

Anticanonical modules of Segre products

by

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Dedicated to Dorin Popescu in honour of his 70th birthday

Abstract

We compute the anticanonical module of the Segre product of standard graded K -algebras and determine its depth.

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1 Introduction

Let R be a standard graded Cohen–Macaulay K -algebra with canonical module ω_R . If R is Gorenstein, then $\omega_R \cong R$. Otherwise, if R happens to be generically Gorenstein, then ω_R may be identified with an ideal of height one. In this case one may study the powers ω_R^a with $a \in \mathbb{Z}$, and ask for which integers a the ideal ω_R^a is Cohen–Macaulay. This question has been answered in [3] for the canonical module of determinantal rings.

The ideal ω_R^{-1} is commonly called the anticanonical ideal. Correspondingly, for any Cohen–Macaulay K -algebra R , one calls the R -dual of ω_R , namely $\omega_R^* = \text{Hom}_R(\omega_R, R)$, the anticanonical module of R . This module is of particular interest. Indeed, assume as before that R is generically Gorenstein. Let S be a Noether normalization of R and $\sigma : Q(R) \rightarrow Q(S)$ a trace map, where $Q(S)$ and $Q(R)$ denote the total ring of fractions of S and R , respectively. Then the complementary module $\mathfrak{C}_{R/S}^\sigma$ is defined and it is shown in [6, Satz 7.20] that $\mathfrak{C}_{R/S}^\sigma$ is isomorphic to the canonical ideal of R . Its inverse is the Dedekind different $\mathfrak{D}^\sigma(R/S)$. In number theory this invariant encodes the ramification of the corresponding field extension, and by definition it is isomorphic to the anticanonical ideal of R .

The anticanonical ideal appears as well in other homological contexts. As observed in [7, Lemma 2.1], $\omega_R^{-1}\omega_R$ describes the non-Gorenstein locus of R . Thus, if R is Gorenstein on the punctured spectrum of R , then $\text{height}(\omega_R^{-1}\omega_R) = \dim R$. On the other hand, the anticanonical module is isomorphic to an ideal of height one. It may be a Cohen–Macaulay ideal or not, while ω_R is always Cohen–Macaulay.

In this paper we study the anticanonical module of the Segre product of standard graded Cohen–Macaulay K -algebras. One motivation for us to study this question came from the study of Hibi rings [8]. Their canonical ideals are well understood, while at present this is not the case for their anticanonical ideals. The Hibi ring of a sum of finite posets is just the Segre product of the Hibi rings of each summand. Thus if we understand how to compute the anticanonical module of Segre products, then for the study of the anticanonical ideal of

a Hibi ring one may restrict oneself to the case that the underlying poset is not a sum of two posets.

Our study of the canonical and anticanonical module of Segre products is based on results from the fundamental paper “On graded rings, I” by Goto and Watanabe [5]. There it is shown [5, Theorem 4.3.1] that if R and S are two standard graded algebras with $\dim R, \dim S \geq 2$ and $T = R\sharp S$, then $\omega_T = \omega_R\sharp\omega_S$. One would then expect that $\omega_T^* \cong \omega_R^*\sharp\omega_S^*$, and of course this would be true if, more generally, for any finitely generated graded R -module M and any finitely generated graded S -module N , it would follow that $(M\sharp N)^* \cong M^*\sharp N^*$. Example 2.2 shows that this is not always the case. Therefore, in the first section of this paper we introduce friendly families of algebras. We say that the family of standard graded K -algebras R_1, \dots, R_m is friendly if the natural map

$$\alpha : R_1(a_1)^*\sharp \cdots \sharp R_m(a_m)^* \rightarrow (R_1(a_1)\sharp \cdots \sharp R_m(a_m))^*$$

is an isomorphism for all integers a_1, \dots, a_m . When $m = 2$ we say that (R_1, R_2) is a friendly pair. We show in Theorem 2.3, that if R_1, \dots, R_m is a friendly family, then $(M_1\sharp \cdots \sharp M_m)^* \cong M_1^*\sharp \cdots \sharp M_m^*$ when M_i is a finitely generated graded R_i -module, $i = 1, \dots, m$. We also prove in this section that if R_1, \dots, R_m are standard graded toric rings with depth $R_i \geq 2$ for $i = 1, \dots, m$, then R_1, \dots, R_m is a friendly family, see Theorem 2.6.

We apply these results in Section 3 to conclude that for any pair (R, S) of standard graded Cohen–Macaulay toric rings with $\dim R, \dim S \geq 2$, we have the desired isomorphism $\omega_T^* \cong \omega_R^*\sharp\omega_S^*$, where $T = R\sharp S$. This follows from Corollary 3.1, where this result is formulated more generally for friendly pairs. In particular, if we assume that R and S are Gorenstein rings of dimension ≥ 2 and with negative a -invariant, we deduce from this result in Proposition 3.4 that the canonical module of the Segre product $T = R\sharp S$ is reflexive, which in turn, as a consequence of [6, Theorem 7.31], implies that the localization T_P is Gorenstein for any height 1 prime ideal P of T .

The concept of a friendly family can be extended to positively graded or multigraded algebras. Results similar to Theorems 2.3 and 2.6 may be formulated in that generality. In view of later applications in Section 3 we restricted to the standard graded case.

The next part of Section 3 is devoted to determine the depth of the anticanonical module for $R\sharp S$ when (R, S) is a friendly pair of standard graded Cohen–Macaulay K -algebras. Inspired by Proposition 4.2.2 in [5], we determine, in terms of a and b , the depth of $R(a)\sharp S(b)$ when R and S are standard graded Gorenstein K -algebras, and use this result in Corollary 3.7 to characterize those pairs of friendly Gorenstein K -algebras for which the anticanonical module of the Segre product $R\sharp S$ is Cohen–Macaulay.

In Theorem 3.8 we generalize Corollary 3.7 to Segre products of finitely many Gorenstein algebras. Namely, given R_1, \dots, R_m a friendly family of standard graded Gorenstein K -algebras of dimension at least two, with $-\rho_i$ denoting the a -invariant of R_i for $i = 1, \dots, m$, Theorem 3.8 describes for which integers a the module

$$T^{\{a\}} = R_1(-a\rho_1)\sharp \cdots \sharp R_m(-a\rho_m)$$

of uniform twists of the canonical module of $T = \sharp_{i=1}^m R_i$ is a Cohen-Macaulay T -module. In particular, assuming $\rho_1 \geq \cdots \geq \rho_m$, the anticanonical module ω_T^* is Cohen-Macaulay if and only if

$$2^{m-1}\rho_m > 2^{m-2}\rho_{m-1} > \cdots > 2\rho_2 > \rho_1,$$

see Corollary 3.10.

