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Positive cones of numerical cycle classes
by
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Dedicated to Dorin Popescu in honour of his 70th birthday

Abstract

We survey recent results on positive cones of numerical cycle classes. The original contribution is a class of singular examples: projective cones from a point over embedded projective varieties, and a relative version of this.

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1 Introduction

Let X be a projective variety of dimension n over an algebraically closed field. The finite dimensional real Néron–Severi space $N^1(X)$ generated by Cartier divisor classes modulo numerical equivalence contains the chain of closed convex cones

$$\overline{\mathrm{Nef}}^1(X) \subseteq \overline{\mathrm{Mov}}^1(X) \subseteq \overline{\mathrm{Eff}}^1(X).$$

The nef cone (the closure of the ample cone) controls all morphisms $X \to \mathbb{P}^N$ for any $N \geq 0$. In many interesting cases, the nef cones of small birational modifications of X give a chamber decomposition of the movable cone. In the dual vector space $N_1(X)$ generated by curve numerical classes we find $\overline{\mathrm{Mov}}_1(X) \subseteq \overline{\mathrm{Eff}}_1(X)$. Kleiman [14] proved that the pseudo-effective cone of curves $\overline{\mathrm{Eff}}_1(X) \subset N_1(X)$ (also known as the Mori cone of X) is dual to $\overline{\mathrm{Nef}}^1(X)$. Many years later, [2] showed that the dual of $\overline{\mathrm{Eff}}^1(X)$ is the cone of movable curves. The dual of $\overline{\mathrm{Mov}}^1(X)$ is more subtle (cf. [16]).

To a Cartier divisor D on X one associates the line bundle $\mathcal{O}_X(D)$ whose first numerical Chern class is $[D] \in N^1(X)$. To study $[D]$ one can draw information from the cohomology of such line bundles, for example from the linear series $|mD|$ for all $m \geq 1$. Results about curve classes are often deduced by duality from the case of divisors.

For cycles of arbitrary codimension there is no such correspondence with vector bundles, and so the cohomology techniques are missing. Several results that hold for divisors or curves are in fact false in arbitrary (co)dimension. [12, 21] construct an example of an inclusion $Y \subset X$ of complex projective manifolds such that the normal bundle $\mathrm{N}_Y X$ is ample, but mY does not move in an algebraic family inside X for any $m \geq 1$. [3, 20] construct examples showing that the inclusion $\overline{\mathrm{Nef}}^1(X) \subseteq \overline{\mathrm{Eff}}^1(X)$ may fail for classes of codimension 2.

The negative outlook has been corrected recently by Lehmann and the author in [8, 9, 6, 7]. The nef cone may display some pathologies (cf. [3, 20]), however it contains interesting large subcones with good properties. One such subcone is the *pliant* cone. Its construction that we review later was inspired by [13]. Being a subcone of the nef cone, many of the properties of the pliant cone transfer to the larger nef cone. For example the pliant cone generates the "dual" numerical space $N^k(X)$ that it naturally sits in. Also, the pliant cone contains complete intersections in its strict interior. Then these hold true for $\overline{\mathrm{Nef}}^k(X) \subset N^k(X)$ as well. We are able to conclude that the *pseudo-effective* cone $\overline{\mathrm{Eff}}_k(X) \subset N_k(X)$ is pointed (contains no linear subspaces), and we understand its functorial behavior under surjective morphisms of projective varieties.

The pliant cone is a higher-codimension analogue of the closure of the cone of semi-ample divisors. For divisors this is of course the nef cone, but in higher codimension it is sometimes smaller. Several other equivalent interpretations of the nef cone of divisors produce interesting cones inside $\overline{\mathrm{Nef}}^k(X)$. Examples of such are the *universally pseudo-effective* and the *basepoint-free* cones. We will review them later.

There are not many examples of pseudo-effective cones computed in literature. Toric varieties are relatively easy to understand. The author computes them in [10] for projective bundles over curves, relating them to the Harder–Narasimhan filtration of a defining vector bundle. In [9] we computed them for $\mathrm{Hilb}^2\mathbb{P}^2$.

Here we compute positive cones for a class of singular examples, that of (projective) cones. If $X \subseteq \mathbb{P}^N \subset \mathbb{P}^{N+1}$ and $o \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$, the projective cone over X is the join of o and X. More generally, if $f : X \to Y$ is a morphism of projective varieties, we construct a cone C over X with "vertex" Y. We show that the pseudo-effective cones of C are generated by effective cycles in the vertex, and by cones over effective cycles in X.

Among other things, we exhibit a normalization morphism $\nu : \tilde{X} \to X$ of a projective variety that has nontrivial kernel for $\nu_* : N_k(\tilde{X}) \to N_k(X)$. This is surprising because finite maps do not contract subvarieties.

Paper organization In the first section we review the general definition of numerical equivalence, valid on projective varieties with arbitrary singularities. We define the numerical spaces $N_k(X)$ and their duals $N^k(X)$ and explore their functorial properties, also listing the known versions of pullbacks. The following section is a survey of positive cones of cycle classes and dual cycle classes. We explain some of the theorems and conjectures that have motivated our work in [8, 7, 6, 9]. Lastly, in what is mostly original contribution, we compute the numerical groups and the pseudo-effective cones for projective cones.

2 Background on numerical equivalence

The reference here is [11]. Let X be a projective variety of dimension n over an algebraically closed field of arbitrary characteristic. A k-cycle on X is a formal linear combination of k-dimensional closed subvarieties of X. The coefficients are considered in $\mathbb{R}$ by default.

The space of k-cycles modulo rational equivalence (cf. [11, Ch 1.]) is denoted $A_k(X)$. If we were working with $\mathbb{Z}$ coefficients, we would call these the Chow groups. If E is a vector bundle and $m \geq 0$, the Chern class $c_m(E)$ induces a linear function (cf. [11, Ch.3])

$$c_m(E) \cap \_ : A_k(X) \to A_{k-m}(X)$$

for all k. By convention $A_k(X) = 0$ when $k < 0$ and for $k > n$. These actions are commutative and associative, and so induce a graded (if we set $\operatorname{wt} c_m(E) := -m$) linear action of the space of polynomials in Chern classes of vector bundles on $A_*(X) = \bigoplus_k A_k(X)$.

Following [11, Ch.19], say that a k-cycle Z is *numerically trivial* if

$$\deg(P \cap Z) = 0$$

for all homogeneous polynomials of weighted degree k in Chern classes of vector bundles on X. Here $\deg : A_0(X) \to \mathbb{R}$ is the point counting function induced linearly from 0-cycles by sending all points to 1. Similarly we say that P is *numerically trivial* if $\deg(P \cap Z) = 0$ for all k-cycles Z. The *numerical space* $N_k(X)$ is the quotient of the space of k-cycles by the numerically trivial ones. It is a finite dimensional vector space by [11, Example 19.1.4]. The *dual numerical space* $N^k(X)$ is the quotient of the space of homogeneous Chern polynomials of weight k by the numerically trivial ones. Directly from the definition we have

$$N^k(X) = N_k(X)^\vee,$$

which justifies the terminology. When X is also nonsingular, [11, Example 19.1.5] verifies that this definition of numerical equivalence agrees with the classical one given by intersecting with subvarieties of complementary dimension. It follows that $N^k(X) \simeq N_{n-k}(X)$.

