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A new efficient and optimal sixteenth-order scheme for simple roots of nonlinear equations

by

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Abstract

In this manuscript, we propose a new highly efficient and optimal scheme of order sixteen for obtaining simple roots of nonlinear equations. The derivation of this scheme is based on the rational approximation approach. The proposed scheme requires four evaluations of the involved function and one evaluation of its first-order derivative, being optimally consistent with the conjecture of Kung-Traub. In addition, we fully investigated theoretical and computational properties of the proposed scheme along with a main theorem describing the order of convergence. Moreover, we find from the numerical experiments that our proposed methods perform better than the existing optimal sixteenth-order methods when we checked the performance in multi precision digits, on a variety of nonlinear equations.

Keywords: Nonlinear equations, Iterative methods, Order of convergence, Basin of attraction.
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1 Introduction

Multi-point iterative belong to the class of most powerful methods that overcome from the theoretical limitations of one-point iterative methods regarding their order of convergence and efficiency (for the details please see [19]). The main advantage of these type of methods is that they only require the value/values of first-order derivative/derivatives of the involved function. This topic become the point of attention of many scholars from the worldwide, when Traub [19] presented the qualitative as well as the quantitative analysis of one-point and multi-point iterative methods. Very recently, Petković et al. [15] also presented the extensive study of the multi-point iterative methods.

In the past, Russian mathematician A.M. Ostrowski was the first person who introduced an optimal fourth-order multi-point method which requires two evaluations of the involved function and one of its firs-order derivative. In addition, other researchers like Jarratt [8, 9] and King [10], had proposed several new optimal fourth-order multi-point methods in 1966 and 1975, respectively. Moreover, King also showed that Ostrowski's method was a particular case of his proposed family.

With the advancement of digital computer and symbolic computation, a large number of optimal eighth-order methods have been proposed by various scholars in [1, 2, 4, 5, 12, 16, 17, 18, 20], within the last two decade. Most of them are the extension of Newton's method or Newton-like method at the expense of additional functional evaluations or increase the sub step of the original methods.

Further, it is often desirable to obtain higher-order and more accurate root-finding techniques for obtaining the roots of nonlinear equations. In 1974, Kung and Traub [11], proposed two general classes of n-point iterative methods with first-order derivative/derivatives of the involved function and without any derivative. Moreover, they also given a remarkable conjecture regarding order of convergence which state that a multi-point iterative method with n-functional evaluations (total evaluations of the involved functions and its derivatives) can have maximum order of convergence 2^{n-1}. Any method which satisfy this conjecture is known as optimal method.

In the past, Neta [14], proposed an optimal sixteenth-order family of multi-point iterative methods. However, Neta did not present an explicit form of the error equation and more recently it was given by Geum and Kim [6]. In the recent years, researchers as Geum and Kim [6, 7], Sharma et al. [16], Ullah et al. [13], have also presented optimal sixteenth-order extension of iterative methods. Nowadays, obtaining new four-step optimal methods of order sixteen is very interesting and challenging task in the field of numerical analysis. On of the reason behind the attention of sixteenth-order iterative methods is the efficiency indices of these methods $E = \sqrt[5]{16} \approx 1.741$, which is far better than the classical Newton's method $E = \sqrt[2]{2} \approx 1.414$.

The principle aim of this manuscript is to propose a more accurate and efficient solution technique of order sixteen as compared to the existing ones. According to the Kung-Traub conjecture, our proposed scheme is optimal. The beauty of the proposed scheme is that we can develop several new optimal methods of order sixteen by considering different types of weight functions The efficiency of the proposed methods is tested on a variety of numerical examples which is a mixture of polynomial, trigonometry, inverse trigonometry, logarithmic and exponential functions. From the numerical experiment, it is observed that our proposed methods perform better than existing optimal methods of order sixteen.

The outline of the paper is as follows. In Section 2, we proposed a new four-point scheme and also demonstrate the convergence analysis which confirm the sixteenth-order convergence of the proposed scheme. Some stability properties are deduced from the basins of attraction of proposed and some known methods on quadratic polynomials and other nonlinear functions in Section 3. Moreover, Section 4 is devoted to the are performance of some numerical experiments where the proposed methods are compared with existing methods of the same order. Section 5 contains the concluding remarks.

2 Design and convergence of an optimal sixteenth-order scheme

In this section, we will propose an optimal sixteenth-order family of iterative methods. Therefore, we rewrite the eighth-order scheme proposed by Artidiello et al. [5], in the following way

$$\begin{aligned} z_n &= w_n - \frac{f(w_n)}{f'(x_n)} P(h), \\ k_n &= z_n - \frac{f(z_n)}{f'(x_n)} S(h,\ t), \end{aligned} \tag{2.1}$$

where w_n is a Newton's stept and weight functions $P : \mathbb{C} \to \mathbb{C}$ and $S : \mathbb{C}^2 \to \mathbb{C}$ are analytic in a neighborhood of (0, 0) with $h = \frac{f(w_n)}{a_1 f(x_n) + a_2 f(w_n)} = O(e_n)$ and $t = \frac{f(z_n)}{f(w_n)} = O(e_n^2)$.

