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Oscillation of Impulsive Neutral Partial Differential Equations with Distributed Deviating Arguments

by

T. Kalaimani[1], T. Raja[1], V. Sadhasivam[1], S. H. Saker[2]

Abstract

In this paper, we will consider a class of even order nonlinear impulsive neutral partial functional differential equations with continuous distributed deviating arguments. Adequate conditions are obtained for the oscillation of solutions by using impulsive differential inequalities and averaging technique with two different boundary conditions. Examples are specified to illustrate the main results.

Key Words: Neutral partial differential equations, Oscillation, Impulse, Distributed deviating arguments
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1 Introduction

The oscillation theory of ordinary differential equations marks its commencement with the manuscript of Sturm [26] in 1836 and for partial differential equations by Hartman and Wintner [9] in 1955. To the best of the authors' knowledge, the work on impulsive delay differential equations was published and initiated in 1989 in [6]. Its consequences were integrated in the monograph [14]. On the other hand there are a few papers has been considered higher order partial differential equations with distributed deviating arguments, we refer the reader to the papers [7], [15], [16], [17]. The primary exertion for impulsive partial differential equations has been started in 1991 in [5]. In recent years the oscillation of parabolic and hyperbolic equations with or without impulse effect has been widely studied in the literature, we refer the reader to the papers [11], [12], [18], [21], [22], [23], [24], [27], [28] and the reference they are cited. In [31], population ecology, generic repression, control theory, climate models, coupled oscillators, viscoelastic materials, and structured population models studied with distributed delay and boundary conditions of the type Dirichlet, Neumann and Robin. From the essence of these mathematical models, we formulated this higher order problem. Distributed delay system models appear in logistics [4], traffic flow [25], microorganism growth [20], and hematopoiesis [1], and [2]. The wide interest on qualitative studies of impulsive ordinary and partial functional differential equations is returned to their varieties of applications in various fields of science and technology [3], [13], [32], and so it is desirable to study these equations scientifically.

In this paper, we will study the following even order nonlinear impulsive neutral partial functional differential equation with continuous distributed deviating arguments

$$\left.\begin{array}{l}\dfrac{\partial^m}{\partial t^m}\left[u(x,t)+\int_a^b g(t,\xi)u(x,\tau(t,\xi))d\eta(\xi)\right]+\int_a^b q(x,t,\xi)f(u(x,\sigma(t,\xi)))d\eta(\xi)\\ =a(t)\Delta u(x,t)+\sum\limits_{j=1}^{n}b_j(t)\Delta u(x,\rho_j(t)),\ t\neq t_k,\quad (x,t)\in\Omega\times(0,+\infty)\equiv G,\\ \dfrac{\partial^{(i)}u(x,t_k^+)}{\partial t^{(i)}}=I_k^{(i)}\left(x,t_k,\dfrac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}}\right),\ t=t_k,\ k=1,2,...,\ i=0,1,2,..,m-1,\end{array}\right\} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in the Euclidean space $\mathbb{R}^N$.

Equation (1.1) is enhancement with one of the subsequent Dirichlet and Robin boundary conditions,

$$u=0,\qquad (x,t)\in\partial\Omega\times(0,+\infty), \tag{1.2}$$

$$\frac{\partial u}{\partial\gamma}+\mu(x,t)u=0,\qquad (x,t)\in\partial\Omega\times(0,+\infty), \tag{1.3}$$

where γ is the outer surface normal vector to $\partial\Omega$ and $\mu(x,t)\in C\left(\partial\Omega\times[0,+\infty),[0,+\infty)\right)$.

In the sequel, we assume that the following hypotheses (H) hold:

(H_1) $a(t),\ b_j(t)\in PC\left([0,+\infty),[0,+\infty)\right)$, $j=1,2,...,n$, where PC represents the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t=t_k,\ k=1,2,...,$ and left continuous at $t=t_k,\ k=1,2,...,$ a,b are non-positive constants with $a<b$.

(H_2) $g(t,\xi)\in C^m([0,+\infty)\times[a,b],[0,+\infty))$, $q(x,t,\xi)\in C(\bar\Omega\times[0,+\infty)\times[a,b],[0,+\infty))$, $Q(t,\xi)=\min\limits_{x\in\bar\Omega}q(x,t,\xi)$, $\rho_j(t)\in C([0,+\infty),\mathbb{R})$, $\lim\limits_{t\to+\infty}\rho_j(t)=+\infty$, $j=1,2,...,n$, $f(u)\in C(\mathbb{R},\mathbb{R})$ is convex in $[0,+\infty)$, $uf(u)>0$ and $\dfrac{f(u)}{u}\geq c>0$ for $u\neq 0$.

(H_3) $\tau(t,\xi),\ \sigma(t,\xi)\in C([0,+\infty)\times[a,b],\mathbb{R})$, $\tau(t,\xi)\leq t$, $\sigma(t,\xi)\leq t$ for $\xi\in[a,b]$, $\tau(t,\xi)$ and $\sigma(t,\xi)$ are nondecreasing with respect to t and ξ respectively and $\liminf\limits_{t\to+\infty,\ \xi\in[a,b]}\tau(t,\xi)=\liminf\limits_{t\to+\infty,\ \xi\in[a,b]}\sigma(t,\xi)=+\infty$.

(H_4) There exist a function $\theta(t)\in C([0,+\infty),[0,+\infty))$ satisfying $\theta(t)\leq\sigma(t,a)$, $\theta'(t)>0$ and $\lim\limits_{t\to+\infty}\theta(t)=+\infty$, $\eta(\xi):[a,b]\to\mathbb{R}$ is nondecreasing and the integral is a Stieltjes integral in (1.1).

(H_5) $\dfrac{\partial^{(i)}u(x,t)}{\partial t^{(i)}}$ are piecewise continuous in t with discontinuities of first kind only at $t=t_k,\ k=1,2,...,$ and left continuous at $t=t_k$, $\dfrac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}}=\dfrac{\partial^{(i)}u(x,t_k^-)}{\partial t^{(i)}}$, $k=1,2,...,\ i=0,1,2,...,m-1$.

(H_6) $I_k^{(i)}\left(x,t_k,\dfrac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}}\right)\in PC(\bar\Omega\times[0,+\infty)\times\mathbb{R},\mathbb{R})$, $k=1,2,\cdots,\ i=0,1,2,...,m-1$, and there exist positive constants $a_k^{(i)},\ b_k^{(i)}$ with $b_k^{(m-1)}\leq a_k^{(0)}$ such that for

$i = 0, 1, 2, ..., m-1, \ k = 1, 2, ...,$

$$a_k^{(i)} \leq \frac{I_k^{(i)}\left(x, t_k, \dfrac{\partial^{(i)} u(x,t_k)}{\partial t^{(i)}}\right)}{\dfrac{\partial^{(i)} u(x,t_k)}{\partial t^{(i)}}} \leq b_k^{(i)}.$$

This paper is organized as follows: In Section 2, we present the definitions and notations that will be needed. In Section 3, we discuss the oscillation of the problem (1.1) and (1.2). In Section 4, we discuss the oscillation of the problem (1.1) and (1.3). In Section 5, we present some examples to illustrate the main results. The results in this paper expand and improve numerous findings in the earlier publications not including impulsive effects. We anticipate that this work acquire the concentration of numerous researchers functioning on the even order impulsive partial functional differential equations.

