Bull. Math. Soc. Sci. Math. Roumanie
Tome 60 (108) No. 2, 2017, 181–193

Completely positive maps on Hilbert modules over pro-C*-algebras

by

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Abstract

We derive Paschke's GNS construction for completely positive maps on unital pro-C*-algebras from the KSGNS construction, presented by M. Joita [J. London Math. Soc. **66** (2002), 421–432], and then we deduce an analogue of Stinespring theorem for Hilbert modules over pro-C*-algebras. Also, we obtain a Radon-Nikodym type theorem for operator valued completely positive maps on Hilbert modules over pro-C*-algebras.

Key Words: Pro-C*-algebras, Hilbert modules, completely positive maps, Stinespring theorem, Radon-Nikodym theorem.
2010 Mathematics Subject Classification: Primary 46L05, Secondary 46L08, 46K10.

1 Introduction

Completely positive maps are the natural generalization of positive linear functionals. These maps are extremely applied in the modern theory of C*-algebras and mathematical model of quantum probability. A completely positive map $\varphi : A \to B$ of C*-algebras is a linear map with the property that $[\varphi(a_{ij})]_{i,j=1}^n$ is a positive element in the C*-algebra $M_n(B)$ of all $n \times n$ matrices with entries in B for all positive matrices $[a_{ij}]_{i,j=1}^n$ in $M_n(A)$, $n \in \mathbb{N}$. Given a C*-algebra A, the Gelfand-Naimark-Segal construction (or GNS-construction) establishes a correspondence between cyclic representations of A on Hilbert spaces and positive linear functionals on A. This fundamental theorem has been generalized for a completely positive linear map from A into $B(H)$ (respectively, from A into a C*-algebra B) to get a representation of A on a Hilbert space (respectively, on a Hilbert B-module) by Stinespring [24] (respectively, Paschke [19]). Stinespring showed that an operator valued completely positive map φ on a unital C*-algebra A is of the form $V_\varphi^*\pi_\varphi(\cdot)V_\varphi$, where π_φ is a representation of A on a Hilbert space H_φ and V_φ is a bounded linear operator. A version of Stinespring theorem for a class of maps on Hilbert modules over unital C*-algebras, which are known operator-valued completely positive maps on Hilbert C*-modules, has been considered by [4, 5]. Skeide [23] has been obtained a very quick proof of the result of [5] by using induced representations of Hilbert C*-modules.

The theory of completely positive maps on pro-C*-algebras has been studied systematically in the book [12] and the paper [9] by Joita. Maliev and Pliev [17] obtained a Stinespring theorem for Hilbert modules over pro-C*-algebras by extending the methods of [5] from the case of C*-algebras to the case of pro-C*-algebras. We generalize the Paschke's

GNS-construction to completely positive maps on pro-C*-algebras which enables us to establish another proof for the Stinespring theorem for Hilbert modules over pro-C*-algebras.

Let us quickly recall the definition of pro-C*-algebras and Hilbert modules over them. A pro-C*-algebra is a complete Hausdorff complex topological $*$-algebra A whose topology is determined by its continuous C*-seminorms in the sense that the net $\{a_i\}_{i\in I}$ converges to 0 if and only if the net $\{p(a_i)\}_{i\in I}$ converges to 0 for every continuous C*-seminorm p on A. For example the algebra $C(X)$ of all continuous complex valued functions on a compactly generated space (or a CW complex X) with the topology of compact convergence and the cartesian product $\prod_{\alpha\in I} A_\alpha$ of C*-algebras A_α with the product topology are pro-C*-algebras [7, §7.6]. Pro-C*-algebras appear in the study of certain aspects of C*-algebras such as tangent algebras of C*-algebras, domain of closed $*$-derivations on C*-algebras, multipliers of Pedersen's ideal, and noncommutative analogous of classical Lie groups. These algebras were first introduced by Inoue [8] who called them locally C*-algebras and studied more in [1, 7, 20] with different names. A (right) *pre-Hilbert module* over a pro-C*-algebra A is a right A-module E, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle\cdot,\cdot\rangle : E\times E\to A\,,\ (x,y)\mapsto\langle x,y\rangle$, which is A-linear in the second variable y and has the properties:

$$\langle x,y\rangle=\langle y,x\rangle^*,\ \text{and}\ \langle x,x\rangle\geq 0\ \text{ with equality if and only if } x=0.$$

A pre-Hilbert A-module E is a Hilbert A-module if E is complete with respect to the topology determined by the family of seminorms $\{\overline{p}_E\}_{p\in S(A)}$ where $\overline{p}_E(\xi)=\sqrt{p(\langle\xi,\xi\rangle)}$, $\xi\in E$. Hilbert modules over pro-C*-algebras have been studied in the book [11] and the papers [20, 21].

There is a natural partial ordering on the set of all operator valued completely positive maps on C*-algebras, defined by $\psi\leq\varphi$ if $\varphi-\psi$ is completely positive. Arveson [3] characterized this relation in terms of the Stinespring construction associated to each completely positive map and introduced a notion of Radon-Nikodym derivative for operator valued completely positive maps on C*-algebras. Indeed, he showed that in the unital case, $\psi\leq\varphi$ if and only if there is a unique positive contraction $\Delta_\varphi(\psi)$ (known as Radon-Nikodym derivative of ψ with respect to φ) in the commutant of $\pi_\varphi(A)$ such that $\psi(\cdot)=V_\varphi^*\Delta_\varphi(\psi)\pi_\varphi(\cdot)V_\varphi\,$, cf. [3, Theorem 1.4.2]. Joita [14] defined a preorder relation in the set of all operator valued completely positive maps on Hilbert C*-modules and extended the Radon-Nikodym type theorem for these maps.

In this paper we first present some definitions and basic facts about pro-C*-algebras and Hilbert modules over them. In Section 3, by using the concept of induced representations of Hilbert C*-modules, we deduce Stinespring representation theorem of pro-C*-algebras from Paschke's GNS-construction. Then we obtain a version of Stinespring representation theorem for Hilbert modules over pro-C*-algebras. Finally in section 4, we generalize the Radon-Nikodym theorem for operator valued completely positive maps on Hilbert modules over pro-C*-algebras.

