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Polynomial minimal surfaces of degree five

by
Ognian Kassabov

Abstract

The problem of finding all minimal surfaces presented in parametric form as polynomials is discussed by many authors. It is known that the classical Enneper surface is (up to position in space and homothety) the only polynomial minimal surface of degree 3 in isothermal parameters. In higher degrees the problem is quite more complicated. Here we find a general form for the functions that generate a polynomial minimal surface of arbitrary degree via the Weierstrass formula and prove that any polynomial minimal surface of degree 5 in isothermal parameters may be considered as belonging to one of three special families.

Key Words: Minimal surface, isothermal parameters, canonical principal parameters, parametric polynomial surface
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1 Introduction

The minimal surfaces are a topic of great interest in many areas as mathematics, computer science, physics, medicine, architecture. The reason is that in small areas they have a minimizing property.

For the applications of minimal surfaces, in particular in computer graphic research, it is important to use minimal surfaces in polynomial form and hence to know all such surfaces in small degrees. In this direction Cosín and Monterde [1] proved that up to position in space and homothety the only polynomial minimal surface of degree three in isothermal parameters is the classical Enneper surface. The case of degree four is considered in [5]. Polynomial minimal surfaces of degrees five and six are studied in [7] and [6], respectively. Theorems about their coefficients-vectors are found and some examples are considered. Unfortunately the systems for the coefficients are very complicated and the general solution is difficult to be found. In [8], polynomial minimal surfaces of arbitrary degree constructed on some special functions are studied, and thus some special surfaces are proposed. It is remarked that in degrees 3 and 5 these surfaces coincide with the Enneper surface and some of the surfaces from [7], respectively.

In the present paper we first show that a polynomial minimal surface in isothermal parameters must be generated via the Weierstrass formula with a polynomial and a rational function (Section 3). Then in Section 4 we determine all polynomial minimal surfaces of degree five but we do not try to solve the system for the coefficients. Instead we use the result from Section 3 and we obtain a list of functions that generate via the Weierstrass formula all such surfaces. The surfaces introduced in [7] belong to one of the obtained families, but these families contain many other surfaces as well.

It is natural to ask whether these families contain different surfaces. In general the problem of comparing surfaces given in different parametric form is very complicated. For minimal surfaces such a method is proposed in [4]. It is based on the canonical parameters introduced in [2] and then solving an ordinary differential equation for finding these parameters.

When trying to investigate the relation between the families obtained in Section 4 we cannot use directly the method from [4], because we cannot find a simple form of the surfaces in canonical parameters. So we change a little the approach and we escape solving the differential equation for the transition to canonical parameters. As a result we find that the obtained three families contain different minimal surfaces except in a special case.

2 Preliminaries

Let S be a regular surface in the Euclidean space defined by the parametric equation

$$\boldsymbol{x} = \boldsymbol{x}(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v)), \qquad (u,v) \in U \subset \mathbb{R}^2.$$

The derivatives of the vector function $\boldsymbol{x} = \boldsymbol{x}(u,v)$ are usualy denoted by $\boldsymbol{x}_u$, $\boldsymbol{x}_v$, $\boldsymbol{x}_{uu}$, etc. Then the coefficients of the first fundamental form are equal to the scalar products

$$E = \boldsymbol{x}_u^2, \qquad F = \boldsymbol{x}_u \boldsymbol{x}_v, \qquad G = \boldsymbol{x}_v^2,$$

The unit normal to the surface is the vector field

$$\boldsymbol{U} = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{|\boldsymbol{x}_u \times \boldsymbol{x}_v|} = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\sqrt{EG - F^2}}.$$

In particular, if $E = G$, $F = 0$, then the parameters (u,v) of the surface are called *isothermal*.

The coefficients of the second fundamental form of S are given by

$$L = \boldsymbol{U}\, \boldsymbol{x}_{uu}, \qquad M = \boldsymbol{U}\, \boldsymbol{x}_{uv}, \qquad N = \boldsymbol{U}\, \boldsymbol{x}_{vv}.$$

The Gauss curvature K and the mean curvature H of S are defined respectively by

$$K = \frac{LN - M^2}{EG - F^2}, \qquad H = \frac{EN - 2FM + GL}{2(EG - F^2)}.$$

Recall that the surface S is called *minimal* if its mean curvature vanishes identically. In this case it follows easily that the Gauss curvature is nonpositive.