2 Preliminaries

In this section, we begin with definitions and well known results which are required throughout this paper.

Definition 1. *A solution u of the problem* (1.1) *is a function $u \in C^m(\bar{\Omega} \times [t_{-1}, +\infty), \mathbb{R}) \cap C(\bar{\Omega} \times [\hat{t}_{-1}, +\infty), \mathbb{R})$ that satisfies* (1.1)*, where*

$$t_{-1} := \min\left\{0, \min_{\xi \in [a,b]}\left\{\inf_{t \geq 0} \tau(t,\xi)\right\}, \min_{\xi \in [a,b]}\left\{\inf_{t \geq 0} \sigma(t,\xi)\right\}\right\},$$
$$\hat{t}_{-1} := \min\left\{0, \min_{1 \leq j \leq n}\left\{\inf_{t \geq 0} \rho_j(t)\right\}\right\}.$$

Definition 2. *The solution u of the problem (1.1), (1.2) [(1.1), (1.3)] is said to be oscillatory in the domain G if for any positive number ℓ there exists a point $(x_0, t_0) \in \Omega \times [\ell, +\infty)$ such that $u(x_0, t_0) = 0$ holds.*

Definition 3. *A function $V(t)$ is said to be eventually positive (negative) if there exists a $t_1 \geq t_0$ such that $V(t) > 0$ (< 0) holds for all $t \geq t_1$.*

It is identified that [29] the least eigenvalue $\lambda_0 > 0$ of the eigenvalue problem

$$\Delta\omega(x) + \lambda\omega(x) = 0, \text{ in } \quad \Omega, \, \omega(x) = 0, \text{ on } \quad \partial\Omega,$$

and the consequent eigenfunction $\Phi(x) > 0$ in Ω.

For each positive solution $u(x,t)$ of the problem (1.1), (1.2) [(1.1), (1.3)] we combine the functions $V(t)$ and $\tilde{V}(t)$ defined by

$$V(t) = K_\Phi \int_\Omega u(x,t)\Phi(x)dx, \qquad \tilde{V}(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx,$$
$$F(t) = M(\theta(t))^{m-2}\theta'(t), \quad \text{and} \quad G(t) = cg_0 \int_a^b Q(t,\xi)d\eta(\xi),$$

where

$$K_\Phi = \left(\int_\Omega \Phi(x)dx \right)^{-1}, \quad |\Omega| = \int_\Omega dx, \quad \text{and} \quad g_0 = 1 - \int_a^b g(\sigma(t,\xi),\xi)d\eta(\xi).$$

Lemma 1. *[10] Let $y(t)$ be a positive and n times differentiable function on $[0,+\infty)$. If $y^{(n)}(t)$ is constant sign and not identically zero on any ray $[t_1,+\infty)$ for $t_1 > 0$, then there exists a $t_y \geq t_1$ and integer l $(0 \leq l \leq n)$, with $n+l$ even for $y(t)y^{(n)}(t) \geq 0$ or $n+l$ odd for $y(t)y^{(n)}(t) \leq 0$; and for $t \geq t_y$, $y(t)y^{(k)}(t) > 0$, $0 \leq k \leq l$; $(-1)^{k-l}y(t)y^{(k)}(t) > 0$, $l \leq k \leq n$.*

Lemma 2. *[19] Suppose that the conditions of Lemma 1 is satisfied, and $y^{(n-1)}(t)y^{(n)}(t) \leq 0$, for $t \geq t_y$. Then there exist constant $\alpha \in (0,1)$ and $M > 0$ such that $\left|y'(\alpha t)\right| \geq Mt^{n-2}\left|y^{(n-1)}(t)\right|$ for sufficiently large t.*

Lemma 3. *[8] If X and Y are nonnegative, then*

$$\begin{aligned} X^\alpha - \alpha XY^{\alpha-1} + (\alpha-1)Y^\alpha &\geq 0, \quad \alpha > 1, \\ X^\alpha - \alpha XY^{\alpha-1} - (1-\alpha)Y^\alpha &\leq 0, \quad 0 < \alpha < 1, \end{aligned}$$

where the equality holds if and only if $X = Y$.

3 Oscillation of the problem (1.1), (1.2)

In this section, we establish some sufficient conditions for the oscillation of all solutions of the problem (1.1), (1.2).

Lemma 4. *If the functional impulsive differential inequality*

$$\left.\begin{aligned} & Z^{(m)}(t) + G(t)Z(\theta(t)) \leq 0, \quad t \neq t_k \\ & a_k^{(i)} \leq \frac{\frac{\partial^{(i)} Z(t_k^+)}{\partial t^{(i)}}}{\frac{\partial^{(i)} Z(t_k)}{\partial t^{(i)}}} \leq b_k^{(i)}, \ k = 1,2,..., \quad i = 0,1,2,...,m-1, \end{aligned}\right\} \tag{3.1}$$

has no eventually positive solution, then every solution of the boundary value problem defined by (1.1), (1.2) is oscillatory in G.

Proof. Assume that there exist a nonoscillatory solution $u(x,t)$ of the boundary value problem (1.1), (1.2) and $u(x,t) > 0$. By the hypothesis (H_3), that there exists a $t_1 > t_0 > 0$ such that $\tau(t,\xi) \geq t_0$, $\sigma(t,\xi) \geq t_0$ for $(t,\xi) \in [t_1,+\infty) \times [a,b]$ and $\rho_j(t) \geq t_0$, $j = 1,2,...,n$ for $t \geq t_1$, then for $t \geq t_1$, $t \neq t_k$, $k = 1,2,...$, we get that

$$\begin{aligned} & u(x,\tau(t,\xi)) > 0, \quad \text{for} \quad (x,t,\xi) \in \Omega \times [t_1,+\infty) \times [a,b], \\ & u(x,\sigma(t,\xi)) > 0, \quad \text{for} \quad (x,t,\xi) \in \Omega \times [t_1,+\infty) \times [a,b], \\ & u(x,\rho_j(t)) > 0, \quad \text{for} \quad (x,t) \in \Omega \times [t_1,+\infty), \quad j = 1,2,...,n. \end{aligned}$$