2 Preliminaries

Let A be a pro-C*-algebra, $S(A)$ be the set of all continuous C*-seminorms on A and $p\in S(A)$. We set $N_p=\{a\in A:\ p(a)=0\}$ then $A_p=A/N_p$ is a C*-algebra in the norm induced by p. For $p, q \in S(A)$ with $p \geq q$, the surjective morphisms $\pi_{pq} : A_p \to A_q$ defined by $\pi_{pq}(a+N_p) = a+N_q$ induce the inverse system $\{A_p; \pi_{pq}\}_{p,q\in S(A),\, p\geq q}$ of C*-algebras and $A = \varprojlim_p A_p$, i.e. the pro-C*-algebra A can be identified with $\varprojlim_p A_p$. The canonical map from A onto A_p is denoted by π_p and a_p is reserved to denote $a + N_p$. A morphism of pro-C*-algebras is a continuous morphism of $*$-algebras. An isomorphism of pro-C*-algebras is a morphism of pro-C*-algebras which possesses an inverse morphism of pro-C*-algebras.

We denote by $M_n(A)$ the set of all $n \times n$ matrices over A. The set $M_n(A)$ with the usual algebraic operations and the topology obtained by regarding it as a direct sum of n^2 copies of A is a pro-C*-algebra. Moreover, it can be identified with $\varprojlim_p M_n(A_p)$. Thus the topology on $M_n(A)$ is determined by the family of C*-seminorms $\{p^{(n)}\}_{p\in S(A)}$, where $p^{(n)}([a_{ij}]) = \|[\pi_p(a_{ij})]\|_{M_n(A_p)}$, $[a_{ij}] \in M_n(A)$.

A representation of a pro-C*-algebra A is a continuous $*$-morphism $\varphi : A \to B(H)$, where $B(H)$ is the C*-algebra of all bounded linear maps on a Hilbert space H. If (φ, H) is a representation of A, then there is $p \in S(A)$ such that $\|\varphi(a)\| \leq p(a)$, for all $a \in A$. The representation (φ_p, H) of A_p, where $\varphi_p \circ \pi_p = \varphi$ is called a representation of A_p associated to (φ, H). We refer to [7, 10] for more detailed information about the representation of pro-C*-algebras.

Suppose E is a Hilbert A-module and $\langle E, E\rangle$ is the closure of linear span of $\{\langle x, y\rangle :\ x, y \in E\}$. The Hilbert A-module E is called *full* if $\langle E, E\rangle = A$. One can always consider any Hilbert A-module as a full Hilbert module over pro-C*-algebra $\langle E, E\rangle$. For each $p \in S(A), N_p^E = \{\xi \in E :\ \bar{p}_E(\xi) = 0\}$ is a closed submodule of E and $E_p = E/N_p^E$ is a Hilbert A_p-module with the action $(\xi + N_p^E)\pi_p(a) = \xi a + N_p^E$ and the inner product $\langle \xi + N_p^E, \eta + N_p^E\rangle = \pi_p(\langle \xi, \eta\rangle)$. The canonical map from E onto E_p is denoted by σ_p^E and ξ_p is reserved to denote $\sigma_p^E(\xi)$. For $p, q \in S(A)$ with $p \geq q$, the surjective morphisms $\sigma_{pq}^E : E_p \to E_q$ defined by $\sigma_{pq}^E(\sigma_p^E(\xi)) = \sigma_q^E(\xi)$ induce the inverse system $\{E_p;\ A_p;\ \sigma_{pq}^E,\ \pi_{pq}\}_{p,q\in S(A),\, p\geq q}$ of Hilbert C*-modules. In this case, $\varprojlim_p E_p$ is a Hilbert A-module which can be identified with E. Let E and F be Hilbert A-modules and $T : E \to F$ be an A-module map. The module map T is called bounded if for each $p \in S(A)$, there is $k_p > 0$ such that $\bar{p}_F(Tx) \leq k_p\, \bar{p}_E(x)$ for all $x \in E$. The set $L_A(E, F)$ of all bounded adjointable A-module maps from E into F becomes a locally convex space with the topology defined by the family of seminorms $\{\tilde{p}\}_{p\in S(A)}$, where $\tilde{p}(T) = \|(\pi_p)_*(T)\|_{L_{A_p}(E_p,F_p)}$ and $(\pi_p)_* : L_A(E, F) \to L_{A_p}(E_p, F_p)$ is defined by $(\pi_p)_*(T)(\xi + N_p^E) = T\xi + N_p^F$, for all $T \in L_A(E, F)$ and $\xi \in E$. For $p, q \in S(A)$ with $p \geq q$, the morphisms $(\pi_{pq})_* : L_{A_p}(E_p, F_p) \to L_{A_q}(E_q, F_q)$ defined by $(\pi_{pq})_*(T_p)(\sigma_q^E(\xi)) = \sigma_{pq}^F(T_p(\sigma_p^E(\xi)))$ induce the inverse system $\{L_{A_p}(E_p, F_p);\ (\pi_{pq})_*\}_{p,q\in S(A),\, p\geq q}$ of Banach spaces such that $\varprojlim_p L_{A_p}(E_p, F_p)$ can be identified to $L_A(E, F)$. In particular, topologizing, $L_A(E, E)$ becomes a pro-C*-algebra which is abbreviated by $L_A(E)$. The set of all compact operators $K_A(E)$ on E is defined as the closed linear subspace of $L_A(E)$ spanned by $\{\theta_{x,y} :\ \theta_{x,y}(\xi) = x\langle y, \xi\rangle$ for all $x, y, \xi \in E\}$. This is a pro-C*-subalgebra and a two sided ideal of $L_A(E)$, moreover, $K_A(E)$ can be identified to $\varprojlim_p K_{A_p}(E_p)$.

Let E and F be Hilbert modules over pro-C*-algebras A and B, respectively, and $\Psi : A \to L_B(F)$ be a continuous $*$-morphism. We can regard F as a left A-module by $(a, y) \to \Psi(a)y,\ a \in A,\ y \in F$. The right B-module $E \otimes_A F$ is a pre-Hilbert module with the inner product given by $\langle x \otimes y, z \otimes t\rangle = \langle y, \Psi(\langle x, z\rangle)t\rangle$. We denote by $E \otimes_\Psi F$ the completion of $E \otimes_A F$, cf. [11] for more detailed information.

