Bull. Math. Soc. Sci. Math. Roumanie
Tome 60 (108) No. 2, 2017, 147–157

Quotient for radial Blaschke-Minkowski homomorphisms

by

Chang-Jian Zhao[(1)], Wing-Sum Cheung[(2)]

Abstract

Some new inequalities for quotient function of quermassintegrals of the radial Blaschke-Minkowski homomorphisms are established. The results in special cases yield some of the recent results on inequalities of this type.

Key Words: quotient function, Radial Blaschke-Minkowski homomorphism, L_p-radial Minkowski addition, L_p-harmonic addition.
2010 Mathematics Subject Classification: Primary 52A20. Secondary 46E30.

1 Introduction

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. The volume of the unit n-ball is denoted by ω_n. We use $V(K)$ for the n-dimensional volume of a body K.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K,\cdot): S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$, by

$$\rho(K,u) = \max\{\lambda \geq 0 : \lambda u \in K\}.$$

If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Let $\mathcal{S}^n$ denote the set of star bodies in $\mathbb{R}^n$. Let $\tilde{\delta}$ denote the radial Hausdorff metric, i.e., if $K, L \in \mathcal{S}^n$, then $\tilde{\delta}(K,L) = |\rho_K - \rho_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

For $K, L \in \mathcal{S}^n$ and $\alpha, \beta \geq 0$, Lutwak [14] defined *the radial Blaschke linear combination*, $\alpha \cdot K \ddot{+} \beta \cdot L$, as the star body whose radial function is given by

$$\rho(\alpha \cdot K \ddot{+} \beta \cdot L, \cdot)^{n-1} = \alpha\rho(K,\cdot)^{n-1} + \beta\rho(L,\cdot)^{n-1}.$$

For $K \in \mathcal{S}^n$, there is a unique star body $\mathbf{I}K$ whose radial function satisfies for $u \in S^{n-1}$,

$$\rho(\mathbf{I}K, u) = v(K \cap E_u),$$

where v is $(n-1)$-dimensional volume and E_u denotes the hyperplane orthogonal to u. It is called the *intersection body* of K. The volume of the intersection body of K is given by

$$V(\mathbf{I}K) = \frac{1}{n}\int_{S^{n-1}} v(K \cap E_u)^n dS(u).$$

The mixed intersection body of $K_1, \ldots, K_{n-1} \in \mathcal{S}^n$, $\mathbf{I}(K_1, \ldots, K_{n-1})$, is defined by

$$\rho(\mathbf{I}(K_1, \ldots, K_{n-1}), u) = \tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u),$$

where $\tilde{v}$ is $(n-1)$-dimensional dual mixed volume (see below for the definition). If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = L$, then $\mathbf{I}(K_1, \ldots, K_{n-1})$ is written as $\mathbf{I}_i(K, L)$. If $L = B$, then $\mathbf{I}_i(K, L)$ is written as $\mathbf{I}_i K$ and called the *ith intersection body* of K. For $\mathbf{I}_0 K$ we simply write $\mathbf{I}K$.

1. Dual mixed volumes

The radial Minkowski linear combination, $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$ is defined by

$$\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r : x_i \in K_i, i = 1, \ldots, r\}, \tag{1.1}$$

for $K_1, \ldots, K_r \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$. It has the following important property (see [14])

$$\rho(\lambda K \tilde{+} \mu L, \cdot) = \lambda \rho(K, \cdot) + \mu \rho(L, \cdot), \tag{1.2}$$

for $K, L \in \mathcal{S}^n$ and $\lambda, \mu \geq 0$. For $K_1, \ldots, K_r \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, the volume of the radial Minkowski linear combination $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$ is a homogeneous polynomial of degree n in the λ_i,

$$V(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r) = \sum \tilde{V}(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}, \tag{1.3}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ whose entries are positive integers not exceeding r. If we require the coefficients of the polynomial in (1.3) to be symmetric in their arguments, then they are uniquely determined. The coefficient $\tilde{V}(K_{i_1}, \ldots, K_{i_n})$ is nonnegative and depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$. It is called the dual mixed volume of $K_{i_1}, \ldots, K_{i_n}$.

If $K_1, \ldots, K_n \in \mathcal{S}^n$, then the dual mixed volume $\tilde{V}(K_1, \ldots, K_n)$ can be represented in the form (see [15])

$$\tilde{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u) dS(u). \tag{1.4}$$

If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, then the dual mixed volume is written as $\tilde{V}_i(K, L)$. If $L = B$, then the dual mixed volume $\tilde{V}_i(K, L) = \tilde{V}_i(K, B)$ is written as $\tilde{W}_i(K)$. For $K, L \in \mathcal{S}^n$, the i-th dual mixed volume of K and L, $\tilde{V}_i(K, L)$, can be extended to all $i \in \mathbb{R}$ by

$$\tilde{V}_i(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} \rho(L, u)^i dS(u). \tag{1.5}$$

Thus, if $K \in \mathcal{S}^n$, then for $i \in \mathbb{R}$

$$\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} dS(u). \tag{1.6}$$

2 Radial Blaschke-Minkowski homomorphisms and L_p-radial addition

Definition 2.1 ([18]) A map $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ is called a radial Blaschke-Minkowski homomorphism, if it satisfies the following conditions:
(a) Ψ is continuous.
(b) For all $K, L \in \mathcal{S}^n$,

$$\Psi(K\ddot{+}L) = \Psi(K)\tilde{+}\Psi(L).$$

(c) For all $K, L \in \mathcal{S}^n$ and every $\vartheta \in SO(n)$,

$$\Psi(\vartheta K) = \vartheta \Psi(K),$$

where $SO(n)$ is the group of rotations in n dimensions.