For any k-cycle Z, denote by $[Z]$ its class in $N_k(X)$. For any X of dimension n, the space $N_0(X) \simeq \mathbb{R}$ is generated by the class of any point, $N_n(X) \simeq \mathbb{R}$ is generated by the fundamental class $[X]$, while $N^1(X)$ is the real Néron–Severi space of Cartier divisors modulo numerical equivalence.

A rationally trivial cycle is also numerically trivial, hence $N_k(X)$ is naturally a quotient of $A_k(X)$. Multiplication of polynomials makes $N^*(X) = \bigoplus_k N^k(X)$ into a ring. Again by convention $N^k(X) = 0$ when $k < 0$ and $k > n$. The action of Chern classes on $A_*(X)$ respects numerical equivalence. It gives $N_*(X) = \bigoplus_k N_k(X)$ the structure of a $N^*(X)$-module. In particular we obtain "cyclification" maps

$$N^k(X) \to N_{n-k}(X) : \beta \mapsto \beta \cap [X].$$

When X is nonsingular these coincide with the isomorphism mentioned above. In the singular case they are usually not isomorphisms because the source and target may have different dimensions. We will see examples of this phenomenon for projective cones. For $k = 1$ the Cartier-to-Weil map $\_ \cap [X] : N^1(X) \to N_{n-1}(X)$ is injective by [11, 19.3.3]. Recall that if X is normal and projective, $\operatorname{Pic} X$ injects in the Chow group of Weil divisors modulo rational equivalence. For numerical equivalence the normality condition is no longer necessary.

The numerical space $N_k(X)$ (hence also $N^k(X)$) is finitely dimensional over $\mathbb{R}$ by [11, Example 19.1.4]. The associations $X \mapsto N_k(X)$ and $X \mapsto N^k(X)$ are functorial. Let $f : X \to Y$ be a morphism of projective varieties. For any k-dimensional closed subvariety $Z \subseteq X$, set

$$f_*Z = \begin{cases} 0 & \text{, if } \dim f(Z) < k \\ (\deg f|_Z) \cdot f(Z) & \text{, otherwise} \end{cases},$$

where the degree of a dominant map is computed as the degree of the corresponding extension of fields of rational functions. This respects rational and numerical equivalence inducing a linear pushforward $f_* : N_k(X) \to N_k(Y)$ and a covariant functor structure for $N_k(X)$. The naturality of Chern classes induces a linear pullback morphism $f^* : N^k(Y) \to N^k(X)$ and a contravariant functor structure for $N^k(X)$. Pushforward and pullback are related by the *projection formula*:

$$f_*(f^*\beta \cap \alpha) = \beta \cap f_*\alpha \quad \forall\ \beta \in N^k(Y),\ \alpha \in N_*(X).$$

There is one more notable case when we can pullback numerical classes, not just dual numerical classes. If f is an l.c.i. morphism (e.g., a regular embedding), the Gysin Chow pullback $f^!$ of [11, Ch.6] respects numerical equivalence (cf. [11, Example 19.2.3]) and induces $f^* : N_*(Y) \to N_*(X)$. For example we may restrict numerical classes to effective Cartier divisors.

Lemma 2.1. *Let X be a projective variety of dimension n over an algebraically closed field. Let $\imath : V \to X$ be a regular embedding of a closed subvariety of codimension d. Define $\imath_! : N^{k-d}(V) \to N^k(X)$ by $\imath_!(\gamma) \cap \alpha = \gamma \cap \imath^*\alpha$ for all $\alpha \in N_k(X)$. Then for any $\gamma \in N^{k-d}(V)$, any $0 \leq m \leq n$ and $\beta \in N_m(X)$ we have $\imath_!\gamma \cap \beta = \imath_*(\gamma \cap \imath^*\beta)$.*

In particular, when $k = d$, the regularly embedded subvariety V determines a dual *class $\imath_![V] \in N^d(X)$ with the property that $\imath_![V] \cap [X] = [V] \in N_{n-d}(X)$.*

Proof. Let P be a Chern polynomial of degree $m-k$. It is enough to prove that $P \cap (\imath_!\gamma \cap \beta) = P \cap \imath_*(\gamma \cap \imath^*\beta)$. The LHS is

$$P \cap (\imath_!\gamma \cap \beta) = \imath_!\gamma \cap (P \cap \beta) = \gamma \cap \imath^*(P \cap \beta) = \gamma \cap \imath^*P \cap \imath^*\beta.$$

The last equality uses [11, Proposition 6.3]. The RHS is

$$P \cap \imath_*(\gamma \cap i^*\beta) = \imath_*(\imath^*P \cap \gamma \cap \imath^*\beta) = \gamma \cap \imath^*P \cap \imath^*\beta.$$

We used that $\imath_* : N_0(V) \to N_0(X)$ is the identity map of $\mathbb{R}$. For the last part of the lemma, the class $[V]$ generates $N^0(V)$ and $\imath^*[X] = [V]$ ([11, Example 6.2.1]). ∎

Remark 2.2. Except possibly for the results on the injectivity of $N^1(X) \to N_{n-1}(X)$ and the classical $\operatorname{Pic} X \to \operatorname{Div} X$, the results above in this section hold true for proper schemes that are embeddable in non-singular varieties.

For Chow groups, hence also for our $A_*(X)$, there is another case when we can pullback. [11, Ch.1.7] constructs $f^* : A_*(Y) \to A_*(X)$ when f is flat. We do not know whether this respects numerical equivalence.

Conjecture 2.3. *Let $f : X \to Y$ be a flat morphism of projective varieties of relative dimension d. Then the flat pullback f^* descends to a linear map $f^* : N_k(Y) \to N_{k+d}(X)$.*

The conjecture holds when Y is also nonsingular, or when f is l.c.i., simply because the flat pullback agrees at the level of Chow groups with f^* of [11, Ch.8] or $f^!$ of [11, Ch.6] respectively (cf. [11, Example 8.3.1]).