3 Stinespring representation theorem

In this section, we first generalize Paschke's GNS-construction [19, Theorem 5.2] to the framework of unital pro-C*-algebras. It is a particular case of the KSGNS construction for completely positive maps on unital pro-C*-algebras, [9, Theorem 4.6]. Then we deduce Stinespring representation theorem in the context of pro-C*-algebras and a version of Stinespring representation theorem for Hilbert modules over pro-C*-algebras. For this aim we briefly restate the concept of induced representations of Hilbert modules over pro-C*-algebras from our recent paper [15].

Given pro-C*-algebras A and B, a linear map $\varphi : A \to B$ is said to be *positive* if $\varphi(a^*a) \geq 0$ for all $a \in A$. If $\varphi^{(n)} : M_n(A) \to M_n(B)$ defined by $\varphi^{(n)}([a_{ij}]_{i,j=1}^n) = [\varphi(a_{ij})]_{i,j=1}^n$ is positive, then φ is said to be n-positive. If φ is n-positive for all natural numbers n, then φ is called a *completely positive map* [6]. Let H be a Hilbert space and $\varphi : A \to B(H)$ be an operator valued completely positive map then the condition of positivity [24] can be written in the form

$$\sum_{i,j=1}^{n} \langle \varphi(a_i^* a_j)h_j, h_i\rangle \geq 0, \text{ for all } h_j \in H,\ a_j \in A,\ j = 1, ..., n, \text{ and all } n \in \mathbb{N}.$$

Let A and B be pro-C*-algebras and let E be a Hilbert B-module. A continuous $*$-morphism from A into $L_A(E)$ is called a *continuous representation* of A on E.

Theorem 1. *Let A and B be two unital pro-C*-algebras and $\varphi : A \to B$ be a continuous completely positive map. There is a Hilbert B-module X, a unital continuous representation π_φ of A on X, $\pi_\varphi : A \to L_B(X)$, and an element $\xi \in X$ such that*

1. *$\varphi(a) = \langle \xi, \pi_\varphi(a)\xi\rangle$ for all $a \in A$;*
2. *the set $\chi_\varphi = span\{\pi_\varphi(a)(\xi b) : a \in A, b \in B\}$ is a dense subspace of X.*

In this case, we say that π_φ is the Paschke's GNS construction associated to completely positive map φ.

The result follows by taking into account $E = B$ and $\xi = V_\varphi(1_B)$ in the proof of [9, Theorem 4.6], where 1_B is the unit of B.

Let E and F be Hilbert modules over pro-C*-algebras A and B, respectively and $\varphi : A \to B$ be a morphism of pro-C*-algebras. A map $\Phi : E \to F$ is said to be a φ-morphism if $\langle \Phi(x), \Phi(y)\rangle = \varphi(\langle x, y\rangle)$, for all $x, y \in E$. A φ-morphism $\Phi : E \to F$ is said to be a completely positive map if $\varphi : A \to B$ be a completely positive map.

Lemma 1. *Let E and F be Hilbert modules over unital pro-C*-algebras A and B, respectively, and let $\Phi : E \to F$ be a completely positive map. Suppose that X, ξ and π_φ are the same as Theorem 1. Then there exists an isometry $v : E \otimes_{\pi_\varphi} X \to F$ such that $v(z \otimes \xi) = \Phi(z)$, for all $z \in E$.*

Proof: For $a, c \in A$, $b, d \in B$ and $z, w \in E$ we have

$$\begin{aligned}\langle z \otimes (\pi_\varphi(a)(\xi b)), w \otimes (\pi_\varphi(c)(\xi d))\rangle &= \langle \pi_\varphi(a)(\xi b), \pi_\varphi(\langle z, w\rangle)(\pi_\varphi(c)(\xi d))\rangle \\ &= b^*\langle \xi, \pi_\varphi(\langle za, wc\rangle)\xi\rangle d \\ &= b^*\varphi(\langle za, wc\rangle)d \\ &= b^*\langle \Phi(za), \Phi(wc)\rangle d \\ &= \langle \Phi(za)b, \Phi(wc)d\rangle.\end{aligned}$$

Since $\text{span}\{\pi_\varphi(a)(\xi b) : a \in A, b \in B\}$ is a dense subspace of X, the map $z \otimes (\pi_\varphi(a)(\xi b)) \mapsto \Phi(za)b$ defines an isometry $v : E \otimes_{\pi_\varphi} X \to F$. In particular, we find $v(z \otimes \xi) = \Phi(z)$ for all $z \in E$, when $a = 1_A$ and $b = 1_B$. □

Let H and K be Hilbert spaces. Then the space $B(H, K)$ of all bounded operators from H into K can be considered as a Hilbert $B(H)$-module with the module action $(T, S) \to TS$, $T \in B(H, K)$ and $S \in B(H)$ and the inner product defined by $\langle T, S\rangle = T^*S$, $T, S \in B(H, K)$. Murphy [18] showed that any Hilbert C*-module can be represented as a submodule of the concrete Hilbert module $B(H, K)$ for some Hilbert spaces H and K. This allows us to extend the notion of a representation from the context of C*-algebras to the context of Hilbert C*-modules. Let E and F be two Hilbert modules over C*-algebras A and B, respectively, and let $\varphi : A \to B$ be a morphism of C*-algebras. A φ-morphism $\Phi : E \to B(H, K)$, where $\varphi : A \to B(H)$ is a representation of A is called a representation of E. When Φ is a representation of E, we assume that an associated representation of A is denoted by the same lowercase letter φ, so we will not explicitly mention φ. Let $\Phi : E \to B(H, K)$ be a representation of a Hilbert A-module E. We say Φ is a non-degenerate representation if $[\Phi(E)(H)] = K$ and $[\Phi(E)^*(K)] = H$, where $[\Phi(E)(H)]$ denotes the closure of $\text{span}\{\Phi(\xi)(h);\ \xi \in E,\ h \in H\}$. Two representations $\Phi_i : E \to B(H_i, K_i)$ of E, $i = 1, 2$ are said to be unitarily equivalent, if there are unitary operators $U_1 : H_1 \to H_2$ and $U_2 : K_1 \to K_2$, such that $U_2\Phi_1(x) = \Phi_2(x)U_1$ for all $x \in E$. Representations of Hilbert modules have been investigated in [2, 22].

Skeide [22] recovered the result of Murphy by embedding of every Hilbert A-module E into a matrix C*-algebra as a lower submodule. He proved that every representation of A induces a representation of E and a representation of $L_A(E)$. We describe his induced representations as follows.