Radial Blaschke-Minkowski homomorphisms are important examples of star body valued valuations. Their natural duals, Blaschke-Minkowski homomorphisms are an important notion in the theory of convex body valued valuations (see, e.g., [6-7, 10-11, 16, 21, 25] and [1-2, 8-9, 12-13, 22-23]). In 2006, Schuster [18] established the following Brunn-Minkowski inequality for radial Blaschke-Minkowski homomorphisms of star bodies. If K and L are star bodies in $\mathbb{R}^n$, then

$$V(\Psi(K\tilde{+}L))^{1/n(n-1)} \leq V(\Psi K)^{1/n(n-1)} + V(\Psi L)^{1/n(n-1)}, \tag{2.1}$$

with equality if and only if K and L are dilates.

If K and L are star bodies in $\mathbb{R}^n$, $p \neq 0$ and $\lambda, \mu \geq 0$, then $\lambda \cdot K\tilde{+}_p\mu \cdot L$, is the star body whose radial function is given by (see e.g., [5])

$$\rho(\lambda \cdot K\tilde{+}_p\mu \cdot L, \cdot)^p = \lambda\rho(K, \cdot)^p + \mu\rho(L, \cdot)^p. \tag{2.2}$$

The addition $\tilde{+}_p$ is called L_p-*radial addition*. The L_p dual Brunn-Minkowski inequality states: If $K, L \in \mathcal{S}^n$ and $0 < p \leq n$, then

$$V(K\tilde{+}_pL)^{p/n} \leq V(K)^{p/n} + V(L)^{p/n},$$

with equality when $p \neq n$ if and only if K and L are dilates. The inequality is reversed when $p > n$ or $p < 0$ (see [5]).

Very recently, an L_p Brunn-Minkowski inequality for radial Blaschke-Minkowski homomorphisms was established in [24]: If K and L are star bodies in $\mathbb{R}^n$, and $0 < p < n-1$, then

$$V(\Psi(K\tilde{+}_pL))^{p/n(n-1)} \leq V(\Psi K)^{p/n(n-1)} + V(\Psi L)^{p/n(n-1)}, \tag{2.3}$$

with equality if and only if K and L are dilates. Taking $p = 1$, (2.3) reduces to (2.1).

Theorem 2.2 (see [18]) *Let $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ be a radial Blaschke-Minkowski homomorphism. There is a continuous operator $\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$, symmetric in its arguments such that, for $K_1, \ldots, K_m \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_m \geq 0$,*

$$\Psi(\lambda_1 K_1\tilde{+}\cdots\tilde{+}\lambda_m K_m) = \sum_{i_1,\ldots,i_{n-1}} \lambda_{i_1}\cdots\lambda_{i_{n-1}}\Psi(K_{i_1}, \ldots, K_{i_{n-1}}). \tag{2.4}$$

Clearly, Theorem 2.2 generalizes the notion of radial Blaschke-Minkowski homomorphisms. We call $\Psi : \mathcal{S}^n \times \cdots \times \mathcal{S}^n \to \mathcal{S}^n$ mixed radial Blaschke-Minkowski homomorphism induced by Ψ. Mixed radial Blaschke-Minkowski homomorphisms were first studied in more detail in [19-20]. If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = L$, we write $\Psi_i(K, L)$ for $\Psi(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{L, \ldots, L}_{i})$. If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = B$, we write $\Psi_i K$ for $\Psi(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{B, \ldots, B}_{i})$ and call $\Psi_i K$ the mixed Blaschke-Minkowski homomorphism of order i of K.

Lemma 2.3 (see [18]) *A map $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ is a radial Blaschke-Minkowski homomorphism if and only if there is a measure $\mu \in \mathcal{M}_+(S^{n-1}, \hat{e})$ such that*

$$\rho(\Psi K, \cdot) = \rho(K, \cdot)^{n-1} * \mu, \tag{2.5}$$

where $\mathcal{M}_+(S^{n-1}, \hat{e})$ denotes the set of nonnegative zonal measures on S^{n-1}.

For the mixed radial Blaschke-Minkowski homomorphism induced by Ψ, Schuster [18] proved that

$$\rho(\Psi(K_1, \ldots, K_{n-1}), \cdot) = \rho(K_1, \cdot) \cdots \rho(K_{n-1}, \cdot) * \mu.$$

We now define the mixed Blaschke-Minkowski homomorphism of order i of K, for all $i \in \mathbb{R}$, by

$$\rho(\Psi_i K, \cdot) = \rho(K, \cdot)^{n-1-i} * \mu. \tag{2.6}$$

This extended definition will be required in the following.

In 2013, the *quotient function* of the volumes was first introduced in [27]: Let K and D be star bodies in $\mathbb{R}^n$, then the dual quermassintegral quotient function of star bodies K and D, $Q_{\tilde{W}_{i,j}(K,D)}$, defined by

$$Q_{\tilde{W}_{i,j}(K,D)} = \frac{\tilde{W}_i(K)}{\tilde{W}_j(D)}, \quad i, j \in \mathbb{R}.$$

The aim of this paper is to establish the following inequalities for quermassintegral of quotient function of radial Blaschke-Minkowski homomorphisms with respect to L_p-radial addition.