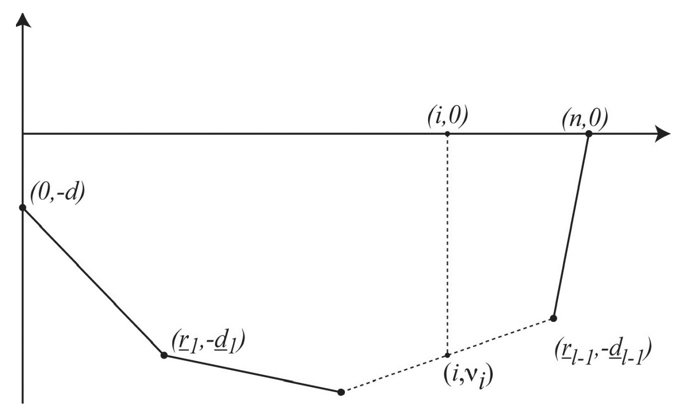
3 Positive cones

3.1 The pseudo-effective cone

Recall that a cycle is called effective if all its coefficients are nonnegative. The *pseudo-effective* cone $\overline{\mathrm{Eff}}_k(X) \subset N_k(X)$ is the closure of the convex cone generated by the classes of the effective k-cycles.

Example 3.1. If $X = X(\Delta)$ is a toric variety, then $\overline{\mathrm{Eff}}_k(X)$ is rational polyhedral, generated by the torus-invariant subvarieties of X. We have one generator for each face of dimension $n-k$ of Δ, but there may be relations.

Example 3.2 (Projective bundles over curves, cf. [10])**.** If C is a smooth projective curve and E is a vector bundle of rank n and degree d on it, the numerical data of ranks and degrees in the Harder–Narasimhan filtration of E with semistable quotients induces a Shatz stratification picture



The slopes of the tilted segments are the slopes (i.e., $\frac{\text{rank}}{\text{degree}}$) of the successive quotients, going from most negative to most positive in the Harder–Narasimhan filtration. The horizontal length of each segment is equal to the rank of the corresponding quotient. Let $\xi \in N^1(\mathbb{P}(E))$ be the class of the relative Serre line bundle, and let $f \in N^1(\mathbb{P}(E))$ be the class of any fiber of the bundle map $\mathbb{P}(E) \to C$. With ν_k being read from the polygonal picture above,

$$\overline{\mathrm{Eff}}_k(\mathbb{P}(E)) = \langle \xi^{n-k} + \nu_k \xi^{n-k-1} f, \xi^{n-k-1} f \rangle.$$

$\square$

If $f : X \to Y$ is a morphism of projective varieties, then clearly $f_* \overline{\mathrm{Eff}}_k(X) \subseteq \overline{\mathrm{Eff}}_k(Y)$.

Theorem 3.3 ([8])**.** *Let X be a projective variety over an algebraically closed field. Then $\overline{\mathrm{Eff}}_k(X)$ is pointed (i.e., it contains no linear subspaces). Furthermore if $f : X \to Y$ is a surjective morphism of projective varieties, then*

$$f_* \overline{\mathrm{Eff}}_k(X) = \overline{\mathrm{Eff}}_k(Y).$$

The main difficulty is the treatment of pseudo-effective classes that are not effective. It is easy to see that $\pm\alpha \in N_k(X) \setminus \{0\}$ cannot both be effective, but it is not so clear why they can't be limits of effective classes. Similarly, while it is easy to prove that $f_* \overline{\mathrm{Eff}}_k(X)$ is dense in $\overline{\mathrm{Eff}}_k(Y)$, equality requires extra effort because linear maps (like f_*) do not usually send closed non-compact sets to closed sets. To prove the theorem, we shifted perspective to the dual numerical space $N^k(X)$.

3.2 The nef cone

The *nef* cone is the dual of the (pseudo)effective cone.

$$\overline{\mathrm{Nef}}^k(X) := \overline{\mathrm{Eff}}_k(X)^\vee := \left\{\beta \in N^k(X) \;\middle|\; \beta \cdot \alpha \geq 0 \;\forall\, \alpha \in \overline{\mathrm{Eff}}_k(X)\right\}.$$

Nefness is preserved by pullback ($f^* \overline{\mathrm{Nef}}^k(Y) \subseteq \overline{\mathrm{Nef}}^k(X)$) as one easily sees from the projection formula.

Example 3.4 (Self products of abelian surfaces, cf. [3])**.** Let (A, θ) be a very general principally polarized abelian surface over $\mathbb{C}$. Consider the self-product $A \times A$. Let θ_1, θ_2 denote the pullbacks of θ via each of the two projections. Let λ denote the class of the Poincaré line bundle in $N^1(A \times A)$. $GL_2(\mathbb{Z})$ acts naturally on $A \times A$ by $(x, y)\begin{pmatrix} a & c \\ b & d \end{pmatrix} :=$ $(ax + by, cx + dy)$, where we use the group structure on A to make sense of the formula on the right. This induces an action of $GL_2(\mathbb{R})$ on $N^k(A \times A)$ for all k. Then $\overline{\mathrm{Eff}}_3(A \times A) =$ $\overline{\mathrm{Nef}}^1(A \times A)$ is the closure of the orbit of θ_1 under the $GL_2(\mathbb{R})$-action. Furthermore

$$\overline{\mathrm{Eff}}_2(A \times A) = \mathrm{Sym}^2 \overline{\mathrm{Nef}}^1(A \times A)$$

is generated (up to closure) by all the complete intersection classes. On the other hand, [3] show that $\overline{\mathrm{Nef}}^2(A \times A) \supsetneq \overline{\mathrm{Eff}}_2(A \times A)$ exhibiting explicit nef classes that are not pseudo-effective. □

Example 3.5 (Spherical varieties, cf. [18])**.** Using a product-type decomposition for the diagonal, [18] proves that nef classes on spherical (e.g., toric) varieties are pseudo-effective.

For a finite dimensional real vector space V and a closed convex cone $C \subset V$, we have that C is pointed if and only if $C^\vee$ generates $V^\vee$. To construct a spanning set of nef classes (thus showing that $\overline{\mathrm{Eff}}_k(X)$ is pointed), the first candidate is to consider complete intersections. These are usually not enough. Take the case of the 4-dimensional Grassmannian $X := G(2, 4)$. Basic Schubert calculus tells us that $N^2(X)$ is 2-dimensional, but $N^1(X)$ is 1-dimensional, so complete intersections cannot generate $N^2(X)$. This is actually an inspirational example.

3.3 The pliant cone

Any effective cycle on a Grassmann variety (or product thereof) can be translated by the group action and placed in general position with respect to any fixed subvariety. In particular it is nef. This Kleiman transversality is a good generalization of Bertini's hyperplane theorem to higher codimension. If for divisors the model positive example is the intersection of X with a linear hyperplane in some $\mathbb{P}^n$, in higher codimension we look at intersections with products of Schubert cycles on products of Grassmann varieties. Complete intersections are what we would get if we just looked at products of projective spaces.