Construction 2. *Let A is a C*-algebra and E be a Hilbert A-module and $\varphi : A \to B(H)$ be a $*$-representation of A. Define a sesquilinear form $\langle ., .\rangle$ on the vector space $E \otimes H$ by $\langle x \otimes h, y \otimes k\rangle = \langle h, \varphi(\langle x, y\rangle)k\rangle_H$, where $\langle ., .\rangle_H$ denotes the inner product on the Hilbert space H. By [22, Proposition 3.8], the sesquilinear form is positive and so $E \otimes H$ is a semi-Hilbert space. Then $(E \otimes H)/N_\varphi$ is a pre-Hilbert space with the inner product defined by*

$$\langle x \otimes h + N_\varphi\ ,\ y \otimes k + N_\varphi\rangle = \langle x \otimes h, y \otimes k\rangle,$$

where N_φ is the vector subspace of $E \otimes H$ generated by $\{x \otimes h \in E \otimes H : \langle x \otimes h, x \otimes h\rangle = 0\}$. Let K be the completion of $(E \otimes H)/N_\varphi$ with respect to the above inner product. We identify the elements $x \otimes h$ with the equivalence classes $x \otimes h + N_\varphi \in K$. Suppose $x \in E$

and $L_x h = x \otimes h$ *then* $\|L_x h\|^2 = \langle h, \varphi(\langle x, x\rangle)h\rangle \le \|h\|^2\|x\|^2$*, i.e.* $L_x \in B(H,K)$*. If* L_x^* *be the adjoint of* L_x *then it is easy to show that* $L_x^*(y \otimes h) = \varphi(\langle x, y\rangle)h$ *for every* $y \in E$ *and* $h \in H$*. We define* $\eta_\varphi : E \to B(H,K)$ *by* $\eta_\varphi(x) = L_x$*. Then for* $x, x' \in E$*,* $h, h' \in H$ *and* $a \in A$ *we have* $\langle \eta_\varphi(x), \eta_\varphi(x')\rangle = \varphi(\langle x, x'\rangle)$ *and* $\eta_\varphi(xa) = \eta_\varphi(x)\varphi(a)$*, and so* η_φ *is a representation of* E*.*

Let $T \in L_A(E)$*. We associate with* T *a map on* $E \otimes H$ *by* $x \otimes h \to Tx \otimes h$*. Since* $\langle x \otimes h, Tx^{'} \otimes h^{'}\rangle = \langle T^* x \otimes h, x^{'} \otimes h^{'}\rangle$*, this map leaves invariant* N_φ *so that it induces a map* $\rho_0(T)$ *on* $(E \otimes H)/N_\varphi$*. By [22, Lemma 3.9],* $\|\rho_0(T)\| = \|T\|$ *and so* $\rho_0(T)$ *is bounded and can be extended to a bounded operator* $\rho(T)$ *on* K*. Therefore* $\rho : L_A(E) \to B(K)$ *defined by* $T \to \rho(T)$ *is a representation of* $L_A(E)$ *on* K*.*

Now, we reformulate representations of Hilbert module from the case of C*-algebras to the case of pro-C*-algebras. Let E and F be two Hilbert modules over pro-C*-algebras A and B, respectively, and $\varphi : A \to B$ be a morphism of pro-C*-algebras. A φ-morphism $\Phi : E \to B(H,K)$, where $\varphi : A \to B(H)$ is a representation of A is called a representation of E. If $p \in S(A)$ and φ_p be a representation of A_p associated to φ, then it is easy to see that the map $\Phi_p : E_p \to B(H,K)$, $\Phi_p(\sigma_p^E(x)) = \Phi(x)$ is a φ_p-morphism. In this case, we say that Φ_p is a representation of E_p associated to Φ. We can define non-degenerate representations and unitarily equivalent representations for Hilbert modules over pro-C*-algebras like Hilbert C*-modules case.

Remark 1. *Suppose* A *is a pro-C*-algebra,* E *a Hilbert* A*-module and* $\varphi : A \to B(H)$ *a representation of* A *on some Hilbert space* H*. Suppose* $p \in S(A)$ *and* φ_p *is a representation of* A_p *associated to* φ*. By the above Construction* φ_p *induces a representation* $\eta_{\varphi_p} : E_p \to B(H,K)$ *of* E_p *where* K *is a Hilbert space associated to* $E_p \otimes H$*. It is easy to see that the map* $\eta_\varphi : E \to B(H,K)$*,* $\eta_\varphi(x) = \eta_{\varphi_p}(\sigma_p^E(x))$ *is a* φ*-morphism, i.e. it is a representation of* E*.*

The following theorem is a version of Stinespring representation theorem for pro-C*-algebras that can be considered as a special case of KSGNS construction for completely positive maps on unital pro-C*-algebras, by setting $B = \mathbb{C}$ in [9, Theorem 4.6]). We prove this theorem by using the concept of induced representations of Hilbert pro-C*-modules.

Theorem 3. *Let* A *be a unital pro-C*-algebras and* $\varphi : A \to B(H)$ *be a continuous operator valued completely positive map. Then there exist a Hilbert space* H_φ*, a unital representation* $\pi_\varphi : A \to B(H_\varphi)$ *and a bounded linear operator* $V_\varphi \in B(H, H_\varphi)$ *such that* $\varphi(a) = V_\varphi^* \pi_\varphi(a) V_\varphi$ *for all* $a \in A$*.*

Proof: Suppose that $\pi_\varphi^{'}$ is Paschke's GNS construction associated to φ and X, χ_φ and ξ are as in Theorem 1. Let ι be the identity map on $B(H)$. If we consider ι as a representation of $B(H)$ on H and apply Construction 2, we get a Hilbert space H_φ (associated to $X \otimes H$), an induced representation $\eta_\iota : X \to B(H, H_\varphi)$ of X and a representation $\rho_\varphi : L_{B(H)}(X) \to B(H_\varphi)$ of $L_{B(H)}(X)$. We define $V_\varphi := \eta_\iota(\xi)$ and $\pi_\varphi := \rho_\varphi \circ \pi_\varphi^{'}$. If $a \in A$ and $h \in H$, we have

$$\begin{aligned} V_\varphi^* \pi_\varphi(a) V_\varphi(h) = V_\varphi^* \pi_\varphi(a)(\xi \otimes h) &= V_\varphi^*(\pi_\varphi^{'}(a)\xi \otimes h) \\ &= \iota(\langle \xi, \pi_\varphi^{'}(a)\xi\rangle)h = \varphi(a)h. \end{aligned}$$

Hence, $\varphi(a) = V_\varphi^* \pi_\varphi(a) V_\varphi$ for all $a \in A$. ◻

In the rest of this section we establish [5, Theorems 2.1 and 2.4] in the context of pro-C*-algebra.

