Bull. Math. Soc. Sci. Math. Roumanie
Tome 60 (108) No. 2, 2017, 159–171

On the solutions and periodicity of some rational systems of difference equations

by

Elsayed M. Elsayed

Abstract

In this paper we deal with the form of the solutions and the periodicity nature of the following systems of nonlinear difference equations

$$x_{n+1} = \frac{x_{n-3}y_{n-2}}{y_n(\pm 1 \pm x_{n-1}y_{n-2}x_{n-3})},\ y_{n+1} = \frac{x_{n-2}y_{n-3}}{x_n(\pm 1 \pm y_{n-1}x_{n-2}y_{n-3})},$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1}$, and y_0 are nonzero real numbers.

Key Words: difference equations, periodic solutions, system of difference equations.

2010 Mathematics Subject Classification: **39A10.**

1 Introduction

Our aim in this paper is to investigate the periodic nature and the solutions of the following systems of nonlinear difference equations

$$x_{n+1} = \frac{x_{n-3}y_{n-2}}{y_n(\pm 1 \pm x_{n-1}y_{n-2}x_{n-3})},\ y_{n+1} = \frac{x_{n-2}y_{n-3}}{x_n(\pm 1 \pm y_{n-1}x_{n-2}y_{n-3})},$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1}$, and y_0 are nonzero real numbers.

Difference equations appear naturally as discrete analogues and as numerical solutions of differential equations having applications in ecology, biology, physics, economy and so on. So, recently there has been an increasing interest in the study of qualitative analysis of rational difference equations and systems of difference equations. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the dynamics of their solutions. see [3]-[6] and the references cited therein.

There are many papers related to systems of difference equations, for examples: the behavior of the positive solutions of the rational difference system

$$x_{n+1} = \frac{m}{y_n},\quad y_{n+1} = \frac{py_n}{x_{n-1}y_{n-1}},$$

has been studied by Cinar [2].

The dynamics of the solutions of the following system

$$x_{n+1}=\frac{x_{n-1}}{1+x_{n-1}y_n},\ y_{n+1}=\frac{y_{n-1}}{1+y_{n-1}x_n},$$

has been studied by Kurbanli et al. [10].

Touafek et al. [11] investigated the periodic nature and gave the form of the solutions of the following systems of difference equations

$$x_{n+1}=\frac{x_{n-3}}{\pm 1\pm x_{n-3}y_{n-1}},\ y_{n+1}=\frac{y_{n-3}}{\pm 1\pm y_{n-3}x_{n-1}}.$$

Yalçınkaya [12] has obtained the sufficient conditions for the global asymptotic stability of the following system of two nonlinear difference equations

$$x_{n+1}=\frac{x_n+y_{n-1}}{x_ny_{n-1}-1},\ y_{n+1}=\frac{y_n+x_{n-1}}{y_nx_{n-1}-1}.$$

In [13], Zhang et al. studied the persistence and the global asymptotic stability of the solutions of the system

$$x_n=A+\frac{1}{y_{n-p}},\qquad y_n=A+\frac{y_{n-1}}{x_{n-r}y_{n-s}}.$$

Similar nonlinear systems of difference equations were investigated see [1], [7], [8].

2 Main Results

Here we obtain the form of the solutions of some systems of difference equations. Also, we deal with periodicity of solutions of the same systems of difference equations.

2.1 On the System: $x_{n+1}=\frac{x_{n-3}y_{n-2}}{y_n(1+x_{n-1}y_{n-2}x_{n-3})},\ y_{n+1}=\frac{x_{n-2}y_{n-3}}{x_n(1+y_{n-1}x_{n-2}y_{n-3})}$

In this section, we study the solutions of the system of two difference equations

$$x_{n+1}=\frac{x_{n-3}y_{n-2}}{y_n(1+x_{n-1}y_{n-2}x_{n-3})},\ y_{n+1}=\frac{x_{n-2}y_{n-3}}{x_n(1+y_{n-1}x_{n-2}y_{n-3})},\qquad n=0,1,...,\tag{1}$$

with nonzero real numbers initials conditions.

Theorem 1. *Suppose that $\{x_n,y_n\}$ are solutions of system (1), then for $n=0,1,2,...,$ we* *see that*

$$
\begin{aligned}
x_{6n-3} &= d\prod_{i=0}^{n-1}\left(\frac{(1+(3i)\,bdg)\,(1+(3i+1)\,beg)}{(1+(3i+2)\,bdg)\,(1+(3i)\,beg)}\right),\\
x_{6n-2} &= c\prod_{i=0}^{n-1}\left(\frac{(1+(3i)\,acf)\,(1+(3i+2)\,cfh)}{(1+(3i+2)\,acf)\,(1+(3i+1)\,cfh)}\right),\\
x_{6n-1} &= b\prod_{i=0}^{n-1}\left(\frac{(1+(3i+1)\,bdg)\,(1+(3i+2)\,beg)}{(1+(3i+3)\,bdg)\,(1+(3i+1)\,beg)}\right),\\
x_{6n} &= a\prod_{i=0}^{n-1}\left(\frac{(1+(3i+1)\,acf)\,(1+(3i+3)\,cfh)}{(1+(3i+3)\,acf)\,(1+(3i+2)\,cfh)}\right),\\
x_{6n+1} &= \frac{dg}{e(1+bdg)}\prod_{i=0}^{n-1}\left(\frac{(1+(3i+2)\,bdg)\,(1+(3i+3)\,beg)}{(1+(3i+4)\,bdg)\,(1+(3i+2)\,beg)}\right),\\
x_{6n+2} &= \frac{af(1+cfh)}{h(1+acf)}\prod_{i=0}^{n-1}\left(\frac{(1+(3i+2)\,acf)\,(1+(3i+4)\,cfh)}{(1+(3i+4)\,acf)\,(1+(3i+3)\,cfh)}\right),
\end{aligned}
$$