Multiplying both sides of equation (1.1) by $K_\Phi \Phi(x) > 0$ and integrating with respect to x over the domain Ω, we obtain

$$\left.\begin{array}{l} \dfrac{d^m}{dt^m}\left[\int_\Omega u(x,t)K_\Phi\Phi(x)dx + \int_\Omega\int_a^b g(t,\xi)u(x,\tau(t,\xi))K_\Phi\Phi(x)d\eta(\xi)dx\right] \\ +\int_\Omega\int_a^b q(x,t,\xi)f(u(x,\sigma(t,\xi)))K_\Phi\Phi(x)d\eta(\xi)dx \\ = a(t)\int_\Omega \Delta u(x,t)K_\Phi\Phi(x)dx + \sum\limits_{j=1}^{n} b_j(t)\int_\Omega \Delta u(x,\rho_j(t))K_\Phi\Phi(x)dx. \end{array}\right\} \quad (3.2)$$

From Green's formula and the boundary condition (1.2), we see that

$$\begin{aligned} K_\Phi\int_\Omega \Delta u(x,t)\Phi(x)dx &= K_\Phi\int_{\partial\Omega}\left[\Phi(x)\frac{\partial u}{\partial\gamma} - u\frac{\partial\Phi(x)}{\partial\gamma}\right]dS + K_\Phi\int_\Omega u(x,t)\Delta\Phi(x)dx \\ &= -\lambda_0 V(t) \le 0, \end{aligned} \quad (3.3)$$

and for $j = 1, 2, ..., n$, we have

$$\begin{aligned} K_\Phi\int_\Omega \Delta u(x,\rho_j(t))\Phi(x)dx &= K_\Phi\int_{\partial\Omega}\left[\Phi(x)\frac{\partial u(x,\rho_j(t))}{\partial\gamma} - u(x,\rho_j(t))\frac{\partial\Phi(x)}{\partial\gamma}\right]dS \\ &\quad + K_\Phi\int_\Omega u(x,\rho_j(t))\Delta\Phi(x)dx \\ &= -\lambda_0 V(\rho_i(t)) \le 0, \end{aligned} \quad (3.4)$$

where dS is surface component on $\partial\Omega$. Furthermore applying Jensen's inequality for convex functions and using the assumptions on (H_2), we get that

$$\begin{aligned} &\int_\Omega\int_a^b q(x,t,\xi)f(u(x,\sigma(t,\xi)))K_\Phi\Phi(x)d\eta(\xi)dx \\ &\qquad \ge \int_a^b Q(t,\xi)\int_\Omega f(u(x,\sigma(t,\xi)))K_\Phi\Phi(x)dxd\eta(\xi) \\ &\qquad \ge c\int_a^b Q(t,\xi)V(\sigma(t,\xi))d\eta(\xi). \end{aligned} \quad (3.5)$$

Combining (3.2)-(3.5), we get that

$$\frac{d^m}{dt^m}\left[V(t) + \int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)\right] + c\int_a^b Q(t,\xi)V(\sigma(t,\xi))d\eta(\xi) \le 0. \quad (3.6)$$

Set $Z(t) = V(t) + \int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)$. Equation (3.6), can be written as

$$Z^{(m)}(t) + c\int_a^b Q(t,\xi)V(\sigma(t,\xi))d\eta(\xi) \le 0, \quad t \ne t_k. \quad (3.7)$$

From the assumption of $\int_a^b g(t,\xi)d\eta(\xi)$ and $Q(t,\xi)$, we have $Z(t) \ge V(t) > 0$ and $Z^{(m)}(t) \le 0$. Simultaneously, we can further prove $Z^{(m-1)}(t) \ge 0,\ t \ge t_2$. In addition, from Lemma 1, there exists a $t_3 \ge t_2$ and a odd number l, $0 \le l \le m-1$, and for $t \ge t_3$, we have

$$Z^{(i)}(t) > 0,\ 0 \le i \le l,\ (-1)^{(i-1)}Z^{(i)}(t) > 0,\ \text{for}\ \ l \le i \le m-1.$$

By choosing $i = 1$, we have $Z^{'}(t) > 0$, since $Z(t) \geq x(t) > 0$, $Z^{'}(t) \geq 0$, we have

$$Z(\sigma(t,\xi)) \geq Z(\sigma(t,\xi) - \tau(t,\xi)) \geq x(\sigma(t,\xi) - \tau(t,\xi)),$$

and thus

$$Z^{(m)}(t) + c\int_a^b Q(t,\xi)Z(\sigma(t,\xi))\left(1 - \int_a^b g(\sigma(t,\xi),\xi)d\eta(\xi)\right)d\eta(\xi) \leq 0. \tag{3.8}$$

From equation (3.7), we get

$$Z^{(m)}(t) + G(t)Z(\sigma(t,\xi)) \leq 0.$$

From (H_3) and (H_4), we have

$$Z(\sigma(t,\xi)) \geq Z(\sigma(t,a)) > 0,\ \xi \in [a,b] \ \text{ and } \ \theta(t) \leq \sigma(t,\xi) \leq t.$$

Thus $Z(\theta(t)) \leq Z(\sigma(t,a))$ for $t \geq t_2$. Then (3.8) can be written as

$$Z^{(m)}(t) + G(t)Z(\theta(t)) \leq 0.$$

Multiplying both sides of the equation (1.1) by $K_\Phi \Phi(x) > 0$, integrating with respect to x over the domain Ω, and from (H_6), we obtain

$$a_k^{(i)} \leq \frac{\dfrac{\partial^{(i)}u(x,t_k^+)}{\partial t^{(i)}}}{\dfrac{\partial^{(i)}u(x,t_k)}{\partial t^{(i)}}} \leq b_k^{(i)}.$$

According to $V(t) = K_\Phi \int_\Omega u(x,t)\Phi(x)dx$, we have

$$a_k^{(i)} \leq \frac{\dfrac{\partial^{(i)}V(x,t_k^+)}{\partial t^{(i)}}}{\dfrac{\partial^{(i)}V(x,t_k)}{\partial t^{(i)}}} \leq b_k^{(i)}.$$

Since $Z(t) = V(t) + \int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)$, we obtain

$$a_k^{(i)} \leq \frac{\dfrac{\partial^{(i)}Z(x,t_k^+)}{\partial t^{(i)}}}{\dfrac{\partial^{(i)}Z(x,t_k)}{\partial t^{(i)}}} \leq b_k^{(i)}.$$

Therefore $Z(t)$ is an eventually positive solution of (3.1). This disagree with the hypothesis. □

Theorem 1. *If there exists a function $\varphi(t) \in C'([0,+\infty),(0,+\infty))$ which is nondecreasing with respect to t, such that*

$$\int_{t_1}^{+\infty} \prod_{t_0 \leq t_k < s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[\varphi(s)G(s) - \frac{(\varphi'(s))^2}{4F(s)\varphi(s)}\right]ds = \infty, \tag{3.9}$$

then every solution of the boundary value problem (1.1), (1.2) *is oscillatory in* G.

