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The hybrid power mean of three-term character sums and two-term exponential sums

by
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Abstract

The main purpose of this paper is using the analytic methods and the properties of Gauss sums to study the computational problem of one kind hybrid power mean of three-term character sums and two-term exponential sums, and give an exact computational formula for it. As an application of this result, we obtained a sharp upper bound estimate and lower bound estimates for the three-term character sums and two-term exponential sums

1 Introduction

Let $q \geq 3$ be a positive integer. For any integers m and n, the two-term exponential sums $E(m, n, k, h; q)$ and three-term character sums $C(m, n, k, h; q)$ are defined as follows:

$$E(m, n, k, h; q) = \sum_{a=1}^{q} e\left(\frac{ma^k + na^h}{q}\right)$$

and

$$C(m, n, k, h; q) = \sum_{a=1}^{q} \chi\left(ma^k + na^h + a\right),$$

where $k > h > 1$ are positive integers, $e(y) = e^{2\pi i y}$, and χ denotes a Dirichlet character mod q.

The estimates of these sums play a very important role in the additive number theory and analytic number theory, so many authors had studied the various properties of $E(m, n, k, h; q)$ and $C(m, n, k, h; q)$, and obtained a series of important results, see [2]-[11]. For example, T. Cochrane and Zheng Zhiyong [4] show for the general sum that

$$\left|\sum_{a=1}^{q} e\left(\frac{ma^k + na}{q}\right)\right| \leq k^{\omega(q)} q^{\frac{1}{2}},$$

where $\omega(q)$ denotes the number of all distinct prime divisors of q.

Zhang Wenpeng and Han Di [10] studied the sixth power mean of the two-term exponential sums, and proved that for any prime $p > 3$ with $(3, p-1) = 1$, one has the identity

$$\sum_{a=1}^{p-1} \left|\sum_{n=0}^{p-1} e\left(\frac{n^3 + an}{p}\right)\right|^6 = 5p^4 - 8p^3 - p^2.$$

On the other hand, Pólya and Vinogradov (see Theorem 8.21 of [1]) proved that for any non-principal character χ mod q, one has the estimate

$$\sum_{a=N+1}^{N+M} \chi(a) \ll q^{\frac{1}{2}} \ln q.$$

If $q = p$ is an odd prime, then Weil (see D. A. Burgess [3]) obtained following important conclusion:

Let χ is a qth-order character mod p, if $f(x)$ is not a perfect qth power mod p, then one has the estimate

$$\sum_{x=N+1}^{N+M} \chi(f(x)) \ll p^{\frac{1}{2}} \ln p \quad \text{and} \quad \sum_{x=0}^{p-1} \chi(f(x)) \ll p^{\frac{1}{2}}, \tag{1.1}$$

where $A \ll B$ denotes $|A| < cB$ for some constant c, which in this case depends only on the degree of $f(x)$.

Zhang Wenpeng and Yi Yuan [7] found some polynomials $f(x) = (x-r)^m(x-s)^n$ such that the identity

$$\left| \sum_{a=1}^{q} \chi\left((a-r)^m (a-s)^n\right) \right| = \sqrt{q}, \tag{1.2}$$

where $(r-s, q) = 1$, m, n and χ also satisfying some special conditions.

It is clear that from (1.2) we know that the estimate (1.1) is the best possible. Some related results can also be found in [2], [5] and [8].

In this paper, we consider the hybrid power mean

$$\sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi\left(ma^3 + na^2 + a\right) \right|^2 \cdot \left| \sum_{b=1}^{p-1} e\left(\frac{mb^3 + nb^2}{p}\right) \right|^2, \tag{1.3}$$

where p be an odd prime.

We are concerned about whether there exists an exact computational formula or asymptotic formula for (1.3)?

For this problem, it seems that none had studied it yet, at least we have not seen any related results before. The mean value is interesting, because the first sum in (1.3) is of multiplicative properties, while the second one in (1.3) is of additive properties. But their hybrid power mean are of orthogonality. They have a good value distribution result. Of course, the mean value (1.3) has also some close relations with the upper bound of $E(m, n, k, h; q)$ and $C(m, n, k, h; q)$.

