Compute $a \in xA^r$ such that $f(y') = d^2a$

10. For $i = 1$ to r, $g_i = a_i + T_i + Q_i$
11. $E := A[Y, T]/(I, g, h)$
12. Compute s the $r \times r$ minor defined by the first r columns of $(\partial g/\partial T)$
13. Compute s' such that $P(y' + dG(y')T) = ds'$
14. return $C := E_{ss'}$ and the canonical map $\pi : B \to C$.

Example 1. Let $N = 10$, $A = \mathbb{Q}[x]_{(x)}$, $A' = \mathbb{Q}[[x]]$, $B = A[Y_1, Y_2, Y_3, Y_4]/(f_1, f_2)$, $f_1 = Y_1^3 - Y_2^2, f_2 = 3{Y_1}^2Y_3 - 2Y_2Y_4$, $r = 2$, $v : B \to \mathbb{Q}[[x]], v(Y_1) = x^2u_1$, $v(Y_2) = x^3u_2, v(Y_3) = v_1, v(Y_4) = xv_2$ where u_1, v_1 are two formal power series from $\mathbb{Q}[[x]]$ which are algebraically independent over $\mathbb{Q}(x)$, $u_1(0) = v_1(0) = 1$ and u_2, v_2 are defined as it follows. By the Implicit Function Theorem there exists $u_2 \in \mathbb{Q}[[x]]$ such that $u_2^2 = u_1^3$. Set $v_2 = (3/2)u_1^2v_1u_2^{-1}$ and $L = 1$. Now we follow the steps of algorithm

We get the following minors: $M_1 = 6Y_1^2Y_4 + 12Y_1Y_2Y_3, M_2 = 9{Y_1}^4, M_3 = 6{Y_1}^2Y_2$, $M_4 = 4{Y_2}^2, M_5 = 0$. Note that $v(M_i) \in (x)$ for all the minors M_i. We choose M_i for which the valuation of $M_i(y)$ is minimum, in this case M_1 which is maped by v in $x^5(6u_1^2v_2 + 12u_1u_2v_1)$. Then $c = 5$ and $2c + 1 > 10$.

Output: y', N are not well chosen.

Example 2. Let $A = \mathbb{Q}[x]_{(x)}$, $A' = \mathbb{Q}[[x]]$, $B = A[Y_1, Y_2, Y_3, Y_4]/(f_1, f_2)$, $f_1 = Y_1^3 - Y_2^2, f_2 = 3{Y_1}^2Y_3 - 2Y_2Y_4$. The only difference from Example 1 consists in the map $v : B \to \mathbb{Q}[[x]]$, $v(Y_1) = u_1, v(Y_2) = u_2$, $v(Y_3) = v_1$, $v(Y_4) = v_2$, where u_1, u_2, v_1, v_2, L are as in Example 1.

We now follow the steps of algorithm. We get the following minors: $M_1 = 6Y_1^2Y_4 + 12Y_1Y_2Y_3, M_2 = 9{Y_1}^4, M_3 = 6{Y_1}^2Y_2, M_4 = 4{Y_2}^2, M_5 = 0$. Note that $v(LM_i) \notin (x)$ for all the minors M_i, $i < 5$. We take $M = 4Y_2^2$, which is mapped by v in $4u_2^2$ and thus $v(M) \notin (x)$ is invertible, that is $c = 0$. So $C = B_M \cong (A[Y_1, Y_2]/(Y_1^3 - Y_2^2))_{4Y_2^2}$ is smooth over A.

Example 3. Let $A = \mathbb{Q}[x]_{(x)}$, $A' = \mathbb{Q}[[x]]$, $B = A[x^2u_1, x^3u_2, v_1, xv_2]$, where u_1,u_2,v_1,v_2 are as in Example 1.

Let $\phi : A[Y_1, Y_2, Y_3, Y_4] \to B$ be given by $Y_1 \to x^2u_1$, $Y_2 \to x^3u_2$, $Y_3 \to v_1$, $Y_4 \to xv_2$. Then Ker ϕ contains $g_1 = Y_1^3 - Y_2^2$, $g_2 = 3Y_1^2Y_3 - 2Y_2Y_4$. Set $a_1 = 27Y_2Y_3^3 - 8Y_4^3$, $a_2 = 9Y_1Y_3^2 - 4Y_4^2$, $a_3 = 2Y_1Y_4 - 3Y_2Y_3$. The minimal prime ideals of (g_1, g_2) are $P_1 = (g_1, g_2, a_1, a_2, a_3)$, $P_2 = (Y_1, Y_2)$, both ideals having the height ≥ 2 because the great common divisor of g_1, g_2 is one. Note that height Ker $\phi = 2$ because $\dim(A[Y]/\text{Ker } \phi) = \dim(B) = 2$, u_1, v_1 being algebraically independent over $\mathbb{Q}[x]$.