$$
\begin{aligned}
y_{6n-3} &= h\prod_{i=0}^{n-1}\left(\frac{(1+(3i+1)\,acf)\,(1+(3i)\,cfh)}{(1+(3i)\,acf)\,(1+(3i+2)\,cfh)}\right),\\
y_{6n-2} &= g\prod_{i=0}^{n-1}\left(\frac{(1+(3i+2)\,bdg)\,(1+(3i)\,beg)}{(1+(3i+1)\,bdg)\,(1+(3i+2)\,beg)}\right),\\
y_{6n-1} &= f\prod_{i=0}^{n-1}\left(\frac{(1+(3i+2)\,acf)\,(1+(3i+1)\,cfh)}{(1+(3i+1)\,acf)\,(1+(3i+3)\,cfh)}\right),\\
y_{6n} &= e\prod_{i=0}^{n-1}\left(\frac{(1+(3i+3)\,bdg)\,(1+(3i+1)\,beg)}{(1+(3i+2)\,bdg)\,(1+(3i+3)\,beg)}\right),\\
y_{6n+1} &= \frac{ch}{a(1+cfh)}\prod_{i=0}^{n-1}\left(\frac{(1+(3i+3)\,acf)\,(1+(3i+2)\,cfh)}{(1+(3i+2)\,acf)\,(1+(3i+4)\,cfh)}\right),\\
y_{6n+2} &= \frac{be(1+bdg)}{d(1+beg)}\prod_{i=0}^{n-1}\left(\frac{(1+(3i+4)\,bdg)\,(1+(3i+2)\,beg)}{(1+(3i+3)\,bdg)\,(1+(3i+4)\,beg)}\right),
\end{aligned}
$$

where $x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a,\ y_{-3} = h,\ y_{-2} = g,\ y_{-1} = f,\ y_0 = e$ *and* $\prod_{i=0}^{-1} A_i = 1$.

Proof. We prove that the forms given are solutions of system (1) by using mathematical induction. First we let $n = 0$, then the result holds. Second we assume that the expressions are satisfied for $n - 1$. Our objective is to show that the expressions are satisfied for n.

That is;

$$\begin{aligned}
x_{6n-9} &= d\prod_{i=0}^{n-2}\left(\frac{(1+(3i)\,bdg)\,(1+(3i+1)\,beg)}{(1+(3i+2)\,bdg)\,(1+(3i)\,beg)}\right),\\
x_{6n-8} &= c\prod_{i=0}^{n-2}\left(\frac{(1+(3i)\,acf)\,(1+(3i+2)\,cfh)}{(1+(3i+2)\,acf)\,(1+(3i+1)\,cfh)}\right),\\
x_{6n-7} &= b\prod_{i=0}^{n-2}\left(\frac{(1+(3i+1)\,bdg)\,(1+(3i+2)\,beg)}{(1+(3i+3)\,bdg)\,(1+(3i+1)\,beg)}\right),\\
x_{6n-6} &= a\prod_{i=0}^{n-2}\left(\frac{(1+(3i+1)\,acf)\,(1+(3i+3)\,cfh)}{(1+(3i+3)\,acf)\,(1+(3i+2)\,cfh)}\right),\\
x_{6n-5} &= \frac{dg}{e(1+bdg)}\prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)\,bdg)\,(1+(3i+3)\,beg)}{(1+(3i+4)\,bdg)\,(1+(3i+2)\,beg)}\right),\\
x_{6n-4} &= \frac{af(1+cfh)}{h(1+acf)}\prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)\,acf)\,(1+(3i+4)\,cfh)}{(1+(3i+4)\,acf)\,(1+(3i+3)\,cfh)}\right),
\end{aligned}$$

$$\begin{aligned}
y_{6n-9} &= h\prod_{i=0}^{n-2}\left(\frac{(1+(3i+1)\,acf)\,(1+(3i)\,cfh)}{(1+(3i)\,acf)\,(1+(3i+2)\,cfh)}\right),\\
y_{6n-8} &= g\prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)\,bdg)\,(1+(3i)\,beg)}{(1+(3i+1)\,bdg)\,(1+(3i+2)\,beg)}\right),\\
y_{6n-7} &= f\prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)\,acf)\,(1+(3i+1)\,cfh)}{(1+(3i+1)\,acf)\,(1+(3i+3)\,cfh)}\right),\\
y_{6n-6} &= e\prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)\,bdg)\,(1+(3i+1)\,beg)}{(1+(3i+2)\,bdg)\,(1+(3i+3)\,beg)}\right),\\
y_{6n-5} &= \frac{ch}{a(1+cfh)}\prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)\,acf)\,(1+(3i+2)\,cfh)}{(1+(3i+2)\,acf)\,(1+(3i+4)\,cfh)}\right),\\
y_{6n-4} &= \frac{be(1+bdg)}{d(1+beg)}\prod_{i=0}^{n-2}\left(\frac{(1+(3i+4)\,bdg)\,(1+(3i+2)\,beg)}{(1+(3i+3)\,bdg)\,(1+(3i+4)\,beg)}\right),
\end{aligned}$$