Proof. Assume that there exist a non-oscillatory solution $u(x,t)$ of the boundary value problem (1.1), (1.2) and $u(x,t)>0$. Proceeding as in the proof of Lemma 4 to get that $Z^{(m)}(t)+G(t)Z(\theta(t))\leq 0$, where $Z(t)=V(t)+\int_a^b g(t,\xi)V(\tau(t,\xi))d\eta(\xi)$ and satisfies $Z^{(m)}(t)\leq 0$, $Z^{(m-1)}(t)\geq 0$ and an odd number l, $0\leq l\leq m-1$, such that $Z^{(i)}(t)>0$, $0\leq i\leq l$, $(-1)^{(i-1)}Z^{(i)}(t)>0$, for $l\leq i\leq m-1$. Define

$$W(t):=\varphi(t)\frac{Z^{(m-1)}(t)}{Z(\theta(t))},\quad \text{for } t\geq t_0,$$

then $W(t)\geq 0$ for $t\geq t_1$, and

$$W^{'}(t)\leq \frac{\varphi^{'}(t)}{\varphi(t)}W(t)+\frac{\varphi(t)Z^{(m)}(t)}{Z(\theta(t))}-\frac{\varphi(t)Z^{(m-1)}(t)Z^{'}(\theta(t))\theta^{'}(t)}{Z(\theta(t))^2}.$$

From $Z^{(m)}(t)\leq 0$, according to Lemma 2, we obtain

$$Z^{'}(\theta(t))\geq M(\theta(t))^{m-2}Z^{(m-1)}(t)\,.$$

Thus

$$W^{'}(t)\leq \frac{\varphi^{'}(t)}{\varphi(t)}W(t)-G(t)\varphi(t)-\frac{F(t)}{\varphi(t)}W^2(t),\ W(t_k^+)\leq \frac{b_k^{(m-1)}}{a_k^{(0)}}W(t_k).$$

Define

$$U(t)=\prod_{t_0\leq t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}W(t).$$

In fact, $W(t)$ is continuous on each interval $(t_k,t_{k+1}]$, and in consideration of $W(t_k^+)\leq (b_k^{(m-1)}/a_k^{(0)})W(t_k)$. It follows for $t\geq t_0$ that

$$U(t_k^+)=\prod_{t_0\leq t_j\leq t_k}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}W(t_k^+)\leq \prod_{t_0\leq t_j<t_k}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}W(t_k)=U(t_k)$$

and for all $t\geq t_0$, we have

$$U(t_k^-)=\prod_{t_0\leq t_j\leq t_{k-1}}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}W(t_k^-)\leq \prod_{t_0\leq t_j<t_k}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}W(t_k)=U(t_k),$$

which implies that $U(t)$ is continuous on $[t_0, +\infty)$ and satisfies

$$\begin{aligned}
&U^{'}(t) + \prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{U^2(t)F(t)}{\varphi(t)} + \prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(t)\varphi(t) - \frac{\varphi^{'}(t)U(t)}{\varphi(t)}\\
&= \prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} W^{'}(t) + \prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-2}\frac{F(t)}{\varphi(t)}W^2(t)\\
&+ \prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(t)\varphi(t) - \prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\frac{\varphi^{'}(t)}{\varphi(t)}W(t)\\
&= \prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[W'(t) + W^2(t)\frac{F(t)}{\varphi(t)} - W(t)\frac{\varphi^{'}(t)}{\varphi(t)} + G(t)\varphi(t)\right] \le 0.
\end{aligned}$$

That is

$$U^{'}(t) \le -\prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{F(t)}{\varphi(t)}U^2(t) + \frac{\varphi^{'}(t)}{\varphi(t)}U(t) - \prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} G(t)\varphi(t). \tag{3.10}$$

Applying Lemma 3 with

$$X = \sqrt{\prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{F(t)}{\varphi(t)}}U(t), \text{ and } \quad Y = \frac{\varphi'(t)}{2}\sqrt{\prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\frac{1}{F(t)\varphi(t)}},$$

we have

$$\frac{\varphi^{'}(t)}{\varphi(t)}U(t) - \prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{F(t)}{\varphi(t)}U^2(t) \le \frac{(\varphi^{'}(t))^2}{4F(t)\varphi(t)}\prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}.$$

Thus

$$U^{'}(t) \le -\prod_{t_0\le t_k<t}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[G(t)\varphi(t) - \frac{(\varphi^{'}(t))^2}{4F(t)\varphi(t)}\right].$$

Integrating both sides from t_1 to t, we have

$$U(t) \le U(t_1) - \int_{t_1}^{t}\prod_{t_0\le t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[G(s)\varphi(s) - \frac{(\varphi^{'}(s))^2}{4F(s)\varphi(s)}\right]ds.$$

Letting $t \to +\infty$, from (3.9), we have $\lim\limits_{t\to+\infty} U(t) = -\infty$, which leads to a contradiction with $U(t) \ge 0$. The proof is complete. □

The following theorem is of Philos type [19] and can be obtained by applying the inequality (3.10) and the Philos technique. The details will be left to the interested reader. To formulate the results we assume that there exist two functions $H(t, s)$, $h(t, s) \in C^1(D, \mathbb{R})$, in which $D = \{(t, s)|t \ge s \ge t_0 > 0\}$, such that

(H_7). $H(t,t)=0,\quad t\geq t_0;\quad H(t,s)>0,\quad t>s\geq t_0$,

(H_8). $H_t^{'}(t,s)\geq 0,\ H_s^{'}(t,s)\leq 0,$

(H_9). $-\dfrac{\partial}{\partial s}[H(t,s)\rho(s)]-H(t,s)\rho(s)\dfrac{\varphi^{'}(s)}{\varphi(s)}=h(t,s).$

Theorem 2. *Assume that there exist functions $\varphi(t)$ and $\rho(s)\in C^1([0,+\infty),(0,+\infty))$ such that $\varphi(t)$ is non-decreasing. If there exist two functions $H(t,s)$, $h(t,s)\in C^1(D,\mathbb{R})$ satisfy $(H_7)-(H_9)$ and*

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\Pi(s)ds=\infty, \tag{3.11}$$

where

$$\Pi(s)=G(s)\varphi(s)H(t,s)\rho(s)-\frac{1}{4}\frac{|h(t,s)|^2\,\varphi(s)}{F(s)H(t,s)\rho(s)},$$

then every solution of the boundary value problem (1.1), (1.2) *is oscillatory in* G.

In Theorem 2, by choosing $\rho(s)=\varphi(s)\equiv 1$, we have the following corollary.

Corollary 1. *Assume that the conditions of Theorem 2 hold, and*

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\Gamma(s)ds=\infty,$$

where

$$\Gamma(s)=G(s)H(t,s)-\frac{1}{4}\frac{|h(t,s)|^2}{F(s)H(t,s)},$$

then every solution of the boundary value problem (1.1), (1.2) *is oscillatory in* G.