The study of minimal surfaces is closely related with some complex curves – those with isotropic tangent vectors. They are called *minimal curves*. Indeed we have the following construction.

Let S be a minimal surface defined in isothermal parameters. Then it can be considered as the real part of a minimal curve. More precisely, let $f(z)$ and $g(z)$ be two holomorphic functions. Define the Weierstrass complex curve $\Psi(z)$ by

$$\boldsymbol{\Psi}(z) = \int_{z_0}^{z} \left(\frac{1}{2} f(z)(1 - g^2(z)), \frac{i}{2} f(z)(1 + g^2(z)), f(z)g(z) \right) dz \,. \tag{2.1}$$

Then $\boldsymbol{\Psi}(z)$ is a minimal curve and its real and imaginary parts $\boldsymbol{x}(u,v)$ and $\boldsymbol{y}(u,v)$ are harmonic functions that define two minimal surfaces in isothermal parametrizations. We say that these two minimal surfaces are *conjugate*. Moreover, every minimal surface can be obtained at least locally as the real (as well as the imaginary) part of a Weierstrass minimal curve.

For any two conjugate minimal surfaces $\boldsymbol{x}(u,v)$ and $\boldsymbol{y}(u,v)$ it is defined the *associated family* $\{S_t\}$, where

$$S_t \quad : \qquad \boldsymbol{x}_t(u,v) = \boldsymbol{x}(u,v)\cos t + \boldsymbol{y}(u,v)\sin t \ .$$

Then for any real number t the surface S_t is also minimal and has the same first fundamental form as $S = S_0$.

Example. Taking $f(z) = 1$, $g(z) = z$, we obtain a Weierstrass minimal curve whose real part is the classical Enneper surface

$$\boldsymbol{x}(u,v) = \left(\frac{u}{2}\big(1+v^2-\frac{u^2}{3}\big), -\frac{v}{2}\big(1+u^2-\frac{v^2}{3}\big), \frac{1}{2}(u^2-v^2)\right) \ .$$

It is well known that the Enneper surface coincide (up to position in space) with any surface in its associated family, see e.g. [3].

In [2] Ganchev introduces *the canonical principal parameters*. If a surface is parametrized with them, the coefficients of its fundamental forms are given by

$$\begin{array}{ccc} E = \dfrac{1}{\nu} \ , & F = 0 \ , & G = \dfrac{1}{\nu} \\ L = 1 \ , & M = 0 \ , & N = -1 \ , \end{array}$$

where $\nu = \sqrt{-K}$ is the *normal curvature* of the surface. Actually a surface in canonical principal parametrization is the real part of a Weierstrass minimal curve generated by some functions $f(z)$, $g(z)$ with $f(z) = -1/g'(z)$, i.e. it is the real part of the special Weierstrass curve

$$\boldsymbol{\Phi}(z) = -\int_{z_0}^{z} \left(\frac{1-g^2(z)}{2g'(z)}, \frac{i(1+g^2(z))}{2g'(z)}, \frac{g(z)}{g'(z)}\right) dz \ .$$

The canonical principal parameters, resp. the normal curvature play a role similar to that of the natural parameters, resp. the curvature and torsion of a space curve. Namely the following theorem holds:

Theorem 1. *[2] If a surface is parametrized with canonical principal parameters, then its normal curvature ν satisfies the differential equation*

$$\Delta \ln \nu + 2\nu = 0.$$

Conversely, for any solution $\nu(u,v)$ of this equation (with $\nu_u \nu_v \neq 0$), there exists an **unique** *(up to position in space) minimal surface with normal curvature $\nu(u,v)$, (u,v) being canonical principal parameters. Moreover, the canonical principal parameters (u,v) are determined uniquely up to the following transformations*

$$\begin{array}{l} u = \varepsilon\bar{u} + a, \\ v = \varepsilon\bar{v} + b, \end{array} \qquad \varepsilon = \pm 1 \ , \ a = const., \ b = const.$$

We will also use the following results:

Theorem 2. *[4] Let S be the minimal surface defined by the real part of the Weierstrass minimal curve (2.1). Any solution of the differential equation*

$$(z'(w))^2 = -\frac{1}{f(z(w))g'(z(w))} \tag{2.2}$$

defines a transformation of the isothermal parameters of S to canonical principal parameters. Moreover the function $\tilde{g}(w)$ that defines S via the Ganchev formula is given by $\tilde{g}(w) = g(z(w))$.