Theorem 2.4 *Let $K, L, D, D' \in \mathcal{S}^n$. If $p \neq 0$, and $i \leq n-1 \leq j \leq n$, then*

$$\left(\frac{\tilde{W}_i(\Psi_{n-1-p}(K \tilde{+}_p L))}{\tilde{W}_j(\Psi_{n-1-p}(D \tilde{+}_p D'))}\right)^{1/(j-i)} \leq \left(\frac{\tilde{W}_i(\Psi_{n-1-p} K)}{\tilde{W}_j(\Psi_{n-1-p} D)}\right)^{1/(j-i)} + \left(\frac{\tilde{W}_i(\Psi_{n-1-p} L)}{\tilde{W}_j(\Psi_{n-1-p} D')}\right)^{1/(j-i)}, \tag{2.7}$$

with equality if and only if $\Psi_{n-1+p} K$ and $\Psi_{n-1+p} L$ are dilates, and $\Psi_{n-1+p} D$ and $\Psi_{n-1+p} D'$ are dilates, and

$$\left(\tilde{W}_i(\Psi_{n-1-p} K)^{1/(n-i)}, \tilde{W}_i(\Psi_{n-1-p} L)^{1/(n-i)}\right) = \mu \left(\tilde{W}_j(\Psi_{n-1-p} D)^{1/(n-j)}, \tilde{W}_j(\Psi_{n-1-p} D')^{1/(n-j)}\right),$$

for some constant.

Remark 2.4 Putting $D = K$ and $D' = L$ in (2.7), (2.7) becomes an inequality established in [28]. Let $D = K$ and $D' = L$, and putting $j = n$ in (2.7), (2.7) becomes the following inequality: If $K, L \in \mathcal{S}^n$, $p \neq 0$, and $i \leq n-1$, then

$$\tilde{W}_i(\Psi_{n-1-p}(K\tilde{+}_pL))^{1/(n-i)} \leq \tilde{W}_i(\Psi_{n-1-p}K)^{1/(n-i)} + \tilde{W}_i(\Psi_{n-1-p}L)^{1/(n-i)},$$

with equality if and only if $\Psi_{n-1-p}K$ *and* $\Psi_{n-1-p}L$ are dilates. Taking $p = n-1$ in (2.7), (2.7) reduces to the following inequality: If $K, L, D, D' \in \mathcal{S}^n$ and $i \leq n-1 \leq j \leq n$, then

$$\left(\frac{\tilde{W}_i(\Psi(K\tilde{+}_{n-1}L))}{\tilde{W}_j(\Psi(D\tilde{+}_{n-1}D'))}\right)^{1/(j-i)} \leq \left(\frac{\tilde{W}_i(\Psi K)}{\tilde{W}_j(\Psi D)}\right)^{1/(j-i)} + \left(\frac{\tilde{W}_i(\Psi L)}{\tilde{W}_j(\Psi D')}\right)^{1/(j-i)}, \tag{2.8}$$

with equality if and only if ΨK and ΨL are dilates, and ΨD and $\Psi D'$ are dilates, and

$$\left(\tilde{W}_i(\Psi K)^{1/(n-i)}, \tilde{W}_i(\Psi L)^{1/(n-i)}\right) = \mu\left(\tilde{W}_j(\Psi D)^{1/(n-j)}, \tilde{W}_j(\Psi D')^{1/(n-j)}\right),$$

for some constant.

Taking $j = n$ in (2.8), (2.8) reduces to the following inequality: If $K, L \in \mathcal{S}^n$ and $i \leq n-1$, then

$$\tilde{W}_i(\Psi(K\tilde{+}_{n-1}L))^{1/(n-i)} \leq \tilde{W}_i(\Psi K)^{1/(n-i)} + \tilde{W}_i(\Psi L)^{1/(n-i)},$$

with equality if and only if ΨK and ΨL are dilates.

3 Radial Blaschke-Minkowski homomorphisms and L_p-harmonic addition

If $K, L \in \mathcal{S}^n$, and $\lambda, \mu \geq 0$ (not both zero), then for $p \geq 1$, the *L_p-harmonic combination,* $\lambda\Diamond K\hat{+}_p\mu\Diamond L \in \mathcal{S}^n$ was defined by

$$\rho(\lambda\Diamond K\hat{+}_p\mu\Diamond L, \cdot)^{-p} = \lambda\rho(K,u)^{-p} + \mu\rho(L,u)^{-p}. \tag{3.1}$$

In 1996, Lutwak [17] established an L_p-Brunn-Minkowski inequality for harmonic addition. If $K, L \in \mathcal{S}^n$ and $p \geq 1$, then

$$V(K\hat{+}_pL)^{-p/n} \geq V(K)^{-p/n} + V(L)^{-p/n}, \tag{3.2}$$

with equality if and only if K and L are dilates.

Another aim of this paper is to establish the following Dresher type inequality for radial Blaschke-Minkowski homomorphisms with respect to L_p-harmonic addition.