Remark 1. *From Theorem 2 and Corollary 1, we can attain various oscillatory criteria by different choices of the weighted function $H(t,s)$. For example, choosing $H(t,s)=(t-s)^{n-1}$, $t\geq s\geq t_0$, in which $n>2$ is an integer, then $h(t,s)=(n-1)(t-s)^{(n-3)/2}$, $t\geq s\geq t_0$. From Corollary 1, we have the following Kamenev type result.*

Corollary 2. *If there exists an integer $n>2$ such that*

$$\limsup_{t\to+\infty}\frac{1}{(t-t_0)^{n-1}}\int_{t_0}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[G(s)(t-s)^{n-1}-\frac{1}{4}\frac{(n-1)^2}{(t-s)^2F(s)}\right]ds=+\infty, \tag{3.12}$$

then every solution of the boundary value problem (1.1), (1.2) *is oscillatory in* G.

Also by applying the method of Philos [19] one can obtain the following new oscillation theorem.

Theorem 3. *Let the functions $H(t,s)$, $h(t,s)$, $\varphi(s)$ and $\rho(s)$ be as defined in Theorem 2. Additionally, suppose that* $0 < \inf_{s\geq t_0}\left\{\liminf_{t\to+\infty}\frac{H(t,s)}{H(t,t_0)}\right\} \leq +\infty$, *and*

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\frac{|h(t,s)|^2\varphi(s)}{F(s)H(t,s)\rho(s)}ds < +\infty.$$

If there exists a function $A(t)\in C([t_0,+\infty),\mathbb{R})$ such that

$$\limsup_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)\frac{F(s)(A_+(s))^2}{\rho(s)\varphi(s)}ds = +\infty,$$

and for every $T\geq t_0$

$$\limsup_{t\to+\infty}\frac{1}{H(t,T)}\int_{T}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[G(s)H(t,s)\varphi(s)\rho(s)-\frac{1}{4}\frac{|h(t,s)|^2\varphi(s)}{F(s)H(t,s)\rho(s)}\right]ds$$
$$\geq A(T),$$

where $A_+(s)=\max\{A(s),0\}$, then every solution of the boundary value problem (1.1), (1.2) *is oscillatory in G.*

In Theorem 3, by choosing $\rho(s)=\varphi(s)\equiv 1$, we get the following corollary.

Corollary 3. *Assume that the conditions of Theorem 3 hold and assume that $\rho(s) = \varphi(s)\equiv 1$. If*

$$\limsup_{t\to+\infty}\frac{1}{H(t,T)}\int_{T}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[G(s)H(t,s)-\frac{1}{4}\frac{|h(t,s)|^2}{F(s)H(t,s)}\right]ds \geq A(T),$$

for every $T\geq t_0$, where $A_+(s)=\max\{A(s),0\}$, then every solution of the boundary value problem (1.1), (1.2) *is oscillatory in G.*

Similar to Corollary 2, we can obtain the following corollary from Corollary 3.

Corollary 4. *Assume that the conditions of Theorem 3 hold, and*

$$\limsup_{t\to+\infty}\frac{1}{(t-t_0)^{n-1}}\int_{t_0}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\frac{(n-1)^2}{(t-s)^2F(s)}ds < \infty.$$

If there exists an integer $n>2$ and function $A(t)\in C([0,+\infty),\mathbb{R})$ such that

$$\limsup_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)F(s)(A_+(s))^2ds = \infty,$$

and for every $T\geq t_0$

$$\limsup_{t\to+\infty}\frac{1}{(t-t_0)^{n-1}}\int_{T}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[G(s)(t-s)^{n-1}-\frac{1}{4}\frac{(n-1)^2}{(t-s)^2F(s)}\right]ds \geq A(T),$$

where $A_+(s)=\max\{A(s),0\}$, then every solution of the boundary value problem (1.1), (1.2) *is oscillatory in G.*

4 Oscillation of the problem (1.1), (1.3)

In this section, we establish sufficient conditions for the oscillation of all solutions of the problem (1.1), (1.3).

Lemma 5. *If the functional impulsive differential inequality*

$$\left.\begin{aligned} &\tilde{Z}^{(m)}(t) + G(t)\tilde{Z}(\theta(t)) \le 0, \quad t \neq t_k, \\ &a_k^{(i)} \le \frac{\dfrac{\partial^{(i)}\tilde{Z}(t_k^+)}{\partial t^{(i)}}}{\dfrac{\partial^{(i)}\tilde{Z}(t_k)}{\partial t^{(i)}}} \le b_k^{(i)}, \quad k = 1,2,..., \quad i = 0,1,2,...,m-1, \end{aligned}\right\} \tag{4.1}$$

has no eventually positive solution, then every solution of the boundary value problem defined by (1.1), (1.3) is oscillatory in G.

Proof. Assume that there exist a nonoscillatory solution $u(x,t)$ of the boundary value problem (1.1), (1.3) and $u(x,t) > 0$. By the assumption (H_3), that there exists a $t_1 > t_0 > 0$ such that $\tau(t,\xi) \ge t_0$, $\sigma(t,\xi) \ge t_0$ for $(t,\xi) \in [t_1,+\infty) \times [a,b]$ and $\rho_j(t) \ge t_0$, $j = 1,2,...,n$ for $t \ge t_1$, then

$$\begin{aligned} &u(x,\tau(t,\xi)) > 0, \quad \text{for} \quad (x,t,\xi) \in \Omega \times [t_1,+\infty) \times [a,b], \\ &u(x,\sigma(t,\xi)) > 0, \quad \text{for} \quad (x,t,\xi) \in \Omega \times [t_1,+\infty) \times [a,b], \\ &u(x,\rho_j(t)) > 0, \quad \text{for} \quad (x,t) \in \Omega \times [t_1,+\infty), \quad j = 1,2,...,n. \end{aligned}$$

Multiplying both sides of equation (1.1) by $1/|\Omega|$ and integrating with respect to x over the domain Ω, we obtain

$$\left.\begin{aligned} &\frac{d^m}{dt^m}\left[\frac{1}{|\Omega|}\int_\Omega u(x,t)dx + \frac{1}{|\Omega|}\int_\Omega\int_a^b g(t,\xi)u(x,\tau(t,\xi))d\eta(\xi)dx\right] \\ &+\frac{1}{|\Omega|}\int_\Omega\int_a^b q(x,t,\xi)f(u(x,\sigma(t,\xi)))d\eta(\xi)dx \\ &= a(t)\frac{1}{|\Omega|}\int_\Omega \Delta u(x,t)dx + \sum_{j=1}^n b_j(t)\frac{1}{|\Omega|}\int_\Omega \Delta u(x,\rho_j(t))dx. \end{aligned}\right\} \tag{4.2}$$

By Green's formula and boundary condition (1.3),

$$\int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega}\frac{\partial u}{\partial\gamma}dS = -\int_{\partial\Omega}\mu(x,t)u(x,t)dS \le 0, \tag{4.3}$$

and for $j = 1,2,...,n$,

$$\int_\Omega \Delta u(x,\rho_j(t))dx = \int_{\partial\Omega}\frac{\partial u(x,\rho_j(t))}{\partial\gamma}dS = -\int_{\partial\Omega}\mu(x,\rho_j(t))u(x,\rho_j(t))dS \le 0 \tag{4.4}$$

where dS is surface element on $\partial\Omega$. Also from (H_2) and Jensen's inequality, we have

$$\begin{aligned}\frac{1}{|\Omega|}\int_\Omega\int_a^b & q(x,t,\xi)f(u(x,\sigma(t,\xi)))d\eta(\xi)dx\\ &\geq \int_a^b Q(t,\xi)\frac{1}{|\Omega|}\int_\Omega f(u(x,\sigma(t,\xi)))dxd\eta(\xi)\\ &\geq c\int_a^b Q(t,\xi)\tilde{V}(\sigma(t,\xi))d\eta(\xi). \qquad (4.5)\end{aligned}$$

