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Accurate numerical solution for a type of astrophysics equations using three classes of Euler functions

by

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Abstract

This paper presents an efficient numerical algorithm based on rational non-orthogonal basis functions derived from Euler polynomials for solving the Lane-Emden type equations as singular initial value problems on the semi-infinite interval $[0, \infty)$. These equations have been used to model several phenomena in theoretical physics, mathematical physics and astrophysics, such as the theory of stellar structure and radiative cooling. The proposed method is based on converting an ordinary nonlinear differential equation into a sequence of linear differential equations through the quasilinearization method and then are solved using the collocation method. This method reduces the solution of these problems to the solution of a system of algebraic equations to simplify the computations. The purpose of this study is solving the Lane-Emden type equations using three classes of Euler functions and providing an accurate comparison of these basis functions. Furthermore, the rational fractional Euler functions (RFE) have been introduced for the first time. Numerical results show that the performance of the three proposed basis functions is nearly equal and the method is reliable, efficient and accuracy more than numerical and analytical methods.

Key Words: Lane-Emden type equations, Nonlinear ODE, Collocation method, Quasilinearization method, Rational fractional Euler functions.
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Secondary 26A33, 65M70

1 Introduction

Most problems of science and engineering are not solvable exactly and therefore should be investigated with the help of semi-analytical or numerical approximation methods. In recent years, considerable attention has been devoted to the solution of the singular initial value problems for nonhomogeneous, nonlinear differential equations. Many powerful methods have been presented to solve problems in unbounded intervals such as:

Semi-analytical methods: Different Analytical methods such as Adomian decomposition method [1], Homotopy perturbation method [2], Variational iteration method [3], Exp-function method [4], and so on.
Numerical methods: These methods include Finite difference approximation method [5], Finite element method (FEM) [5], Meshfree methods [6], and Spectral methods [7].

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We attempt to introduce a Spectral method based on the rational Euler functions, fractional order of rational Euler functions, and rational fractional Euler functions to solve the problems on semi-infinite interval. The purpose of the present investigation is to obtain an approximation numerical of the Lane-Emden type equations by using basic functions that are originated from Euler polynomials.

Introduction of Lane-Emden equations

Many problems in the literature of mathematical physics and astrophysics can be modelled by Lane-Emden type initial value problems, defined in the form [8]

$$y''(x) + \frac{2}{x}y'(x) + f(x, y) = g(x), \quad x > 0,$$

with the boundary conditions:

$$y(0) = A, \quad y'(0) = B.$$

where A and B are constants and $f(x, y)$ is a nonlinear function of x and y. These equations were applied to describe various phenomena including the thermal history of a spherical cloud of gas, thermionic currents, isothermal gas spheres and the theory of stellar structure [8]. The main difficulty of the Lane-Emden type equations is the singular behavior that occurs at $x = 0$. This problem has been studied by many researchers and has been solved by the different techniques, that number of them is shown in Table 1.

In the present paper, we intend to extend the application of Euler polynomials to solve Lane-Emden type equations. These problems using the quasilinearization method (QLM) converts to the sequence of linear ordinary differential equations to obtain the solution. In addition, the equation will be solved on a semi-infinite interval by taking rational Euler functions (RE), the fractional order of rational Euler functions (FRE) [18] and rational fractional Euler functions (RFE) as basis functions for the collocation method. The objective of this paper is comparison between Three Spectral approaches based on rational non-orthogonal basis functions for the solution of singular initial value problems.
The organization of the current paper is as follows: in section 2, the proposed trial basis functions and their properties on the semi-infinite interval are defined. In section 3, the general procedure of method together with an algorithm is explained. In section 4, we report our numerical findings and demonstrate the validity, accuracy and applicability of the numerical methods by considering examples. Also, a conclusion is given in the last section.