Theorem 4. *Let A be a unital pro-C*-algebra and $\varphi : A \to B(H)$ be a continuous completely positive map. Let E be a Hilbert A-module and $\Phi : E \to B(H, K)$ be a φ-morphism. Then there exist triples $(\pi_\varphi, V_\varphi, H_\varphi)$ and $(\pi_\Phi, W_\Phi, K_\Phi)$, where*

1. *H_φ and K_Φ are Hilbert spaces;*
2. *$\pi_\varphi : A \to B(H_\varphi)$ is a unital representation of A;*
3. *$\pi_\Phi : E \to B(H_\varphi, K_\Phi)$ is a π_φ-morphism;*
4. *$V_\varphi : H \to H_\varphi$ and $W_\Phi : K \to K_\Phi$ are bounded linear operators such that $\varphi(a) = V_\varphi^* \pi_\varphi(a) V_\varphi$, for all $a \in A$ and $\Phi(z) = W_\Phi^* \pi_\Phi(z) V_\varphi$, for all $z \in E$.*

Proof: Let $\pi_\varphi^{'} : A \to L_{B(H)}(X)$ be the Paschke's GNS construction associated to φ. By continuity of $\pi_\varphi^{'}$, there exists $M > 0$ and $p \in S(A)$ such that $\|\pi_\varphi^{'}(a)\| \leq Mp(a)$, for all $a \in A$. Let $(\pi_\varphi, V_\varphi, H_\varphi)$ be the Stinespring triple for φ as obtained in Theorem 3. Since $\pi_\varphi = \rho_\varphi \circ \pi_\varphi^{'}$, we may consider $(\pi_\varphi)_p$ as a representation of A_p associated to π_φ. By Remark 1, the Stinespring representation π_φ induces a representation $\pi_\Phi : E \to B(H_\varphi, K_\Phi)$ of E, where K_Φ is the Hilbert space associated to $E_p \otimes H_\varphi$. Moreover, Lemma 1 implies the existence of an isometry $v : E \otimes_{\pi_\varphi^{'}} X \to B(H, K)$ which is defined by $v(x \otimes \xi) = \Phi(x)$ for all $x \in E$. We consider the linear map $W_0 : (E_p \otimes X) \otimes H \to K$ defined by $W_0((\sigma_p^E(z) \otimes x) \otimes h) = v(z \otimes x)h$, where $z \in E$, $x \in X$ and $h \in H$. Let $z \in E$ and $\sigma_p^E(z) = 0$. Since $\|v(z \otimes x)\|^2 = \|z \otimes x\|^2 = \langle z \otimes x, z \otimes x\rangle = \langle x, \pi_\varphi^{'}(\langle z, z\rangle)\rangle$, we have $v(z \otimes x) = 0$ which shows that W_0 is well-defined. Moreover,

$$\begin{aligned}
\|\sum_{i=1}^n (\sigma_p^E(z_i) \otimes x_i) \otimes h_i\|^2 &= \sum_{i,j=1}^n \langle h_i, \langle \sigma_p^E(z_i) \otimes x_i, \sigma_p^E(z_j) \otimes x_j\rangle h_j\rangle \\
&= \sum_{i,j=1}^n \langle h_i, \langle x_i, \pi_\varphi^{'}(\langle z_i, z_j\rangle) x_j\rangle h_j\rangle \\
&= \sum_{i,j=1}^n \langle h_i, \langle z_i \otimes x_i, z_j \otimes x_j\rangle h_j\rangle \\
&= \sum_{i,j=1}^n \langle h_i, \langle v(z_i \otimes x_i), v(z_j \otimes x_j)\rangle h_j\rangle \\
&= \sum_{i,j=1}^n \langle v(z_i \otimes x_i) h_i, v(z_j \otimes x_j) h_j\rangle \\
&= \|\sum_{i=1}^n v(z_i \otimes x_i) h_i\|^2
\end{aligned}$$

which implies that W_0 is an isometry. Since H_φ is the Hilbert space associated to $X \otimes H$, W_0 can be extended to a bounded linear operator $W : K_\Phi \to K$. We define $W_\Phi := W^*$, then

$$\begin{aligned} W_\Phi^* \pi_\Phi(z) V_\varphi(h) &= W\pi_\Phi(z)(\xi \otimes h) \\ &= W(\pi_\Phi)_p(\sigma_p^E(z))(\xi \otimes h) \\ &= W(\sigma_p^E(z) \otimes (\xi \otimes h)) \\ &= W((\sigma_p^E(z) \otimes \xi) \otimes h) \\ &= v(z \otimes \xi)h = \Phi(z)h, \end{aligned}$$

for all $z \in E$ and $h \in H$. Hence, $\Phi(z) = W_\Phi^* \pi_\Phi(z) V_\varphi$ for all $z \in E$. $\square$

Remark 2. *Let φ and Φ be as in Theorem 4 and $q \in S(A)$.*

(1) In the proof of Theorem 4, if $(\pi_\varphi)_q$ be a representation of A_q associated to π_φ then we obtain a representation $\tilde{\pi}_\Phi : E \to B(H_\varphi, \tilde{K}_\Phi)$, where $\tilde{K}_\Phi$ is a Hilbert space associated to $E_q \otimes H_\varphi$. It is easy to show that π_Φ and $\tilde{\pi}_\Phi$ are two unitarily equivalent representations of E.