In view of (4.2)-(4.5), yield

$$\frac{d^m}{dt^m}\left[\tilde{V}(t)+\int_a^b g(t,\xi)\tilde{V}(\tau(t,\xi))d\eta(\xi)\right]+c\int_a^b Q(t,\xi)\tilde{V}(\sigma(t,\xi))d\eta(\xi)\leq 0. \qquad (4.6)$$

Set $\tilde{Z}(t)=\tilde{V}(t)+\int_a^b g(t,\xi)\tilde{V}(\tau(t,\xi))d\eta(\xi)$. Equation (4.6), can be written as

$$\tilde{Z}^{(m)}(t)+c\int_a^b Q(t,\xi)\tilde{V}(\sigma(t,\xi))d\eta(\xi)\leq 0, \quad t\neq t_k.$$

The rest of the proof is similar to the proof of Lemma 4, and hence the details are omitted. □

As in the proofs of the results in Section 3, we can also obtain the following results for (1.1), (1.3).

Theorem 4. *If there exists a function $\tilde{\varphi}(t)\in C'([0,+\infty),(0,+\infty))$ which is nondecreasing with respect to t, such that*

$$\int_{t_1}^{+\infty}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\left[\tilde{\varphi}(s)G(s)-\frac{(\tilde{\varphi}'(s))^2}{4F(s)\tilde{\varphi}(s)}\right]ds=\infty,$$

then every solution of the boundary value problem (1.1), (1.3) *is oscillatory in G.*

Theorem 5. *Assume that there exist functions $\tilde{\varphi}(t)$ and $\tilde{\rho}(s)\in C^1([0,+\infty),(0,+\infty))$ such that $\tilde{\varphi}(t)$ is nondecreasing. Assume that there exist two functions $H(t,s)$, $h(t,s)\in C^1(D,\mathbb{R})$, in which $D=\{(t,s)|t\geq s\geq t_0>0\}$, such that $(H_7)-(H_9)$ hold. If*

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\prod_{t_0\leq t_k<s}\left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1}\tilde{\Pi}(s)ds=\infty,$$

where

$$\tilde{\Pi}(s)=G(s)\tilde{\varphi}(s)H(t,s)\tilde{\rho}(s)-\frac{1}{4}\frac{|h(t,s)|^2\tilde{\varphi}(s)}{F(s)H(t,s)\tilde{\rho}(s)},$$

then every solution of the boundary value problem (1.1), (1.3) *is oscillatory in G.*

By choosing $\tilde{\rho}(s) = \tilde{\varphi}(s) \equiv 1$, we have the following corollary.

Corollary 5. *Assume that the conditions $(H_7) - (H_9)$ hold, and*

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \prod_{t_0 \le t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \Gamma(s)ds = \infty,$$

then every solution of the boundary value problem (1.1), (1.3) *is oscillatory in G.*

Remark 2. *From Theorem 5 and Corollary 5, we can attain various oscillatory criteria by different choices of the weighted function $H(t,s)$. For example, choosing $H(t,s) = (t-s)^{n-1}$, $t \ge s \ge t_0$, in which $n > 2$ is an integer, then $h(t,s) = (n-1)(t-s)^{(n-3)/2}$, $t \ge s \ge t_0$. From Corollary 5, we get the following result.*

Corollary 6. *If there exists an integer $n > 2$ such that*

$$\limsup_{t\to+\infty} \frac{1}{(t-t_0)^{n-1}} \int_{t_0}^{t} \prod_{t_0 \le t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left[G(s)(t-s)^{n-1} - \frac{1}{4} \frac{(n-1)^2}{(t-s)^2 F(s)} \right] ds = +\infty, \tag{4.7}$$

then every solution of the boundary value problem (1.1), (1.3) *is oscillatory in G.*

Theorem 6. *Let the functions $H(t,s)$, $h(t,s)$, $\tilde{\varphi}(s)$ and $\tilde{\rho}(s)$ be as defined in Theorem 5. Additionally, suppose that $0 < \inf_{s \ge t_0} \left\{ \liminf_{t\to+\infty} \frac{H(t,s)}{H(t,t_0)} \right\} \le +\infty$, and*

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \prod_{t_0 \le t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \frac{|h(t,s)|^2 \tilde{\varphi}(s)}{F(s)H(t,s)\tilde{\rho}(s)} ds < +\infty.$$

If there exists a function $\tilde{A}(t) \in C([t_0,+\infty),\mathbb{R})$ such that

$$\limsup_{t\to+\infty} \int_{t_0}^{t} \prod_{t_0 \le t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right) \frac{F(s)(\tilde{A}_+(s))^2}{\tilde{\rho}(s)\tilde{\varphi}(s)} ds = +\infty,$$

and for every $T \ge t_0$

$$\limsup_{t\to+\infty} \frac{1}{H(t,T)} \int_{T}^{t} \prod_{t_0 \le t_k < s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}} \right)^{-1} \left[G(s)H(t,s)\tilde{\varphi}(s)\tilde{\rho}(s) - \frac{1}{4} \frac{|h(t,s)|^2 \tilde{\varphi}(s)}{F(s)H(t,s)\tilde{\rho}(s)} \right] ds$$
$$\ge \tilde{A}(T),$$

where $\tilde{A}_+(s) = \max\{\tilde{A}(s), 0\}$, then every solution of the boundary value problem (1.1), (1.3) *is oscillatory in G.*

By choosing $\tilde{\rho}(s) = \tilde{\varphi}(s) \equiv 1$, we attain the following corollary.

Corollary 7. *Assume that the conditions of Theorem 6 hold and assume that* $\tilde{\rho}(s) = \tilde{\varphi}(s) \equiv 1$*. If*

$$\limsup_{t\to+\infty} \frac{1}{H(t,T)} \int_T^t \prod_{t_0\le t_k<s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \left[G(s)H(t,s) - \frac{1}{4}\frac{|h(t,s)|^2}{F(s)H(t,s)}\right] ds \ge \tilde{A}(T),$$

for every $T \ge t_0$*, then every solution of the boundary value problem* (1.1), (1.3) *is oscillatory in* G*.*

Similar to Corollary 6, we can obtain the following corollary from Corollary 7.