2 Main ideas of the trial basis functions

The Lane-Emden type equations are nonlinear ordinary differential equations on the semi-infinite interval. As discussed before, we can apply different Spectral basis that are used to solve problems in the semi-infinite interval. Since the classical Euler polynomials are defined on the interval [0,1], the rational Euler functions (RE) on semi-infinite interval have been defined as follows [18]:

$$RE_n(x, L) = E_n(\frac{x}{x + L}),$$

Table 1: Lane-Emden type equations bibliography

Year	Ref	Comment
1986	Horedt [9]	The exact calculation equations to 7 decimal places.
1996	Liu [10]	Providing an approximate analytic solution of equation
2001	Wazwaz [1]	ADM with the modified structure
2003	He [3]	Ritz method
2007	Ramos [11]	Piecewise-adaptive decomposition methods
2009	Singh et al [12]	The modified homotopy analysis method
2011	Boyd [13]	Chebyshev Spectral Methods
2012	Mutsa & Shateyi [14]	A successive linearization method
2013	Parand et al [15]	Using Bessel orthogonal functions collocation method
2015	Azarnavid et al [16]	Picard-Reproducing kernel Hilbert space method
2017	Parand & Delkhosh [17]	Generalized fractional order of the Chebyshev functions

and an analytical form of $RE_n(x,L)$ for $n=0,1,\cdots$ is as follows:

$$RE_n(x,L)=\frac{1}{n+1}\sum_{k=1}^{n+1}(2-2^{k+1})\binom{n+1}{k}B_k\left(\frac{x}{x+L}\right)^{n+1-k}.$$

Also, the fractional order of rational Euler functions (FRE) on semi-infinite interval have been defined as follows [18]:

$$FRE_n^{\alpha}(x,L)=\frac{1}{n+1}\sum_{k=1}^{n+1}(2-2^{k+1})\binom{n+1}{k}B_k\left(\frac{x^\alpha}{x^\alpha+L}\right)^{n+1-k},$$

in order to solve the problems on the semi-infinite interval.
A new Spectral basis, namely rational fractional Euler functions, denoted by RFE, on the semi-infinite interval is constructed. By applying the algebraic mapping $x\to(\frac{x}{x+L})^\alpha$ to the Euler function, a new Spectral basis RFE on the interval $[0,\infty)$ is defined as follows:

$$RFE_n^{\alpha}(x,L)=FE_n(\frac{x}{x+L})=E_n\big((\frac{x}{x+L})^\alpha\big), \tag{2.1}$$

an analytical form of $RFE_n(x,L)$ for $n=0,1,\cdots$ is presented as follows:

$$RFE_n^{\alpha}(x,L)=\frac{1}{n+1}\sum_{k=1}^{n+1}(2-2^{k+1})\binom{n+1}{k}B_k\left(\frac{x}{x+L}\right)^{\alpha(n+1-k)},$$

where L is a constant parameter and sets the length scale of the mapping.

Now, $\Gamma=\{x|0\le x\le\infty\}$ is defined and
$L^2_w(\Gamma)=\{\mu:\Gamma\longrightarrow\mathbb{R}|\ \mu\ \ \textit{is measurable and}\ \|\mu\|_w<\infty\}$, where

$$\|\mu\|_w^2=\int_0^\infty|\mu(x)|^2w(x)\ dx,\qquad w(x)=\frac{L\alpha x^{\alpha-1}}{(x+L)^{\alpha+1}}, \tag{2.2}$$

is the norm induced by the inner product of the space $L^2(\Gamma)$,

$$\langle\nu,\mu\rangle_w=\int_0^\infty\nu(x)\mu(x)w(x)\ dx. \tag{2.3}$$

Now, we assume $\mathcal{RFB}_n = \{RFE_0^\alpha(x), RFE_1^\alpha(x), \cdots, RFE_n^\alpha(x)\}$, is finite dimensional subspace, therefore $\mathcal{RFB}_n$ is a complete subspace of $L^2(\Gamma)$. The interpolating function of a smooth function ν on a semi-infinite interval is denoted by $\epsilon_n \nu$. It is an element of $\mathcal{RFB}_n$ and

$$\epsilon_n \nu = \sum_{k=0}^{n} a_k RFE_k^\alpha(x),$$

If $\epsilon_n \nu$ is the best projection of ν upon $\mathcal{RFB}_n$ with respect to the inner product Eq. (2.3) and the norm Eq. (2.2). Then, we have

$$\langle \xi_n \nu - \nu, RFE_i^\alpha(x) \rangle = 0 \qquad \forall\ RFE_i^\alpha(x) \in \mathcal{RFB}_n.$$