(2) The bounded linear operator $W_\Phi : K \to K_\Phi$ is a coisometry. Indeed, for $z \in E,\ x \in X$ and $h \in H$ we have

$$\begin{aligned} \langle W_\Phi^*(\sigma_p^E(z) \otimes x \otimes h), W_\Phi^*(\sigma_p^E(z) \otimes x \otimes h)\rangle &= \langle v(z \otimes x)h, v(z \otimes x)h\rangle \\ &= \langle v(z \otimes x)^* v(z \otimes x)h, h\rangle \\ &= \langle\langle v(z \otimes x), v(z \otimes x)\rangle h, h\rangle \\ &= \langle\langle z \otimes x, z \otimes x\rangle h, h\rangle \\ &= \langle h, \langle x, \pi'_\varphi(\langle z, z\rangle)x\rangle h\rangle \\ &= \langle x \otimes h, \pi'_\varphi(\langle z, z\rangle)x \otimes h\rangle \\ &= \langle x \otimes h, (\rho_\varphi \circ \pi'_\varphi)(\langle z, z\rangle)(x \otimes h)\rangle \\ &= \langle x \otimes h, \pi_\varphi(\langle z, z\rangle)(x \otimes h)\rangle \\ &= \langle x \otimes h, (\pi_\varphi)_p(\langle \sigma_p^E(z), \sigma_p^E(z)\rangle)(x \otimes h)\rangle \\ &= \langle \sigma_p^E(z) \otimes x \otimes h, \sigma_p^E(z) \otimes x \otimes h\rangle \end{aligned}$$

(3) If E is full then $\pi_\Phi : E \to B(H_\varphi, K_\Phi)$ is a non-degenerate representation of E. To see this, let $z \in E$ and $h_\varphi \in H_\varphi$ then $\pi_\Phi(z)(h_\varphi) = \sigma_p^E(z) \otimes h_\varphi$. Since K_Φ is a Hilbert space associated to $E_p \otimes H_\varphi$, $[\pi_\Phi(E)(H_\varphi)] = K_\Phi$. Moreover, for $w \in E, x \in X$ and $h \in H$ we have

$$\pi_\Phi(z)^*(\sigma_p^E(w) \otimes x \otimes h) = \pi_\varphi(\langle z, z\rangle)(x \otimes h) = \pi'_p(\langle z, z\rangle)(x) \otimes h.$$

Since E is full, $[\pi'_\varphi(A)(X)] = X$. The Hilbert space H_φ is associated to $X \otimes H$ which follows that $\pi_\Phi(E)^*(K_\Phi) = H_\varphi$.

Definition 1. *Let φ and Φ be as in Theorem 4. We say that the pair $((\pi_\varphi, V_\varphi, H_\varphi), (\pi_\Phi, W_\Phi, K_\Phi))$ is a Stinespring representation of (φ, Φ) if conditions (1)-(3) of Theorem 4 are fulfilled. Such a representation is said to be minimal if*

1. $[\pi_\varphi(A)V_\varphi H] = H_\varphi$, *and*
2. $[\pi_\Phi(E)V_\varphi H] = K_\Phi$.

Remark 3. *The pair* $((\pi_\varphi, V_\varphi, H_\varphi), (\pi_\Phi, W_\Phi, K_\Phi))$ *obtained in Theorem 4 is a minimal representation for* (φ, Φ) *since*

$$\begin{aligned}[\pi_\varphi(A)V_\varphi H] &= [(\rho_\varphi \circ \pi'_\varphi)(A)(\xi \otimes H)] \\ &= [(\pi'_\varphi(A)(\xi)) \otimes H] \\ &= [\chi_\varphi \otimes H] = H_\varphi\end{aligned}$$

and

$$\begin{aligned}[\pi_\Phi(E)V_\varphi H] = [\pi_\Phi(E)\pi_\varphi(A)V_\varphi H] &= [(\pi_\Phi)_p(E_p)H_\varphi] \\ &= [E_p \otimes H_\varphi] = K_\Phi.\end{aligned}$$

The following result shows that the minimal Stinespring representation is unique up to the unitarily equivalency.

Proposition 1. *Let* φ *and* Φ *be as in Theorem 4 and* $((\pi_A, V', H'), (\pi_E, W', K'))$ *be a minimal representation for* (φ, Φ). *Then there are two unitary operators* $U_1 : H_\varphi \to H'$ *and* $U_2 : K_\Phi \to K'$ *such that*

1. $V' = U_1 V_\varphi$, $U_1\pi_\varphi(a) = \pi_A(a)U_1$, *for all* $a \in A$ *and*
2. $W' = U_2 W_\Phi$, $U_2\pi_\Phi(z) = \pi_E(z)U_1$, *for all* $z \in E$.

Proof: Existence U_1 and the statement (1) follow from [9, Theorem 4.6 (2)]. As in the proof of [5, Theorem 2.4], we define the linear map $U_2 : \mathrm{span}(\pi_\Phi(E)V_\varphi H) \to \mathrm{span}(\pi_E(E)V'H)$ by

$$U_2(\sum_{i=1}^n \pi_\Phi(z_i)V_\varphi h_i) = \sum_{i=1}^n \pi_E(z_i)V'h_i,$$

for $z_i \in E$, $h_i \in H$ and $n \geq 1$. Then U_2 is a well-defined isometry and so it can be extended to a unitary U_2 from K_Φ onto K' which satisfies the statement (2). ∎

4 Radon-Nikodym derivatives

A Radon-Nikodym-type theorem for operator valued completely positive maps on Hilbert C*-modules has been demonstrated in [14] by Joita. We are going to generalize her definitions and results to the case of Hilbert modules over pro-C*-algebras. Let E be a full Hilbert module over a pro-C*-algebra A and H, K be two Hilbert spaces. The set of all completely positive maps of E into $B(H, K)$ will be denoted by $CP(E, B(H, K))$. There is an equivalence relation on $CP(E, B(H, K))$ as follows.

Definition 2. *Let* Φ *and* Ψ *be in* $CP(E, B(H, K))$. *We say that* Φ *is equivalent to* Ψ, *denoted by* $\Phi \sim \Psi$, *if* $\Phi(x)^*\Phi(x) = \Psi(x)^*\Psi(x)$ *for all* $x \in E$.

Definition 3. *Let Φ and Ψ be in $CP(E, B(H, K))$. We say that Ψ is dominated by Φ, denoted by $\Psi \preceq \Phi$, if $\Psi(x)^*\Psi(x) \leq \Phi(x)^*\Phi(x)$ for all $x \in E$.*

Remark 4. *The relation " $\preceq$ " is reflexive and transitive and so is a preorder relation on $CP(E, B(H, K))$. Moreover, if $\Phi, \Psi \in CP(E, B(H, K))$ then $\Phi \preceq \Psi$ and $\Psi \preceq \Phi$ if and only if $\Phi \sim \Psi$.*

In [2], Arambašić extended the definition of the commutant of a C*-algebra to a Hilbert C*-module. We define a similar notion for Hilbert modules over pro-C*-algebras.