In Proposition 3.11 we see that if all ρ_i 's are positive and not all equal (i.e. T is Cohen-Macaulay, and not a Gorenstein ring), then the set of a 's such that $T^{\{a\}}$ is Cohen-Macaulay represents a bounded interval which is explicitly described.

If we assume, moreover, that $\otimes_{i=1}^m R_i$ is a domain, which is for instance the case when the R_i 's are toric rings, then the canonical module and its dual may be identified with ideals in T . In Proposition 3.13, which is the last result in this paper, we describe which powers ω_T^a are Cohen-Macaulay, using the previous Proposition 3.11.

2 Friendly families of standard graded algebras

Let R be a standard graded K -algebra, where K is a field, and M a finitely generated graded R -module. Then the dual of M , $M^* = \text{Hom}_R(M, R)$ is graded by $M^* = \oplus_{i \in \mathbb{Z}} \text{Hom}_R^i(M, R)$ where

$$\text{Hom}_R^i(M, R) = \{\varphi \in \text{Hom}_R(M, R) : \varphi(M_k) \subseteq R_{k+i} \text{ for all } k\}.$$

In what follows, we will also consider the graded K -dual of M , namely

$$M^\vee = \text{Hom}_K(M, K) = \oplus_{i \in \mathbb{Z}} \text{Hom}_K(M_{-i}, K).$$

Let S be another standard graded K -algebra and N a finitely generated graded S -module. Let $T = R \# S$ be the Segre product of R and S . The algebra T is also standard graded with the grading $T = \oplus_{i \geq 0} (R_i \otimes_K S_i)$. The Segre product $M \# N$ is a graded T -module with $M \# N = \oplus_{j \in \mathbb{Z}} (M_j \otimes_K N_j)$.

There is a natural homogeneous T -module homomorphism $\alpha : M^* \# N^* \rightarrow (M \# N)^*$ defined as follows. An element of $(M^* \# N^*)_i = M_i^* \otimes N_i^*$ is a sum of elements of the form $\varphi \otimes \psi$ with $\varphi \in M_i^*, \psi \in N_i^*$. Here M_i^* denotes the i -th homogeneous component of M^* . Then $\alpha(\varphi \otimes \psi) \in (M \# N)_i^*$ is the map $(M \# N)_k \rightarrow R_{k+i} \otimes S_{k+i}$ given by $m \otimes n \mapsto \varphi(m) \otimes \psi(n)$.

We can iterate this construction. In general, given R_1, \dots, R_m standard graded algebras, and M_i a graded R_i -module, $i = 1, \dots, m$, we can construct a natural map

$$\alpha : M_1^* \# \dots \# M_m^* \rightarrow (M_1 \# \dots \# M_m)^*.$$

Definition 2.1. The family of algebras R_1, \dots, R_m is called *friendly* if the natural map

$$\alpha : R_1(a_1)^* \# \dots \# R_m(a_m)^* \rightarrow (R_1(a_1) \# \dots \# R_m(a_m))^*$$

is an isomorphism for all integers a_1, \dots, a_m .

In case $m = 2$ we say that (R_1, R_2) is a friendly pair.

Not all families of standard graded K -algebras are friendly, as the following example shows.

Example 2.2. Let x and y be indeterminates over K and let $R = K[x]/(x^3)$ and $S = K[y]/(y^2)$. Then $R \# S = K \oplus Kxy$. One has $R(2) \# S(1) = Kx \oplus Kx^2y$, with x sitting in degree -1 , therefore $R(2) \# S(1) \cong (R \# S)(1)$. On the other hand, $R(2)^* \# S(1)^* \cong R(-2) \# S(-1) = Ky$.

Thus $(R(2) \# S(1))^* \cong (R \# S)(-1)$ has two nonzero components, while $R(2)^* \# S(1)^*$ has only one nonzero component. Hence these two modules cannot be isomorphic and (R, S) is not a friendly pair.

However, we show in Theorem 2.6 that any finite collection of standard graded toric rings is friendly. The following result will be useful to that purpose. At the same time, Theorem 2.3 gives a wider class of modules for which the map α used before is an isomorphism.

Theorem 2.3. *Let R_1, \dots, R_m be a friendly family of standard graded K -algebras, and $T = \#_{i=1}^m R_i$. If M_i is a finitely generated graded R_i -module for $i = 1, \dots, m$, then the natural map $\alpha : \#_{i=1}^m M_i^* \rightarrow (\#_{i=1}^m M_i)^*$ is an isomorphism of graded T -modules.*

Proof. The statement is proved by induction on

$$r = |\{i : M_i \text{ is not free}, i = 1, \dots, m\}|.$$

If $r = 0$, by the definition of a friendly family and taking into account that the dual functor and the Segre product commute with finite direct sums, it follows that α is an isomorphism.

Let $r > 0$ and assume that the conclusion of the theorem holds when at most $r - 1$ of the M_i 's are not free. Without loss of generality we may assume M_1 is not free. Let

$$G \rightarrow F \rightarrow M_1 \rightarrow 0 \tag{1}$$

be a presentation of M_1 by finitely generated graded free R_1 -modules. We dualize (1) and then apply the exact functor $-\#(\#_{i>1} M_i)$ to obtain the exact sequence

$$0 \rightarrow \#_{i=1}^m M_i^* \rightarrow F^* \#(\#_{i>1} M_i) \rightarrow G^* \#(\#_{i>1} M_i). \tag{2}$$

Similarly, if we first apply $-\#(\#_{i>1} M_i)$ to (1) and then we dualize, we have the exact sequence

$$0 \rightarrow (\#_{i=1}^m M_i)^* \rightarrow (F \#(\#_{i>1} M_i))^* \rightarrow (G \#(\#_{i>1} M_i))^*. \tag{3}$$

The maps α induce a map between the chains (2) and (3). By the inductive hypothesis, the two rightmost maps α are isomorphisms, hence using the 5-Lemma ([12, Exercise 1.3.3]) we get that the leftmost map α is an isomorphism, as well. This finishes the proof. \square

Let us make a simple remark before proceeding to the main result of this section.