Theorem 3. *[4] Let $g(z)$ be a holomorphic function that generates a minimal surface in canonical principal parameters, i.e. via the Ganchev formula. Then, for an arbitrary complex number α, and for an arbitrary real number φ, any of the functions*

$$e^{i\varphi}\frac{\alpha + g(z)}{1 - \bar{\alpha} g(z)}\,, \qquad \frac{e^{i\varphi}}{g(z)}$$

generates the same surface in canonical principal parameters (up to position in space). Conversely, any function that generates this surface (up to position in space) in canonical principal parameters has one of the above forms.

In sections 4 and 5 we shall consider minimal polynomial surfaces of degree five. An interesting study of such surfaces is presented in [7]. First of all it is proved that the harmonic condition implies that such a surface must have the form

$$\begin{aligned} \boldsymbol{r}(u,v) \;=\;& \boldsymbol{a}\,(u^5 - 10u^3v^2 + 5uv^4) + \boldsymbol{b}\,(v^5 - 10u^2v^3 + 5u^4v) \\ &+\boldsymbol{c}\,(u^4 - 6u^2v^2 + v^4) + \boldsymbol{d}\,uv(u^2 - v^2) + \boldsymbol{e}\,u(u^2 - 3v^2) \\ &+\boldsymbol{f}\,v(v^2 - 3u^2) + \boldsymbol{g}\,(u^2 - v^2) + \boldsymbol{h}\,uv + \boldsymbol{i}\,u + \boldsymbol{j}\,v + \boldsymbol{k} \end{aligned} \tag{2.3}$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$, $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}$, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are coefficient vectors. For these coefficients the following holds, see [7]:

Theorem 4. *The harmonic polynomial surface (2.1) is minimal if and only if its coefficient* *vectors satisfy the following system of equations*

$$\left\{\begin{array}{l}
\boldsymbol{a}^2 = \boldsymbol{b}^2 \\
\boldsymbol{a}.\boldsymbol{b} = 0 \\
4\,\boldsymbol{a}.\boldsymbol{c} - \boldsymbol{b}.\boldsymbol{d} = 0 \\
\boldsymbol{a}.\boldsymbol{d} + 4\boldsymbol{b}.\boldsymbol{c} = 0 \\
16\,\boldsymbol{c}^2 - \boldsymbol{d}^2 + 30\,\boldsymbol{a}.\boldsymbol{e} + 30\,\boldsymbol{b}.\boldsymbol{f} = 0 \\
4\,\boldsymbol{d}.\boldsymbol{c} + 15\,\boldsymbol{b}.\boldsymbol{e} - 15\boldsymbol{a}.\boldsymbol{f} = 0 \\
9\,\boldsymbol{e}^2 - 9\boldsymbol{f}^2 + 16\,\boldsymbol{c}.\boldsymbol{g} - 2\,\boldsymbol{d}.\boldsymbol{h} + 10\,\boldsymbol{a}.\boldsymbol{i} - 10\,\boldsymbol{b}.\boldsymbol{j} = 0 \\
9\,\boldsymbol{e}.\boldsymbol{f} - 4\,\boldsymbol{c}.\boldsymbol{h} - 2\,\boldsymbol{d}.\boldsymbol{g} - 5\,\boldsymbol{b}.\boldsymbol{i} - 5\,\boldsymbol{a}.\boldsymbol{j} = 0 \\
4\,\boldsymbol{g}^2 - \boldsymbol{h}^2 + 6\,\boldsymbol{e}.\boldsymbol{i} + 6\,\boldsymbol{f}.\boldsymbol{j} = 0 \\
2\,\boldsymbol{g}.\boldsymbol{h} - 3\,\boldsymbol{f}.\boldsymbol{i} + 3\,\boldsymbol{e}.\boldsymbol{j} = 0 \\
5\,\boldsymbol{a}.\boldsymbol{h} + 10\,\boldsymbol{b}.\boldsymbol{g} - 12\,\boldsymbol{c}.\boldsymbol{f} + 3\,\boldsymbol{d}.\boldsymbol{e} = 0 \\
5\,\boldsymbol{b}.\boldsymbol{h} - 10\,\boldsymbol{a}.\boldsymbol{g} - 3\,\boldsymbol{d}.\boldsymbol{f} - 12\,\boldsymbol{c}.\boldsymbol{e} = 0 \\
6\,\boldsymbol{e}.\boldsymbol{g} + 3\boldsymbol{f}.\boldsymbol{h} + 4\boldsymbol{c}.\boldsymbol{i} - \boldsymbol{d}.\boldsymbol{j} = 0 \\
6\,\boldsymbol{f}.\boldsymbol{g} - 3\boldsymbol{e}.\boldsymbol{h} - \boldsymbol{d}.\boldsymbol{i} - 4\boldsymbol{c}.\boldsymbol{j} = 0 \\
\boldsymbol{h}.\boldsymbol{i} + 2\boldsymbol{g}.\boldsymbol{j} = 0 \\
2\,\boldsymbol{g}.\boldsymbol{i} - \boldsymbol{h}.\boldsymbol{j} = 0 \\
\boldsymbol{i}^2 = \boldsymbol{j}^2 \\
\boldsymbol{i}.\boldsymbol{j} = 0\ .
\end{array}\right. \tag{2.4}$$