We have Ker $\phi \supset P_1$ since Ker $\phi \supset (g_1, g_2)$ and $Y_1 \notin$ Ker ϕ. In fact Ker $\phi = P_1$ because height$P_1 \geq$ height Ker ϕ as above. Note that $Y_2^2a_2 \in (g_1, g_2)$, $Y_2a_3 \in (g_1, g_2)$, $Y_2^3a_1 \in (g_1, g_2)$, which implies $Y_2^3\text{Ker } \phi \subset (g_1, g_2)$ and so $Y_2^3 \in ((g_1, g_2) : \text{Ker } \phi)$. Here $L = Y_2^3$.

The Jacobian matrix $(\partial g_k/\partial Y_i)$ contains a 2×2-minor $M = 4Y_2^2$ and it follows that $Y_2 \in H_{B/A}$. So $\phi(M((g_1, g_2) : \text{Ker } \phi))$ contains $\phi(Y_2^5) = 4x^{15}u_2^5$. Taking $c = 15$ we see that $(\phi(M((g_1, g_2) : \text{Ker } \phi))) = (x^c)$.

Take $E = A[u_1, u_2, v_1, v_2]$. The kernel of the map $\psi : A[U_1, U_2, V_1, V_2] \to E$ given by $U_i \to u_i$, $V_i \to v_i$, contains the polynomials $h_1 = U_1^3 - U_2^2$, $h_2 = 3{U_1}^2V_1 - 2U_2V_2$ and we set

$b_1 = 27U_2V_1^3 - 8V_2^3$, $b_2 = 9U_1V_1^2 - 4V_2^2$, $b_3 = 2U_1V_2 - 3U_2V_1$. The minimal prime ideals of (h_1, h_2) are $Q_1 = (h_1, h_2, b_1, b_2, b_3)$, $Q_2 = (U_1, U_2)$. As above we see that Ker $\psi = Q_1$ since $U_1 \notin$ Ker ψ and so ψ induces the isomorphism $A[U_1, U_2, V_1, V_2]/(h_1, h_2, b_1, b_2, b_3) \cong E$. Note that $U_2^2 b_2 \in (h_1, h_2)$, $U_2 b_3 \in (h_1, h_2)$, $U_2^3 b_1 \in (h_1, h_2)$ which implies U_2^3Ker $\psi \subset (h_1, h_2)$ and so $U_2^3 \in ((h_1, h_2) : \text{Ker } \psi)$.

Now the Jacobian matrix $\partial h/\partial(U_i, V_i)$ contains a minor $M' = 4U_2^2$ and so $U_2 \in H_{E/A}$. Note that $\psi(M'((h_1, h_2) : \text{Ker } \phi))$ contains $\psi(4U_2^5)$ which is mapped by ψ in $4u_2^5 \notin (x)$. Then v factors through the smooth A-algebra

$$C = (A[U_1, U_2, V_1, V_2]/(h_1, h_2))_{U_2} \cong (A[U_1, U_2, V_1]/(h_1))_{U_2}$$

because v is the composite map

$$B \cong A[Y_1, Y_2, Y_3, Y_4]/(g_1, g_2, a_1, a_2, a_3) \xrightarrow{\rho} E \to C \to \mathbb{Q}[[x]],$$

where ρ is given by $Y_1 \to x^2u_1$,$Y_2 \to x^3u_2$, $Y_3 \to v_1$, $Y_4 \to xv_2$.

Remark 2. In Example 3, we gave a General Néron Desingularization using no algorithm. Applying our algorithm we will get a more complicated General Néron Desingularization. Example 4 illustrates the whole construction of the proof from Theorem 2.