Now, it follows from Eq.(1) that

$$x_{6n-3}=\frac{x_{6n-7}y_{6n-6}}{y_{6n-4}(1+x_{6n-5}y_{6n-6}x_{6n-7})}$$

$$
\begin{aligned}
&= \frac{\genfrac{}{}{0pt}{}{\left(b\prod_{i=0}^{n-2}\left(\frac{(1+(3i+1)\,bdg)\,(1+(3i+2)\,beg)}{(1+(3i+3)\,bdg)\,(1+(3i+1)\,beg)}\right)\right)}{\left(e\prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)\,bdg)\,(1+(3i+1)\,beg)}{(1+(3i+2)\,bdg)\,(1+(3i+3)\,beg)}\right)\right)}}{\genfrac{}{}{0pt}{}{\left(\frac{be(1+bdg)}{d(1+beg)}\prod_{i=0}^{n-2}\left(\frac{(1+(3i+4)\,bdg)\,(1+(3i+2)\,beg)}{(1+(3i+3)\,bdg)\,(1+(3i+4)\,beg)}\right)\right)}{\left(1+\left(\genfrac{}{}{0pt}{}{\frac{dg}{e(1+bdg)}\prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)\,bdg)\,(1+(3i+3)\,beg)}{(1+(3i+4)\,bdg)\,(1+(3i+2)\,beg)}\right)}{\genfrac{}{}{0pt}{}{\left(b\prod_{i=0}^{n-2}\left(\frac{(1+(3i+1)\,bdg)\,(1+(3i+2)\,beg)}{(1+(3i+3)\,bdg)\,(1+(3i+1)\,beg)}\right)\right)}{\left(e\prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)\,bdg)\,(1+(3i+1)\,beg)}{(1+(3i+2)\,bdg)\,(1+(3i+3)\,beg)}\right)\right)}}\right)\right)}} \\
&= \frac{d(1+beg)\left(\prod_{i=0}^{n-2}\left(\frac{(1+(3i+1)\,bdg)\,(1+(3i+3)\,bdg)}{(1+(3i+2)\,bdg)\,(1+(3i+3)\,beg)}\right)\right)}{(1+bdg)\prod_{i=0}^{n-2}\frac{(1+(3i+4)\,bdg)}{(1+(3i+4)\,beg)}\left(1+\frac{bdg}{(1+bdg)}\prod_{i=0}^{n-2}\frac{1+(3i+1)\,bdg}{1+(3i+4)\,bdg}\right)} \\
&= \frac{d(1+beg)\left(\prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)\,bdg)\,(1+(3i+4)\,beg)}{(1+(3i+2)\,bdg)\,(1+(3i+3)\,beg)}\right)\right)}{(1+(3n-2)bdg)\left(1+\left(\frac{bdg}{(1+(3n-2)bdg)}\right)\right)} \\
&= \frac{d(1+beg)\prod_{i=0}^{n-2}\frac{(1+(3i+3)\,bdg)\,(1+(3i+4)\,beg)}{(1+(3i+2)\,bdg)\,(1+(3i+3)\,beg)}}{(1+(3n-2)bdg+bdg)} \\
&= d\prod_{i=0}^{n-1}\left(\frac{(1+(3i)\,bdg)\,(1+(3i+1)\,beg)}{(1+(3i+2)\,bdg)\,(1+(3i)\,beg)}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
y_{6n-3} &= \frac{y_{6n-7}x_{6n-6}}{x_{6n-4}(1+y_{6n-5}x_{6n-6}y_{6n-7})} \\
&= \frac{\left(f\prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)acf)(1+(3i+1)cfh)}{(1+(3i+1)acf)(1+(3i+3)cfh)}\right)\right)\left(a\prod_{i=0}^{n-2}\left(\frac{(1+(3i+1)acf)(1+(3i+3)cfh)}{(1+(3i+3)acf)(1+(3i+2)cfh)}\right)\right)}{\genfrac{}{}{0pt}{}{\left(\frac{af(1+cfh)}{h(1+acf)}\prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)acf)(1+(3i+4)cfh)}{(1+(3i+4)acf)(1+(3i+3)cfh)}\right)\right)}{\left(1+\left(\genfrac{}{}{0pt}{}{\left(\frac{ch}{a(1+cfh)}\prod_{i=0}^{n-2}\left(\frac{(1+(3i+3)acf)(1+(3i+2)cfh)}{(1+(3i+2)acf)(1+(3i+4)cfh)}\right)\right)}{\genfrac{}{}{0pt}{}{\left(a\prod_{i=0}^{n-2}\left(\frac{(1+(3i+1)acf)(1+(3i+3)cfh)}{(1+(3i+3)acf)(1+(3i+2)cfh)}\right)\right)}{\left(f\prod_{i=0}^{n-2}\left(\frac{(1+(3i+2)acf)(1+(3i+1)cfh)}{(1+(3i+1)acf)(1+(3i+3)cfh)}\right)\right)}}\right)\right)}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{h(1+acf)\left(\prod_{i=0}^{n-2}\frac{(1+(3i+4)acf)(1+(3i+3)cfh)}{(1+(3i+3)acf)(1+(3i+2)cfh)}\prod_{i=0}^{n-2}\left(\frac{(1+(3i+1)cfh)}{(1+(3i+4)cfh)}\right)\right)}{(1+cfh)\left(1+\frac{cfh}{(1+cfh)}\prod_{i=0}^{n-2}\frac{(1+(3i+1)cfh)}{(1+(3i+4)cfh)}\right)} \\
&= \frac{\left(\frac{1}{(1+(3n-2)cfh)}\right)h(1+acf)\left(\prod_{i=0}^{n-2}\frac{(1+(3i+4)acf)(1+(3i+3)cfh)}{(1+(3i+3)acf)(1+(3i+2)cfh)}\right)}{\left(1+\frac{cfh}{(1+(3n-2)cfh)}\right)} \\
&= \frac{h(1+acf)\left(\prod_{i=0}^{n-2}\frac{(1+(3i+4)acf)(1+(3i+3)cfh)}{(1+(3i+3)acf)(1+(3i+2)cfh)}\right)}{(1+(3n-2)cfh+cfh)} \\
&= \frac{h(1+acf)}{(1+(3n-1)cfh)}\prod_{i=0}^{n-2}\frac{(1+(3i+4)\,acf)\,(1+(3i+3)\,cfh)}{(1+(3i+3)\,acf)\,(1+(3i+2)\,cfh)} \\
&= h\prod_{i=0}^{n-1}\left(\frac{(1+(3i+1)\,acf)\,(1+(3i)\,cfh)}{(1+(3i)\,acf)\,(1+(3i+2)\,cfh)}\right).
\end{aligned}
$$