Theorem 3.1 *Let $K, L, D, D' \in \mathcal{S}^n$. If $p \geq 1$, and $i \leq n-1 \leq j \leq n$, then*

$$\left(\frac{\tilde{W}_i(\Psi_{n-1+p}(K\hat{+}_pL))}{\tilde{W}_j(\Psi_{n-1+p}(D\hat{+}_pD'))}\right)^{1/(j-i)} \leq \left(\frac{\tilde{W}_i(\Psi_{n-1+p}K)}{\tilde{W}_j(\Psi_{n-1+p}D)}\right)^{1/(j-i)} + \left(\frac{\tilde{W}_i(\Psi_{n-1+p}L)}{\tilde{W}_j(\Psi_{n-1+p}D')}\right)^{1/(j-i)}, \tag{3.3}$$

with equality if and only if $\Psi_{n-1+p}K$ and $\Psi_{n-1+p}L$ are dilates, and $\Psi_{n-1+p}D$ and $\Psi_{n-1+p}D'$ are dilates, and

$$\left(\tilde{W}_i(\Psi_{n-1+p}K)^{1/(n-i)}, \tilde{W}_i(\Psi_{n-1+p}L)^{1/(n-i)}\right) = \mu\left(\tilde{W}_j(\Psi_{n-1+p}D)^{1/(n-j)}, \tilde{W}_j(\Psi_{n-1+p}D')^{1/(n-j)}\right),$$

for some constant.

Remark 3.2 Putting $D = K$ and $D' = L$ in (3.3), (3.3) becomes an inequality established in [28]. Let $D = K$ and $D' = L$, and putting $j = n$ in (3.3), (3.3) becomes the following inequality: If $K, L \in \mathcal{S}^n$, $p \geq 1$, and $i \leq n-1$, then

$$\tilde{W}_i(\Psi_{n-1+p}(K \hat{+}_p L))^{1/(n-i)} \leq \tilde{W}_i(\Psi_{n-1+p}K)^{1/(n-i)} + \tilde{W}_i(\Psi_{n-1+p}L)^{1/(n-i)},$$

with equality if and only if $\Psi_{n-1+p}K$ and $\Psi_{n-1+p}L$ are dilates. Taking $i = 0$ in (3.3), (3.3) becomes the following inequality: If $K, L, D, D' \in \mathcal{S}^n$ and $n-1 \leq j \leq n$, then

$$\left(\frac{V(\Psi_{n-1+p}(K \hat{+}_p L))}{\tilde{W}_j(\Psi_{n-1+p}(D \hat{+}_p D'))}\right)^{1/j} \leq \left(\frac{V(\Psi_{n-1+p}K)}{\tilde{W}_j(\Psi_{n-1+p}D)}\right)^{1/j} + \left(\frac{V(\Psi_{n-1+p}L)}{\tilde{W}_j(\Psi_{n-1+p}D')}\right)^{1/j}, \qquad (3.3)$$

with equality if and only if $\Psi_{n-1+p}K$ and $\Psi_{n-1+p}L$ are dilates, and $\Psi_{n-1+p}D$ and $\Psi_{n-1+p}D'$ are dilates, and

$$\left(V(\Psi_{n-1+p}K)^{1/(n-i)}, V(\Psi_{n-1+p}L)^{1/(n-i)}\right) = \mu\left(\tilde{W}_j(\Psi_{n-1+p}D)^{1/(n-j)}, \tilde{W}_j(\Psi_{n-1+p}D')^{1/(n-j)}\right),$$

for some constant.

4. Inequalities for radial Blaschke-Minkowski homomorphisms

An extension of Beckenbach's inequality (see [3], p.27) was obtained by Dresher [4] by means of moment-space techniques: If $p \geq 1 \geq r \geq 0$, $f, g \geq 0$, and ϕ is a distribution function, then

$$\left(\frac{\int (f+g)^p d\phi}{\int (f+g)^r d\phi}\right)^{1/(p-r)} \leq \left(\frac{\int f^p d\phi}{\int f^r d\phi}\right)^{1/(p-r)} + \left(\frac{\int g^p d\phi}{\int g^r d\phi}\right)^{1/(p-r)}.$$

Recently, a new Dresher type inequality was derived in [26] as follows.

Lemma 4.1 *Let E be a bounded measurable subset of $\mathbb{R}^n$, let ϕ be a distribution function and let $f_1, f_2, g_1, g_2 : E \to \mathbb{R}^+$. If $p \geq 1 \geq r \geq 0$, then*

$$\left(\frac{\int_E (f_1+f_2)^p d\phi}{\int_E (g_1+g_2)^r d\phi}\right)^{\frac{1}{p-r}} \leq \left(\frac{\int_E f_1^p d\phi}{\int_E g_1^r d\phi}\right)^{\frac{1}{p-r}} + \left(\frac{\int_E f_2^p d\phi}{\int_E g_2^r d\phi}\right)^{\frac{1}{p-r}} \qquad (4.1)$$

with equality if and only if $f_1 = k_1 f_2$, $g_1 = k_2 g_2$ and $(\|f_1\|_p, \|f_2\|_p) = \mu(\|g_1\|_r, \|g_2\|_r)$ where k_1, k_2, μ are constants.