Is seems impossible to find the general solution of the system (2.4). So in [7] some special solutions are considered and several interesting properties are proved for the obtained surfaces. Using a different approach we shall find all polynomial minimal surfaces of degree five.

3 Polynomial minimal surfaces of arbitrary degree

As is said in Introduction, polynomial minimal surfaces of arbitrary degree are studied in [8]. The following construction is proposed. Consider the functions

$$P_n = \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} (-1)^k \binom{n}{2k} u^{n-2k} v^{2k}$$

$$Q_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} u^{n-2k-1} v^{2k+1}\ ,$$

where $\lceil x \rceil$ denotes the smallest integer not less than x and $\lfloor x \rfloor$ denotes the largest integer not greater than x.

Then it is proved that for any real number ω the polynomial surface of degree n defined by

$$\boldsymbol{x}(u,v) = \big(- P_n + \omega P_{n-2},\ Q_n + \omega Q_{n-2},\ \frac{2\sqrt{n(n-2)\omega}}{n-1} P_{n-1} \big)$$

is minimal. Of course this large family is very interesting. But it is important also to know if these are all the possible polynomial minimal surfaces and if not to find other families. To solve the last problem we propose the following approach.

Let

$$S \ : \qquad \boldsymbol{x}(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$$

be a polynomial minimal surface of degree n in isothermal parameters. Then $x_i(u,v)$ are polynomials of degree $\leq n$, and at least for one $i = 1,2,3$ there is an equality. Suppose that the parametrization is isothermal and S is defined in an open subset of $\mathbb{R}^2$, containing $(0,0)$. From Lemma 22.25 in [3] it follows that (up to translation) $\boldsymbol{x}(u,v)$ is the real part of the minimal curve

$$\boldsymbol{\Psi}(z) = 2\boldsymbol{x}\left(\frac{z}{2}, \frac{z}{2i}\right) .$$

So this minimal curve is also polynomial of degree n. Then

$$\boldsymbol{\Psi}'(z) = (\phi_1(z), \phi_2(z), \phi_3(z)) = \left(\frac{1}{2}f(z)(1-g^2(z)), \frac{i}{2}f(z)(1+g^2(z)), f(z)g(z)\right)$$

for some functions $f(z)$, $g(z)$ and the coordinate functions $\phi_i(z)$ are polynomials of degree $\leq n-1$ so that at least for one i the degree of $\phi_i(z)$ is exactly $n-1$. Hence every one of the functions

$$f(z)(1-g^2(z)) = 2\phi_1(z) \qquad f(z)(1+g^2(z)) = -2i\phi_2(z) \qquad f(z)g(z) = \phi_3(z)$$

is a polynomial and so

$$f(z) = \phi_1(z) - i\phi_2(z), \quad f(z)g^2(z) = -(\phi_1(z) + i\phi_2(z)), \quad f(z)g(z) = \phi_3(z) \tag{3.1}$$

are polynomials of degree $\leq n-1$ and at least for one i the degree in (3.1) is exactly $n-1$. So $f(z)$ is a polynomial of degree $\leq n-1$. Now the third equality in (3.1) implies that $g(z)$ is a rational function of the form

$$g(z) = \frac{P_p(z)}{Q_q(z)} , \tag{3.2}$$

where $P_p(z)$ and $Q_q(z)$ are polynomials (of degrees p and q, respectively) with no common zeros. According to the second equality in (3.1) the function $f(z)g^2(z)$ is also a polynomial, so