Similarly we can prove the other relations. The proof is complete. □

The following cases can be proved similarly.

2.2 On the System: $x_{n+1}=\frac{x_{n-3}y_{n-2}}{y_n(1+x_{n-1}y_{n-2}x_{n-3})}$, $y_{n+1}=\frac{x_{n-2}y_{n-3}}{x_n(-1+y_{n-1}x_{n-2}y_{n-3})}$

In this section, we study the solutions of the system of two difference equations

$$x_{n+1}=\frac{x_{n-3}y_{n-2}}{y_n(1+x_{n-1}y_{n-2}x_{n-3})},\ y_{n+1}=\frac{x_{n-2}y_{n-3}}{x_n(-1+y_{n-1}x_{n-2}y_{n-3})}, \tag{2}$$

with a nonzero real numbers initial conditions and $x_{-3}y_{-2}x_{-1}$, $x_{-2}y_{-1}x_0\neq 1$, $y_{-2}x_{-1}y_0$, $y_{-3}x_{-2}y_{-1}\neq -1$.

Theorem 2. *Suppose that $\{x_n, y_n\}$ are solutions of system (2). For $n = 0, 1, 2, \ldots$, we have*

$$
\begin{aligned}
x_{12n-3} &= d\prod_{i=0}^{n-1}\left(\frac{(1+(6i)bdg)(1+(6i+3)bdg)}{(1+(6i+2)bdg)(1+(6i+5)bdg)}\right),\\
x_{12n-2} &= c\prod_{i=0}^{n-1}\left(\frac{(1+(6i)acf)(1+(6i+3)acf)}{(1+(6i+2)acf)(1+(6i+5)acf)}\right),\\
x_{12n-1} &= b\prod_{i=0}^{n-1}\left(\frac{(1+(6i+1)bdg)(1+(6i+4)bdg)}{(1+(6i+3)bdg)(1+(6i+6)bdg)}\right),\\
x_{12n} &= a\prod_{i=0}^{n-1}\left(\frac{(1+(6i+1)acf)(1+(6i+4)acf)}{(1+(6i+3)acf)(1+(6i+6)acf)}\right),\\
x_{12n+1} &= \frac{dg}{e(1+bdg)}\prod_{i=0}^{n-1}\left(\frac{(1+(6i+2)bdg)(1+(6i+5)bdg)}{(1+(6i+4)bdg)(1+(6i+7)bdg)}\right),\\
x_{12n+2} &= \frac{af(-1+cfh)}{h(1+acf)}\prod_{i=0}^{n-1}\left(\frac{(1+(6i+2)acf)(1+(6i+5)acf)}{(1+(6i+4)acf)(1+(6i+7)acf)}\right),\\
x_{12n+3} &= \frac{d(-1+beg)}{(1+2bdg)}\prod_{i=0}^{n-1}\left(\frac{(1+(6i+3)bdg)(1+(6i+6)bdg)}{(1+(6i+5)bdg)(1+(6i+8)bdg)}\right),\\
x_{12n+4} &= \frac{c}{(-1+cfh)(1+2acf)}\prod_{i=0}^{n-1}\left(\frac{(1+(6i+3)acf)(1+(6i+6)acf)}{(1+(6i+5)acf)(1+(6i+8)acf)}\right),\\
x_{12n+5} &= \frac{b(1+bdg)}{(-1+beg)(1+3bdg)}\prod_{i=0}^{n-1}\left(\frac{(1+(6i+4)\,bdg)\,(1+(6i+7)\,bdg)}{(1+(6i+6)\,bdg)\,(1+(6i+9)\,bdg)}\right),\\
x_{12n+6} &= \frac{a(-1+cfh)(1+acf)}{(1+3acf)}\prod_{i=0}^{n-1}\left(\frac{(1+(6i+4)\,acf)\,(1+(6i+7)\,acf)}{(1+(6i+6)\,acf)\,(1+(6i+9)\,acf)}\right),\\
x_{12n+7} &= \frac{dg(-1+beg)(1+2bdg)}{e(1+bdg)(1+4bdg)}\prod_{i=0}^{n-1}\left(\frac{(1+(6i+5)\,bdg)\,(1+(6i+8)\,bdg)}{(1+(6i+7)\,bdg)\,(1+(6i+10)\,bdg)}\right),\\
x_{12n+8} &= \frac{af(1+2acf)}{h(1+acf)(1+4acf)}\prod_{i=0}^{n-1}\left(\frac{(1+(6i+5)\,acf)\,(1+(6i+8)\,acf)}{(1+(6i+7)\,acf)\,(1+(6i+10)\,acf)}\right),
\end{aligned}
$$