We prove now Theorem 3.1. The following statement is just a slight reformulation of it:

Theorem 4.2 *Let $K, L, D, D' \in \mathcal{S}^n$. If $p \geq 1$, and $s, t \in \mathbb{R}$ satisfy $s \geq 1 \geq t \geq 0$, then*

$$\left(\frac{\tilde{W}_{n-s}(\Psi_{n-1+p}(K \hat{+}_p L))}{\tilde{W}_{n-t}(\Psi_{n-1+p}(D \tilde{+}_p D'))}\right)^{1/(s-t)} \leq \left(\frac{\tilde{W}_{n-s}(\Psi_{n-1+p}K)}{\tilde{W}_{n-t}(\Psi_{n-1+p}D)}\right)^{1/(s-t)} + \left(\frac{\tilde{W}_{n-s}(\Psi_{n-1+p}L)}{\tilde{W}_{n-t}(\Psi_{n-1+p}D')}\right)^{1/(s-t)}, \qquad (4.2)$$

with equality if and only if $\Psi_{n-1+p}K$ and $\Psi_{n-1+p}L$ are dilates, and $\Psi_{n-1+p}D$ and $\Psi_{n-1+p}D'$ are dilates, and

$$\left(\tilde{W}_{n-s}(\Psi_{n-1+p}K)^{1/s}, \tilde{W}_{n-s}(\Psi_{n-1+p}L)^{1/s}\right) = \mu\left(\tilde{W}_{n-t}(\Psi_{n-1+p}D)^{1/t}, \tilde{W}_{n-t}(\Psi_{n-1+p}D')^{1/t}\right).$$

Proof From (3.1), we have for $p \geq 1$

$$\rho(K\hat{+}_pL,\cdot)^{-p} * \mu = \rho(K,\cdot)^{-p} * \mu + \rho(L,\cdot)^{-p} * \mu,$$

where μ is the generating measure of Ψ from Lemma 2.3. Hence, from (2.6), we obtain

$$\rho(\Psi_{n-1+p}(K\hat{+}_pL),\cdot) = \rho(\Psi_{n-1+p}K,\cdot) + \rho(\Psi_{n-1+p}L,\cdot).$$

Therefore, by (1.6), we have

$$\tilde{W}_{n-s}(\Psi_{n-1+p}(K\hat{+}_pL)) = \frac{1}{n}\int_{S^{n-1}} (\rho(\Psi_{n-1+p}K,u) + \rho(\Psi_{n-1+p}L,u))^s \, dS(u) \tag{4.3}$$

and

$$\tilde{W}_{n-t}(\Psi_{n-1+p}(D\hat{+}_pD')) = \frac{1}{n}\int_{S^{n-1}} (\rho(\Psi_{n-1+p}D,u) + \rho(\Psi_{n-1+p}D',u))^t \, dS(u). \tag{4.4}$$

From (4.3), (4.4) and Lemma 4.1, we obtain

$$\begin{aligned}\left(\frac{\tilde{W}_{n-s}(\Psi_{n-1+p}(K\hat{+}_pL))}{\tilde{W}_{n-t}(\Psi_{n-1+p}(D\tilde{+}_pD'))}\right)^{1/(s-t)} &= \left(\frac{\int_{S^{n-1}} (\rho(\Psi_{n-1+p}K,u) + \rho(\Psi_{n-1+p}L,u))^s \, dS(u)}{\int_{S^{n-1}} (\rho(\Psi_{n-1+p}D,u) + \rho(\Psi_{n-1+p}D',u))^t \, dS(u)}\right)^{1/(s-t)} \\ &\leq \left(\frac{\int_{S^{n-1}} \rho(\Psi_{n-1+p}K,u)^s dS(u)}{\int_{S^{n-1}} \rho(\Psi_{n-1+p}D,u)^t dS(u)}\right)^{1/(s-t)} + \left(\frac{\int_{S^{n-1}} \rho(\Psi_{n-1+p}L,u)^s dS(u)}{\int_{S^{n-1}} \rho(\Psi_{n-1+p}D',u)^t dS(u)}\right)^{1/(s-t)} \\ &= \left(\frac{\tilde{W}_{n-s}(\Psi_{n-1+p}K)}{\tilde{W}_{n-t}(\Psi_{n-1+p}D)}\right)^{1/(s-t)} + \left(\frac{\tilde{W}_{n-s}(\Psi_{n-1+p}L)}{\tilde{W}_{n-t}(\Psi_{n-1+p}D')}\right)^{1/(s-t)}.\end{aligned}$$

From the equality condition of Lemma 4.1, equality in (4.2) holds if and only if the functions $\rho(\Psi_{n-1+p}K,u)$ and $\rho(\Psi_{n-1+p}L,u)$ are proportional, and $\rho(\Psi_{n-1+p}D,u)$ and $\rho(\Psi_{n-1+p}D',u)$ are proportional, and

$$\left(\tilde{W}_{n-s}(\Psi_{n-1+p}K)^{1/s}, \tilde{W}_{n-s}(\Psi_{n-1+p}L)^{1/s}\right) = \mu\left(\tilde{W}_{n-t}(\Psi_{n-1+p}D)^{1/t}, \tilde{W}_{n-t}(\Psi_{n-1+p}D')^{1/t}\right).$$

Taking $s = n-i$ and $t = n-j$ in Theorem 4.2, Theorem 4.2 becomes Theorem 3.1 stated in Section 3.