Example 4. Let $N = 7$ be a bound. Let $A = \mathbb{Q}[x]_{(x)}$, $A' = \mathbb{Q}[[x]]$, $B = A[Y_1, Y_2]/(f)$, $f = Y_1^3 - Y_2^2$, and $v : B \to A'$ be a morphism given by $Y_1 \to x^2u_1$, $Y_2 \to x^3u_2$, where u_1, u_2, are as in Example 1. Let $u_1' = \sum_{i=0}^{7} \frac{x^i}{i!}$ and compute u_2' as in the previous example. Let y' be given by u_1' and u_2'. Now we follow the steps of algorithm:

1. $f = f$
2. $Y = (Y_1, Y_2)$. Actually the order taken in the algorithm was (Y_2, Y_1). Among the minors $M_1 = 3Y_1^2, M_2 = 2Y_2$, we choose $M = 2Y_2$; $L = 1$ and $v(LM) = 2x^3u_2' \notin (x)^7$
3. $P = 2Y_2$, $P(y') = 2x^3u_2'$, to avoid complexity we take $d = x^3$; $c = 3$.
4. $2c + 1 = 7 = N$
5. $H = \begin{pmatrix} 3Y_1^2 & -2Y_2 \\ 1 & 0 \end{pmatrix}$
6. $G = LG' = \begin{pmatrix} 0 & 2Y_2 \\ -1 & 3Y_1^2 \end{pmatrix}$
7. $h_1 = Y_1 - x^2u_1' - 2x^6u_2'T_2$,
   $h_2 = Y_2 - x^3u_2' + x^3T_1 - 3x^7u_1'^2T_2$
8. $Q = -2T_1^2 + 12x^4u_1'^2T_1T_2 + (24x^8u_2'^2u_1' - 18x^8u_1'^4)T_2^2 + 48x^{12}u_2'^3T_2^3$
9. $f(y') = x^6 \cdot a$ where $a = x\alpha \in xA$

10. $g = x\alpha + 2u_2'T_1 - 2T_1^2 + 12x^4u_1'^2T_1T_2 + (24x^8u_2'^2u_1' - 18x^8u_1'^4)T_2^2 + 48x^{12}u_2'^3T_2^3$

11. $E = A[Y_1, Y_2, T_1, T_2]/(f, g, h_1, h_2)$

12. $s = 2u_2' - 4T_1 + 12x^4u_1'^2T_2$

13. $s' = 2u_2' - T_1 + 3x^4u_1'^2T_2$

14. $C = E_{ss'} \cong (A[T]/g)_{ss'}$

Set $b := Y - h \in A[T]^2$. Then the above isomorphism is induced by the A-morphism $A[Y, T] \to A[T]$, $Y \to b$.

We can also compute the Example 4 in Singular using `GND.lib` given in [10] but the result is harder.

Example 5. Let $N = 31$ be a bound. Let $A = \mathbb{Q}[x]_{(x)}$, $A' = \mathbb{Q}[[x]]$, $B = A[x^2u_1, x^3u_2, v_1, xv_2]$, where u_1, u_2, v_1, v_2 are as in Example 1. Suppose that $u_1 = \sum_{i=0}^{\infty} \frac{x^i}{i!}$, $v_1 = \sum_{i=0}^{\infty} i!x^i$. Suppose also that $u_1' = \sum_{i=0}^{31} \frac{x^i}{i!}$ and $v_1' = \sum_{i=0}^{31} i!x^i$ and we will get u_2' and v_2' according to the relations in Example 1. Let y' be given by u_1', u_2', v_1', v_2'. Let $v : B \to A'$ be the inclusion, that is in fact the map $B \cong A[Y_1, Y_2, Y_3, Y_4]/(f_1, f_2, f_3, f_4, f_5) \to A'$ given by $Y_1 \to x^2u_1$, $Y_2 \to x^3u_2$, $Y_3 \to v_1$, $Y_4 \to xv_2$ where $f_1 = Y_1^3 - Y_2^2$, $f_2 = 3Y_1^2Y_3 - 2Y_2Y_4$, $f_3 = 27Y_2Y_3^3 - 8Y_4^3$, $f_4 = 9Y_1Y_3^2 - 4Y_4^2$, $f_5 = 2Y_1Y_4 - 3Y_2Y_3$ same as in Example 3. Now we follow the steps of algorithm:

1. $f = (f_1, f_2)$.