$$\begin{aligned}
y_{12n-3} &= h\prod_{i=0}^{n-1}\left(\frac{(1+(6i+1)\,acf)\,(1+(6i+4)\,acf)}{(1+(6i)\,acf)\,(1+(6i+3)\,acf)}\right),\\
y_{12n-2} &= g\prod_{i=0}^{n-1}\left(\frac{(1+(6i+2)\,bdg)\,(1+(6i+5)\,bdg)}{(1+(6i+1)\,bdg)\,(1+(6i+4)\,bdg)}\right),\\
y_{12n-1} &= f\prod_{i=0}^{n-1}\left(\frac{(1+(6i+2)\,acf)\,(1+(6i+5)\,acf)}{(1+(6i+1)\,acf)\,(1+(6i+4)\,acf)}\right),\\
y_{12n} &= e\prod_{i=0}^{n-1}\left(\frac{(1+(6i+3)\,bdg)\,(1+(6i+6)\,bdg)}{(1+(6i+2)\,bdg)\,(1+(6i+5)\,bdg)}\right),\\
y_{12n+1} &= \frac{ch}{a(-1+cfh)}\prod_{i=0}^{n-1}\left(\frac{(1+(6i+3)\,acf)\,(1+(6i+6)\,acf)}{(1+(6i+2)\,acf)\,(1+(6i+5)\,acf)}\right),\\
y_{12n+2} &= \frac{be(1+bdg)}{d(-1+beg)}\prod_{i=0}^{n-1}\left(\frac{(1+(6i+4)\,bdg)\,(1+(6i+7)\,bdg)}{(1+(6i+3)\,bdg)\,(1+(6i+6)\,bdg)}\right),\\
y_{12n+3} &= h(1+acf)\prod_{i=0}^{n-1}\left(\frac{(1+(6i+4)\,acf)\,(1+(6i+7)\,acf)}{(1+(6i+3)\,acf)\,(1+(6i+6)\,acf)}\right),\\
y_{12n+4} &= \frac{g(1+2bdg)}{(1+bdg)}\prod_{i=0}^{n-1}\left(\frac{(1+(6i+5)\,bdg)\,(1+(6i+8)\,bdg)}{(1+(6i+4)\,bdg)\,(1+(6i+7)\,bdg)}\right),\\
y_{12n+5} &= \frac{f(1+2acf)}{(1+acf)}\prod_{i=0}^{n-1}\left(\frac{(1+(6i+5)\,acf)\,(1+(6i+8)\,acf)}{(1+(6i+4)\,acf)\,(1+(6i+7)\,acf)}\right),\\
y_{12n+6} &= \frac{e(1+3bdg)}{(1+2bdg)}\prod_{i=0}^{n-1}\left(\frac{(1+(6i+6)\,bdg)\,(1+(6i+9)\,bdg)}{(1+(6i+5)\,bdg)\,(1+(6i+8)\,bdg)}\right),\\
y_{12n+7} &= \frac{ch(1+3acf)}{a(-1+cfh)(1+2acf)}\prod_{i=0}^{n-1}\left(\frac{(1+(6i+6)\,acf)\,(1+(6i+9)\,acf)}{(1+(6i+5)\,acf)\,(1+(6i+8)\,acf)}\right),\\
y_{12n+8} &= \frac{be(1+bdg)(1+4bdg)}{d(-1+beg)(1+3bdg)}\prod_{i=0}^{n-1}\left(\frac{(1+(6i+7)\,bdg)\,(1+(6i+10)\,bdg)}{(1+(6i+6)\,bdg)\,(1+(6i+9)\,bdg)}\right).
\end{aligned}$$

2.3 On the System: $x_{n+1}=\frac{x_{n-3}y_{n-2}}{y_n(-1+x_{n-1}y_{n-2}x_{n-3})},\ y_{n+1}=\frac{x_{n-2}y_{n-3}}{x_n(-1+y_{n-1}x_{n-2}y_{n-3})}$

In this section, we study the solutions of the system of two difference equations

$$x_{n+1}=\frac{x_{n-3}y_{n-2}}{y_n(-1+x_{n-1}y_{n-2}x_{n-3})},\ y_{n+1}=\frac{x_{n-2}y_{n-3}}{x_n(-1+y_{n-1}x_{n-2}y_{n-3})}, \tag{3}$$

with a nonzero real numbers initial conditions with $x_{-3}y_{-2}x_{-1},\ y_{-2}x_{-1}y_0,\ x_{-2}y_{-1}x_0,$ $y_{-3}x_{-2}y_{-1}\neq 1$.

Theorem 3. *Suppose that $\{x_n, y_n\}$ are solutions of system (3). Also, assume that x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1} and y_0 are arbitrary nonzero real numbers with $x_{-3}y_{-2}x_{-1}$, $y_{-2}x_{-1}y_0$, $x_{-2}y_{-1}x_0$, $y_{-3}x_{-2}y_{-1} \neq 1$. Then all solutions of the system are periodic with period twelve and takes the form*

$$\begin{aligned}
x_{12n-3} &= d,\ x_{12n-2} = c,\ x_{12n-1} = b,\ x_{12n} = a, \\
x_{12n+1} &= \tfrac{dg}{e(-1+bdg)},\quad x_{12n+2} = \tfrac{af(-1+cfh)}{h(-1+acf)},\ x_{12n+3} = d(-1+beg), \\
x_{12n+4} &= \tfrac{c}{(-1+cfh)},\quad x_{12n+5} = \tfrac{b}{(-1+beg)},\quad x_{12n+6} = a(-1+cfh), \\
x_{12n+7} &= \tfrac{dg(-1+beg)}{e(-1+bdg)},\ x_{12n+8} = \tfrac{af}{h(-1+acf)}, \\
y_{12n-3} &= h,\ y_{12n-2} = g,\ y_{12n-1} = f,\ y_{12n} = e, \\
y_{12n+1} &= \tfrac{ch}{a(-1+cfh)},\quad y_{12n+2} = \tfrac{be(-1+bdg)}{d(-1+beg)},\ y_{12n+3} = h(-1+acf), \\
y_{12n+4} &= \tfrac{g}{(-1+bdg)},\ y_{12n+5} = \tfrac{f}{(-1+acf)}, \\
y_{12n+6} &= e(-1+bdg),\ y_{12n+7} = \tfrac{ch(-1+acf)}{a(-1+cfh)},\ y_{12n+8} = \tfrac{be}{d(-1+beg)},
\end{aligned}$$