Corollary 8. *Assume that the conditions of Theorem 6 hold, and*

$$\limsup_{t\to+\infty} \frac{1}{(t-t_0)^{n-1}} \int_{t_0}^t \prod_{t_0\le t_k<s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \frac{(n-1)^2}{(t-s)^2F(s)} ds < \infty.$$

If there exists an integer $n > 2$ *and function* $\tilde{A}(t) \in C([0,+\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to+\infty} \int_{t_0}^t \prod_{t_0\le t_k<s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right) F(s)(\tilde{A}_+(s))^2 ds = \infty,$$

and for every $T \ge t_0$

$$\limsup_{t\to+\infty} \frac{1}{(t-t_0)^{n-1}} \int_T^t \prod_{t_0\le t_k<s} \left(\frac{b_k^{(m-1)}}{a_k^{(0)}}\right)^{-1} \left[G(s)(t-s)^{n-1} - \frac{1}{4}\frac{(n-1)^2}{(t-s)^2F(s)}\right] ds \ge \tilde{A}(T),$$

then every solution of the boundary value problem (1.1), (1.3) *is oscillatory in* G*.*

5 Examples

In this section, we present couple of examples to point up our results established in Section 3 and Section 4.

Example 1. *Consider the following equation*

$$\left.\begin{aligned} &\frac{\partial^4}{\partial t^4}\left(u(x,t) + \frac{2}{3}\int_{-\pi/2}^{-\pi/4} u(x,t+2\xi)d\xi\right) + \frac{5}{3}\int_{-\pi/2}^{-\pi/4} u(x,t+2\xi)d\xi \\ &\qquad = \frac{1}{6}\Delta u(x,t) + \frac{7}{6}\Delta u(x,t-\tfrac{3\pi}{2}), \quad t>1,\ t\ne t_k,\ k=1,2,..., \\ &u(x,(t_k)^+) = \frac{k+1}{k}u(x,t_k), \\ &\frac{\partial^{(i)}}{\partial t^{(i)}}u(x,(t_k)^+) = \frac{\partial^{(i)}}{\partial t^{(i)}}u(x,t_k), \quad i=1,2,3, \quad k=1,2,..., \end{aligned}\right\} \tag{5.1}$$

for $(x,t) \in (0,\pi)\times[0,+\infty)$*, with the boundary condition*

$$u(0,t) = u(\pi,t) = 0, \quad t \ne t_k. \tag{5.2}$$

Here $\Omega=(0,\pi),\ m=4,\ n=1,\ a_k^{(0)}=b_k^{(0)}=\dfrac{k+1}{k},\ a_k^{(i)}=b_k^{(i)}=1,\ i=1,2,3,$

$$g(t,\xi)=\frac{2}{3},\ Q(t,\xi)=\frac{5}{3},\ f(u)=u,\tau(t,\xi)=\sigma(t,\xi)=t+2\xi,\ a(t)=\frac{1}{6},$$

$$b_1(t)=\frac{7}{6},\quad \rho_1(t)=t-\frac{3\pi}{2},\quad \eta(\xi)=\xi,\ \theta(t)=t,\quad \theta'(t)=1,\quad c=1.$$

Also $G(s)=\dfrac{5\pi}{12}-\dfrac{5\pi^2}{72},\ F(s)=s^2$. *Since* $t_0=1,\ t_k=2^k,\ g_0=1-\dfrac{\pi}{6}$, *we see from the above assumption that the hypotheses* $(H_1)-(H_6)$ *hold, moreover*

$$\begin{aligned}
&\lim_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0\le t_k<s}\frac{a_k^{(0)}}{b_k^{(i)}}ds=\int_1^{+\infty}\prod_{1<t_k<s}\frac{k}{k+1}ds\\
&=\int_1^{t_1}\prod_{1<t_k<s}\frac{k}{k+1}ds+\int_{t_1^+}^{t_2}\prod_{1<t_k<s}\frac{k}{k+1}ds+\int_{t_2^+}^{t_3}\prod_{1<t_k<s}\frac{k}{k+1}ds+\cdots\\
&=1+\frac{1}{2}\times 2+\frac{1}{2}\times\frac{2}{3}\times 2^2+\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times 2^3+\cdots\\
&=\sum_{n=0}^{+\infty}\frac{2^n}{n+1}=\infty.
\end{aligned}$$

Now, the condition (3.12) *reads,*

$$\limsup_{t\to+\infty}\frac{1}{(t-1)^2}\left\{\int_1^t\prod_{1<t_k<s}\frac{k}{k+1}\left[\left(\frac{5\pi}{12}-\frac{5\pi^2}{72}\right)(t-s)^2-\frac{1}{s^2(t-s)^2}\right]ds\right\}=+\infty.$$

Therefore all the conditions of the Corollary 2 are satisfied. Therefore, every solution of equation (5.1)-(5.2) *is oscillatory in* G. *In fact* $u(x,t)=\sin x\cos t$ *is such a solution.*

Example 2. *Consider the following equation of the form*

$$\left.\begin{aligned}
&\frac{\partial^2}{\partial t^2}\left(u(x,t)+\frac{1}{2(t+1)}\int_{-\pi}^{0}u(x,t+\xi)d\xi\right)+\frac{1}{2(t+1)}\int_{-\pi}^{0}u(x,t+\xi)d\xi\\
&\quad=\left(1+\frac{2}{(t+1)^2}\right)\Delta u(x,t)+\frac{2}{(t+1)^3}\Delta u(x,t-\tfrac{7\pi}{2}),\quad t>1,\ t\ne t_k,\\
&u(x,(t_k)^+)=\frac{k+1}{k}u(x,t_k),\ k=1,2,...,\\
&\frac{\partial}{\partial t}u(x,(t_k)^+)=\frac{\partial}{\partial t}u(x,t_k),\quad k=1,2,...,
\end{aligned}\right\}\tag{5.3}$$

for $(x,t)\in(0,\pi)\times[0,+\infty)$, *with the boundary condition*

$$u_x(0,t)=u_x(\pi,t)=0,\quad t\ne t_k.\tag{5.4}$$

Here $\Omega=(0,\pi),\ m=2,\ n=1,\ \mu(x,t)=1,\ a_k^{(0)}=b_k^{(0)}=\dfrac{k+1}{k},\ a_k^{(i)}=b_k^{(i)}=1,\ i=1,$

$$g(t,\xi)=\frac{1}{2(t+1)},\ Q(t,\xi)=\frac{1}{2(t+1)},\quad f(u)=u,\quad F(s)=2s,$$

$$\tau(t,\xi)=\sigma(t,\xi)=t+\xi, \quad a(t)=1+\frac{2}{(t+1)^2}, \; b_1(t)=\frac{2}{(t+1)^3},$$

and $\rho_1(t)=t-\frac{7\pi}{2}$, $\eta(\xi)=\xi$, $\theta(t)=t^2$, $\theta'(t)=2t$, $c=1$. *Since* $t_0=1$, $t_k=2^k$,

$$g_0=1-\frac{1}{2}\log\left(\frac{t+1}{t+1-\pi}\right), \quad G(s)=\frac{\pi}{2(t+1)}\left(1-\frac{1}{2}\log\left(\frac{t+1}{t+1-\pi}\right)\right).$$