If $\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ is the mixed intersection operator $\mathbf{I} : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ in (4.2) and $n-s = i$ and $n-t = j$, we obtain the following result: If $K, L, D, D' \in \mathcal{S}^n$, $p \geq 1$ and $i \leq n-1 \leq j \leq n$, then

$$\left(\frac{\tilde{W}_i(\mathbf{I}_{n-1+p}(K\hat{+}_pL))}{\tilde{W}_j(\mathbf{I}_{n-1+p}(D\tilde{+}_pD'))}\right)^{1/(j-i)} \leq \left(\frac{\tilde{W}_i(\mathbf{I}_{n-1+p}K)}{\tilde{W}_j(\mathbf{I}_{n-1+p}D)}\right)^{1/(j-i)} + \left(\frac{\tilde{W}_i(\mathbf{I}_{n-1+p}L)}{\tilde{W}_j(\mathbf{I}_{n-1+p}D')}\right)^{1/(j-i)}, \tag{4.5}$$

with equality if and only if $\mathbf{I}_{n-1+p}K$ and $\mathbf{I}_{n-1+p}L$ are dilates, $\mathbf{I}_{n-1+p}D$ and $\mathbf{I}_{n-1+p}D'$ are dilates, and

$$\left(\tilde{W}_i(\mathbf{I}_{n-1+p}K)^{1/(n-i)}, \tilde{W}_i(\mathbf{I}_{n-1+p}L)^{1/(n-i)}\right) = \mu\left(\tilde{W}_j(\mathbf{I}_{n-1+p}D)^{1/(n-j)}, \tilde{W}_j(\mathbf{I}_{n-1+p}D')^{1/(n-j)}\right).$$

Taking $j = n$ in (4.5) and noting that $\tilde{W}_n(K) = \int_{S^{n-1}} dS(u) = n\omega_n$, (4.5) becomes the following inequality: If $K, L \in \mathcal{S}^n$, $p \geq 1$, and $i \leq n-1$, then

$$\tilde{W}_i(\mathbf{I}_{n-1+p}(K\hat{+}_pL))^{1/(n-i)} \leq \tilde{W}_i(\mathbf{I}_{n-1+p}K)^{1/(n-i)} + \tilde{W}_i(\mathbf{I}_{n-1+p}L)^{1/(n-i)},$$

with equality if and only if $\mathbf{I}_{n-1+p}K$ and $\mathbf{I}_{n-1+p}L$ are dilates.

Theorem 4.3 *Let $K, L, D, D' \in \mathcal{S}^n$. If $p \neq 0$, and $s, t \in \mathbb{R}$ satisfy $s \geq 1 \geq t \geq 0$, then*

$$\left(\frac{\tilde{W}_{n-s}(\Psi_{n-1-p}(K\tilde{+}_pL))}{\tilde{W}_{n-t}(\Psi_{n-1-p}(D\tilde{+}_pD'))}\right)^{1/(s-t)} \leq \left(\frac{\tilde{W}_{n-s}(\Psi_{n-1-p}K)}{\tilde{W}_{n-t}(\Psi_{n-1-p}D)}\right)^{1/(s-t)} + \left(\frac{\tilde{W}_{n-s}(\Psi_{n-1-p}L)}{\tilde{W}_{n-t}(\Psi_{n-1-p}D')}\right)^{1/(s-t)}, \tag{4.6}$$

with equality if and only if $\Psi_{n-1+p}K$ and $\Psi_{n-1+p}L$ are dilates, and $\Psi_{n-1+p}D$ and $\Psi_{n-1+p}D'$ are dilates, and

$$\left(\tilde{W}_{n-s}(\Psi_{n-1+p}K)^{1/s}, \tilde{W}_{n-s}(\Psi_{n-1+p}L)^{1/s}\right) = \mu\left(\tilde{W}_{n-t}(\Psi_{n-1+p}D)^{1/t}, \tilde{W}_{n-t}(\Psi_{n-1+p}D')^{1/t}\right).$$

Proof From (2.2), we have for $p \neq 0$

$$\rho(K\tilde{+}_pL, \cdot)^p * \mu = \rho(K, \cdot)^p * \mu + \rho(L, \cdot)^p * \mu.$$

Hence, from (2.6), we obtain

$$\rho(\Psi_{n-1-p}(K\tilde{+}_pL), \cdot) = \rho(\Psi_{n-1-p}K, \cdot) + \rho(\Psi_{n-1-p}L, \cdot).$$

By (1.6), we have

$$\tilde{W}_{n-s}(\Psi_{n-1-p}(K\tilde{+}_pL)) = \frac{1}{n}\int_{S^{n-1}} \left(\rho(\Psi_{n-1-p}K, u) + \rho(\Psi_{n-1-p}L, u)\right)^s dS(u) \tag{4.7}$$

and

$$\tilde{W}_{n-t}(\Psi_{n-1-p}(D\hat{+}_pD')) = \frac{1}{n}\int_{S^{n-1}} \left(\rho(\Psi_{n-1-p}D, u) + \rho(\Psi_{n-1-p}D', u)\right)^t dS(u). \tag{4.8}$$