3 The solution technique

In this section, a reliable algorithm that consists of two distinct approaches to handle in a realistic and efficient way the Lane-Emden type equations is introduced. In general the Lane-Emden type equations are formulated as

$$y''(x) + \tfrac{k}{x} y'(x) + f(x,y) = h(x), \qquad k, x \geqslant 0, \tag{3.1}$$
$$y(0) = A, \quad y'(0) = B,$$

where k, A and B are real constants and $f(x,y)$ and $h(x)$ are some given functions.
By using the QLM , the solution general Lane-Emden type equations determines the $(r+1)th$ iterative approximation $y_{r+1}(x)$ as a solution of the linear differential equation:

$$y_{r+1}^{''} + \frac{k}{x}\, y_r' + f(x, y_r) - h(x) - (y_{r+1} - y_r)\, f_y(x,y) + \frac{k}{x}\left(y_{r+1}' - y_r'\right) = 0, \tag{3.2}$$

with the boundary conditions:

$$y_{r+1}(0) = A, \qquad y_{r+1}'(0) = B. \tag{3.3}$$

In the other approach, the rational Euler collocation (REC) method, the rational fractional Euler collocation (RFEC) method, and the fractional order of rational Euler collocation (FREC) method are employed for the approximate solutions of the Lane-Emden type equations.
Suppose that $\phi_n(x, L)$ is one of the functions of RE, FRE, or RFE. We employ the $\phi_n(x, L)$ collocation method for solving the linear differential equations at each iteration in Eq. (3.2) with the boundary conditions Eq. (3.3).
We suppose that $y_0(x) \equiv A$. At the first step, the trial solution for the $(r+1)th$ iterative has been constructed as follows:

$$u_{n,r}(x) = \sum_{i=0}^{n} a_{i,r}\ \phi_i(x, L), \tag{3.4}$$

where n is a positive integer and our goal is to find the coefficients $a_{i,r}$.
In the next step, the boundary conditions in Eq. (3.3) are satisfied:

$$y_{n,r+1}(x) = A + Bx + x^2\ u_{n,r}(x). \tag{3.5}$$

By substituting Eq. (3.5) in Eq. (3.2), we form residual function at each iteration QLM as follows:

$$Res_r(x) = y''_{n,r+1} + \frac{k}{x}\, y'_{n,r} + f(x, y_{n,r}) - h - (y_{n,r+1} - y_{n,r}(x))\, f_y(x,y) + \frac{k}{x}(y'_{n,r+1} - y'_{n,r}). \quad (3.6)$$

The main goal in the collocation method is to minimize the residual function for calculating unknown coefficients . By putting the arbitrary collocation points $\{x_j\}_{j=0}^{n}$ in Eq. (3.6), we have a system of the $n+1$ linear equations at each iteration that can be solved by the Newton's method for unknown coefficients $a_{i,r}$. The described method is summarized in the following algorithm:

Begin
1. Input: n, L, $max\_iteration$, and α for FRE and RFE .
2. Selection an initial guess (We suppose that $y_0(x) \equiv$ A).
For $r = 0, 1, 2, \cdots, max\_iteration$ do
3. Calculation a linear combination of trial functions
4. Satisfy the boundary conditions by calculating function
5. Calculation residual function $Res_r(x)$
6. Creating of the system of equations by putting the points $\{x_j\}_{j=0}^{n}$ in $Res_r(x)$.
7. Solving the system of linear equations.
End For
End

In steps 1 and 2, the order of complexity is $O(1)$. In step 3, according to Eq. (3.4), the order of complexity is $O(n)$. In step 4, the order of complexity is $O(1)$. In step 5, according to Eq. (3.6), the order of complexity is $O(n^2)$. The order of complexity in step 6 is $O(n)$. The order of complexity in step 7 is dependent on the method of choice for solving these systems, we have applied the Newton's method, so the order of complexity in this step is $O(n^2)$. The order of complexity in the FOR loop in $O(n)$. Thus, the order of complexity in the above algorithm is at least $O(n^3)$.

4 Solution of nonlinear initial value problems: Comparison of REC, FREC, and RFEC

In what follows several numerical examples are given to illustrate the performance and reliability of the present methods of solution. The results are tabulated and compared with the accurate results of other methods.
In this study, the roots of the generalized fractional order of the Chebyshev functions of first kind on the interval $[0, L]$ (i.e. $x_j = L((1 - \cos(\frac{(2j-1)\pi}{2(n+1)})/2)^{\frac{1}{\alpha}},\ \ j = 1, 2, \cdots, n+1)$ have been used as collocation points and all of the computations have been done by software Maple 18.