$$f(z) = (Q_q(z))^2 R_r(z) \tag{3.3}$$

with a polynomial $R_r(z)$ (of degree r). Moreover since the polynomials

$$\begin{aligned} f(z) &= (Q_q(z))^2 R_r(z) , \\ f(z)g^2(z) &= P_p^2(z) R_r(z) , \\ f(z)g(z) &= P_p(z) Q_q(z) R_r(z) \end{aligned}$$

are of degree $\leq n-1$, then $2q + r \leq n-1$, $2p + r \leq n-1$, $p + q + r \leq n-1$ and at least once there is an equality.

Conversely it is easy to see that for arbitrary polynomials $P_p(z)$, $Q_q(z)$, $R_r(z)$ with the above restrictions on p, q, r, the functions (3.2), (3.3) generate a minimal polynomial surface of degree n via the Weierstrass formula.

So we have

Theorem 5. *Any polynomial minimal surface of degree n in isothermal parameters is generated via the Weierstrass formula by two functions of the form*

$$f(z) = (Q_q(z))^2 R_r(z) \qquad\qquad g(z) = \frac{P_p(z)}{Q_q(z)}$$

where $P_p(z)$, $Q_q(z)$, $R_r(z)$ are polynomials of degree p, q, r, respectively and $2q + r \leq n - 1$, $2p + r \leq n - 1$, $p + q + r \leq n - 1$ with at least one equality. Conversely any two functions $f(z)$, $g(z)$ with the above form generate a minimal polynomial surface of degree n via the Weierstrass formula.

4 Consequences for polynomial minimal surfaces of degree five

With the notations of the previous section we assume $n = 5$. Then

$$2q + r \leq 4, \qquad 2p + r \leq 4, \qquad p + q + r \leq 4$$

with at least one equality. According to the first two equations in Theorem 4 we may assume that (up to a change of the coordinate system) $\mathbf{a} = (a_1, a_2, 0)$, $\mathbf{b} = (-a_2, a_1, 0)$ and hence

$$2q + r \leq 4, \qquad 2p + r \leq 4, \qquad p + q + r \leq 3 \ . \tag{4.1}$$

Then $(4.1)_1$ and $(4.1)_2$ imply $q \leq 2$, $p \leq 2$, so the following cases can appear:

1. $p = 2$. Then $r = 0$.

1.1. $q = 0$, i.e. $f(z) = a$, $g(z) = Az^2 + Bz + C$, where a, $A \neq 0$. To obtain another functions defining the surface we can use a consequence of the following assertion:

Proposition 1. *[4] Suppose the pairs $(\tilde{f}(z), \tilde{g}(z))$ and $(f(w), g(w))$ generate two minimal surfaces via the Weierstrass formula. Then these surfaces coincide (up to translation) iff there exists a function $w = w(z)$, such that*

$$\tilde{f}(z) = f(w(z))w'(z) \qquad \text{and} \qquad \tilde{g}(z) = g(w(z)) \ .$$

Corollary 1. *Suppose the pair $(f(z), g(z))$ generates a minimal surface via the Weierstrass formula. Then for arbitrary numbers α $(\alpha \neq 0)$, β the pair*

$$\tilde{f}(z) = \alpha f(\alpha z + \beta) \ , \qquad \tilde{g}(z) = g(\alpha z + \beta)$$

generates the same minimal surface (up to translation).

Using Corollary 1 with $\alpha = \sqrt{A}$, $\beta = \frac{B}{2\sqrt{A}}$ we can see that the surface is generated by two functions of the form

1.1. $f(z) = a$, $g(z) = z^2 + b$, with $a \neq 0$.

Analogously we obtain the cases:

1.2. $p = 2$, $q = 1$, $r = 0$ and $f(z) = a(z + b)^2$, $g(z) = \dfrac{cz^2 + d}{z + b}$, with $a, c \neq 0$.