Remark 2.4. Let R and S be standard graded algebras such that the natural map

$$\alpha : R(a)^* \# S \rightarrow (R(a) \# S)^*$$

is an isomorphism for all $a \in \mathbb{Z}$. Then

$$\begin{aligned} (R(a) \# S(b))^* &= ((R(a-b) \# S)(b))^* = (R(a-b) \# S)^*(-b) \\ &\cong ((R(a-b))^* \# S)(-b) = (R(b-a) \# S)(-b) \\ &= R(-a) \# S(-b) \\ &= R(a)^* \# S(b)^* \quad \text{for all } a, b \in \mathbb{Z}. \end{aligned}$$

Thus (R, S) is a friendly pair in the sense of Definition 2.1.

We note that, by [9, Corollary 16], the Segre product of two standard graded toric rings R, S is a standard graded toric ring, as well. For the convenience of the reader we include a short proof of this statement.

Proposition 2.5. *Tensor products and Segre products of standard graded toric rings are standard graded toric rings, too. In particular, they are domains.*

Proof. Let $R = K[t^{a_1}, \dots, t^{a_n}] = K[A]$ where A is an integer matrix with column vectors a_1, \dots, a_n and $S = K[s^{b_1}, \dots, s^{b_m}] = K[B]$ where B is an integer matrix with column vectors b_1, \dots, b_m . Let I be the kernel of the morphism $K[x_1, \dots, x_n] \rightarrow K[A]$ induced by $x_i \mapsto t^{a_i}$ for $1 \leq i \leq n$, and J the kernel of the morphism $K[y_1, \dots, y_m] \rightarrow K[B]$ induced by $y_j \mapsto s^{b_j}$ for $1 \leq j \leq m$. Then

$$R \otimes_K S \cong K[x, y]/IK[x, y] + JK[x, y]$$

where $K[x, y]$ denotes the polynomial ring in the variables $x_1, \dots, x_n, y_1, \dots, y_m$.

Let L_1, L_2 be the following lattices in \mathbb{Z}^n : $L_1 = \{c \in \mathbb{Z}^n : Ac = 0\}$ and $L_2 = \{d \in \mathbb{Z}^m : Bd = 0\}$. Then $I = I_{L_1}$ and $J = I_{L_2}$ where I_{L_j} is the lattice ideal of L_j for $j = 1, 2$.

Let

$$L = \left\{ \begin{pmatrix} c \\ d \end{pmatrix} \in \mathbb{Z}^n \oplus \mathbb{Z}^m : \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = 0 \right\}.$$

Then $I_L = (I_{L_1} + I_{L_2})K[x, y]$. As \mathbb{Z}^n/L_1 and \mathbb{Z}^m/L_2 are torsion free, $\mathbb{Z}^n \oplus \mathbb{Z}^m/I_L$ is torsion free as well, hence I_L is a prime ideal ([11, Theorem 8.2.2] or [4, Theorem 2.1]) and

$$R \otimes_K S \cong K[x, y]/I_L = K[C],$$

where $C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

This implies that the morphism $R \otimes_K S \rightarrow K[t^{a_1}, \dots, t^{a_n}, s^{b_1}, \dots, s^{b_m}] = K[C]$ induced by $t^{a_i} \otimes 1 \mapsto t^{a_i}$ and $1 \otimes s^{b_j} \mapsto s^{b_j}$ is an isomorphism. Since R and S are standard graded, we get

$$R \sharp S = K[t^{a_i} s^{b_j} : 1 \leq i \leq n, 1 \leq j \leq m].$$

□

Theorem 2.6. *Any finite family of standard graded toric rings of depth at least two is friendly.*

Proof. For the proof we make a series of reductions. Firstly, we note that it is enough to prove the statement for friendly pairs, and then argue by induction on the cardinality of the family of algebras. Indeed, if R_1, \dots, R_m are a friendly family of standard graded toric rings of depth at least 2, and $m \geq 3$, then for all $a_1, \dots, a_m \in \mathbb{Z}$ we have

$$\begin{aligned} (\sharp_{i=1}^m R_i(a_i))^* &\cong ((\sharp_{i=1}^{m-1} R_i(a_i)) \sharp R_m(a_m))^* \\ &\cong (\sharp_{i=1}^{m-1} R_i(a_i))^* \sharp R_m(a_m)^* \\ &\cong (\sharp_{i=1}^{m-1} R_i(a_i))^* \sharp R_m(a_m)^* \\ &\cong \sharp_{i=1}^m R_i(a_i)^*. \end{aligned} \tag{4}$$

Here, for the isomorphism in (4) we used Theorem 2.3 applied to the algebras $\sharp_{i=1}^{m-1} R_i$ and R_m that are toric (cf. Proposition 2.5) and form a friendly pair, see the base case which is discussed in the next paragraph.

For the induction step to work we prove the base case $m = 2$ in a slightly more general setup. We let R and S be toric standard graded algebras with depth $R \geq 2$ or depth $S \geq 2$.

Without loss of generality, we may assume that $\text{depth } S \geq 2$. According to Remark 2.4, for checking that (R, S) is a friendly pair it suffices to show that $(R(a)\sharp S)^* \cong R(a)^*\sharp S^*$ or, equivalently, $(R(a)\sharp S)^* \cong R(-a)\sharp S$ for all $a \in \mathbb{Z}$. Suppose that we have proved the latter isomorphism for $a > 0$, and let $a < 0$. Then

$$(R(a)\sharp S)^* = ((R\sharp S(-a))(a))^* = (R\sharp S(-a))^*(-a) \cong (R\sharp S(a))(-a) = R(-a)\sharp S.$$

Thus, in order to complete the proof, we need to show that if $a > 0$, then

$$(R(-a)\sharp S)^* \cong R(a)\sharp S.$$

As a $T = R\sharp S$ -module, $R(-a)\sharp S$ is generated by $1 \otimes S_a$. Let f be a monomial in R_a . Since R is a domain, the multiplication map $R(-a) \xrightarrow{f} R$ is injective, and since $\sharp S$ is an exact functor, the induced map $R(-a)\sharp S \rightarrow R\sharp S$ is injective. Thus, the map $R(-a)\sharp S \rightarrow R \otimes_K S$ is injective as well. It follows that $R(-a)\sharp S$ is isomorphic to the ideal J of T generated by $f \otimes S_a$. Therefore, up to a shift, $(R(-a)\sharp S)^*$ is isomorphic to the inverse J^{-1} of J .