The general definition of numerical equivalence that we use involves characteristic classes of vector bundles. The Schubert cycles on a Grassmann variety $\mathbb{G}$ are represented by the Schur polynomial classes of the tautological (globally generated) quotient bundle Q. Recall that if λ is a partition of k, the Schur polynomial of a vector bundle E is the following determinant in Chern classes

$$s_\lambda(E) := \begin{vmatrix} c_{\lambda_1} & c_{\lambda_1+1} & \cdots & c_{\lambda_1+k-1} \\ c_{\lambda_2-1} & c_{\lambda_2} & \cdots & c_{\lambda_2+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\lambda_k-k+1} & c_{\lambda_k-k+2} & \cdots & c_{\lambda_k} \end{vmatrix} (E).$$

Monomials in Schur polynomial classes of globally generated vector bundles correspond via Gauss maps[1] pullbacks of products of Schubert cycles on products of Grassmann varieties. They also appear in work of Fulton and Lazarsfeld [13].

Definition 3.6. The closure of the cone in $N^k(X)$ generated by monomials in Schur polynomials of globally generated vector bundles on X is the *pliant* cone $\mathrm{PL}^k(X)$.

For divisors, the only Schur polynomial of weight 1 is c_1. In this case $\mathrm{PL}^1(X) = \overline{\mathrm{Nef}}^1(X)$.

Example 3.7 ([8])**.** If X is a Grassmann variety or a product thereof, then $\mathrm{PL}^k(X) = \overline{\mathrm{Eff}}_{n-k}(X) = \overline{\mathrm{Nef}}^k(X)$.

Considering just monomials in Chern classes of globally generated vector bundles instead of monomials in Schur polynomials is sufficient to generate $N^k(X)$, but we do not know if the theorem below would hold for the resulting cone. It is a parallel to the fact that ample divisors are in the strict interior of $\overline{\mathrm{Nef}}^1(X)$.

Theorem 3.8 ([8])**.** *If h is an ample divisor class on X, then h^k is in the strict interior of $\mathrm{PL}^k(X)$. In particular it is in the strict interior of the nef cone. In particular, if $\alpha \in \overline{\mathrm{Eff}}_k(X)$, then $\alpha \neq 0$ if and only if $h^k \cdot \alpha \neq 0$.*

We do not know of a direct proof of this fact that avoids the pliant cone. A very useful consequence is a geometric construction of a norm on $N_k(X)$. Since h^k is in the interior of $\overline{\mathrm{Nef}}^k(X)$, we can write it as $h^k = \beta_1 + \ldots + \beta_r$ where $\{\beta_1, \ldots \beta_r\}$ is a basis of $N^k(X)$ contained in $\overline{\mathrm{Nef}}^k(X)$. For any $\alpha \in N_k(X)$ simply define

$$\| \alpha \| := \sum_{i=1}^{r} |\beta_i \cdot \alpha|.$$

[1] If V is a finite-dimensional vector space over our base field, then to any locally free quotient $V \otimes \mathcal{O}_X \twoheadrightarrow E$ of rank r, one associates the Gauss map $X \xrightarrow{\gamma} G(V, r) : x \mapsto (V \twoheadrightarrow E_x)$ to the Grassmann variety of r-dimensional quotients of V.

Observe that if $\alpha \in \overline{\mathrm{Eff}}_k(X)$, then $\| \alpha \| = \sum_{i=1}^r \beta_i \cdot \alpha = h^k \cdot \alpha$. We call the latter the degree of α with respect to h and denote it $\deg_h \alpha$. In particular $\| \cdot \|$ is an extension to $N_k(X)$ of the degree function on $\overline{\mathrm{Eff}}_k(X)$. In [8] we use this in a fundamental way in the proof of the second part of Theorem 3.3.

3.4 Morphisms and faces of the pseudo-effective cone

See [4, 7, 6] for references. If Z is an effective k-cycle on X, and $f : X \to Y$ is a morphism, then clearly $f_*[Z] = 0$ if and only if the support Supp Z is contracted by f. Without positivity assumptions it is easy to construct cycles Z with $f_*Z = 0$, such that Z is not numerically equivalent to a cycle with contracted support. For example $Z := \Delta_C - (C \times \{\mathrm{pt}\})$ pushes to 0 under the first projection from $C \times C$, where C is a projective curve of positive genus. But Z is not numerically equivalent to any multiple of $\{\mathrm{pt}\} \times C$.

In [6] we study morphisms f for which this does not happen. We say that f satisfies the GK property if $\ker f_*$ is generated by f-contracted subvarieties. We show for example that if Y is smooth projective over $\mathbb{C}$ and the general fibers of f are rationally chain connected, then f satisfies the GK property.

Returning to arbitrary morphisms f, we note that while understanding the effective classes that vanish under pushforward was easy, the pseudo-effective case is open.

Conjecture 3.9 ([4])**.** *Let $f : X \to Y$ be a morphism of projective varieties. Let $\alpha \in \overline{\mathrm{Eff}}_k(X)$ satisfy $f_*\alpha = 0$. Then α is a limit of classes of effective cycles whose supports are contracted by f.*

In other words, the face of the cone $\overline{\mathrm{Eff}}_k(X) \cap \ker f_*$ should be generated up to closure by f-contracted subvarieties. A weaker, but still open, version of the conjecture is asking to prove that α is a linear combination of f-contracted subvarieties. It allows negative coefficients in the limiting sequence. Note that if f has the GK property, then the weak conjecture follows without the positivity assumption on α.

The case of curve classes for Conjecture 3.9 over any $k = \bar{k}$ is settled by [4]. When the characteristic is 0, they also handle the case of divisors. In particular, over $\mathbb{C}$, the fist cases of interest are $k = 2$, $\dim X = 4$, and $\dim f(X) \in \{2, 3, 4\}$. Together with Lehmann we settled the cases $(\dim X, \dim f(Y)) \in \{(4, 3), (4, 4)\}$ in [7]. The $(4, 4)$ case is done by restricting to the case $\dim X \leq 3$. When f has relative dimension 1, the situation is more complicated. We will return to it below.

3.5 Zariski decompositions for cycle classes

Recall that if D is an effective divisor on a smooth projective surface, a *Zariski decomposition* is an expression $D = P + N$ where

- P is a nef $\mathbb{Q}$-divisor,
- N is an effective $\mathbb{Q}$-divisor such that the intersection form between the irreducible components of its support is negative definite
- $P \cdot N = 0$.

Such decompositions exist also for pseudo-effective $\mathbb{R}$-divisors (in this case P and N are possibly $\mathbb{R}$-divisors) and are unique. The "positive part" P captures all the sections of D in the sense that the natural inclusion of sections induces an equality $H^0(X, \mathcal{O}_X(\lfloor mP \rfloor)) = H^0(X, \mathcal{O}_X(\lfloor mD \rfloor))$.

In particular, we have $\mathrm{vol}\,(D) = \mathrm{vol}\,(P)$. Recall ([15, 5]) that for a divisor D on a projective variety of dimension n, the *volume* measures the asymptotic rate of growth of spaces of sections of multiples of D

$$\mathrm{vol}\,(D) := n! \cdot \limsup_{m \to \infty} \frac{\dim H^0(X, \mathcal{O}_X(\lfloor mD \rfloor))}{m^n}.$$

A divisor D is called *big* if $\mathrm{vol}\,(D) > 0$.