From (4.7), (4.8) and Lemma 4.1, we obtain

$$\left(\frac{\tilde{W}_{n-s}(\Psi_{n-1-p}(K\tilde{+}_pL))}{\tilde{W}_{n-t}(\Psi_{n-1-p}(D\tilde{+}_pD'))}\right)^{1/(s-t)} = \left(\frac{\int_{S^{n-1}} \left(\rho(\Psi_{n-1-p}K, u) + \rho(\Psi_{n-1-p}L, u)\right)^s dS(u)}{\int_{S^{n-1}} \left(\rho(\Psi_{n-1-p}D, u) + \rho(\Psi_{n-1-p}D', u)\right)^t dS(u)}\right)^{1/(s-t)}$$

$$\leq \left(\frac{\int_{S^{n-1}} \rho(\Psi_{n-1-p}K, u)^s dS(u)}{\int_{S^{n-1}} \rho(\Psi_{n-1-p}D, u)^t dS(u)}\right)^{1/(s-t)} + \left(\frac{\int_{S^{n-1}} \rho(\Psi_{n-1-p}L, u)^s dS(u)}{\int_{S^{n-1}} \rho(\Psi_{n-1-p}D'L, u)^t dS(u)}\right)^{1/(s-t)}$$

$$= \left(\frac{\tilde{W}_{n-s}(\Psi_{n-1-p}K)}{\tilde{W}_{n-t}(\Psi_{n-1-p}D)} \right)^{1/(s-t)} + \left(\frac{\tilde{W}_{n-s}(\Psi_{n-1-p}L)}{\tilde{W}_{n-t}(\Psi_{n-1-p}D')} \right)^{1/(s-t)}.$$

From the equality condition of Lemma 4.1, equality in (4.6) holds if and only if the functions $\rho(\Psi_{n-1+p}K, u)$ and $\rho(\Psi_{n-1+p}L, u)$ are proportional, and $\rho(\Psi_{n-1+p}D, u)$ and $\rho(\Psi_{n-1+p}D', u)$ are proportional, and

$$\left(\tilde{W}_{n-s}(\Psi_{n-1+p}K)^{1/s}, \tilde{W}_{n-s}(\Psi_{n-1+p}L)^{1/s} \right) = \mu \left(\tilde{W}_{n-t}(\Psi_{n-1+p}D)^{1/t}, \tilde{W}_{n-t}(\Psi_{n-1+p}D')^{1/t} \right).$$

Taking $s = n-i$ and $t = n-j$ in Theorem 4.3, Theorem 4.3 becomes Theorem 2.4 stated in Section 2.

If $\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ is the mixed intersection operator $\mathbf{I} : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ in (4.6) and $i = n-s$ and $j = n-t$, we obtain the following result: If $K, L, D, D' \in \mathcal{S}^n$, $p \neq 0$ and $i \leq n-1 \leq j \leq n$, then

$$\left(\frac{\tilde{W}_i(\mathbf{I}_{n-1-p}(K \tilde{+}_p L))}{\tilde{W}_j(\mathbf{I}_{n-1-p}(D \tilde{+}_p D'))} \right)^{1/(j-i)} \leq \left(\frac{\tilde{W}_i(\mathbf{I}_{n-1-p}K)}{\tilde{W}_j(\mathbf{I}_{n-1-p}D)} \right)^{1/(j-i)} + \left(\frac{\tilde{W}_i(\mathbf{I}_{n-1-p}L)}{\tilde{W}_j(\mathbf{I}_{n-1-p}D')} \right)^{1/(j-i)}, \tag{4.9}$$

with equality if and only if $\mathbf{I}_{n-1-p}K$ and $\mathbf{I}_{n-1-p}L$ are dilates, and $\mathbf{I}_{n-1-p}D$ and $\mathbf{I}_{n-1-p}D'$ are dilates, and

$$\left(\tilde{W}_{n-s}(\mathbf{I}_{n-1-p}K)^{1/s}, \tilde{W}_{n-s}(\mathbf{I}_{n-1-p}L)^{1/s} \right) = \mu \left(\tilde{W}_{n-t}(\mathbf{I}_{n-1-p}D)^{1/t}, \tilde{W}_{n-t}(\mathbf{I}_{n-1-p}D')^{1/t} \right).$$

Taking $j = n$ in (4.9) and noting that $\tilde{W}_n(K) = \int_{S^{n-1}} dS(u) = n\omega_n$, (4.9) becomes the following inequality: If $K, L \in \mathcal{S}^n$, $p \neq 0$, and $i \leq n-1$, then

$$\tilde{W}_i(\mathbf{I}_{n-1-p}(K \tilde{+}_p L))^{1/(n-i)} \leq \tilde{W}_i(\mathbf{I}_{n-1-p}K)^{1/(n-i)} + \tilde{W}_i(\mathbf{I}_{n-1-p}L)^{1/(n-i)}, \tag{4.10}$$

with equality if and only if $\mathbf{I}_{n-1-p}K$ and $\mathbf{I}_{n-1-p}L$ are dilates.

Taking $p = n-1$ in (4.10), (4.10) reduces to the following inequality: If $K, L \in \mathcal{S}^n$ and $i \leq n-1$, then

$$\tilde{W}_i(\mathbf{I}(K \tilde{+}_{n-1} L))^{1/(n-i)} \leq \tilde{W}_i(\mathbf{I}K)^{1/(n-i)} + \tilde{W}_i(\mathbf{I}L)^{1/(n-i)},$$

with equality if and only if $\mathbf{I}K$ and $\mathbf{I}L$ are dilates. Taking $p = n-1$ in (4.9), (4.9) reduces to the following inequality: If $K, L, D, D' \in \mathcal{S}^n$ and $i \leq n-1 \leq j \leq n$, then

$$\left(\frac{\tilde{W}_i(\mathbf{I}(K \tilde{+}_{n-1} L))}{\tilde{W}_j(\mathbf{I}(D \tilde{+}_{n-1} D'))} \right)^{1/(j-i)} \leq \left(\frac{\tilde{W}_i(\mathbf{I}K)}{\tilde{W}_j(\mathbf{I}D)} \right)^{1/(j-i)} + \left(\frac{\tilde{W}_i(\Psi L)}{\tilde{W}_j(\Psi D')} \right)^{1/(j-i)},$$

with equality if and only if $\mathbf{I}K$ and $\mathbf{I}L$ are dilates, and $\mathbf{I}D$ and $\mathbf{I}D'$ are dilates, and

$$\left(\tilde{W}_{n-s}(\mathbf{I}K)^{1/s}, \tilde{W}_{n-s}(\mathbf{I}L)^{1/s} \right) = \mu \left(\tilde{W}_{n-t}(\mathbf{I}D)^{1/t}, \tilde{W}_{n-t}(\mathbf{I}D')^{1/t} \right).$$