or

$$\begin{aligned}
\{x_n\}_{n=-3}^{\infty} &= \left\{ \begin{array}{c} d, c, b, a, \frac{dg}{e(-1+bdg)}, \frac{af(-1+cfh)}{h(-1+acf)}, d(-1+beg), \frac{c}{(-1+cfh)}, \\ \frac{b}{(-1+beg)}, a(-1+cfh), \frac{dg(-1+beg)}{e(-1+bdg)}, \frac{af}{h(-1+acf)}, d, c, b, a, ... \end{array} \right\}, \\
\{y_n\}_{n=-3}^{\infty} &= \left\{ \begin{array}{c} h, g, f, e, \frac{ch}{a(-1+cfh)}, \frac{be(-1+bdg)}{d(-1+beg)}, h(-1+acf), \frac{g}{(-1+bdg)}, \\ \frac{f}{(-1+acf)}, e(-1+bdg), \frac{ch(-1+acf)}{a(-1+cfh)}, \frac{be}{d(-1+beg)}, h, g, f, e, ... \end{array} \right\}.
\end{aligned}$$

Lemma 1. *All solutions of System (3) are periodic of period six if and only if $x_{-3}y_{-2}x_{-1} = y_{-2}x_{-1}y_0 = x_{-2}y_{-1}x_0 = y_{-3}x_{-2}y_{-1} = 2$ and has the form*

$$\begin{aligned}
\{x_n\} &= \left\{d, c, b, a, \frac{dg}{e}, \frac{af}{h}, d, ...\right\} \\
\{y_n\} &= \left\{h, g, f, e, \frac{ch}{a}, \frac{be}{d}, h, ...\right\}.
\end{aligned}$$

Proof. First suppose that there exists a prime period six solution of System (3) of the form $\{x_n\} = \left\{d, c, b, a, \frac{dg}{e}, \frac{af}{h}, d, ...\right\}$, $\{y_n\} = \left\{h, g, f, e, \frac{ch}{a}, \frac{be}{d}, h, ...\right\}$.

By substituting in the obtained form of the solutions of System (3) in previous Theorem, we get

$$\begin{aligned}
\tfrac{dg}{e} &= \tfrac{dg}{e(-1+bdg)},\ \tfrac{af}{h} = \tfrac{af(-1+cfh)}{h(-1+acf)},\ d = d(-1+beg),\ c = \tfrac{c}{(-1+cfh)}, \\
b &= \tfrac{b}{(-1+beg)},\ a = a(-1+cfh),\quad \tfrac{dg}{e} = \tfrac{dg(-1+beg)}{e(-1+bdg)},\quad \tfrac{af}{h} = \tfrac{af}{h(-1+acf)}, \\
\tfrac{ch}{a} &= \tfrac{ch}{a(-1+cfh)},\ \tfrac{be}{d} = \tfrac{be(-1+bdg)}{d(-1+beg)},\ h = h(-1+acf),\quad g = \tfrac{g}{(-1+bdg)}, \\
f &= \tfrac{f}{(-1+acf)},\ e = e(-1+bdg),\quad \tfrac{ch}{a} = \tfrac{ch(-1+acf)}{a(-1+cfh)},\quad \tfrac{be}{d} = \tfrac{be}{d(-1+beg)},
\end{aligned}$$

Then it is easy to see that $acf = cfh = beg = bdg = 2$.Thus the conditions are satisfied.

Second suppose that $x_{-3}y_{-2}x_{-1} = y_{-2}x_{-1}y_0 = x_{-2}y_{-1}x_0 = y_{-3}x_{-2}y_{-1} = 2$. It follows from the form of solutions of system (3) that

$$\begin{aligned}
x_{12n-3} &= d,\ x_{12n-2} = c,\ x_{12n-1} = b,\ x_{12n} = a,\ x_{12n+1} = \tfrac{dg}{e},\ x_{12n+2} = \tfrac{af}{h},\\
x_{12n+3} &= d,\ x_{12n+4} = c,\ x_{12n+5} = b,\ x_{12n+6} = a,\ x_{12n+7} = \tfrac{dg}{e},\ x_{12n+8} = \tfrac{af}{h},\\
y_{12n-3} &= h,\ y_{12n-2} = g,\ y_{12n-1} = f,\ y_{12n} = e,\ y_{12n+1} = \tfrac{ch}{a},\ y_{12n+2} = \tfrac{be}{d},\\
y_{12n+3} &= h,\ y_{12n+4} = g,\ y_{12n+5} = f,\ y_{12n+6} = e,\ y_{12n+7} = \tfrac{ch}{a},\ y_{12n+8} = \tfrac{be}{d},
\end{aligned}$$

which gives period six solutions and then the proof is completed. □

Lemma 2. *All solutions of the system (3) are periodic of period three if and only if* $x_{-3}y_{-2}x_{-1} = y_{-2}x_{-1}y_0 = x_{-2}y_{-1}x_0 = y_{-3}x_{-2}y_{-1} = 2$, *and* $x_{-3} = x_0,\ x_{-2} = y_{-2},\ x_{-1} = y_{-1}$ *and has the form* $\{x_n\}_{n=-3}^{+\infty} = \{d, c, b, d, c, b, ...\}$ *and* $\{y_n\}_{n=-3}^{+\infty} = \{h, g, f, h, g, f, ...\}$.