2.1. $p = 0$, $q = 2$, $r = 0$ and $f(z) = a(z^2 + b)^2$, $g(z) = \dfrac{1}{z^2 + b}$, with $a \neq 0$;

2.2. $p = 1$, $q = 2$, $r = 0$ and $f(z) = a(bz^2 + c)^2$, $g(z) = \dfrac{z + d}{bz^2 + c}$, with $a, b \neq 0$;

3. $p = 1$, $q = 0$, $r = 2$ and $f(z) = az^2 + b$, $g(z) = z + c$, with $a \neq 0$;

4. $p = 0$, $q = 1$, $r = 2$ and $f(z) = (az^2 + b)(z + c)^2$, $g(z) = \dfrac{1}{z + c}$, with $a \neq 0$.

We will denote the corresponding surfaces $\boldsymbol{r}_{11}[a, b](u, v)$, $\boldsymbol{r}_{12}[a, b, c, d](u, v)$ etc., respectively.

Remark 1. *The case $p = q = r = 1$ is not interesting, because in this case the surface is not of degree 5.*

Now we note that the following can be easily proved:

Proposition 2. *Consider the surfaces*

$$S_1 \ : \ \boldsymbol{x}_1(u, v) = \mathbf{Re} \int_{z_0}^{z} \left(\frac{f_1(z)}{2}(1 - g_1^2(z)), \frac{if_1(z)}{2}(1 + g_1^2(z)), f_1(z)g_1(z) \right) dz \ ,$$

$$S_2 \ : \ \boldsymbol{x}_2(u, v) = \mathbf{Re} \int_{w_0}^{w} \left(\frac{f_2(w)}{2}(1 - g_2^2(w)), \frac{if_2(w)}{2}(1 + g_2^2(w)), f_2(w)g_2(w) \right) dw \ .$$

Denote by

$$S_2^s \ : \ \boldsymbol{x}_2^s(u, v) = \mathbf{Re} \int_{w_0}^{w} \left(\frac{f_2(w)}{2}(g_2^2(w) - 1), \frac{if_2(w)}{2}(1 + g_2^2(w)), f_2(w)g_2(w) \right) dw$$

the surface, symmetric of S_2 about the plane Oyz. Then S_1 and S_2^s coincide if and only if

$$f_2(w) = f_1(Z(w))g_1^2(Z(w))Z'(w) \qquad\qquad g_2(w) = \frac{1}{g_1(Z(w))}$$

for some function $Z(w)$.

Using this proposition we see that the surfaces from cases 2.1, 2.2 and 4 can be viewed as symmetric to those in cases 1.1, 1.2 and 3, respectively. Consequently we have

Theorem 6. *Any polynomial minimal surface of degree 5 in isothermal parameters coincides up to position in space and symmetry with a surface generated via the Weierstrass formula with the pair of functions*

1.1. $f(z) = a$, $g(z) = z^2 + b$, *with* $a \neq 0$;

1.2. $f(z) = a(z + b)^2$, $g(z) = \dfrac{cz^2 + d}{z + b}$, *with* $a, c \neq 0$;

3. $f(z) = az^2 + b$, $g(z) = z + c$, *with* $a \neq 0$,

where a, b, c are complex numbers.

Remark 2. *The family of surfaces introduced in [7] belongs to the case 1.2. More precisely the family from [7] is defined by*

$$r(u,v) = \big(X(u,v), Y(u,v), Z(u,v)\big)$$

with

$$\begin{aligned}
X(u,v) &= a_1(u^5 - 10u^3v^2 + 5uv^4) - a_2(v^5 - 10v^3u^2 + 5vu^4) \\
&\qquad + e_1 u(u^2 - 3v^2) - e_2 v(v^2 - 3u^2) \\
Y(u,v) &= a_2(u^5 - 10u^3v^2 + 5uv^4) + a_1(v^5 - 10v^3u^2 + 5vu^4) \\
&\qquad + e_2 u(u^2 - 3v^2) + e_1 v(v^2 - 3u^2) \\
Z(u,v) &= \tfrac{\sqrt{30}}{4}\sqrt{\sqrt{(a_1^2+a_2^2)(e_1^2+e_2^2)} - (a_1e_1 + a_2e_2)}\,(u^4 - 6u^2v^2 + v^4) \\
&\qquad - \sqrt{30}\sqrt{\sqrt{(a_1^2+a_2^2)(e_1^2+e_2^2)} + (a_1e_1 + a_2e_2)}\,uv(u^2 - v^2)\ ,
\end{aligned}$$

where a_1, a_2, e_1, e_2 are real parameters. For $a_2e_1 - a_1e_2 < 0$ the surfaces are minimal. Such a surface is generated by the Weierstrass formula with

$$\begin{aligned}
f(z) &= 6(e_1 - ie_2)z^2 \\
g(z) &= \sqrt{\tfrac{5}{6}}\,\frac{\sqrt{\sqrt{(a_1^2+a_2^2)(e_1^2+e_2^2)} - a_1e_1 - a_2e_2} + i\sqrt{\sqrt{(a_1^2+a_2^2)(e_1^2+e_2^2)} + a_1e_1 + a_2e_2}}{e_1 - ie_2}\, z
\end{aligned}$$

so it belongs to the case 1.2 with $b = d = 0$.