2. $Y = (Y_1, Y_2, Y_3, Y_4)$. Among the minors $M_1 = 6Y_1^2Y_4 + 12Y_1Y_2Y_3$, $M_2 = 9{Y_1}^4, M_3 = 6{Y_1}^2Y_2, M_4 = 4{Y_2}^2, M_5 = 0$ we choose $M = 4Y_2^2$. $L = Y_2^3$; and $v(LM) = 4x^{15}{u_2}^5 \notin (x)^{31}$.

3. $P = 4Y_2^5$, $P(y') = 4x^{15}u_2'^5$, $d = x^{15}$ and $c = 15$.

4. $2c + 1 = 31$.

5. $H = \begin{pmatrix} 3Y_1^2 & 0 & 0 & -2Y_2 \\ 6Y_1Y_3 & 3Y_1^2 & -2Y_2 & -2Y_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

6. $G = LG' = \begin{pmatrix} 0 & 0 & -4Y_2^5 & 0 \\ 0 & 0 & 0 & -4Y_2^5 \\ -2Y_2^3Y_4 & 2Y_2^4 & -12Y_1Y_2^4Y_3 + 6Y_1^2Y_2^3Y_4 & -6Y_1^2Y_2^4 \\ 2Y_2^4 & 0 & -6Y_1^2Y_2^4 & 0 \end{pmatrix}$

We stop here with the algorithm since the computations of h, g, s, s' are difficult already.

4 A Constructive General Néron Desingularization in a special case

Let $(A, \mathfrak{m})$ be a local Artinian ring, $(A', \mathfrak{m}')$ a Noetherian complete local ring of dimension one such that $k = A/\mathfrak{m} \cong A'/\mathfrak{m}'$, and $u : A \to A'$ be a regular morphism. Suppose that $k \subset A$. Then $\bar{A}' = A'/\mathfrak{m}A'$ is a discrete valuation ring. Choose $x \in A'$ such that its class modulo $\mathfrak{m}A'$ is a local parameter of $\bar{A}'$, that is, it generates $\mathfrak{m}'\bar{A}'$. Let $B = A[Y]/I$, $Y = (Y_1, \dots, Y_n)$.

Theorem 3 (Khalid-Kosar [3])**.** *Then any morphism $v : B \to A'$ factors through a smooth A-algebra C.*

Proof. Here we recall in sketch the proof from [3] because we need it in the next algorithm. Let $A_1 = A[x]_{(x)}$ and u_1 be the inclusion $A_1 \subset A'$. Then u_1 is a regular morphism. Let $B_1 = A_1 \otimes_A B$ and $v_1 : B_1 \to A'$ be the map $a_1 \otimes b \mapsto u_1(a_1) \cdot v(b)$.

There exists a certain s such that $\mathfrak{m}^s = 0$ because A is an Artinian local ring and so A has the form $A = k[T]/\mathfrak{a}$, $T = (T_1, \dots, T_m)$, and the maximal ideal of A is generated by T. Then for all $i \in [m] = \{1, \dots, m\}$, ${T_i}^s \in \mathfrak{a}$ and $A' = k[[x]][T]/(\mathfrak{a}) \cong A \otimes_k k[[x]]$. Note that $v(Y_i) = \hat{y}_i$ has the form $\sum_{\alpha \in \mathbb{N}^m, |\alpha| < s} y_{i\alpha} T^\alpha$, $T^\alpha = {T_1}^{\alpha_1} \cdots {T_m}^{\alpha_m}$, $|\alpha| = \alpha_1 + \dots + \alpha_m$ and $y_{i\alpha} \in \bar{A}' = k[[x]]$.

Set $\bar{B}_1 = \bar{A}_1[(y_{i\alpha})_\alpha] \subset k[[x]]$ and let $\bar{v}_1$ be this inclusion. Then v factors through $B_1 = A \otimes_k \bar{B}_1 \subset A'$, that is v is the composite map $B \xrightarrow{q} B_1 \xrightarrow{A \otimes_k \bar{v}_1} A'$, where q is defined by $Y_i \to \sum_\alpha T^\alpha \otimes y_{i\alpha}$. Applying Theorem 1 to the case $\bar{A}_1 = k[x]_{(x)}$, $\bar{A}'$, $\bar{B}_1$ and $\bar{v}_1 = \bar{A}_1 \otimes_{A_1} v_1$ we see that $\bar{v}_1$ factors through a smooth k-algebra $\bar{C}$. Then $A \otimes_k \bar{v}_1$ factors through $A \otimes_k \bar{C}$. It follows that v factors through a smooth A-algebra C (see e.g. [3, Lemma 1]). ∎

5 A special Algorithm

In our next algorithm we will use the Néron Desingularization algorithm given in Section 3.