Definition 4. *Let A be a pro-C*-algebra and $\Phi : E \to B(H, K)$ be a representation of a Hilbert A- module E. The commutant of $\Phi(E)$, which is denoted by $\Phi(E)^{'}$, is defined by*

$$\{T \oplus S \in B(H \oplus K) : T \in B(H), S \in B(K), \Phi(z)T = S\Phi(z), \Phi(z)^*S = T\Phi(z)^*, z \in E\}$$

in which, $(T \oplus S)(h \oplus k) := Th \oplus Sk$.

If $T \oplus S \in \Phi(E)^{'}$, then $T \in \varphi(A)^{'}$, cf. [2, Lemma 4.4]. If Φ is non-degenerate, then S is uniquely determined by T, cf. [2, Note 4.6].

Lemma 2. *Let $\Phi \in CP(E, B(H, K))$ and $((\pi_\varphi, V_\varphi, H_\varphi), (\pi_\Phi, W_\Phi, K_\Phi))$ be the Stinespring representation of (φ, Φ). If $T \oplus S$ be a positive linear operator in $\pi_\Phi(E)^{'}$, then the map $\Phi_{T\oplus S} : E \to B(H, K)$ defined by $\Phi_{T\oplus S}(x) = W_\Phi^* \sqrt{T} \pi_\Phi(x) \sqrt{S} V_\varphi$ is completely positive.*

Proof: As in proof of [14, Lemma 2.10], $\Phi_{T\oplus S}(x)^*\Phi_{T\oplus S}(y) = V_\varphi^* T^2 \pi_\varphi(\langle x, y\rangle)V_\varphi$, for all $x, y \in E$. Using [12, Lemma 3.4.1] and the fact that $T^2 \in \pi_\varphi(A)^{'}$, we find
$\Phi_{T\oplus S}(x)^*\Phi_{T\oplus S}(y) = \varphi_{T^2}(\langle x, y\rangle)$. Indeed, the completely positive map associated to $\Phi_{T\oplus S}$ is φ_{T^2}. □

Theorem 5. *Let $\Psi, \Phi \in CP(E, B(H, K))$. If $\Psi \preceq \Phi$, then there is a unique positive linear operator $\Delta_\Phi(\Psi)$ in $\pi_\Phi(E)^{'}$ such that $\Psi \sim \Phi_{\sqrt{\Delta_\Phi(\Psi)}}$.*

Proof: Let $((\pi_\varphi, V_\varphi, H_\varphi), (\pi_\Phi, W_\Phi, K_\Phi))$ be the Stinespring representation of (φ, Φ). Continuity of φ and ψ implies that there exist $p, q \in S(A)$ and $M, N > 0$ such that $\|\varphi(a)\| \leq Mp(a)$ and $\|\psi(a)\| \leq Nq(a)$, for all $a \in A$. Let $r \in S(A)$ and $r \geq p, q$. The linear maps $\varphi_r : A_r \to B(H), \varphi_r(\pi_r(a)) = \varphi(a)$ and $\psi_r : A_r \to B(H), \psi_r(\pi_r(a)) = \psi(a)$ are completely positive maps since, $\sum_{i,j=1}^n \langle \varphi_r(\pi_r(a_i)^*\pi_r(a_j))x_j, x_i\rangle = \sum_{i,j=1}^n \langle \varphi(a_i^* a_j)x_j, x_i\rangle \geq 0$, for all $a_i \in A$, $x_i \in H$ and $1 \leq i \leq n$.

The maps $\Phi_r : \sigma_r^E(x) \mapsto \Phi(x)$ and $\Psi_r : \sigma_r^E(x) \mapsto \Psi(x)$ are in $CP(E_r, B(H, K))$ and $\Psi_r \preceq \Phi_r$. Let $((\pi_{\varphi_r}, V_{\varphi_r}, H_{\varphi_r}), (\pi_{\Phi_r}, W_{\Phi_r}, K_{\Phi_r}))$ be the Stinespring representation of (φ_r, Φ_r). By the proof of [14, Theorem 2.12], there are unique positive linear operators $\Delta_{1\Phi_r}(\Psi_r) \in B(H_{\varphi_r})$ and $\Delta_{2\Phi_r}(\Psi_r) \in B(K_{\Phi_r})$ such that $\Psi_r \sim \Phi_{r_{\sqrt{\Delta_{\Phi_r}(\Psi_r)}}}$, where $\Delta_{\Phi_r}(\Psi_r) = \Delta_{1\Phi_r}(\Psi_r) \oplus \Delta_{2\Phi_r}(\Psi_r) \in \pi_{\Phi_r}(E)^{'}$ is the Radon-Nikodym derivative of Ψ_r with respect to Φ_r. The pairs $((\pi_{\varphi_r} \circ \pi_r, V_{\varphi_r}, H_{\varphi_r}), (\pi_{\Phi_r} \circ \sigma_r^E, W_{\Phi_r}, K_{\Phi_r}))$ and
$((\pi_\varphi, V_\varphi, H_\varphi), (\pi_\Phi, W_\Phi, K_\Phi))$ are two minimal Stinespring representations of (φ, Φ) and so, by Proposition 1, there are two unitary operators $U_1 : H_\varphi \to H_{\varphi_r}$ and $U_2 : K_\Phi \to K_{\Phi_r}$ such that $V_{\varphi_r} = U_1 V_\varphi$, $U_1\pi_\varphi(a) = (\pi_{\varphi_r} \circ \pi_r)(a)U_1$ for all $a \in A$, $W_{\Phi_r} = U_2 W_\Phi$ and $U_2\pi_\Phi(z) =$

$(\pi_{\Phi_r} \circ \sigma_r)(z)U_1$ for all $z \in E$. Let $\Delta_{1\Phi}(\Psi) = U_1^* \Delta_{1\Phi_r}(\Psi_r)U_1$ and $\Delta_{2\Phi}(\Psi) = U_2^* \Delta_{2\Phi_r}(\Psi_r)U_2$. It is easy to see that $\Delta_\Phi(\Psi) = \Delta_{1\Phi}(\Psi) \oplus \Delta_{2\Phi}(\Psi)$ is a positive operator in $\pi_\Phi(E)^{'}$. For every $a \in A$, we have

$$\begin{aligned} \psi(a) = \psi_r(\pi_r(a)) &= V_{\varphi_r}^* \Delta_{1\Phi_r}(\Psi_r)\pi_{\varphi_r}(\pi_r(a))V_{\varphi_r} \\ &= V_\varphi^* U_1^* \Delta_{1\Phi_r}(\Psi_r)U_1 \pi_\varphi(a) U_1^* U_1 V_\varphi \\ &= V_\varphi^* \Delta_{1\Phi}(\Psi)\pi_\varphi(a)V_\varphi = \varphi_{\Delta_{1\Phi}(\Psi)}(a). \end{aligned}$$