In this paper, we use the analytic method and the properties of Gauss sums to study this problem, and give an exact computational formula for (1.3). As an application of our result, we obtained a sharp upper bound estimate and lower bound estimates for the three-term character sums and two-term exponential sums. That is, we shall prove the following conclusion:

Theorem. Let p be an odd prime with $(3, p-1) = 1$. Then for any non-principal character χ mod p, we have the identity

$$\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(ma^3+na^2+a\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+nb^2}{p}\right)\right|^2$$

$$= \begin{cases} p(p-1)(p^2-2p+2), & \text{if } \chi \text{ is a non-real odd character mod } p,\\ p\left(p^3-5p^2+10p-2\right), & \text{if } \chi \text{ is a non-real even character mod } p,\\ p^2(p-1)(p-2), & \text{if } \chi \text{ is the Legendre symbol and } p\equiv 3 \bmod 4,\\ p^2\left(p^2-5p+8\right), & \text{if } \chi \text{ is the Legendre symbol and } p\equiv 1 \bmod 4.\end{cases}$$

Some notes: In fact, our ultimate goal in this paper is to give an exact calculating formula for the mean square error

$$\begin{aligned}&\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left(|E(m,n,3,2;p)|^2-|C(m,n,3,2;p)|^2\right)^2\\ =\ &\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}|E(m,n,3,2;p)|^4+\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}|C(m,n,3,2;p)|^4\\ &-2\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}|E(m,n,3,2;p)|^2\cdot|C(m,n,3,2;p)|^2.\end{aligned}$$

If this can be done, then in a lot of number theory problems, we can replace each other with the $|E(m,n,3,2;p)|^2$ and $|C(m,n,3,2;p)|^2$. But it is a pity that we can not give an exact computational formula for the fourth power mean

$$\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(ma^3+na^2+a\right)\right|^4 \quad\text{or}\quad \sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+nb^2}{p}\right)\right|^4.$$

These problems will be our further research contents. Our theorem is just a preliminary result.

For a general integer $h > 1$, whether there exists an exact computational formula or asymptotic formula for the hybrid power mean

$$\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(ma^3+na^2+a\right)\right|^{2h}\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+nb^2}{p}\right)\right|^{2h}$$

is an open problem, where p be an odd prime.

2 Several lemmas

In this section, we will give some lemmas which are necessary in the proof of our theorem. Hereinafter, we will use many properties of trigonometric sums, character sums, reduced residue system and congruence mod p, all of which can be found in [1], so they will not be repeated here. First we have the following:

Lemma 1. Let p be an odd prime with $(3, p-1) = 1$, χ be any non-principal character mod p. Then we have the identity

$$\sum_{\substack{u=1 \\ ua^3-vb^3\equiv c^3-1 \bmod p \\ ua^2-vb^2\equiv c^2-1 \bmod p}}^{p-1}\sum_{v=1}^{p-1}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(u\overline{v}) = \begin{cases} (p-1)(p-2), & \text{if } \chi \text{ is not a Legendre symbol of } p, \\ p(p-1), & \text{if } \chi \text{ is not a Legendre symbol of } p. \end{cases}$$

Proof. Since $(3, p-1) = 1$, so if u passes through a reduced residue system mod p, then for all $1 \le v \le p-1$, uv also pass through a reduced residue system mod p. So from this and the properties of character sums mod p we have

$$\begin{aligned} &\sum_{\substack{u=1 \\ ua^3-vb^3\equiv c^3-1 \bmod p \\ ua^2-vb^2\equiv c^2-1 \bmod p}}^{p-1}\sum_{v=1}^{p-1}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(u\overline{v}) = \sum_{u=1}^{p-1}\overline{\chi}(u) \sum_{\substack{v=1 \\ vb^3(ua^3-1)\equiv c^3-1 \bmod p \\ vb^2(ua^2-1)\equiv c^2-1 \bmod p}}^{p-1}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} 1 \\ = &\sum_{u=1}^{p-1}\overline{\chi}(u) \sum_{\substack{v=1 \\ vb(ua^3-1)\equiv c^3-1 \bmod p \\ v(ua^2-1)\equiv c^2-1 \bmod p}}^{p-1}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} 1 = \sum_{u=1}^{p-1}\overline{\chi}(u) \sum_{\substack{v=1 \\ b(ua^3-1)\equiv c^3-1 \bmod p \\ v(ua^2-1)\equiv c^2-1 \bmod p}}^{p-1}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} 1. \end{aligned} \tag{2.1}$$