In higher dimension, decompositions that verify analogues of all the properties for P and N listed above do not always exist for divisors. One solution is to notice that on smooth surfaces *nef divisor* and *movable curve* in the sense of [2] are equivalent notions. Movability generalizes for cycles of arbitrary dimension.

Definition 3.10. Let W be an algebraic variety and let U be a cycle of dimension $\dim W + k$ on $W \times X$. For general $w \in W$ we can define a cycle U_w on X by intersecting with X_w, the vertical fiber of $W \times X \to W$ over w. The class $[U_w]$ is independent of the general w chosen. If U is effective and every component of its support dominates X via the second projection, we say that U determines a *strictly moving* family of k-cycles on X. We say that the class $[U_w]$, which is independent of the choice of general $w \in W$, is represented by this family.

The *movable* cone $\overline{\mathrm{Mov}}_k(X)$ is the closure of the cone generated by classes represented by strictly moving families of k-cycles.

For curve classes we recover the movable cone of [2]. If X is smooth, then for divisors we recover the classical definition: $\overline{\mathrm{Mov}}^1(X)$ is the closure of the cone generated by divisors that move in linear series without fixed divisorial components. Returning to Zariski decompositions, if X is smooth projective and D a big $\mathbb{R}$-divisor, a *divisorial Zariski decomposition* in the sense of Nakayama [19] is the unique decomposition

$$D = P_\sigma(D) + N_\sigma(D)$$

with $P_\sigma(D)$ movable, $N_\sigma(D)$ effective, and $H^0(X, \mathcal{O}_X(\lfloor mP_\sigma(D) \rfloor)) = H^0(X, \mathcal{O}_X(\lfloor mD \rfloor))$ for all $m \geq 0$. We do not review the specifics of the construction here, but mention that if D is big, then $N_\sigma(D)$ is obtained asymptotically by considering the fixed divisorial components of $\lfloor mD \rfloor$.

In [9], together with Lehmann, we showed that for a big $\mathbb{R}$-divisor D on a smooth projective variety, the divisorial Zariski decomposition is the unique decomposition $D = P + N$ with P movable, N effective, and $\mathrm{vol}\,(D) = \mathrm{vol}\,(P)$. In particular we can replace the condition on the equality of Hilbert functions for D and P with the weaker asymptotic condition of equality of volumes.

Together with Kollár we apply this in [5] to study when the monotonicity of the volume function in effective directions is not strict. It is easy to see that if D is big and E is effective, then $\mathrm{vol}\,(D - E) \leq \mathrm{vol}\,(D) \leq \mathrm{vol}\,(D + E)$. In [5] we determined when either equality holds. Specifically, $\mathrm{vol}\,(D - E) = \mathrm{vol}\,(D)$ if and only if $D - E$ and D have the same Hilbert function, and if and only if $N_\sigma(D)$ is componentwise bigger than E. For the other inequality, equality is also equivalent to the equality of Hilbert functions, and to Supp E being contained in the non-ample locus $\mathbf{B}_+(D)$ in the sense of [5, Definition 5.1]. For example if D is a big and nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, then $\mathrm{vol}\,(D+E) = \mathrm{vol}\,D$ if and only if $D^{n-1}E = 0$.

For cycles of dimension k we similarly prove in [9] that any $\alpha \in \overline{\mathrm{Eff}}_k(X)$ can be expressed as $\alpha = P + N$ for some $P \in \overline{\mathrm{Mov}}_k(X)$ and some pseudo-effective $N \in \overline{\mathrm{Eff}}_k(X)$ with $\mathrm{mob}\,(\alpha) = \mathrm{mob}\,(P)$. Here $\mathrm{mob} : \overline{\mathrm{Eff}}_k(X) \to \mathbb{R}_{\geq 0}$ is a volume-like function defined by Lehmann [17]. It is homogeneous, continuous, and vanishes precisely on the boundary of $\overline{\mathrm{Eff}}_k(X)$. For a big class α it counts asymptotically the maximal number N such that through every N general points on X there exists an effective cycle of class α.

We call a decomposition $\alpha = P + N$ as above a *Zariski decomposition.* In [9] we exhibit non-trivial decompositions for surface classes on $\mathrm{Hilb}^2\mathbb{P}^2$ and for some pseudo-effective cycles on projective bundles over curves.

We do not know if Zariski decompositions are unique for *big* classes (i.e., in the strict interior of $\overline{\mathrm{Eff}}_k(X)$). For classes on the boundary it may happen that the decomposition is not unique, and that for at least some decomposition the negative part is not effective.

We used the perspective of Zariski decompositions in [7] to approach the case of Conjecture 3.9 where $f : X \to Y$ is a surjective morphism from a 4-fold to a 3-fold over $\mathbb{C}$, and $k = 2$. We reduce the question to the case when X and Y are smooth, f has irreducible general fiber, and α is movable in a sequential sense. Using the philosophy that a movability-like condition should allow us to avoid the locus of reducible fibers, we prove that $\alpha = \pi^*\beta$ for some $\beta \in \overline{\mathrm{Mov}}^1(Y)$.

3.6 Dual positivity notions

We defined $\overline{\mathrm{Nef}}^k(X)$ as the dual of $\overline{\mathrm{Eff}}_k(X)$, in analogy to Kleiman's result for $k = 1$. In this subsection we see that different characterizations of nef divisors lead to different positivity notions in $N^k(X)$. We already defined the pliant cone $\mathrm{PL}^k(X) \subseteq \overline{\mathrm{Nef}}^k(X)$ starting from the perspective that $\overline{\mathrm{Nef}}^1(X)$ is the closure of the cone of classes of semi-ample divisors.

Nefness is preserved by pullback, and nef divisors are pseudo-effective (being limits of ample divisors). Conversely, if a class $\beta \in N^1(X)$ stays pseudo-effective upon arbitrary pullbacks, then by restricting to curves we see that β is nef.

Definition 3.11. Let $\beta \in N^k(X)$. We say that β is *universally pseudo-effective* if for all morphisms $f : Y \to X$ with Y a projective variety we have that $f^*\beta \cap [Y]$ is a pseudo-effective class on Y. The cone of all such β is closed, and denoted $\mathrm{Upsef}^k(X)$.

Note that $\mathrm{Upsef}^k(X) \subseteq \overline{\mathrm{Nef}}^k(X)$. Since nef classes may fail to be pseudo-effective (cf. [3, 20]), the inclusion may be strict. In [8] we prove the equality $\mathrm{Upsef}^k(X) = \overline{\mathrm{Nef}}^k(X)$ when X is spherical (e.g., toric), or a projective bundle over a curve. On homogeneous spaces (e.g., G/P spaces or abelian varieties) we have $\mathrm{Upsef}^k(X) = \overline{\mathrm{Eff}}_{n-k}(X)$. From this, since the $\mathrm{Upsef}^k(X)$ cone is tautologically invariant under pullback, or directly by Kleiman transversality, we deduce that $\mathrm{PL}^k(X) \subseteq \mathrm{Upsef}^k(X)$ for all X.