Indeed by the uniqueness of Radon-Nikodym derivative ([12, Theorem 3.4.5]), $\Delta_{1\Phi}(\Psi)$ is the Radon-Nikodym derivative of ψ with respect to φ. Consequently,

$$\Phi^*_{\sqrt{\Delta_\Phi(\Psi)}}(x)\,\Phi_{\sqrt{\Delta_\Phi(\Psi)}}(x) = \varphi_{\Delta_{1\Phi}(\Psi)}(\langle x, x\rangle) = \psi(\langle x, x\rangle) = \Psi(x)^*\Psi(x)$$

for every $x \in E$, which implies $\Psi \sim \Phi_{\sqrt{\Delta_\Phi(\Psi)}}$. Let $T \oplus S$ be another positive linear map in $\pi_\Phi(E)^{'}$ such that $\Psi \sim \Phi_{\sqrt{T\oplus S}}$. Then $\Phi_{\sqrt{\Delta_\Phi(\Psi)}} \sim \Phi_{\sqrt{T\oplus S}}$ and so $\varphi_{\Delta_{1\Phi}(\Psi)} = \varphi_T$. By [12, Theorem 3.4.5], we deduce that $\Delta_{1\Phi}(\Psi) = T$. Since π_Φ is non-degenerate (Remark 2 (3)), $\Delta_{2\Phi}(\Psi)$ and S are uniquely determined by $\Delta_{1\Phi}(\Psi)$ and T, respectively. Consequently, $\Delta_{2\Phi}(\Psi) = S$ and so $\Delta_\Phi(\Psi) = T \oplus S$. $\square$

Suppose that $\Phi \in CP(E, B(H,K))$, $\hat{\Phi} = \{\Psi \in CP(E, B(H,K)) : \Phi \sim \Psi\}$ and $\Phi, \Psi \in CP(E, B(H,K))$. We write $\hat{\Psi} \le \hat{\Phi}$, if $\Psi \preceq \Phi$. We define

$$[0, \hat{\Phi}] := \{\hat{\Psi} : \Psi \in CP(E, B(H,K)), \Psi \preceq \Phi\}$$

and

$$[0, I]_\Phi := \{T \oplus S \in \pi_\Phi(E)^{'} : 0 \le T \oplus S \le I\}.$$

The following theorem can be thought as a Radon-Nikodym type theorem for operator valued completely positive maps on Hilbert modules over pro-C*-algebras.

Theorem 6. *Let $\Phi \in CP(E, B(H,K))$. The map $\hat{\Psi} \to \Delta_\Phi(\Psi)$ from $[0, \hat{\Phi}]$ to $[0, I]_\Phi$ is an order-preserving isomorphism.*

Proof: The map is well-defined by Theorem 5. Let $\hat{\Psi}_1, \hat{\Psi}_2 \in [0, \hat{\Phi}]$ and $\Delta_\Phi(\Psi_1) = \Delta_\Phi(\Psi_2)$. Then $\Psi_1 \sim \Phi_{\Delta_\Phi(\Psi_1)} = \Phi_{\Delta_\Phi(\Psi_2)} \sim \Psi_2$ and so it is injective. Let $T \oplus S \in [0, I]_\Phi$ then $\Phi_{\sqrt{T\oplus S}} \in CP(E, B(H,K))$. Since $T \oplus S \in \pi_\Phi(E)^{'}$, $T \in \pi_\varphi(A)^{'}$ and so by [12, Theorem 3.4.5], $\Phi_{\sqrt{T\oplus S}}(x)^*\Phi_{\sqrt{T\oplus S}}(x) = \varphi_T(\langle x, x\rangle) \le \varphi(\langle x, x\rangle) = \Phi(x)^*\Phi(x)$ for all $x \in E$. Thus $\Phi_{\sqrt{T\oplus S}} \preceq \Phi$. Since $\Delta(\varphi_T) = T$, $\Delta_\Phi(\Phi_{\sqrt{T\oplus S}}) = T \oplus S$, i.e., the map is surjective.

If $\hat{\Psi}_1, \hat{\Psi}_2 \in [0, \hat{\Phi}]$ and $\hat{\Psi}_1 \le \hat{\Psi}_2$, then $\Psi_1 \preceq \Psi_2$ and so $\psi_1 \le \psi_2$. By [12, Theorem 3.4.5], we have $\Delta_{1\Phi}(\Psi_1) \le \Delta_{1\Phi}(\Psi_2)$. Since π_Φ is non-degenerate (Remark 2 (3)), $\Delta_{2\Phi}(\Psi_1)$ and $\Delta_{2\Phi}(\Psi_2)$ are uniquely determined by $\Delta_{1\Phi}(\Psi_1)$ and $\Delta_{1\Phi}(\Psi_2)$, respectively. Consequently, $\Delta_{2\Phi}(\Psi_1) \le \Delta_{2\Phi}(\Psi_2)$ and so $\Delta_\Phi(\Psi_1) \le \Delta_\Phi(\Psi_2)$. Conversely, let $T_1 \oplus S_1, T_2 \oplus S_2 \in [0, I]_\Phi$ and $T_1 \oplus S_1 \le T_2 \oplus S_2$ then $T_1, T_2 \in [0, I]_\varphi$ and $T_1 \le T_2$. By [12, Theorem 3.4.5], $\varphi_{T_1} \le \varphi_{T_2}$ and so $\Phi_{\sqrt{T_1\oplus S_1}} \preceq \Phi_{\sqrt{T_2\oplus S_2}}$. $\square$

Acknowledgement: This research was done while the second author stayed at the Mathematisches Institut of the Westfälische Wilhelms-Universität in Münster, Germany, in 2015. He would like to express his thanks to Wend Werner and his colleagues in functional analysis and operator algebras group for warm hospitality and great scientific atmosphere. The second author gratefully acknowledges financial support from the Alexander von Humboldt-Foundation. The second author also would like to thank Maria Joita who sent him some copies of her recent books. The authors would like to thank the referee for his/her careful reading and useful comments.

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Received: 10.04.2016
Revised: 03.10.2016
Accepted: 31.10.2016

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