Since J is a monomial ideal, it follows that J^{-1} is a fractionary monomial ideal. Let $x = \frac{g_1 \otimes h_1}{g_2 \otimes h_2} = \frac{g_1}{g_2} \otimes \frac{h_1}{h_2} \in J^{-1}$ where g_1, g_2, h_1, h_2 are monomials with $\deg g_1 = \deg h_1 = i$ and $\deg g_2 = \deg h_2 = j$. Then $f \frac{g_1}{g_2} \otimes \frac{h_1}{h_2} S_a \subseteq R\sharp S$. This implies that $h_1/h_2 \in U^{-1}$, where U denotes the ideal of S generated by its homogeneous component of degree a . Since $\text{depth } S \geq 2$, and U is a primary ideal in S with radical the maximal graded ideal of S , we have $\text{grade } U \geq 2$, which implies that $U^{-1} \cong S$ by [2, Exercise 1.2.24]. Hence $h_1/h_2 \in S$ which means $i - j \geq 0$. Then $f g_1/g_2 \in R_{a+i-j}$, that is, $g_1/g_2 \in f^{-1}R_{a+i-j}$. We obtain $x f \in (R(a)\sharp S)$. Therefore, we have proved that $J^{-1} \subseteq f^{-1}(R(a)\sharp S)$.

On the other hand, $J = f(R(-a)\sharp S)$ and $f^{-1}(R(a)\sharp S)f(R(-a)\sharp S) \subseteq R\sharp S$, which implies that $f^{-1}(R(a)\sharp S) \subseteq J^{-1}$. Hence $J^{-1} = f^{-1}(R(a)\sharp S)$, which proves our claim. \square

3 The anticanonical module and its twists

Let R be a standard graded K -algebra of $\dim R = r$ with maximal graded ideal \mathfrak{m} . The module $\omega_R = (H_{\mathfrak{m}}^r(R))^\vee \cong \text{Hom}_K(H_{\mathfrak{m}}^r(R), K)$ is called the canonical module of R . Here $H_{\mathfrak{m}}^r(R)$ denotes the r -th local cohomology module of R . We refer the reader to [5] for several properties of ω_R .

Let ω_R^* be the dual of ω_R , that is, $\omega_R^* = \text{Hom}_R(\omega_R, R)$. The module ω_R^* is called the anticanonical module of R .

When R is a Cohen-Macaulay normal domain with canonical ideal ω_R , it is customary to define the anticanonical module of R as the element in the divisor class group $\text{Cl}(R)$ which is the inverse to the canonical class $[\omega_R]$. Then $-[\omega_R] = [\omega_R^{-1}]$ by the way $\text{Cl}(R)$ is defined. By [2, Corollary 3.3.19] one has that ω_R is a divisorial ideal, i.e. $(\omega_R^{-1})^{-1} = \omega_R$. It follows that $\omega_R^{-1} = ((\omega_R^{-1})^{-1})^{-1}$. On the other hand, $((\omega_R^{-1})^{-1})^{-1}$ is a divisorial ideal of R , see [1, Lemma 4.48(c)]. This implies that ω_R^{-1} is a divisorial ideal, as well. We have $\omega_R^{-1} \cong \omega_R^*$, since $\text{grade } \omega_R > 0$, see [10, Lemma 3.14]. This shows that $-[\omega_R] = [\omega_R^*]$ and our definition for the anticanonical module of R agrees with the classical one when R is normal.

Theorem 2.3 has several consequences.

Corollary 3.1. *Let (R, S) be a friendly pair of standard graded algebras with $\text{depth } R \geq 2$, $\text{depth } S \geq 2$ and $T = R\sharp S$. Then*

$$\omega_T^* \cong \omega_R^* \sharp \omega_S^*.$$

In particular, if S is Gorenstein with a -invariant σ , then

$$\omega_T^* \cong \omega_R^* \sharp S(-\sigma).$$

Moreover, if ω_R^ is generated by w_1, \dots, w_m with $\text{deg } w_i = \rho_i$ for $1 \leq i \leq m$ and $\sigma \leq \min_{1 \leq i \leq m} \rho_i$, then ω_T^* is generated by $w_i \otimes S_{\rho_i - \sigma}, 1 \leq i \leq m$.*

Proof. By [5, Theorem 4.3.1], we have $\omega_T \cong \omega_R \sharp \omega_S$. Hence, the desired isomorphism follows by applying Theorem 2.3. The second isomorphism follows since $\omega_S = S(\sigma)$ if S is Gorenstein with a -invariant σ . \square

Remark 3.2. With notation as in Corollary 3.1, it is not true in general that ω_T^* and $\omega_R^* \sharp \omega_S^*$ are isomorphic. Indeed, for the algebras $R = K[x]/(x^3)$ and $S = K[y]/(y^2)$ in Example 2.2 one has $\omega_R \cong R(2)$, $\omega_S \cong S(1)$, hence $\omega_R^* \sharp \omega_S^* = R(-2) \sharp S(-1) = Ky$. We note that the ring $R \sharp S$ is graded isomorphic with the ring S , hence ω_T^* has two nonzero components. This shows that ω_T^* and $\omega_R^* \sharp \omega_S^*$ are not isomorphic.

Corollary 3.3. *Let (R, S) be a pair of friendly standard graded algebras and $T = R \sharp S$. Assume that R, S are Gorenstein of a -invariants ρ , respectively, σ , and $\dim R, \dim S \geq 2$. Then $\omega_T^* \cong R(-\rho) \sharp S(-\sigma)$, hence ω_T^* is isomorphic to $R(\sigma) \sharp S(\rho)$ up to a shift.*

Proof. Under our hypothesis, we have $\omega_R = R(\rho)$ and $\omega_S = S(\sigma)$. It follows that $\omega_T \cong R(\rho) \sharp S(\sigma)$, thus $\omega_T^* \cong R(-\rho) \sharp S(-\sigma) = (R(\sigma) \sharp S(\rho))(-\rho - \sigma)$. \square

Proposition 3.4. *Let (R, S) be a pair of friendly standard graded algebras and $T = R \sharp S$. Assume that R, S are Gorenstein of a -invariants $\rho < 0$, respectively, $\sigma < 0$, and $\dim R, \dim S \geq 2$. Then*

- (i) *the canonical module ω_T is reflexive;*
- (ii) *the localization T_P is Gorenstein for any height 1 prime ideal P of T .*

Proof. (i). We have $(R(\rho) \sharp S(\sigma))^{**} \cong (R(-\rho) \sharp S(-\sigma))^* \cong R(\rho) \sharp S(\sigma)$, thus the canonical module of T is reflexive.