4.1 Example 1 (The standard Lane-Emden equation)

For $k = 2$, $f(x, y) = y^m(x)$, $h(x) = 0$, $A = 1$ and $B = 0$, Eq. (3.1) has been defined the standard Lane-Emden equation that was used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics.

$$y''(x) + \frac{2}{x}y'(x) + y^m(x) = 0 \qquad x \geq 0, \tag{4.1}$$
$$y(0) = 1, \qquad y'(0) = 0,$$

where m is a real constant. In Eq. (4.1) for $m = 0, 1$, and 5 exist the exact solution, but in other cases, there is not any exact analytical solution.
By using Eqs. (3.2) and (3.3), we have:

$$y''_{r+1}(x) + \frac{2}{x}\ y'_{r+1}(x) + m\ y_{r+1}(x)y_r^{(m-1)}(x) - (m-1)y_r^{(m)}(x) = 0, \tag{4.2}$$

with the boundary conditions:

$$y_{r+1}(0) = 1, \qquad y'_{r+1}(0) = 0, \qquad r = 0, 1, 2, \cdots max\_iteration. \tag{4.3}$$

The QLM iteration requires an initialization or "initial guess" $y_0(x)$. We suppose that $y_0(x) \equiv 1$. According to the algorithm that was presented, we form residual function at each iteration QLM as follows:

$$Res_r(x) = y''_{n,r+1}(x) + \frac{2}{x}\ y'_{n,r+1}(x) + m\ y_{n,r+1}(x)y_{n,r}^{(m-1)}(x) - (m-1)y_{n,r}^{(m)}(x). \tag{4.4}$$

In this equation, the FRE and RFE collocation methods with $\alpha = \frac{1}{2}$ are considered. The results show that the method works with all three basis functions effectively to solve the standard Lane-Emden equation. Accurate numerical results for integer and fractional values of the nonlinear exponent m to 22 and 13 decimal places are reported, respectively. Tables 2 and 3 show the comparison of the first zeros of standard Lane-Emden equation, for the REC, FREC and RFEC and numerical results given by Hordet [9], Boyd [13], Motsa & Shateyi [14], and Seidov [19], for $m = 0.5, 1, 1.5,\ 2, 2.5, 3, 3.5$, and 4 with N = 45, $10th$ iteration, and $\alpha = 0.5$. Tables 4 and 5 have presented the approximations of $y(x)$ for the standard Lane-Emden equation for $m = 1.5, 2, 2.5, 3, 3.5$, and 4 with $n = 45$, $10th$ iteration, and $\alpha = 0.5$ by using the proposed methods in this paper and Horedt [9].

Table 2: Comparison of the first zeros of standard Lane-Emden equation.

Method	$m = 2$	$m = 3$	$m = 4$
	$L = 4.35$	$L = 5.35$	$L = 14.97$
REC	4.3528745959461246769 7357	6.8968486193769603754544	14.971546348838095117
FREC	4.352874595946124676 97354	6.8968486193769603754545	14.971546348838095096
RFEC	4.3528745959461246769 7357	6.8968486193769603754545	14.971546348838095125
Ref. [13]	4.352874595946125	6.896848619376960	14.971546348838095
Ref. [14]	4.352874595946125	6.896848619376960	14.971546348838095
Ref. [19]	4.352874595946124	6.896848619376960	14.971546348838093
Ref. [9]	4.35287460	6.89684862	14.9715463

Table 3: Comparison of the first zeros of standard Lane-Emden equations.

$Method$	$m = 0.5$	$m = 1.5$	$m = 2.5$	$m = 3.5$
	$L = 2.75$	$L = 3.65$	$L = 5.35$	$L = 9.53$
REC	2.75269802904624	3.65375373627426	5.35527545901001	9.53580534424485
$FREC$	2.75269807561046	3.65375373626527	5.35527545900993	9.53580534424485
$RFEC$	2.75269801114752	3.65375373691605	5.35527545900690	9.53580534424487
Ref. [13]	2.752698054065	3.653753736219	5.355275459011	9.535805344245
Ref. [14]	2.752698054065	3.653753736219	5.355275459011	9.535805344245
Ref. [9]	2.75269805	3.65375374	5.35527546	9.53580534

Table 4: Obtained values of $y(x)$ for standard Lane-Emden $m = 2, 3$, and 4.