5 Relations among the families in Theorem 6

For some special values of the parameters the surfaces from Theorem 6 obviously coincide. Namely if $d = -b^2c$ in $\boldsymbol{r}_{12}[a,b,c,d](u,v)$, the surface is of type 3. On the other hand even when this equality is not satisfied, the corresponding surfaces may look very similar, as Fig. 1 and Fig. 2 show.

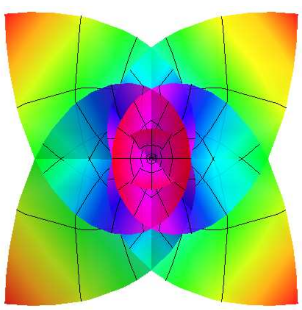

Fig. 1: $\boldsymbol{r}_{12}[1,0,1,1](u,v), |u|,|v| \le 4$

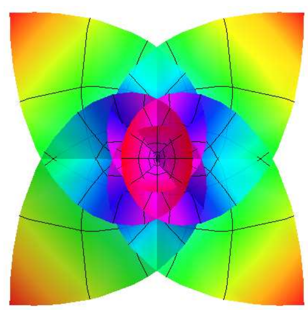

Fig. 2: $\boldsymbol{r}_{3}[1,1,0](u,v), |u|,|v| \le 4$.

We will see that despite the resemblance these two surfaces are different as well as that in general the families $\boldsymbol{r}_{11}$, $\boldsymbol{r}_{12}$ and $\boldsymbol{r}_3$ give different surfaces.

Suppose that a surface $\boldsymbol{r}_{12}[a,b,c,d](u,v)$ generated via the Weierstrass formula by the functions

$$f_{12}(z) = a(z+b)^2 \qquad\qquad g_{12}(z) = \frac{cz^2 + d}{z + b} \tag{5.1}$$

coincides (up to position in space) with $\boldsymbol{r}_3[A,B,C](u,v)$ generated by

$$f_3(z) = Az^2 + B \qquad\qquad g_3(z) = z + C \ . \tag{5.2}$$

Denote $z_{12}(w)$, $z_3(w)$ solutions of the respective equations (2.2), so that (according to Theorem 3) the generating functions in canonical principal parameters

$$\tilde{g}_{12}(w) = g_{12}(z_{12}(w)) \qquad\qquad \tilde{g}_3(w) = g_3(z_3(w)) \tag{5.3}$$

are related by

$$\tilde{g}_3(w) = e^{i\varphi}\frac{\alpha + \tilde{g}_{12}(w)}{1 - \bar{\alpha}\,\tilde{g}_{12}(w)} \qquad \text{or} \qquad \tilde{g}_3(w) = \frac{e^{i\varphi}}{\tilde{g}_{12}(w)} \ . \tag{5.4}$$

We will consider only the first possibility. The second can be considered analogously. Note that according to the equation (2.2) the functions $z_{12}(w)$ and $z_3(w)$ are related by

$$f_{12}(z_{12}(w))g_{12}'(z_{12}(w))(z_{12}')^2 = f_3(z_3(w))g_3'(z_3(w))(z_3')^2 \ . \tag{5.5}$$

From the last equality, using (5.1)–(5.4) and comparing the coefficients of $z_{12}(w)$ (note that $z_{12}(w)$ may not be constant) we may derive

$$\alpha = 0 \qquad a = Ac^3e^{4i\varphi} \qquad C + 2\,b\,c\,e^{i\varphi} = 0 \qquad B = 0 \qquad b^2c + d = 0.$$

So the surfaces can coincide only if $b^2c + d = 0$. In particular the surfaces defined by $\boldsymbol{r}_{12}[1,0,1,1](u,v)$ and $\boldsymbol{r}_3[1,1,0](u,v)$ are different despite the resemblance in Figures 5.1 and 5.2. Actually in a smaller neighborhood of $(u,v) = (0,0)$ (with the same viewpoint as for Figures 5.1 and 5.2) the difference is clear, see Fig. 3 and Fig. 4.