Proof. The proof follows from Lemma 1 and so will be omitted. □

2.4 On the System: $x_{n+1} = \frac{x_{n-3}y_{n-2}}{y_n(-1-x_{n-1}y_{n-2}x_{n-3})}$, $y_{n+1} = \frac{x_{n-2}y_{n-3}}{x_n(1-y_{n-1}x_{n-2}y_{n-3})}$

In this section, we study the solutions of the system of two difference equations

$$x_{n+1} = \frac{x_{n-3}y_{n-2}}{y_n(-1 - x_{n-1}y_{n-2}x_{n-3})},\ y_{n+1} = \frac{x_{n-2}y_{n-3}}{x_n(1 - y_{n-1}x_{n-2}y_{n-3})}, \tag{4}$$

with a nonzero real numbers initial conditions and $x_{-3}y_{-2}x_{-1}$, $x_{-2}y_{-1}x_0 \neq -1$, $y_{-2}x_{-1}y_0$, $y_{-3}x_{-2}y_{-1} \neq 1$.

Theorem 4. *Let* $\{x_n, y_n\}$ *be solutions of system (4). Then*

$$\begin{aligned}
x_{6n-3} &= d\prod_{i=0}^{n-1}\left(\tfrac{-1+(3i+1)beg}{-1+(3i)beg}\right), \quad x_{6n-2} = c\prod_{i=0}^{n-1}\left(\tfrac{-1+(3i+2)cfh}{-1+(3i+1)cfh}\right),\\
x_{6n-1} &= b\prod_{i=0}^{n-1}\left(\tfrac{-1+(3i+2)beg}{-1+(3i+1)beg}\right), \quad x_{6n} = a\prod_{i=0}^{n-1}\left(\tfrac{-1+(3i+3)cfh}{-1+(3i+2)cfh}\right),\\
x_{6n+1} &= \tfrac{-dg}{e(1+bdg)}\prod_{i=0}^{n-1}\left(\tfrac{-1+(3i+3)beg}{-1+(3i+2)beg}\right),\\
x_{6n+2} &= \tfrac{af(-1+cfh)}{h(1+acf)}\prod_{i=0}^{n-1}\left(\tfrac{-1+(3i+4)cfh}{-1+(3i+3)cfh}\right),\\
y_{12n-3} &= h\prod_{i=0}^{n-1}\left(\tfrac{(1-(6i)cfh)(1-(6i+3)cfh)}{(1-(6i+2)cfh)(1-(6i+5)cfh)}\right),
\end{aligned}$$

$$\begin{aligned}
y_{12n-2} &= g\prod_{i=0}^{n-1}\left(\frac{(1-(6i)beg)(1-(6i+3)beg)}{(1-(6i+2)beg)(1-(6i+5)beg)}\right),\\
y_{12n-1} &= f\prod_{i=0}^{n-1}\left(\frac{(1-(6i+1)cfh)(1-(6i+4)cfh)}{(1-(6i+3)cfh)(1-(6i+6)cfh)}\right),\\
y_{12n} &= e\prod_{i=0}^{n-1}\left(\frac{(1-(6i+1)beg)(1-(6i+4)beg)}{(1-(6i+3)beg)(1-(6i+6)beg)}\right),\\
y_{12n+1} &= \frac{ch}{a(1-cfh)}\prod_{i=0}^{n-1}\left(\frac{(1-(6i+2)cfh)(1-(6i+5)cfh)}{(1-(6i+4)cfh)(1-(6i+7)cfh)}\right),\\
y_{12n+2} &= \frac{be(1+bdg)}{d(-1+beg)}\prod_{i=0}^{n-1}\left(\frac{(1-(6i+2)beg)(1-(6i+5)beg)}{(1-(6i+4)beg)(1-(6i+7)beg)}\right),
\end{aligned}$$

$$\begin{aligned}
y_{12n+3} &= \frac{h(1+acf)}{(-1+2cfh)}\prod_{i=0}^{n-1}\left(\frac{(1-(6i+3)cfh)(1-(6i+6)cfh)}{(1-(6i+5)cfh)(1-(6i+8)cfh)}\right),\\
y_{12n+4} &= \frac{g}{(1+bdg)(-1+2beg)}\prod_{i=0}^{n-1}\left(\frac{(1-(6i+3)beg)(1-(6i+6)beg)}{(1-(6i+5)beg)(1-(6i+8)beg)}\right),\\
y_{12n+5} &= \frac{f(1-cfh)}{(1+acf)(-1+3cfh)}\prod_{i=0}^{n-1}\left(\frac{(1-(6i+4)cfh)(1-(6i+7)cfh)}{(1-(6i+6)cfh)(1-(6i+9)cfh)}\right),\\
y_{12n+6} &= \frac{e(-1+beg)(1+bdg)}{(1-3beg)}\prod_{i=0}^{n-1}\left(\frac{(1-(6i+4)beg)(1-(6i+7)beg)}{(1-(6i+6)beg)(1-(6i+9)beg)}\right),\\
y_{12n+7} &= \frac{ch(1+acf)(1-2cfh)}{a(-1+cfh)(1-4cfh)}\prod_{i=0}^{n-1}\left(\frac{(1-(6i+5)cfh)(1-(6i+8)cfh)}{(1-(6i+7)cfh)(1-(6i+10)cfh)}\right),\\
y_{12n+8} &= \frac{be(1-2beg)}{d(1-beg)(1-4beg)}\prod_{i=0}^{n-1}\left(\frac{(1-(6i+5)beg)(1-(6i+8)beg)}{(1-(6i+7)beg)(1-(6i+10)beg)}\right).
\end{aligned}$$

2.5 Numerical Examples

For confirming the results of this paper, we consider some numerical examples which represent different types of solutions for the systems (1) - (4).