Going back to the interpretation of the nef cone as the closure of the cone generated by semi-ample divisors, we may also see it as the closure of the cone of divisors in basepoint-free linear series.

Definition 3.12. Let $p : U \to W$ be a projective morphism with equidimensional fibers of dimension $n-k$ between quasi-projective varieties. Assume that it admits a flat morphism $s : U \to X$. The class $\alpha := s|_{U_w *}[U_w] \in N_{n-k}(X)$ is independent of the choice of general $w \in W$. Here U_w is the cycle associated to the scheme theoretic fiber $p^{-1}w$. The closure of the cone generated by all such α is called the *basepoint-free* cone in $N_{n-k}(X)$, and denoted $\mathrm{BPF}_{n-k}(X)$.

When X is smooth, via the identification $N_{n-k}(X) = N^k(X)$, we may denote it $\mathrm{BPF}^k(X)$.

While in the singular case the basepoint-free cone lives in $N_{n-k}(X)$, not in $N^k(X)$, in the smooth case it is invariant under pullback which makes the dual class perspective preferable. Also, the baspoint-free cone is preserved by flat pushforward (which was helpful in the proof of the main result of [7]), but not by arbitrary projective pushforwards. The terminology is justified by the fact that for $p : U \to W$ and flat $s : U \to X$ as above, any subvariety $Z \subseteq X$ intersects some $s(U_w)$ properly as we show in [8]. In particular $\mathrm{BPF}^k(X) \subseteq \overline{\mathrm{Nef}}^k(X)$ if X is smooth.

Under the same assumption we also have that $\mathrm{BPF}^k(X) \subseteq \mathrm{Upsef}^k(X)$. The inclusion may be strict. In [8, Example 5.12] we see this for a particular surface ($k = 2$) class on an explicit toric 4-fold constructed in [1]. When X is smooth projective we hence have inclusions

$$\mathrm{PL}^k(X) \subseteq \mathrm{BPF}^k(X) \subseteq \mathrm{Upsef}^k(X) \subseteq \overline{\mathrm{Nef}}^k(X).$$

Of these, only the first is potentially always an equality. In general we don't have sufficient tools to compute $\mathrm{PL}^k(X)$. One way of disproving the equality is showing that pliancy is not preserved by flat pushforward. Each of these positivity notions has its advantages. The generators of the pliant or basepoint-free cone are explicit and offer good geometric tools. The universally pseudo-effective cone offers better bounds for $\overline{\mathrm{Eff}}_k(X)$ because it is closer to $\overline{\mathrm{Nef}}^k(X)$, but we do not have explicit generators. The definition that takes into account all morphisms $f : Y \to X$ is also too hard to verify. In [8] we show that it is enough to verify it when f is generically finite (in characteristic 0 birational suffices) onto its image, which may be a proper closed subset of X. We do not know if considering inclusions $f : Y \hookrightarrow X$ of closed subvarieties is enough.

4 Cones over morphisms of projective varieties

For an embedding $X \subseteq \mathbb{P}^N$, it is classical to consider the affine cone over it $C_X \subseteq \mathbb{A}^{N+1}$. It is unfortunate terminology that this variety shares its name with our main objects of interest, convex cones in real vector spaces. If $o \in C_X$ denotes the vertex corresponding to the origin, it is known that $\mathrm{Bl}_o C_X$ has the structure of a geometric line bundle over X. We also consider the compactified version of this.

4.1 The classical case

Let (X, H) be a polarized projective variety of dimension n with H very ample. Consider the projective bundle of quotients

$$Z := \mathbb{P}_X(\mathcal{O}_X(H) \oplus \mathcal{O}_X) \xrightarrow{\pi} X.$$

Let $\xi \in N^1(Z)$ denote the class of the relative $\mathcal{O}(1)$. It verifies the Groethendieck relation

$$\xi(\xi - \pi^* h) = 0.$$

We use small letters to denote the classes of the divisors represented by the corresponding capital letter. The surjection on the second factor $\mathcal{O}_X(H) \oplus \mathcal{O}_X \to \mathcal{O}_X$ induces an inclusion $X \subset Z$ whose image we denote X_0. We have $\xi|_{X_0} = 0$, so $[X_0] = \xi - \pi^* h$. The projection on the first component induces another inclusion denoted X_h with $[X_h] = \xi$. Consider the embedding $X \subset \mathbb{P}^N$ given by H. Then Z sits naturally in $\mathbb{P}_{\mathbb{P}^N}(\mathcal{O}(1) \oplus \mathcal{O})$. The latter is the blow-up of $\mathbb{P}^{N+1}$ at one point with exceptional divisor given by the projection $\mathcal{O}(1) \oplus \mathcal{O} \to \mathcal{O}$. The image of Z is the projective cone C over X. Note that C depends on H. The blown-up point is the vertex o of C. Denote by

$$\sigma : Z \to C$$

the blow-up morphism. In [8] we computed the numerical spaces of C. We review the description here:

Proposition 4.1. *The map $\sigma_* \pi^* : N_{k-1}(X) \to N_k(C)$ is an isomorphism for all $1 \leq k \leq n+1$.*

This says that any cycle on C is numerically equivalent to the cone over a cycle on X (of dimension one less).

Proof. The geometric intuition is that the inverse is given by intersecting with the hyperplane at infinity X_h. Since Z is the projectivization of a rank 2 vector bundle over X, we have

$$N_k(Z) \simeq \pi^* N_{k-1}(X) \oplus [X_0] \cdot \pi^* N_k(X).$$

Clearly σ_* is a surjection and it annihilates any class coming from the exceptional X_0. It follows that $\sigma_* \pi^* : N_{k-1}(X) \to N_k(C)$ is surjective. Note that σ is an isomorphism around X_h, and so we can see X_h as a Cartier divisor on C. Denote this embedding by $\imath : X'_h \hookrightarrow C$, and the embedding of X as X_h in Z by $\jmath_h : X \hookrightarrow Z$. Since $\jmath_h$ is a section of π, we have $\jmath_h^* \pi^* = id$. At the level of Chow groups, it is easy to prove by localizing outside X_0 that $\jmath_h^* = \imath^* \sigma_*$. Then the same holds for numerical classes. If $\sigma_* \pi^* \alpha = 0$, then $\alpha = \jmath_h^* \pi^* \alpha = \imath^* \sigma_* \pi^* \alpha = 0$. Therefore $\sigma_* \pi^* : N_{k-1}(Y) \to N_k(C)$ is also injective. □