It is clear that in (2.1), if $c = 1$, then $u = a = 1$. This time, the sums as $\sum_{v=1}^{p-1}\sum_{b=1}^{p-1} 1 = (p-1)^2$. If $c = p-1$, then $ua^2 \equiv 1 \bmod p$ and $(ua^3 - 1, p) \ne p$. Since $(3, p-1) = 1$, so the congruence equation $c^3 - 1 \equiv 0 \bmod p$ has only solution $c \equiv 1 \bmod p$. This time, the sums as $(p-1)\sum_{a=2}^{p-1}\chi^2(a)$. If $2 \le c \le p-2$, then we have $(c^2-1, p) = (c^3-1, p) = 1$, so we must have $(ua^2-1, p) = (ua^3-1, p) = 1$. This time, v and b are uniquely determined by the congruence equations $b\left(ua^3-1\right) \equiv c^3 - 1 \bmod p$ and $v(ua^2-1) \equiv c^2-1 \bmod p$. Therefore, the sums as $\sum\limits_{c=2}^{p-2}\sum\limits_{a=1}^{p-1}\left(\sum\limits_{\substack{u=1\\ u\neq\overline{a}^2,\ \overline{a}^3}}^{p-1}\overline{\chi}(u)\right)$. From (2.1) with above explanations we have

$$\begin{aligned}
&\sum_{\substack{u=1\\ ua^3-vb^3\equiv c^3-1 \bmod p\\ ua^2-vb^2\equiv c^2-1 \bmod p}}^{p-1}\sum_{v=1}^{p-1}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(u\overline{v})\\
=\;& (p-1)^2+(p-1)\sum_{a=2}^{p-1}\chi^2(a)+\sum_{c=2}^{p-2}\sum_{a=1}^{p-1}\left(\sum_{\substack{u=1\\ u\neq\overline{a}^2,\ \overline{a}^3}}^{p-1}\overline{\chi}(u)\right)\\
=\;& (p-1)^2+(p-1)\sum_{a=2}^{p-1}\chi^2(a)+(p-3)\left(\sum_{a=1}^{p-1}\sum_{u=1}^{p-1}\overline{\chi}(u)-\sum_{a=1}^{p-1}\chi^2(a)-\sum_{a=1}^{p-1}\chi^3(a)\right)\\
=\;& \begin{cases} (p-1)(p-2), & \text{if } \chi \text{ is not a Legendre symbol mod } p,\\ p(p-1), & \text{if } \chi \text{ is a Legendre symbol mod } p.\end{cases}
\end{aligned}$$

This proves Lemma 1.

Lemma 2. Let p be an odd prime with $(3,p-1)=1$, χ be any non-principal character mod p. Then we have the identity

$$\sum_{\substack{u=1\\ ua^3-vb^3\equiv c^3-1 \bmod p\\ ua^2-vb^2\equiv c^2-1 \bmod p\\ ua-vb\equiv 0 \bmod p}}^{p-1}\sum_{v=1}^{p-1}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(u\overline{v})=\begin{cases}(p-1)^2, & \text{if } \chi(-1)=-1,\\ p^2-4p+7, & \text{if } \chi(-1)=1.\end{cases}$$

Proof. From the properties of reduced residue system mod p and the method of proving Lemma 1 we can deduce this conclusion, the detailed process is omitted.

3 Proofs of the theorems

In this section, we shall complete the proof of our theorem. Note that the trigonometric identity

$$\sum_{m=0}^{p-1}e\left(\frac{nm}{p}\right)=\begin{cases} p, & \text{if } (p,n)=p;\\ 0, & \text{if } (p,n)=1\end{cases}$$

and the properties of Gauss sums $|\tau(\chi)|=\sqrt{p}$ and

$$\chi(n)=\frac{1}{\tau(\overline{\chi})}\sum_{b=1}^{p-1}\overline{\chi}(b)e\left(\frac{bn}{p}\right)$$

we have

$$\begin{aligned}
&\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(ma^3+na^2+a\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+nb^2}{p}\right)\right|^2\\
=\;&\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left|\frac{1}{\tau\left(\overline{\chi}\right)}\sum_{u=1}^{p-1}\overline{\chi}(u)\sum_{a=1}^{p-1}e\left(\frac{u\left(ma^3+na^2+a\right)}{p}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+nb^2}{p}\right)\right|^2\\
=\;&\frac{1}{p}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\overline{\chi}(u\overline{v})\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{m=0}^{p-1}e\left(\frac{m\left(ua^3-vb^3-c^3+d^3\right)}{p}\right)\\
&\times\sum_{n=0}^{p-1}e\left(\frac{n(ua^2-vb^2-c^2+d^2)+ua-vb}{p}\right)\\
=\;&p\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\overline{\chi}(u\overline{v})\sum_{\substack{a=1\\ ua^3-vb^3\equiv c^3-d^3 \bmod p\\ ua^2-vb^2\equiv c^2-d^2 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}e\left(\frac{ua-vb}{p}\right)\\
=\;&p\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\overline{\chi}(u\overline{v})\sum_{\substack{a=1\\ ua^3-vb^3\equiv c^3-1 \bmod p\\ ua^2-vb^2\equiv c^2-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}e\left(\frac{d(ua-vb)}{p}\right)\\
=\;&p^2\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\overline{\chi}(u\overline{v})\sum_{\substack{a=1\\ ua^3-vb^3\equiv c^3-1 \bmod p\\ ua^2-vb^2\equiv c^2-1 \bmod p\\ ua-vb\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}1\\
&-p\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\overline{\chi}(u\overline{v})\sum_{\substack{a=1\\ ua^3-vb^3\equiv c^3-1 \bmod p\\ ua^2-vb^2\equiv c^2-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}1. \qquad (3.1)
\end{aligned}$$