From the above assumptions it is easy to see that the hypotheses $(H_1)-(H_6)$ *hold. Still to show that the condition* (4.7) *is satisfied. In fact this condition reads*

$$\limsup_{t\to+\infty}\frac{1}{(t-1)^2}\left\{\int_1^t\prod_{1<t_k<s}\frac{k}{k+1}\left[\frac{\pi}{2(t+1)}\left(1-\frac{1}{2}\log\left(\frac{t+1}{t+1-\pi}\right)\right)(t-s)^2-\frac{1}{2s(t-s)^2}\right]ds\right\} = +\infty.$$

Therefore all the conditions of the Corollary 6 are satisfied. Therefore, every solution of equation (5.3)-(5.4) *is oscillatory in G. In fact* $u(x,t)=\cos x\sin t$ *is such a solution.*

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References

[1] M. Adimy, F. Crauste, Global stability of a partial differential equation with distributed delay due to cellular replication, *Nonlinear Anal.,* **54** (2003), 1469-1491.

[2] M. Adimy, F. Crauste, Modeling and asymptotic stability of a growth factor dependent stem cell dynamics model with distributed delay, *Disc. Con. Dyn. Sys.,* **8** (1) (2007), 19-38.

[3] D. D. Bainov, D. P. Mishev, *Oscillation Theory for Neutral Differential Equations with Delay, Adam Hilger, New York*, 1991.

[4] L. Berezansky, E. Braverman, Oscillation properties of a logistic equation with distributed delay, *Nonlinear Anal. Real World Appl.,* **4** (2003), 1-19.

[5] L. Erbe, H. Freedman, X. Z. Liu, J. H. Wu, Comparison principles for impulsive parabolic equations with application to models of single species growth, *J. Aust. Math. Soc.,* **32** (1991), 382-400.

[6] K. Gopalsamy, B. G. Zhang, On delay differential equations with impulses, *J. Math. Anal. Appl.,* **139** (1989), 110-122.

[7] G. Gui, Z. Xu, Oscillation of even order partial differential equations with distributed deviating arguments, *J. Comput. Appl. Math.,* **228** (2009), 20-29.

[8] G. H. Hardy, J. E. Littlewood, G. Polya, Inequalities, *Cambridge University Press, Cambridge, UK,* 1988.

[9] P. Hartman, A. Wintner, On a comparison theorem for self-adjoint partial differential equations of elliptic type, *Proc. Amer. Math. Soc.,* **6** (1955), 862-865.

[10] I. T. Kiguradze, On the oscillation of solutions of the equation $\frac{d^m u}{dt^m} + a(t)\,|u|^n\, sgnu = 0$, *Math. Sb.,* **65** (1964), 172-187 (in Russian).

[11] I. Kubiaczyk, S. H. Saker, Kamenev-type oscillation criteria for hyperbolic nonlinear delay difference equations, *Demonst. Math.,* **36** (1) (2003), 113-122.

[12] I. Kubiaczyk, S. H. Saker, Oscillation of parabolic delay differential equations, *Demonst. Math.*, **35** (4) (2002), 781-792.

[13] G. S. Ladde, V. Lakshmikantham, B. G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments, Marcel Dekker, Inc, New York,* 1987.

[14] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, *Theory of Impulsive Differential Equations*, *World Scientific Publishers, Singapore,* 1989.

[15] W. N. Li, L. Debnath, Oscillation of higher-order neutral partial functional differential equations, *Appl. Math. Lett.,* **16** (2003), 525-530.

[16] W. N. Li, W. Sheng, Oscillation of certain higher-order neutral partial functional differential equations, *Springer Plus,* (2016), 1-8.

[17] W. X. Lin, Some oscillation theorems for systems of even order quasilinear partial differential equations, *Appl. Math. Comput.,* **152** (2004), 337-349.

[18] G. J. Liu, C. Y. Wang, Forced oscillation of neutral impulsive parabolic partial differential equations with continuous distributed deviating arguments, *Open Access Library Journal,* **1** (2014), 1-8.

[19] Ch. G. Philos, A new criterion for the oscillatory and asymptotic behavior of delay differential equations, *Bull. Acad. Pol. Sci. Ser. Sci. Math.,* **39** (1981), 61-64.

[20] V. S. H. Rao, P. R. S. Rao, Global stability in chemostat models involving time delays and wall growth, *Nonlinear Anal. Real World appl.,* **5** (2004), 141-158.

[21] V. Sadhasivam, J. Kavitha, T. Raja, Forced oscillation of nonlinear impulsive hyperbolic partial differential equation with several delays, *Journal of Applied Mathematics and Physics,* **3** (2015), 1491-1505.

[22] V. Sadhasivam, J. Kavitha, T. Raja, Forced oscillation of impulsive neutral hyperbolic differential equations, *International Journal of Applied Engineering Research,* **11** (1) (2016), 58-63.

[23] S. H. Saker, Oscillation of hyperbolic nonlinear differential equations with deviating arguments, Publ. Math. Debr. **62** (2003), 165-185.

[24] S. H. Saker, Oscillation and global attractivity of impulsive periodic delay respiratory dynamics model, Chinese Annals of Math. **26 B** (2005), 511-522.

[25] R. Sipahi, F. M. Atay, S-I. Niculescu, Stability of traffic flow behavior with distributed delays modeling the memory effects of the drivers, *SIAM J. Appl. Math.,* **68** (3) (2007), 738-759.

[26] C. Sturm, Surles équations différentielles linéaries du second ordre, *J. Math. Pure Appl.,* **1** (1836), 106-186.

[27] S. Tanaka, N. Yoshida, Forced oscillation of certain hyperbolic equations with continuous distributed deviating arguments, *Ann. Polon. Math.,* **85** (2005), 37-54.

[28] E. Thandapani, R. Savithri, On oscillation of a neutral partial functional differential equations, *Bull. Inst. Math. Acad. Sin.,* **31** (4) (2003), 273-292.

[29] V. S. Vladimirov, *Equations of Mathematical Physics*, Nauka, Moscow, 1981.

[30] P. G. Wang, Y. H. Yu, L. Caccetta, Oscillation criteria for boundary value problems of high-order partial functional differential equations, *J. Comput. Appl. Math.,* **206** (2007), 567-577.

[31] J. H. Wu, *Theory and Applications of Partial Functional Differential Equations*, *Springer, New York,* 1996.

[32] N. Yoshida, *Oscillation Theory of Partial Differential Equations*, *World Scientific, Singapore,* 2008.

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[1] PG and Research Department of Mathematics,
Thiruvalluvar Government Arts College,
(Affiliated with Periyar University, Salem - 636 011)
Rasipuram - 637 401, Namakkal (Dt), Tamil Nadu, India.
E-mail: kalaimaths4@gmail.com, trmaths19@gmail.com
ovsadha@gmail.com
and
[2] Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt,
e-mail: shsaker@mans.edu.eg