m	x	REC	FREC	RFEC	Hordet
2	0.10	0.9983349985461484173	0.9983349985461484173	0.9983349985461484181	0.9983350
	0.50	0.9593527158033827008	0.9593527158033827008	0.9593527158033827009	0.9593527
	1.00	0.8486541114082496769	0.8486541114082496769	0.8486541114082496769	0.8486541
	3.00	0.2418240830523409167	0.2418240830523409167	0.2418240830523409167	0.2418241
	4.30	6.8109432742058300e-3	6.81094327420583009e-3	6.81094327420583009e-3	6.810943e-3
	4.35	3.6603017936128573e-4	3.66030179361285732e-4	3.66030179361285732e-4	3.660302e-4
3	0.10	0.9983358295691694936	0.9983358295691694935	0.9983358295691694935	0.9983358
	0.50	0.9598390699448517235	0.9598390699448517233	0.9598390699448517233	0.9598391
	1.00	0.8550575685886263114	0.8550575685886263113	0.8550575685886263113	0.8550576
	5.00	0.1108198351396255988	0.1108198351396255988	0.1108198351396255988	0.1108198
	6.80	4.16778936545346001e-3	4.16778936545346001e-3	4.16778936545346001e-3	4.167789e-3
	6.896	3.60111454367096332e-5	3.60111454367096332e-5	3.60111454367096332e-5	3.601115e-5
4	0.10	0.9983366595395740477	0.9983366595395740127	0.9983366595395740127	0.9.983367
	0.20	0.9933862135323675949	0.9933862135323689176	0.9983366595395742818	0.9.933862
	0.50	0.9603109023422213094	0.9603109023422199490	0.9933862135323710561	0.9.603109
	5.00	0.2359227310424867981	0.2359227310424857399	0.2359227310424867992	0.2.359227
	10.0	5.96727415894893291e-2	5.96727415894893320e-2	5.96727415894893288e-2	5.967274e-2
	14.0	8.33052669542489561e-3	8.33052669542489575e-3	8.33052669542489562e-3	8.330527e-4
	14.9	5.76418866213546803e-4	5.76418866213546761e-4	5.76418866213546867e-4	5.764189e-4

Table 5: Obtained values of $y(x)$ for standard Lane-Emden $m = 0.5, 1.5, 2.5, 3.5$ by the present methods for Example 1 (with $n = 45$, 10th iteration, and $\alpha = 0.5$)

m	x	REC	FREC	RFEC	Hordet [9]
0.5	0.10	0.998333750	0.998362576	0.998795459	0.9983338
	0.50	0.958594277	0.958485222	0.958551872	0.9585943
	2.00	0.402579409	0.402579396	0.402578963	0.4025795
	2.70	2.6741151e-2	2.6741176e-2	2.6741534e-2	2.6741e-2
	2.75	1.3502687e-3	1.3502921e-3	1.3502432e-3	1.3502e-3
1.5	0.10	0.998334582	0.998336066	0.997209141	0.9983346
	0.50	0.959103856	0.959104042	0.959356797	0.9591039
	3.00	0.158857608	0.158857607	0.158857608	0.1588576
	3.60	1.1090994e-2	1.1090994e-2	1.1090994e-2	1.1090e-2
	3.65	7.6392419e-4	7.6392419e-4	7.6392430e-4	7.6392e-4
2.5	0.10	0.99833541418937	0.99833541551590	0.99833046983239	0.9983354
	0.50	0.95959775446377	0.95959775498864	0.95959509587174	0.9595978
	4.00	0.13768073302111	0.13768073302075	0.13768073303326	0.1376807
	5.00	2.901918664896e-2	2.901918664896e-2	2.901918664742e-2	2.901919e-2
	5.30	4.259543533953e-3	4.259543533945e-3	4.259543533952e-3	4.259544e-3
3.5	0.10	0.99833624468580	0.99833624468580	0.99833624174357	0.9983362
	0.50	0.96007675581495	0.96007675581502	0.96007675602508	0.9600768
	5.00	0.17868426574896	0.17868426574896	0.17868426575023	0.1786843
	9.00	1.180312152959e-2	1.180312152959e-2	1.180312152959e-2	1.180312e-2
	9.50	7.472340753339e-4	7.472340753339e-4	7.472340753346e-4	7.472341e-4
	9.53	1.207723444758e-4	1.207723444758e-4	1.207723444763e-4	1.207723e-4