If χ is not a non-real odd character mod p, then combining (3.1), Lemma 1 and Lemma 2 we have the identity

$$\begin{aligned}
&\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(ma^3+na^2+a\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+nb^2}{p}\right)\right|^2\\
=\;&p(p-1)(p^2-2p+2). \qquad (3.2)
\end{aligned}$$

If χ is a non-real even character mod p, then combining (3.1), Lemma 1 and Lemma 2 we have

$$\begin{aligned}
&\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(ma^3+na^2+a\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+nb^2}{p}\right)\right|^2\\
=\;&p\left(p^3-4p^2+10p-2\right). \qquad (3.3)
\end{aligned}$$

If $\chi = \left(\frac{*}{p}\right)$ is the Legendre symbol mod p, then from the properties of Legendre symbol we know that $\left(\frac{-1}{p}\right) = -1$, if $p \equiv 3 \bmod 4$, and $\left(\frac{-1}{p}\right) = 1$, if $p \equiv 1 \bmod 4$. So from (3.1), Lemma 1 and Lemma 2 we have the identity

$$\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(ma^3+na^2+a\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+nb^2}{p}\right)\right|^2$$
$$= \begin{cases} p^2(p-1)(p-2), & \text{if } \chi \text{ is the Legendre symbol and } p\equiv 3 \bmod 4, \\ p^2\left(p^2-5p+8\right), & \text{if } \chi \text{ is the Legendre symbol and } p\equiv 1 \bmod 4. \end{cases} \tag{3.4}$$

Combining (3.2), (3.3) and (3.4) we may immediately deduce the identity

$$\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(ma^3+na^2+a\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+nb^2}{p}\right)\right|^2$$
$$= \begin{cases} p(p-1)(p^2-2p+2), & \text{if } \chi \text{ is a non-real odd character mod } p, \\ p\left(p^3-5p^2+10p-2\right), & \text{if } \chi \text{ is a non-real even character mod } p, \\ p^2(p-1)(p-2), & \text{if } \chi \text{ is the Legendre symbol and } p\equiv 3 \bmod 4, \\ p^2\left(p^2-5p+8\right), & \text{if } \chi \text{ is the Legendre symbol and } p\equiv 1 \bmod 4. \end{cases}$$

This completes the proof of our Theorem.

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References

[1] Tom M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.

[2] D. A. Burgess, On character sums and primitive roots, Proc. London Math. Soc., 12(1962), 179-192.

[3] D. A. Burgess, On Dirichlet characters of polynomials, Proc. London Math. Soc., 13(1963), 537-548.

[4] T. Cochrane and Z. Zheng, Bounds for certain exponential sums, Asian J. Math. 4 (2000), 757-774.

[5] A. Granville and K. Soundararajan, Large character sums: Pretentious characters and the Pólya-Vinogradov theorem, J. Amer. Math. Soc., 20 (2007), 357-384.

[6] Zhang Wenpeng and Wang Tingting, On the Dirichlet characters of polynomials, Mathematica Slovaca,

[7] Zhang Wenpeng and Yi Yuan, On Dirichlet characters of polynomials, Bull. London Math. Soc., 34(2002), 469-473.

[8] Zhang Wenpeng and Yao Weili, A note on the Dirichlet characters of polynomials, Acta Arith., 115 (2004), 225-229.

[9] Zhu Minhui and Han Di, On the fourth power mean of the two-term exponential sums, Journal of Inequalities and Applications 2014: 116, page 7.

[10] Zhang Wenpeng and Han Di, On the sixth power mean of the two-term exponential sums, Journal of Number Theory, 136(2014), 403-413.

[11] A. Weil, On some exponential sums, Proc. Nat. Acad. Sci. U.S.A., 34(1948), 204-207.

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