Statement (ii) follows by [6, Theorem 7.31]. \square

Remark 3.5. Note that, in the setting of Corollary 3.3, ω_T is generated by $1 \otimes S_{\sigma - \rho}$ if $\rho < \sigma$ and ω_T^* is generated by $R_{\sigma - \rho} \otimes 1$. This shows that one can easily find examples of rings R, S such that $\mu(\omega_T) > \mu(\omega_T^*)$ or $\mu(\omega_T^*) > \mu(\omega_T)$. By $\mu(M)$ we denote the minimal number of homogeneous generators of a finitely generated graded module M .

It would be interesting to see under which conditions the anticanonical module of Corollary 3.3 is Cohen-Macaulay. To this aim, we first prove a slightly more general result, inspired by [5, Proposition 4.2.2].

Proposition 3.6. *Let R, S be Gorenstein standard graded algebras with $\dim R = r \geq 1$ and $\dim S = s \geq 1$. Let ρ be the a -invariant of R and σ the a -invariant of S . Let $M = R(a) \sharp S(b)$ for some integers a, b . The following statements hold:*

(i) If $r = s > 1$, then $\text{depth } M = r$ if $a - b \geq -\sigma$ or $a - b \leq \rho$, and M is Cohen-Macaulay if $-\sigma > a - b > \rho$.

(ii) If $r > s > 1$, then

$$\text{depth } M = \begin{cases} s, & \text{if } b - a \leq \sigma, \\ r, & \text{if } a - b \leq \rho \text{ and } a - b < -\sigma, \end{cases}$$

and M is Cohen-Macaulay in all the other cases, that is, if $-\sigma > a - b > \rho$.

(iii) If $r > s = 1$, then $\text{depth } M = 1$ if $b - a \leq \sigma$ and M is Cohen-Macaulay otherwise.

(iv) If $r = s = 1$, then M is Cohen-Macaulay for any a, b .

Proof. We first note that (iv) follows by [5, Theorem 4.2.3]. Let us now prove statements (i)–(iii).

By [5, Theorem 4.2.3], $\dim R\sharp S = r + s - 1$. Let $\mathfrak{m}, \mathfrak{n}$, and \mathfrak{p} be the maximal graded ideals of R, S , and, respectively, $R\sharp S$. Since $\dim R > 1$ or $\dim S > 1$, by [5, Theorem 4.1.5], we have

$$H_{\mathfrak{p}}^q(M) \cong (R(a)\sharp H_{\mathfrak{n}}^q(S(b))) \oplus (H_{\mathfrak{m}}^q(R(a))\sharp S(b)) \oplus (\oplus_{i+j=q+1} H_{\mathfrak{m}}^i(R(a))\sharp H_{\mathfrak{n}}^j(S(b))), \quad (5)$$

for every $q \geq 0$. By local duality [2, Theorem 3.6.19], we have $H_{\mathfrak{m}}^q(R(\rho)) \cong R^\vee$ if $q = r$ and $H_{\mathfrak{m}}^q(R(\rho)) = 0$ if $q \neq r$. Hence, $H_{\mathfrak{m}}^q(R(a)) = H_{\mathfrak{m}}^q(R(\rho))(a - \rho) = R^\vee(a - \rho)$ if $q = r$ and $H_{\mathfrak{m}}^q(R(a)) = 0$, otherwise. Similarly, we get $H_{\mathfrak{n}}^q(S(b)) = S^\vee(b - \sigma)$ if $q = s$ and $H_{\mathfrak{n}}^q(S(b)) = 0$, otherwise.

(i). Let $r = s > 1$. By using (5), we get $H_{\mathfrak{p}}^q(M) = 0$ if $q \neq r, q < 2r + 1$, and

$$H_{\mathfrak{p}}^r(M) \cong (R(a)\sharp H_{\mathfrak{n}}^r(S(b))) \oplus (H_{\mathfrak{m}}^r(R(a))\sharp S(b)) \cong (R(a)\sharp S^\vee(b - \sigma)) \oplus (R^\vee(a - \rho)\sharp S(b)).$$

Thus $H_{\mathfrak{p}}^r(M) \neq 0$ if and only if there exists i such that $(R(a)\sharp S^\vee(b - \sigma))_i \neq 0$ or $(R^\vee(a - \rho)\sharp S(b))_i \neq 0$. This is equivalent to $(a + i \geq 0$ and $b - \sigma + i \leq 0)$ or $(a - \rho + i \leq 0$ and $b + i \geq 0)$ which means $-a \leq i \leq \sigma - b$ or $-b \leq i \leq \rho - a$. Therefore, $H_{\mathfrak{p}}^r(M) \neq 0$ if and only if $b - a \leq \sigma$ or $a - b \leq \rho$. This implies (i).

(ii). Let $r > s > 1$. By using (5), we get $H_{\mathfrak{p}}^s(M) \cong (R(a)\sharp H_{\mathfrak{n}}^s(S(b)))$. Thus, $H_{\mathfrak{p}}^s(M) \neq 0$ if and only if $R(a)\sharp S^\vee(b - \sigma)_i \neq 0$. With similar calculations as in case (i), we get $H_{\mathfrak{p}}^s(M) \neq 0$ if and only if $b - a \leq \sigma$. Next, $H_{\mathfrak{p}}^r(M) \cong H_{\mathfrak{m}}^r(R(a))\sharp S(b) = R^\vee(a - \rho)\sharp S(b)$. Thus $H_{\mathfrak{p}}^r(M) \neq 0$ if and only if there exists i with $a - \rho + i \leq \sigma$ and $b + i \geq 0$. Hence, $H_{\mathfrak{p}}^r(M) \neq 0$ if and only if $a - b \leq \rho$ and the conclusion of (ii) follows.

(iii). Let $r > s = 1$. By (5), we get $H_{\mathfrak{p}}^1(M) \cong R(a)\sharp H_{\mathfrak{n}}^1(S(b)) \cong R(a)\sharp S^\vee(b - \sigma)$. Thus, $H_{\mathfrak{p}}^1(M) \neq 0$ if and only if there exists i such that $i + a \geq 0$ and $b - \sigma + i \leq 0$. This implies that $H_{\mathfrak{p}}^1(M) \neq 0$ if and only if $b - a \leq \sigma$. In this case, $\text{depth } M = 1$. \square

Corollary 3.7. *Let (R, S) be a pair of friendly standard graded algebras and $T = R\sharp S$. Assume that R, S are Gorenstein of a -invariants ρ, σ , respectively, and $\dim R, \dim S \geq 2$. Then, the anticanonical module ω_T^* is Cohen-Macaulay if and only if $\sigma > 2\rho$ and $\rho > 2\sigma$.*

Proof. By Corollary 3.3, we have $\omega_T^* \cong R(-\rho)\sharp S(-\sigma)$. The conclusion follows by applying Proposition 3.6 for $a = -\rho$ and $b = -\sigma$. \square

We can generalize the above corollary to any friendly family R_1, \dots, R_m of Gorenstein standard graded algebras of dimension at least two.