Table 6: Obtained values of $y(x)$ for the isothermal gas spheres equation by the present methods for Example 2 (with $n=40$,$\alpha=0.75$, and 10th iteration)

x	ADM	REC	FREC	RFEC
0.1	−0.0016658339	−0.00166583386206	−0.00166583386210	−0.00166583386210
0.2	−0.0066533671	−0.00665336710042	−0.00665336710046	−0.00665336710046
0.5	−0.0411539568	−0.04115395729271	−0.04115395729275	−0.04115395729275
1.0	−0.1588273537	−0.15882767752444	−0.15882767752442	−0.15882767752443
1.5	−0.3380131103	−0.33801942476081	−0.33801942476082	−0.33801942476082
2.0	−0.5599626601	−0.55982300433553	−0.55982300433555	−0.55982300433556
2.5	−0.8100196713	−0.80634087059839	−0.80634087059840	−0.80634087059840

4.2 Example 2 (The isothermal gas spheres equation)

According to Eq. (3.1), if $k=2$, $f(x,y)=e^{y(x)}$, $h(x)=0$, $A=0$ and $B=0$ the isothermal gas spheres equation has been defined as follows:

$$y''(x)+\tfrac{2}{x}y'(x)+e^{y(x)}=0 \qquad x\geq 0, \tag{4.5}$$

$$y(0)=0, \qquad y'(0)=0, \tag{4.6}$$

Davis [8] and Van Gorder [20] have discussed about Eq. (4.5) that can be used to view the isothermal gas spheres, where the temperature remains constant. This equation has been solved by some researchers, for example Wazwaz [1] by using Adomian decomposition method (ADM), and Parand et al. [15] by using Bessel orthogonal functions collocation method. A semi-analytical solution have investigated by Wazwaz [1] and Singh et al. [12] by using ADM and modified Homotopy analysis method (MHAM), respectively:

$$y(x)\simeq -\frac{1}{6}x^2+\frac{1}{5!}x^4-\frac{8}{21.6!}x^6+\frac{122}{81.8!}x^8-\frac{61.67}{459.10!}x^{10}+\cdots. \tag{4.7}$$

According to the algorithm that was presented, we form residual function at each iteration QLM as follows:

$$Res_r(x)=y^{''}_{n,r+1}+\frac{2}{x}\ y'_{n,r+1}+e^{y_{n,r+1}}\big(y_{n,r+1}-y_{n,r}+1\big) \tag{4.8}$$

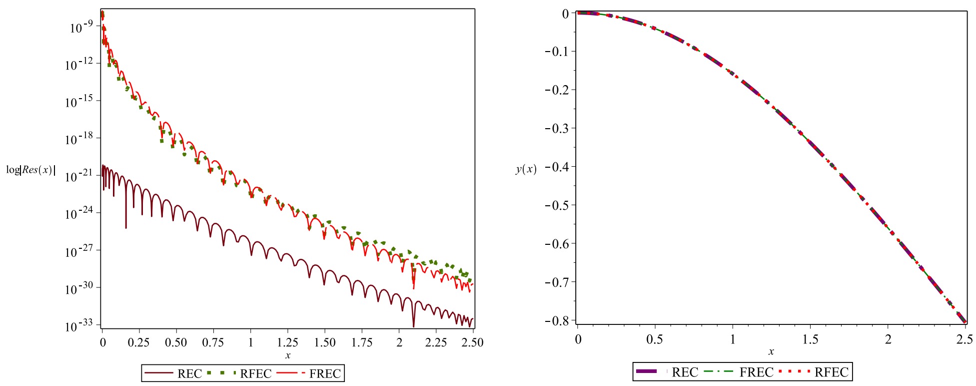
In this equation, the FRE and RFE collocation methods with $\alpha=0.75$, $n=40$, $10th$ iteration, and $L=2.5$ are considered.
Table 6 shows the comparison of $y(x)$ obtained by the proposed methods in this paper and the obtained values by Wazwaz [1]. The resulting graph of the isothermal gas spheres equation in comparison to the presented methods and the Log graph of the residual error of the approximate solution of the isothermal gas spheres equation are shown in Fig. 1.