4.2 The relative case

We generalize the construction of C by working in the relative setting. Seeing it as a cone associated to the surjective morphism $X \to \mathrm{pt}$, we consider the case of an arbitrary surjective projective morphism $f : X \to Y$. Embed $X \subset \mathbb{P}^N_Y$ for some N, and let H be a

Cartier divisor on X such that $\mathcal{O}_X(H)$ is the pullback of $\mathcal{O}_{\mathbb{P}^N}(1)$ to X. As before, we find a diagram

$$\begin{tikzcd} Z \arrow[r, "\pi"] \arrow[d, "\sigma"'] & X \arrow[l, bend right, "\jmath_0"'] \arrow[l, bend right=60, "\jmath_h"'] \arrow[dl, "\imath"] \arrow[d, "f"] \\ C \arrow[r] & Y \arrow[l, bend left, "\omega"] \end{tikzcd}$$

with τ the lower horizontal arrow $C \to Y$ — such that every fiber C_y of τ over a closed point $y \in Y$ is the cone over the fiber X_y, and $Z_y \to C_y$ is the blow-up of the vertex of this cone. In the diagram, $f\pi = \tau\sigma$. The sections $\jmath_0$ and $\jmath_h$ of π are determined by projections $\mathcal{O}_X(H) \oplus \mathcal{O}_X$ onto the two factors $\mathcal{O}_X$ and $\mathcal{O}_X(H)$ respectively, and $\imath$ is the embedding induced by X'_h, so $\imath = \sigma\jmath_h$ and $\tau\imath = f$. The section ω of τ satisfies $\omega f = \sigma\jmath_0$.

We also constructed C in [7]. Note that it depends on the choice of the polarization H.

From this point on we assume that $f : X \to Y$ is a surjective morphism of projective varieties. The compactness assumption allows us to work with numerical equivalence.

Proposition 4.2. *Let $f : X \to Y$ be a surjective morphism as above with $\dim X = n$. Then for all $1 \leq k \leq n+1$ there is a direct sum decomposition*

$$N_k(C) = \sigma_*\pi^*N_{k-1}(X) \oplus \omega_*N_k(Y).$$

inducing an isomorphism $N_k(C) \simeq N_{k-1}(X) \oplus N_k(Y)$.

Proof. As in the case where Y is a point, $\sigma_* : \pi^*N_{k-1}(X) \oplus [X_0] \cdot \pi^*N_k(X) = N_k(Z) \to N_k(C)$ is surjective. Furthermore

$$\sigma_*([X_0] \cdot \pi^*\beta) = (\sigma_*\jmath_{0*}) \circ ({\jmath_0}^*\pi^*)(\beta) = \omega_*f_*id^*\beta = \omega_*f_*\beta.$$

Since $f_*(N_k(X)) = N_k(Y)$, find $N_k(C) = \sigma_*\pi^*N_{k-1}(X) + \omega_*N_k(Y)$. We still need to justify that the sum is direct.

Assume $\sigma_*(\pi^*\alpha + [X_0] \cdot \pi^*\beta) = 0$. Pulling back by $\imath$, which is the inclusion of the Cartier divisor X'_h in C, and using that $X_0 \cap X_h = \emptyset$ in Z, we find $\alpha = 0 \in N_{k-1}(X)$. From previous computation, $\omega_*f_*\beta = 0$. Applying τ_* and using $\tau\omega = id_Y$, we find $f_*\beta = 0$. The conclusion follows. □

Example 4.3 (Normalization morphisms with nontrivial pushforwards)**.** Let X be a smooth projective variety of dimension n, and let $f : X \to \mathbb{P}^n$ be a finite and surjective morphism provided by a Noether normalization. Then σ is finite birational with Z smooth, therefore it is the normalization of C. From the description above, we see that $\ker \sigma_*$ identifies with $\ker f_*$. Since the numerical spaces of $\mathbb{P}^n$ are 1-dimensional, $\ker f_*$ can be nontrivial. Thus we find examples of normalization morphisms σ where σ_* has nontrivial kernel. In particular these morphisms may fail the GK property of [6].

4.3 Pseudo-effective and nef cones

Corollary 4.4. *Assuming that H is very ample on X, the isomorphism of Proposition 4.2 induces a decomposition* $\overline{\mathrm{Eff}}_k(C) \simeq \overline{\mathrm{Eff}}_{k-1}(X) \oplus \overline{\mathrm{Eff}}_k(Y)$.

Proof. Since π is smooth, the pullback π^* preserves (pseudo)effectivity. It follows that if $\alpha \in \overline{\mathrm{Eff}}_{k-1}(X)$ and $\beta \in \overline{\mathrm{Eff}}_k(Y)$, then $\sigma_*\pi^*\alpha + \omega_*\beta \in \overline{\mathrm{Eff}}_k(C)$. Assume now $\gamma := \sigma_*\pi^*\alpha + \omega_*\beta \in \overline{\mathrm{Eff}}_k(C)$. We have $\overline{\mathrm{Eff}}_k(Y) \ni \tau_*\gamma = f_*\pi_*\pi^*\alpha + \beta = \beta$. By the lemma below, $[X'_h]$ is nef. Restricting to X'_h then preserves positivity (as in [7, Lemma 4.10]). As before, in the Chow groups of Z and C, by restricting away from X_0 we prove $\imath^*\gamma = \imath^*\sigma_*\pi^*\alpha = \alpha$. We deduce that α is pseudo-effective. ∎

Lemma 4.5. *Assuming that H is very ample, the class $[X'_h] \in N^1(C)$ is nef.*

Proof. Since $[X_h] = \xi \in N^1(Z)$ and $\mathcal{O}_X(H) \oplus \mathcal{O}_X$ is globally generated, it follows that ξ is nef. However $\xi = \sigma^*[X'_h]$, and nefness can be verified on any dominant cover. ∎

The spaces of Cartier divisors, or more generally the dual numerical spaces $N^k(X)$ are also interesting. Note that C is singular in general, so we do not expect $N^k(C) \simeq N_{n+1-k}(C)$.

Proposition 4.6. *Associating to $\gamma \in N^{k-1}(X)$ the map $N_k(C) \ni \alpha \mapsto \gamma \cap \imath^*\alpha \in \mathbb{R}$ defines $\imath_! : N^{k-1}(X) \to N^k(C)$. Then*

$$N^k(C) = \imath_! N^{k-1}(X) \oplus \tau^* N^k(Y)$$

induces an isomorphism $N^k(C) \simeq N^{k-1}(X) \oplus N^k(Y)$. In particular $N^1(C) = \mathbb{R}[X'_h] \oplus \tau^* N^1(Y)$.