Theorem 3.8. *Let R_1, \dots, R_m be a friendly family of standard graded algebras with $\dim R_i = d_i \geq 2$ for $1 \leq i \leq m$. Assume that R_i is Gorenstein of a -invariant $-\rho_i$ for $1 \leq i \leq m$, with $\rho_1 \geq \rho_2 \geq \dots \geq \rho_m$. Set $T = R_1 \sharp \dots \sharp R_m$ and let a be any integer.*

Then the T -module $M = \sharp_{i=1}^m R_i(-a\rho_i)$ is Cohen-Macaulay if and only if

$$\begin{aligned} (1-a)\rho_{\ell+1} &> -a\rho_\ell, && \text{for } \ell = 1, \dots, m-1, \text{ if } a \leq 0, \text{ or} \\ a\rho_{\ell+1} &> (a-1)\rho_\ell, && \text{for } \ell = 1, \dots, m-1, \text{ if } a > 0. \end{aligned}$$

In particular, if M is a Cohen-Macaulay module then T is a Cohen-Macaulay ring.

Proof. Let $M_i = R_i(-a\rho_i)$ and \mathfrak{m}_i the graded maximal ideal of R_i for $1 \leq i \leq m$. Let \mathfrak{m} be the graded maximal ideal of $\sharp_{i=1}^m R_i$. By applying induction on m and [5, Proposition 4.4.3, Theorem 4.1.5], we obtain:

$$\dim \sharp_{i=1}^m M_i = \sum_{i=1}^m d_i - (m-1) \tag{6}$$

and

$$H_{\mathfrak{m}}^q(M) \cong \bigoplus_{\ell=1}^m \bigoplus_{\substack{1 \leq i_1 < \dots < i_\ell \leq m \\ j_1 + \dots + j_\ell = q + (\ell-1)}} M_1 \sharp \dots \sharp H_{\mathfrak{m}_{i_1}}^{j_1}(M_{i_1}) \sharp \dots \sharp H_{\mathfrak{m}_{i_\ell}}^{j_\ell}(M_{i_\ell}) \sharp \dots \sharp M_m. \tag{7}$$

Using [2, Theorems 3.5.7 and 3.6.18] we get

$$\begin{aligned} H_{\mathfrak{m}_i}^j(M_i) &= H_{\mathfrak{m}_i}^j(R_i(-a\rho_i)) = 0 \text{ if } j \neq d_i \text{ and} \\ H_{\mathfrak{m}_i}^{d_i}(M_i) &= H_{\mathfrak{m}_i}^{d_i}(R_i(-a\rho_i)) \cong H_{\mathfrak{m}_i}^{d_i}(R_i)(-a\rho_i) \cong (\omega_{R_i}^\vee)(-a\rho_i) \\ &\cong (R_i(-\rho_i))^\vee(-a\rho_i) \cong R_i^\vee((1-a)\rho_i). \end{aligned} \tag{8}$$

Set $d = \sum_{i=1}^m d_i - (m-1)$. We know that M is Cohen-Macaulay if and only if $H_{\mathfrak{m}}^q(M) = 0$ for all $q < d$. When $q = d$, each summand in the RHS of (7) corresponds to a choice of ℓ with $1 \leq \ell \leq m$, of $1 \leq i_1 < \dots < i_\ell \leq m$ and j_1, \dots, j_ℓ such that $j_1 + \dots + j_\ell = \sum_{i=1}^m (d_i - 1) + \ell$. The latter equation implies $\sum_{i=1}^\ell (j_i - 1) = \sum_{i=1}^m (d_i - 1)$. Since $j_i \leq d_i$ we must have $\ell = m$, $j_i = d_i$ for $i = 1, \dots, m$ and $i_1 = 1, \dots, i_m = m$, hence

$$H_{\mathfrak{m}}^d(M) \cong \sharp_{i=1}^m H_{\mathfrak{m}_i}^{d_i}(M_i).$$

On the other hand, by (8), for any nonzero summand in the RHS of (7) with $\ell = m$ we must have $i_1 = 1, \dots, i_m = m$ and $j_1 = d_1, \dots, j_m = d_m$, therefore $q = \sum_{i=1}^m d_i - (m-1) = d$.

Therefore, M is Cohen-Macaulay if and only if, for $1 \leq \ell \leq m-1$, and for all $1 \leq i_1 < \dots < i_\ell \leq m$, we have

$$M_1 \sharp \dots \sharp H_{\mathfrak{m}_{i_1}}^{d_{i_1}}(M_{i_1}) \sharp \dots \sharp H_{\mathfrak{m}_{i_\ell}}^{d_{i_\ell}}(M_{i_\ell}) \sharp \dots \sharp M_m = 0.$$

Note that, by (8),

$$M_1 \sharp \dots \sharp H_{\mathfrak{m}_{i_1}}^{d_{i_1}}(M_{i_1}) \sharp \dots \sharp H_{\mathfrak{m}_{i_\ell}}^{d_{i_\ell}}(M_{i_\ell}) \sharp \dots \sharp M_m \neq 0$$

if and only if there exists k such that $k - a\rho_i \geq 0$ for $i \notin \{i_1, \dots, i_\ell\}$ and $(1 - a)\rho_i + k \leq 0$ for $i \in \{i_1, \dots, i_\ell\}$. This is equivalent to the inequalities

$$\min\{(a - 1)\rho_i : i \in \{i_1, \dots, i_\ell\}\} \geq k \geq \max\{a\rho_i : i \notin \{i_1, \dots, i_\ell\}\}.$$

Thus, we derive that

$$M_1 \# \dots \# H_{m_{i_1}}^{d_{i_1}}(M_{i_1}) \# \dots \# H_{m_{i_\ell}}^{d_{i_\ell}}(M_{i_\ell}) \# \dots \# M_m = 0$$

if and only if

$$\max\{a\rho_i : i \notin \{i_1, \dots, i_\ell\}\} > \min\{(a - 1)\rho_i : i \in \{i_1, \dots, i_\ell\}\}. \tag{9}$$

Consequently, M is Cohen-Macaulay if and only if, for $1 \leq \ell \leq m - 1$, and for all $1 \leq i_1 < \dots < i_\ell \leq m$, inequality (9) holds.