4.3 Example 3

According to Eq. (3.1), if $f(x,y)=\sinh(y(x))$, $A=1$ and $B=0$ the equation has been defined as follows:

$$y''(x)+\frac{2}{x}y'(x)+\sinh(y(x))=0 \qquad x\geq 0, \tag{4.9}$$

$$y(0)=1, \qquad y'(0)=0,$$



(a) Graphs of residual error (b) Graph of solutions

Figure 1: Graphs of solution and the residual error for Example 2

Table 7: Obtained values of $y(x)$ for the Lane-Emden equation by the present method for Example 3 (with $n = 50$, $\alpha = 0.75$, and $10th$ iteration)

x	ADM	REC	FREC	RFEC
0.1	0.9980428414	0.998042841444807	0.998042841444802	0.998042841444801
0.2	0.9921894348	0.992189434812197	0.992189434812192	0.992189434812192
0.5	0.9519611019	0.951961092744912	0.951961092744907	0.951961092744907
1.0	0.8182516669	0.818242928490512	0.818242928490518	0.818242928490518
1.5	0.6258916077	0.625438763484943	0.625438763484941	0.625438763484940
2.0	0.4136691039	0.406622887545649	0.406622887545648	0.406622887545648

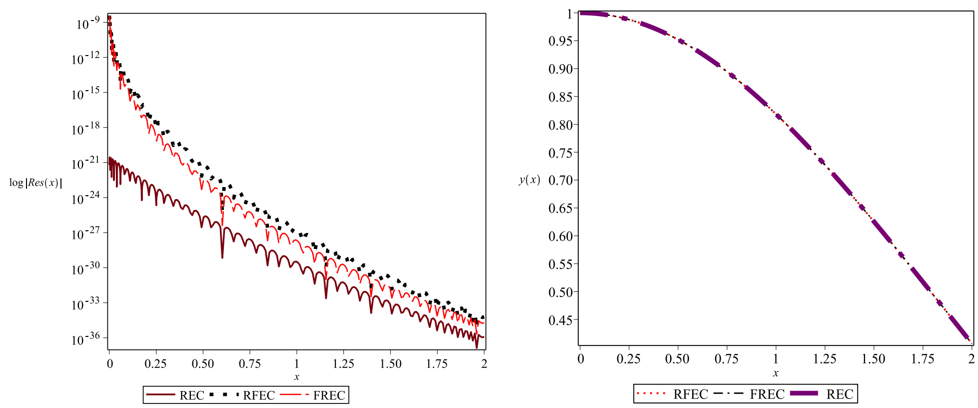
Using Adomian decomposition method, Wazwaz [1] has calculated

$$y(x) \simeq 1 - \frac{(e^2+1)x^2}{12e} + \frac{1}{480}\frac{(e^4-1)x^4}{e^2} - \frac{1}{30240}\frac{(2e^6+3e^2-3e^4-2)x^6}{e^3} + \cdots .$$

According to the algorithm that was presented, we form residual function at each iteration QLM as follows:

$$Res_r(x) = y''_{n,r+1} + \frac{2}{x}\, y'_{n,r+1} + \sinh(y_{n,r}) + \big(y_{n,r+1} - y_{n,r}\big)\sinh(y_{n,r}). \tag{4.10}$$

In this equation, the FRE and RFE collocation methods with $\alpha = 0.75$, $n = 50$, $10th$ iteration, and $L = 2$ are used.
Table 7 shows the comparison of $y(x)$ obtained by the proposed methods in this paper and the obtained values by Wazwaz [1]. The resulting graph of the isothermal gas spheres equation in comparison to the presented methods and the Log graph of the residual error of the approximate solution of the equation are shown in Fig. 2.



(a) Graphs of residual error

(b) Graph of solutions

Figure 2: Graphs of solution and the residual error for Example 3

5 Conclusions

In this paper, a powerful and efficient technique is presented to solve the Lane-Emden type equations that model many phenomena in mathematical physics and astrophysics. They are nonlinear differential equations and have a singularity at the origin. A numerical method is suggested based on a hybrid of quasilinearization method and collocation method. A comparison was made between three collocation methods using the Euler functions. The comparison indicates that the performance is almost equal to the three basic functions and give very accurate results and performs better than other numerical methods that have previously been applied to solve these problems. Moreover, for the first time, the rational fractional Euler functions (RFE) have been introduced. We can conclude that the combination of quasilinearization method and the collocation method based on rational basis functions originated from Euler polynomials is an efficient and accurate method. Accurate of the approximate results to 22 decimal places are reported for integer values of the nonlinear exponent m in the standard Lane-Emden equation. The results presented in this paper can easily be extended to other initial and boundary value problems which are difficult to solve using other numerical methods.

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