Example 1. *We consider interesting numerical example for the difference system (1) with the initial conditions* $x_{-3} = .8,\ x_{-2} = -4,\ x_{-1} = 3.5,\ x_0 = 5,\ y_{-3} = 3,\ y_{-2} = -1.9,\ y_{-1} = 6$ *and* $y_0 = 2.6$. *(See Fig. 1).*

Example 2. *See Figure 2, when we take the initial conditions* $x_{-3} = .8,\ x_{-2} = -1.4,\ x_{-1} = 1.1,\ x_0 = .5,\ y_{-3} = 1.9,\ y_{-2} = -2,\ y_{-1} = .26$ *and* $y_0 = -.7$ *for System (3).*

Example 3. *Figure 3 shows the periodicity with period six of System of difference equations (3) with the initial conditions* $x_{-3} = 3,\ x_{-2} = -5,\ x_{-1} = -1/6,\ x_0 = -1/15,\ y_{-3} = -1/15,\ y_{-2} = -4,\ y_{-1} = 6$ *and* $y_0 = 3$.

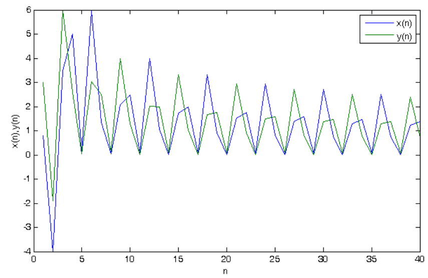

Figure 1

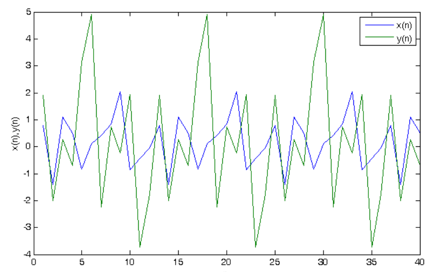

Figure 2

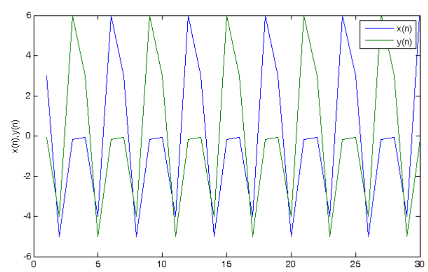

Figure 3

References

[1] A. Asiri, E. M. Elsayed, and M. M. El-Dessoky, On the solutions and periodic nature of some systems of difference equations, *Journal of Computational and Theoretical Nanoscience*, **12 (10)** (2015), 3697-3704.

[2] C. Cinar, I. Yalcinkaya and R. Karatas, On the positive solutions of the difference equation system $x_{n+1} = m/y_n,\ y_{n+1} = py_n/x_{n-1}y_{n-1}$, *J. Inst. Math. Comp. Sci.*, **18** (2005), 135-136.

[3] Q. Din, and E. M. Elsayed, Stability analysis of a discrete ecological model, *Computational Ecology and Software*, **4 (2)** (2014), 89–103.

[4] E. M. Elsayed, On the solutions and periodic nature of some systems of difference equations, *International Journal of Biomathematics*, **7 (6)** (2014), 1450067, (26 pages).

[5] E. M. Elsayed, Behavior and expression of the solutions of some rational difference equations, *Journal of Computational Analysis and Applications*, **15 (1)** (2013), 73-81.

[6] E. M. Elsayed, New method to obtain periodic solutions of period two and three of a rational difference equation, *Nonlinear Dynamics,* **79 (1)** (2015), 241-250.

[7] E. M. Elsayed, Dynamics and Behavior of a Higher Order Rational Difference Equation, *The Journal of Nonlinear Science and Applications*, **9 (4)** (2016), 1463-1474.

[8] E. M. Elsayed and H. El-Metwally, Global behavior and periodicity of some difference equations, *Journal of Computational Analysis and Applications*, **19 (2)** (2015), 298-309.

[9] D. Jana and E. M. Elsayed, Interplay between strong Allee effect, harvesting and hydra effect of a single population discrete-time system, *International Journal of Biomathematics*, **9 (1)** (2016), 1650004, (25 pages).

[10] A. S. Kurbanli, C. Cinar and I. Yalcinkaya, On the behavior of positive solutions of the system of rational difference equations, *Mathematical and Computer Modelling*, **53** (2011), 1261-1267.

[11] N. Touafek and E. M. Elsayed, On the solutions of systems of rational difference equations, *Math. Comput. Mod.*, **55** (2012), 1987–1997.

[12] I. Yalcinkaya, On the global asymptotic behavior of a system of two nonlinear difference equations, *ARS Combinatoria*, **95** (2010), 151-159.

[13] Y. Zhang, X. Yang, G. M. Megson and D. J. Evans, On the system of rational difference equations, *Applied Mathematics and Computation*, **176** (2006), 403–408.

Received: 15.11.2015
Revised: 22.01.2016
Accepted: 25.01.2016

King Abdulaziz University,
Faculty of Science,
Mathematics Department,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.
and
Department of Mathematics,
Faculty of Science,
Mansoura University,
Mansoura 35516, Egypt.
E-mail: `emmelsayed@yahoo.com`