Case $a < 0$. The inequalities (9) are equivalent to

$$(1 - a) \max\{\rho_i : i \in \{i_1, \dots, i_\ell\}\} > -a \min\{\rho_i : i \notin \{i_1, \dots, i_\ell\}\} \tag{10}$$

for all ℓ and i_1, \dots, i_ℓ as above.

Let $1 \leq \ell \leq m - 1$. By (10), we have

$$(1 - a) \max\{\rho_{m-\ell+1}, \dots, \rho_m\} > -a \min\{\rho_1, \dots, \rho_{m-\ell}\}$$

which implies $(1 - a)\rho_{m-\ell+1} > -a\rho_{m-\ell}$. Equivalently, $(1 - a)\rho_{\ell+1} > -a\rho_\ell$ for $\ell = 1, \dots, m - 1$.

Conversely, if $(1 - a)\rho_{m-\ell+1} > -a\rho_{m-\ell}$ then, for any $1 \leq i_1 < \dots < i_\ell \leq m$, we have

$$\begin{aligned} (1 - a) \max\{\rho_i : i \in \{i_1, \dots, i_\ell\}\} &\geq (1 - a)\rho_{m-\ell+1} \\ &> -a\rho_{m-\ell} \geq -a \min\{\rho_i : i \notin \{i_1, \dots, i_\ell\}\}, \end{aligned}$$

and (10) holds.

Case $a = 0$. According to (9), M is Cohen-Macaulay if and only if

$$0 > - \max\{\rho_i : i \in \{i_1, \dots, i_\ell\}\} = -\rho_{i_1},$$

for all $\ell < m$ and $i_1 < \dots < i_\ell$, which is equivalent to requiring $\rho_i > 0$ for $i = 1, \dots, m$.

In case $a = 1$, the inequalities (9) imply that M is Cohen-Macaulay if and only if $\rho_i > 0$ for $i = 1, \dots, m$.

Case $a > 1$. Now (9) implies that M is Cohen-Macaulay if and only if

$$a \max\{\rho_i : i \notin \{i_1, \dots, i_\ell\}\} > (a - 1) \min\{\rho_i : i \in \{i_1, \dots, i_\ell\}\} \tag{11}$$

for all $1 \leq \ell < m$ and $1 \leq i_1 < \dots < i_\ell \leq m$. In particular, for any $1 \leq \ell < m$ one has

$$a \max\{\rho_{\ell+1}, \dots, \rho_m\} > (a - 1) \min\{\rho_1, \dots, \rho_\ell\},$$

i.e. $a\rho_{\ell+1} > (a - 1)\rho_\ell$ for $1 \leq \ell < m$. We check that the latter inequalities imply (11). Indeed, for $\ell < m$ and $i_1 < \dots < i_\ell$ we have

$$\begin{aligned} a \max\{\rho_i : i \notin \{i_1, \dots, i_\ell\}\} &\geq a\{\rho_{\ell+1}, \dots, \rho_m\} = a\rho_{\ell+1} \\ &> (a - 1)\rho_\ell = (a - 1) \min\{\rho_1, \dots, \rho_\ell\} \\ &\geq (a - 1) \min\{\rho_i : i \in \{i_1, \dots, i_\ell\}\}. \end{aligned}$$

Let us assume that M is a Cohen-Macaulay T -module. To prove that T is a Cohen-Macaulay ring is equivalent to the a -invariants of R_i be negative for all i , see [5, Theorem 4.4.1(i)], or equivalently that $\rho_m > 0$. This was checked above for $0 \leq a \leq 1$.

If $a > 1$, the inequality $a\rho_m > (a-1)\rho_{m-1}$ gives $\rho_m > (a-1)(\rho_{m-1}-\rho_m) \geq 0$. Similarly, when $a < 0$, the inequality $(1-a)\rho_m > a\rho_{m-1}$ implies that $\rho_m > a(\rho_m - \rho_{m-1}) \geq 0$, as desired. \square

Remark 3.9. In the setting of Theorem 3.8, for $a = 0$ we get $M = \#_{i=1}^m R_i = T$, and if $a = 1$ we get $M = \#_{i=1}^m R_i(-\rho_i) \cong \#_{i=1}^m \omega_{R_i} \cong \omega_T$, see [5, Theorem 4.3.1].

Corollary 3.10. *In the setting of Theorem 3.8, assume $a \neq 0, 1$. Denote $C = (a/(a-1))^{sgn(a)}$. Then M is a Cohen-Macaulay T -module if and only if*

$$C^{m-1}\rho_m > C^{m-2}\rho_{m-1} > \dots > C\rho_2 > \rho_1.$$

In particular, the anticanonical module ω_T^ is Cohen-Macaulay if and only if*

$$2^{m-1}\rho_m > 2^{m-2}\rho_{m-1} > \dots > 2\rho_2 > \rho_1.$$

Proof. The first part follows immediately from Theorem 3.8. For the computation of the anticanonical module ω_T^* we use that $\omega_{R_i} = R_i(-\rho_i)$, hence $\omega_{R_i}^* = R_i(\rho_i)$ for all i . Moreover, since the algebras R_1, \dots, R_m are friendly, [5, Theorem 4.3.1] yields

$$\omega_T^* \cong (\#_{i=1}^m \omega_{R_i})^* \cong \#_{i=1}^m \omega_{R_i}^* \cong \#_{i=1}^m R_i(\rho_i).$$

\square

Proposition 3.11. *Let R_1, \dots, R_m be a friendly family of standard graded algebras with $\dim R_i \geq 2$ for $1 \leq i \leq m$. Assume that R_i is Gorenstein of a -invariant $-\rho_i$ for $1 \leq i \leq m$ with $\rho_1 \geq \rho_2 \geq \dots \geq \rho_m > 0$. Let $\rho = \max\{\rho_i/\rho_{i+1} : i = 1, \dots, m-1\}$. For any integer a we set $T^{\{a\}} = \#_{i=1}^m R_i(-a\rho_i)$ and we let $T = T^{\{0\}}$.*

Then the T -module $T^{\{a\}}$ is Cohen-Macaulay if and only if either $\rho = 1$ or

$$\frac{1}{1-\rho} < a < \frac{\rho}{\rho-1}.$$

Proof. Since $\rho_i > 0$ for all i , it follows that T is a Cohen-Macaulay ring, cf. [5, Theorem 4.4.4(i)]. Our hypothesis on the ρ_i 's implies that $\rho \geq 1$. Equality holds if and only if $\rho_1 = \dots = \rho_m$, i.e. $T^{\{a\}}$ is a shifted copy of T , hence Cohen-Macaulay. As a matter of facts, the case $\rho = 1$ is equivalent to T being Gorenstein, by [5, Theorem 4.4.7].