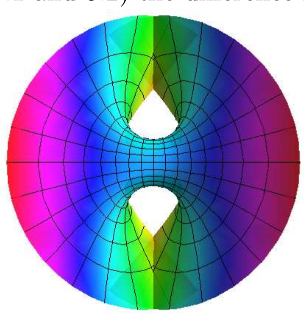

Fig. 3: $\boldsymbol{r}_{12}[1,0,1,1](u,v), |u|, |v| \leq 1$

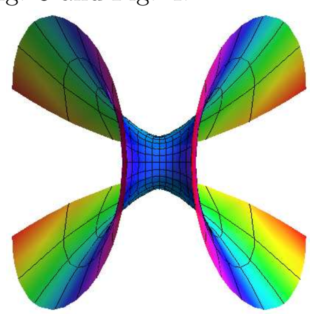

Fig. 4: $\boldsymbol{r}_3[1,1,0](u,v), |u|, |v| \leq 1$.

We use a similar idea to investigate a possible coincidence of surfaces from cases 1.1 and 1.2. Let the surface $S_{11} : \boldsymbol{r}_{11}[A,B](u,v)$ be generated via the Weierstrass formula by

$$f_{11}(z) = A \qquad\qquad g_{11}(z) = z^2 + B \ . \tag{5.6}$$

and suppose that it coincides with $S_{12} : \boldsymbol{r}_{12}[a,b,c,d](u,v)$. Then some functions $\tilde{g}_{11}$ and $\tilde{g}_{12}$ that generate them in canonical parameters are related by

$$\tilde{g}_{11}(w) = e^{i\varphi}\frac{\alpha + \tilde{g}_{12}(w)}{1 - \bar{\alpha}\,\tilde{g}_{12}(w)} \qquad \text{or} \qquad \tilde{g}_{11}(w) = \frac{e^{i\varphi}}{\tilde{g}_{12}(w)} \ . \tag{5.7}$$

As before, we consider only the first possibility. Denote $z_{11}(w)$, $z_{12}(w)$ respective solutions of the equation (2.2). Then

$$\tilde{g}_{11}(w) = (z_{11}(w))^2 + B \tag{5.8}$$

and hence

$$2z_{11}(w)z'_{11} = \tilde{g}'_{11}(w)\ . \tag{5.9}$$

On the other hand analogously to (5.5)

$$\big(f_{11}(z_{11}(w))g'_{11}(z_{11}(w))\big)^2(z'_{11})^4 = \big(f_{12}(z_{12}(w))g'_{12}(z_{12}(w))\big)^2(z'_{12})^4$$

holds. Applying (5.6)–(5.9) we can find the left hand side of the above equality as a function of $z_{12}(w)$. Then looking at the coefficients of $z_{12}(w)$ we conclude that this equality implies a contradiction. So a surface S_{11} can not coincide with a surface S_{12}.

Applying the same arguments we may prove that a surface S_{11} can not coincide with a surface S_3. So we have

Theorem 7. *The families from Theorem 6 contain different surfaces except if* $b^2c + d = 0$ *in case 2.1 and then the surface belongs also to the case 3.*

Remark 3. *Analogously we can see that for example a surface of the type 1.1 can not coincide with a surface of the type 2.2 or 4.*

Remark 4. *The surfaces generated via the Weierstrass formula by the pairs of functions*

$$(f(z), g(z)) \qquad \text{and} \qquad (Cf(z), g(z))$$

are homothetic for any positive real number C. On the other hand, if C is not real, these surfaces are different in general. More precisely let $C = |C|e^{i\varphi}$ for a real number φ. The pairs $(f(z), g(z))$ and $(|C|f(z), g(z))$ generate two homothetic surfaces, but the surface generated by $(e^{i\varphi}|C|f(z), g(z))$ belongs to the associated family of the surface generated by $(|C|f(z), g(z))$. Thus we see that if a surface belongs to a family from Theorem 6 then its homothetic surfaces, as well as their associated surfaces, belong to the same family.

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Department of Mathematics
and Informatics,
University of Transport, Sofia, Bulgaria
E-mail: okassabov@abv.bg