Special-Neron-Desingularization

Input: $N \in \mathbb{Z}_{>0}$ a bound
$A = k[T]/(a)$, $a = (a_1, \dots, a_e)$ $T = (T_1, \dots, T_m), {T_i}^s \in (a)$, $A' = k[[x]][T]/(a)$, $B = A[Y]/I$, $I = (g_1, \dots, g_l), g_i \in k[T, Y], Y = (Y_1, \dots, Y_n)$, integers q, α. $v : B \to A'$ an $A-$morphism given by $v(Y_i) = \hat{y}_i = \sum_{\alpha \in \mathbb{N}^m, |\alpha| < s} y_{i\alpha} \cdot T^\alpha$, $y_{i\alpha} \in k[[x]]$, $\bar{A}' = k[[x]]$.

Output: A Néron Desingularization (C, π) of $v : B \to A'$ or the message "the algorithm fails since the bound is too small".

1. $\bar{A}_1 = k[x]_{(x)}$, $\bar{B}_1 := \bar{A}_1[(y_{i\alpha})_\alpha]$, $\bar{v}_1$ is the inclusion $\bar{B}_1 \subset \bar{A}'$
2. Write $(\bar{C}, \bar{\pi}) :=$ Neron-Desingularization_Dim1 for $N, \bar{A}_1, \bar{A}', \bar{B}_1, \bar{v}_1$
3. $\bar{C} := E_{ss'} = (\bar{A}_1[(Y_{i\alpha})_\alpha, T]/L)_{ss'}$ where $L =< (l_i) >$

4. $\bar{C} := \bar{A}_1 \otimes_k \tilde{C}$ where $\tilde{C} = ((k[x,(Y_{i\alpha})_\alpha, T]/\tilde{L})_{\tilde{s}\tilde{s}'})_\eta$ and $\tilde{s} = s \cdot \eta$, $\tilde{s}' = s' \cdot \eta$, $\tilde{L} =< (\tilde{l}_i) >, \tilde{l}_i = l_i \cdot \eta$, where $\eta \in k[x] \setminus (x)$

5. $C := A \otimes_k \tilde{C}$, π is induced by $\bar{\pi}$

6. return (C, π).

Remark 3. Here we give two examples for the same rings. Example 6 gives a Néron Desingularization which comes from the direct computations, while Example 7 gives a smooth $A-$algebra C which we get by following the algorithm of Section 3.

Example 6. Let $A = \mathbb{Q}[t]/(t^2)$, $A' = \mathbb{Q}[[x]][t]/(t^2)$, $A_1 = \mathbb{Q}[x]_{(x)}[t]/(t^2)$, $B = A[Y_1, Y_2]/(Y_1^3 - Y_2^2)$ and u_1, v_1 two formal power series from $\mathbb{Q}[[x]]$ which are algebraically independent over $\mathbb{Q}(x)$ and $u_1(0) = v_1(0) = 1$. By the Implicit Function Theorem there exists $u_2 \in \mathbb{Q}[[x]]$ such that $u_2^2 = u_1^3$. Set $v_2 = (3/2)xu_1^2v_1u_2^{-1}$, $\hat{y}_1 = x^2u_1 + tv_1$, $\hat{y}_2 = x^3u_2 + txv_2$. We have $g(\hat{y}_1, \hat{y}_2) = x^6(u_1^3 - u_2^2) + tx^4(3u_1^2v_1 - 2u_2v_2) = 0$ and we may define $v : B \to A'$ by $Y \to (\hat{y}_1, \hat{y}_2)$.

Take $\bar{B}_1 = \bar{A}_1[x, x^2u_1, x^3u_2, v_1, xv_2]$. Then $\bar{v} = \mathbb{Q} \otimes_A v : B/tB \to \mathbb{Q}[[x]]$ factors through $\bar{B}_1$. Now $\bar{B}_1 = \bar{A}_1[(Y_{i\alpha})_\alpha]/J$ as in Example 3. Since $\bar{A}_1 = k[x]_{(x)}$, $\bar{A}' = k[[x]]$, $\bar{B}_1 = \bar{A}_1[(Y_{i\alpha})_\alpha]/J$ so applying the algorithm from Section 3 for $\bar{A}_1, \bar{A}', \bar{B}_1$ we get $\bar{C} = (\bar{A}_1[U_1, U_2, V_1]/(h_1))_{U_2} = \bar{A}_1 \otimes_k \tilde{C}$ where $\tilde{C} = (k[U_1, U_2, V_1]/(h_1))_{U_2}$ and U_1, U_2, V_1, h_1 are as in Example 3. So $C = A \otimes_k \tilde{C} \cong (A[U_1, U_2, V_1]/(h_1))_{U_2}$.