Proof. The isomorphism follows from the description of $N_k(C)$. For the direct sum decomposition, we verify $\imath_! N^{k-1}(X) \cap \tau^* N^k(Y) = 0$. The subspaces then also generate $N^k(C)$ for dimension reasons. Assume $\imath_!\gamma = \tau^* v$. Then for any $\sigma_*\pi^*\alpha + \omega_*\beta \in N_k(C)$ we have $\gamma \cap \imath^*(\sigma_*\pi^*\alpha + \omega_*\beta) = v \cap \tau_*(\sigma_*\pi^*\alpha + \omega_*\beta)$. The LHS is $\gamma \cap (\jmath_h^*\pi^*\alpha + \imath^*\omega_*\beta) = \gamma \cap \alpha$. The RHS is $v \cap (f_*\pi_*\pi^*\alpha + (\tau \circ \omega)_*\beta) = v \cap \beta$. Since α and β are arbitrary, it follows that $\gamma = 0$ and $v = 0$. ∎

Corollary 4.7. *Assume that H is very ample on X. We have a decomposition*

$$\overline{\mathrm{Nef}}^k(C) = \imath_! \overline{\mathrm{Nef}}^{k-1}(X) \oplus \tau^* \overline{\mathrm{Nef}}^k(Y).$$

In particular $\overline{\mathrm{Nef}}^1(C) = \mathbb{R}_{\geq 0}[X'_h] \oplus \tau^* \overline{\mathrm{Nef}}^1(Y)$ *and* $\overline{\mathrm{Nef}}^n(C) = \imath_! \overline{\mathrm{Nef}}^{n-1}(X) \oplus \tau^* \overline{\mathrm{Nef}}^n(Y)$.

Note that when f is not generically finite, $\overline{\mathrm{Nef}}^n(Y) = 0$.

Proof. If $\gamma \in \overline{\mathrm{Nef}}^{k-1}(X)$, then since $\imath^*$ preserves pseudo-effectivity, we find that $\imath_!\gamma$ is nef. Clearly τ^* preserves nefness. Then $\overline{\mathrm{Nef}}^k(C) \supseteq \imath_! \overline{\mathrm{Nef}}^{k-1}(X) \oplus \tau^* \overline{\mathrm{Nef}}^k(Y)$. If $\imath_!\gamma + \tau^* v$ is nef, then by pairing with classes $\sigma_*\pi^* \overline{\mathrm{Eff}}_{k-1}(X)$ we find that γ is nef. Similarly by pairing with $\omega_* \overline{\mathrm{Eff}}_k(Y)$ we find that v is nef. These explain the reverse inclusion. ∎

4.4 The movable cone of curves and the pseudo-effective cone of divisors

Lemma 4.8. *Assuming that H is very ample, we have*

$$\pi^* \overline{\mathrm{Mov}}_{k-1}(X) \oplus [X_h] \cdot \pi^* \overline{\mathrm{Mov}}_k(X) \subseteq \overline{\mathrm{Mov}}_k(Z) \subseteq \pi^* \overline{\mathrm{Eff}}_{k-1}(X) \oplus [X_h] \cdot \pi^* \overline{\mathrm{Mov}}_k(X).$$

Moreover $\overline{\mathrm{Mov}}_1(Z) = \mathbb{R}_{\geq 0}\pi^*[\mathrm{pt}] \oplus [X_h] \cdot \pi^* \overline{\mathrm{Mov}}_1(X)$.

Note that the direct sum decomposition $N_k(Z) = \pi^* N_{k-1}(X) \oplus [X_h] \cdot \pi^* N_k(X)$ differs from our usual one that replaces $[X_h]$ by $[X_0]$. The relation is $[X_h] = [X_0] + \pi^* h$. However, since $[X_h] = \xi$, this is the standard Groethendieck decomposition.

Proof. Since π is smooth, pulling back by it preserves movability. Since $[X_h] \in N^1(Z)$ is nef, we also find that $[X_h] \cdot \pi^* \overline{\mathrm{Mov}}_1(X) \subseteq \overline{\mathrm{Mov}}_1(Z)$. These prove the first inclusion. If $\pi^*\alpha + [X_h] \cdot \pi^*\beta$ is movable, then by pushing to X we find that β is movable. By intersecting with X_0 we find that α is at least pseudo-effective.

When $k = 1$, the equalities $\overline{\mathrm{Eff}}_0(X) = \overline{\mathrm{Mov}}_0(X) = \mathbb{R}_{\geq 0}[\mathrm{pt}]$ provide the assertion. ∎

Corollary 4.9. *Assuming that H is very ample, we have*

$$\overline{\mathrm{Mov}}_1(C) = \left\{ \sigma_* \pi^*(a[\mathrm{pt}] + h \cdot \beta) + \omega_* f_* \beta \;\middle|\; a \geq 0 \quad \textit{and} \quad \beta \in \overline{\mathrm{Mov}}_1(X) \right\}.$$

Proof. The only new ingredients are the equalities $\sigma_* \overline{\mathrm{Mov}}_1(Z) = \overline{\mathrm{Mov}}_1(C)$ and $f_* \overline{\mathrm{Mov}}_1(X) = \overline{\mathrm{Mov}}_1(Y)$ from [9], together with the relation $[X_h] = [X_0] + \pi^* h$. ∎

Corollary 4.10. *Assuming that H is very ample,*

$$\overline{\mathrm{Eff}}^1(C) = \left\{ a[X'_h] + \tau^* v \;\middle|\; a \geq 0 \quad \textit{and} \quad f^* v + ah \in \overline{\mathrm{Eff}}^1(X) \right\}.$$

Proof. The pseudo-effective cone of divisors is dual to the movable cone of curves by [2, 0.2 Theorem]. ∎

Remark 4.11. Via the cyclification morphism $N^k(X) \overset{\cap[C]}{\to} N_{n+1-k}(C)$ we have

$$\begin{aligned} \imath_! \gamma \cap [C] + \tau^* v \cap [C] &= \imath_*(\gamma \cap \imath^*[C]) + \tau^* v \cap [C] \\ &= \imath_*(\gamma \cap [X]) + \tau^* v \cap [C] \\ &= \sigma_* \pi^*((h \cdot \gamma + f^* v) \cap [X]) + \omega_* f_*(\gamma \cap [X]) \end{aligned}$$

The key identity in the first equality is provided by Lemma 2.1. For the second, we use $\imath^*[C] = [X]$, which is [11, Proposition 2.6.(d)]. The coordinates in $N_{n-k}(X)$ and $N_{n-k+1}(Y)$ are computed by applying $\imath^*$ (same as intersecting with X'_h), and τ_* respectively. Here we also use the self-intersection formula $\imath^* \imath_* \alpha = \imath^*[X'_h] \cap \alpha$ from [11, Proposition 2.6.(c)].

The image of the nef cone is generated by $\imath_! \gamma \cap [C]$ and $\tau^* v \cap [C]$ with $\gamma \in \overline{\mathrm{Nef}}^{k-1}(X)$ and $v \in \overline{\mathrm{Nef}}^k(Y)$.

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