Acknowledgement. The first author expresses his grateful thanks to Prof. Gangsong Leng for his many suggestions.
This research was supported by the National Natural Science Foundation of China (11371334) and by HKU Seed Grant for Basic Research.

References

[1] J. Abardia, A. Bernig, Projection bodies in complex vector spaces, *Adv. Math.*, **227**, 830-846 (2011).

[2] S. Alesker, A. Bernig, F. E. Schuster, Harmonic Analysis of translation invariant valuations, *Geom. Funct. Anal.,* **21**, 751-773 (2011).

[3] E. F. Beckenbach, R. Bellman, *Inequalities*, Springer-Verlag, Berlin-Göttingen, Heidelberg (1961).

[4] M. Dresher, Moment space and inequalities, *Duke. Math. J.*, **20**, 261-271 (1953).

[5] R. J. Gardner, The Brunn-Minkowski inequality, *Bull. Amer. Math. Soc.,* **39**, 355-405 (2002).

[6] C. Haberl, Star body valued valuations, *Indiana Univ. Math. J.*, **58**, 2253-2276 (2009).

[7] C. Haberl, F. E. Schuster, General L_p affine isoperimetric inequalities, *J. Diff. Geom.*, **83**, 1-26 (2009).

[8] C. Haberl, L_p intersection bodies, *Adv. Math.,* **217**, 2599-2624 (2008).

[9] C. Haberl, M. Ludwig, A characterization of L_p intersection bodies, *Int. Math. Res. Not.,* Art ID 10548, 29 pages.

[10] M. Kiderlen, Blaschke- and Minkowski-Endomorphisms of convex bodies, *Trans. Amer. Math. Soc.,* **358**, 5539-5564 (2006).

[11] M. Ludwig, Minkowski valuations, *Trans. Amer. Math. Soc.,* **357**(10), 4191-4213 (2005).

[12] M. Ludwig, General affine surface areas, *Adv. Math.,* **224**, 2346-2360 (2010).

[13] M. Ludwig, Minkowski areas and valuations, *J. Diff. Geom.,* **86**, 133-161 (2010).

[14] E. Lutwak, Intersection bodies and dual mixed volumes, *Adv. Math.*, **71**, 232-261 (1988).

[15] E. Lutwak, Dual mixed volumes, *Pacific J. Math.*, **58**, 531-538 (1975).

[16] E. Lutwak, Mixed projection inequalities, *Trans. Amer. Math. Soc.,* **287**(1), 91-105 (1985).

[17] E. Lutwak, The Brunn-Minkowski-Firey theory II. Affine and geominimal surface areas. *Adv. Math.,* **118**, 244-294 (1996).

[18] F. E. Schuster, Volume inequalities and additive maps of star bodies, *Mathematika*, **53**, 211-234 (2006).

[19] F. E. Schuster, Valuations and Busemann-Petty type problems, *Adv. Math.*, **219**, 344- 368 (2008).

[20] F. E. Schuster, Convolutions and multiplier transformations of star bodies, *Trans. Amer. Math. Soc.*, **359**, 5567-5591 (2007).

[21] F. E. Schuster, Crofton measures and Minkowski valuations, *Duke Math. J.,* **154**, 1-30 (2010).

[22] F. E. Schuster, T. Wannerer, $GL(n)$ contravariant Minkowski valuations, *Trans. Amer. Math. Soc.,* **364**, 815-826 (2012).

[23] T. Wannerer, $GL(n)$ equivariant Minkowski valuations, *Indiana Univ. Math. J.,* **60**, 1655-1672 (2011).

[24] W. Wang, L_p Brunn-Minkowski type inequalities for Blaschke-Minkowski homomorphisms, *Geom. Dedicata*, **164**, 273-285 (2013).

[25] C. J. Zhao, On Blaschke-Minkowski homomorphisms, *Geom. Dedicata*, **149**, 373-378 (2010).

[26] C. J. Zhao, Quotient functions of dual quermassintegrals, *Quaestions Math.*, **38**(2), 271-283 (2015).

[27] C. J. Zhao, On Blaschke-Minkowski homomorphisms, *Indagationes Math.*, **24**, 57-67 (2013).

[28] C. J. Zhao, W. S. Cheung, Dresher type inequalities for radial Blaschke-Minkowski homomorphisms, *Moscow J. Math.*, 2017, in press.

Received: 06.01.2016
Accepted: 29.05.2016

(1) Department of Mathematics
China Jiliang University
Hangzhou 310018, P. R. China
E-mail: `chjzhao@163.com`
`chjzhao@aliyun.com`

(2) Department of Mathematics
The University of Hong Kong
Pokfulam Road, Hong Kong
E-mail: `wscheung@hku.hk`