Assume $\rho > 1$. Then $1/(1-\rho) < 0$ and $1 < \rho/(\rho-1)$. Since T is Cohen-Macaulay it follows from Theorem 3.8 that $T^{\{1\}}$ is Cohen-Macaulay, too.

If $a > 1$, by Theorem 3.8 we see that $T^{\{a\}}$ is Cohen-Macaulay if and only if $a/(a-1) > \rho_\ell/\rho_{\ell+1}$ for $\ell = 1, \dots, m-1$, i.e. $a/(a-1) > \rho$, which is equivalent to $1 + 1/(a-1) > \rho$, and to $1/(a-1) > \rho - 1$. Thus $a < \rho/(\rho-1)$.

If $a < 0$, by Theorem 3.8 it follows that $T^{\{a\}}$ is Cohen-Macaulay if and only if $(a-1)/a > \rho$, which is equivalent to $a > 1/(1-\rho)$. \square

Corollary 3.12. *In the setting of Proposition 3.11, assume $\rho > 1$. Then $T^{\{a\}}$ is Cohen-Macaulay only for $a = 0, 1$ if and only if $\rho \geq 2$. In particular, the anticanonical module ω_T^* is not Cohen-Macaulay when $\rho \geq 2$.*

Proof. It is clear that $1/(1 - \rho) < 0$ and $\rho/(\rho - 1) = 1 + 1/(\rho - 1) > 1$, hence $T^{\{a\}}$ is Cohen-Macaulay for $a = 0, 1$. That these are the only values, by Proposition 3.11, it is equivalent to having $-1 \leq 1/(1 - \rho)$ and $\rho/(\rho - 1) \leq 2$. It is easy to check that both conditions mean $\rho \geq 2$.

As noted in the proof of Corollary 3.10, the anticanonical module $\omega_T^* \cong T^{\{-1\}}$ and by the above arguments it is not Cohen-Macaulay when $\rho \geq 2$. \square

By [2, Proposition 3.3.18], for a standard graded Cohen-Macaulay algebra R which is generically Gorenstein (for instance, a domain), its canonical module may be identified with an ideal in R , that we call a canonical ideal of R . If I and J are two canonical ideals of R , they are isomorphic and there exists an element x invertible in $Q(R)$ such that $I = xJ$. Hence, for any integer a the T -modules I^a and J^a are isomorphic, and Cohen-Macaulay or not at the same time.

The next result describes which powers of the canonical ideal of a Segre product of some friendly Gorenstein algebras are Cohen-Macaulay.

Proposition 3.13. *Let R_1, \dots, R_m be a friendly family of standard graded Gorenstein K -algebras of dimension at least two such that $R_1 \otimes_K \dots \otimes_K R_m$ is a domain. Let $-\rho_i$ be the a -invariant of R_i for $i = 1, \dots, m$. Assume $\rho_1 \geq \rho_2 \geq \dots \geq \rho_m > 0$ and that $\rho = \max\{\rho_i/\rho_{i+1} : i = 1, \dots, m - 1\} > 1$.*

Let $T = \#_{i=1}^m R_i$, ω_T a canonical ideal of T , and a any integer. Then ω_T^a is a Cohen-Macaulay T -module if and only if

$$\frac{1}{1 - \rho} < a < \frac{\rho}{\rho - 1}.$$

Proof. The Segre product T is a Cohen-Macaulay ring which is not Gorenstein, since all ρ_i 's are positive and $\rho > 1$. By [5, Theorem 4.3.1], the canonical module of T is isomorphic to $\#_{i=1}^m R_i(-\rho_i)$, which is generated by $1 \otimes R_{2, \rho_1 - \rho_2} \otimes \dots \otimes R_{m, \rho_1 - \rho_m}$. Here $R_{i,j}$ denotes the j th homogeneous component of R_i for $i = 1, \dots, m$ and $j \geq 0$. Pick $f = u_1 \otimes \dots \otimes u_{m-1} \otimes 1$ with $0 \neq u_i \in R_{i, \rho_i - \rho_m}$ for $i = 1, \dots, m - 1$. Since $\#_{i=1}^m R_i$ is a domain it follows that T is a domain, as well. Therefore, the map $\phi : \#_{i=1}^m R_i(-\rho_i) \xrightarrow{f} T$ is injective. Without loss of generality we may assume that ω_T is the image of ϕ . Hence

$$\omega_T = (u_1 \otimes u_2 R_{2, \rho_1 - \rho_2} \otimes \dots \otimes u_{m-1} R_{m-1, \rho_1 - \rho_{m-1}} \otimes R_{m, \rho_1 - \rho_m})T.$$

Assume $a > 0$. Since the algebras R_i are standard graded, we get

$$\omega_T^a = (u_1^a \otimes u_2^a R_{2, a(\rho_1 - \rho_2)} \otimes \dots \otimes u_{m-1}^a R_{m-1, a(\rho_1 - \rho_{m-1})} \otimes R_{m, a(\rho_1 - \rho_m)})T.$$

Arguing as above, we have that ω_T^a is isomorphic to the T -module $\#_{i=1}^m R_i(-a\rho_i)$, for all $a > 0$.

With a similar argument we may identify the anticanonical module $\omega_T^* \cong \#_{i=1}^m R_i(\rho_i)$ with the ideal

$$\omega_T^* = (R_{1, \rho_1 - \rho_m} \otimes v_2 R_{2, \rho_2 - \rho_m} \otimes \dots \otimes v_{m-1} R_{m-1, \rho_1 - \rho_{m-1}} \otimes v_m)T,$$

where $0 \neq v_i \in R_{i, \rho_1 - \rho_i}$ for $i = 2, \dots, m$. A similar discussion shows the isomorphism $\omega_T^a \cong \#_{i=1}^m R_i(-a\rho_i)$ for negative a , as well.

The conclusion now follows by Theorem 3.11. \square

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