Example 7. Considering everything as in Example 6 until we apply the algorithm from Section 3, we get $\bar{C} = E_{ss'}$ where $E_{ss'}$ is the same as in Example 4. So $C = A \otimes_k \tilde{C}$ where $\tilde{C}$ can be obtained as in Example 6.

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References

[1] M. Artin, Algebraic approximation of structures over complete local rings, *Publ. Math. IHES*, **36** (1969), 23-58.

[2] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann: Singular 3-1-6 — A computer algebra system for polynomial computations. `http://www.singular.uni-kl.de` (2012).

[3] A. Khalid, Z. Kosar, An Easy proof of the General Neron Desingularization in dimension 1, *Bull. Math. Soc. Sci. Math. Roumanie*, **59 (107)**, 2016, 349-353.

[4] A. Khalid, G. Pfister, D. Popescu, An uniform General Neron Desingularization in dimension one, J. of Algebra and Appl. (2018), arXiv:AC/1612.03416.

[5] H. Matsumura, *Commutative Ring Theory*, Cambridge Univ. Press, Cambridge, 1986.

[6] A. Néron, Modeles minimaux des varietes abeliennes sur les corps locaux et globaux, *Publ. Math. IHES*, **21**, 1964, 5-128.

[7] G. Pfister, D. Popescu, Die strenge Approximationseigenschaft lokaler Ringe, *Inventiones Math*, **30**, (1975), 145-174.

[8] G. Pfister, D. Popescu, Constructive General Neron Desingularization for one dimensional local rings, *Journal of Symbolic Computation*, **80**, (2017), 570-580, arXiv:AC/1512.08435v1.

[9] A. Ploski, Note on a theorem of M. Artin, *Bull. Acad. Polon. des Sci.*, t. XXII, **11** (1974), 1107-1110.

[10] A. Popescu, D. Popescu, *A method to compute the General Neron Desingularization in the frame of one dimensional local domains*, to appear in "Singularities and Computer Algebra - Festschrift for Gert-Martin Greuel On the Occasion of his 70th Birthday", Editors Wolfram Decker, Gerhard Pfister, Mathias Schulze, Springer Monograph., DOI 10.1007/978-3-319-28829-1, arXiv:AC/1508.05511.

[11] D. Popescu, A strong approximation theorem over discrete valuation rings, *Rev. Roum. Math. Pures et Appl.*, **20**, (1975), 659-692.

[12] D. Popescu, General Neron Desingularization and approximation, *Nagoya Math. J.*, **104** (1986), 85-115.

[13] D. Popescu, *Artin Approximation*, in "Handbook of Algebra", vol. 2, Ed. M. Hazewinkel, Elsevier, 2000, 321-355.

[14] D. Popescu, *Lecture 3, Neron Desingularization and its generalizations* in the Conference **Algebraic versus Analytic Geometry** organized by H. Hauser in Erwin Schrödinger Institut for Mathematical Physics from Wien, November-December 2011,(see http://www.esi.ac.at/activities/events/2011/algebraic-versus-analytic-geometry).

[15] D. Popescu, Around General Neron Desingularization, *Journal of Algebra and Its Applications*, **16**, No. 2 (2017), doi: 10.1142/S0219498817500724, arXiv:1504.06938.

[16] R. Swan, *Neron-Popescu desingularization*, in "Algebra and Geometry", Ed. M. Kang, International Press, Cambridge, (1998), 135-192.

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(1) Abdus Salam School of Mathematical Sciences
GC University, Lahore, Pakistan
E-mail: asmakhalid768@gmail.com

(2) E-mail: adrian.mihail.popescu@gmail.com

(3) Simion Stoilow Institute of Mathematics
of the Romanian Academy
Research unit 5, University of Bucharest
P.O.Box 1-764, Bucharest 014700, Romania
E-mail: dorin.popescu@imar.ro