Let us consider the rational function, which is given by

$$Q(x) = \frac{(x - x_n) + b_1}{b_2(x - x_n)^3 + b_3(x - x_n)^2 + b_4(x - x_n) + b_5}, \tag{2.2}$$

where $b_1,\ b_2,\ \ b_3,\ b_4$ and b_5 are arbitrary parameters. We can determine these parameters by imposing the following tangency conditions

$$Q(x_n) = f(x_n),\ \ Q'(x_n) = f'(x_n),\ \ Q(w_n) = f(w_n),\ \ Q(z_n) = f(z_n),\ \ Q(k_n) = f(k_n).$$

In addition, we assume that the above rational function meets the $x-$ axis at the point $x = x_{n+1}$ to obtain the next approximation x_{n+1}, which is given by

$$Q(x_{n+1}) = 0, \tag{2.3}$$

which further yields $x_{n+1} = x_n - b_1$. In this way, we obtain the next approximation x_{n+1}. We will impose the first two tangency conditions to obtain the value of disposable parameter b_1. Then, we have

$$b_1 = b_5 f(x_n),\ \ b_4 = \frac{1 - b_5 f'(x_n)}{f(x_n)}. \tag{2.4}$$

Moreover, we will impose the last three tangency conditions to obtain the value of b_5. Then, we obtain the following three equations involving b_2, b_3 and b_5,

$$\begin{aligned} & f(w_n)\big[f'(x_n)\left(f'(x_n)\left(2b_5 f'(x_n) - 1\right) + b_3 f(x_n)^2\right) - b_2 f(x_n)^3\big] = f'(x_n)^2 f(x_n)\left(b_5 f'(x_n) - 1\right), \\ & f(z_n)\left[\frac{(1 - b_5 f'(x_n))(z_n - x_n)}{f(x_n)} + b_2(z_n - x_n)^3 + b_3(x_n - z_n)^2 + b_5\right] = b_5 f(x_n) + z_n - x_n, \\ & f(k_n)\left[\frac{(1 - b_5 f'(x_n))(k_n - x_n)}{f(x_n)} + b_2(k_n - x_n)^3 + b_3(k_n - x_n)^2 + b_5\right] = b_5 f(x_n) + k_n - x_n. \end{aligned}$$

By eliminating b_2 and b_3 from the above equations, we obtain the following value of b_5

$$b_5 = \frac{(x_n - z_n)(k_n - x_n)\big(\theta_1 f(x_n)^2 f(w_n) + \theta_2 f'(x_n) f(k_n) f(z_n)\big)}{\theta_3 f(x_n)^3 + \theta_4 f'(x_n) f(k_n) f(z_n)}, \tag{2.5}$$

where

$$\begin{aligned}
\theta_1 &= f(k_n)\big((k_n-x_n)^2 f'(x_n)+(k_n-x_n)f(x_n)-(k_n-z_n)f(z_n)\big)\\
&\quad +(x_n-z_n)\big(f(x_n)-(x_n-z_n)f'(x_n)\big)f(z_n),\\
\theta_2 &= (x_n-z_n)(k_n-x_n)(k_n-z_n)f'(x_n)\big(f(w_n)-f(x_n)\big)\\
&\quad +(k_n-z_n)f(w_n)f(x_n)(2x_n-z_n-k_n),\\
\theta_3 &= f(w_n)\big[(k_n-x_n)f(k_n)\left((k_n-x_n)^2 f'(x_n)+(k_n-x_n)f(x_n)-(k_n-z_n)f(z_n)\right)\\
&\quad +\big((x_n-z_n)^3 f'(x_n)+(k_n-z_n)(x_n-z_n)f(k_n)-(x_n-z_n)^2 f(x_n)\big)f(z_n)\big],\\
\theta_4 &= (x_n-z_n)^2(k_n-x_n)^2(k_n-z_n)f'(x_n)^2\big(2f(w_n)-f(x_n)\big)\\
&\quad +(x_n-z_n)(k_n-x_n)(k_n-z_n)(2x_n-z_n-k_n)f'(x_n)f(w_n)f(x_n)\\
&\quad +(k_n-z_n)(2k_nx_n-k_n^2+2x_nz_n-2x_n^2-z_n^2)f(w_n)f(x_n)^2.
\end{aligned}$$

Now, by using the equations (2.1), (2.4) and (2.5), we yield

$$\begin{aligned}
z_n &= w_n-\frac{f(w_n)}{f'(x_n)}P(h),\\
k_n &= z_n-\frac{f(z_n)}{f'(x_n)}S(h,\ k),\\
x_{n+1} &= x_n-b_5 f(x_n),
\end{aligned}\tag{2.6}$$

where w_n is a Newton's step and b_5 is previously defined in equation (2.5). The following theorem demonstrates that the order of convergence reachs at the optimal sixteenth-order without using any additional functional evaluations.

Theorem 1. *Let us assume that $f:\mathbb{C}\to\mathbb{C}$ has a simple zero r and analytic function in the region containing the zero r. In addition, we consider that an initial approximation $x=x_0$ is sufficiently close to r for the guaranteed convergence. Then, the iterative scheme defined by* (2.6)*, reaches an optimal sixteenth-order convergence, when the following conditions are satisfied,*

$$\begin{aligned}
&P(0)=1,\ \ P'(0)=2a_1,\ \ P''(0)=2a_1(2a_1+a_2),\ \ S_{00}=1,\\
&S_{01}=1,\ \ S_{10}=2a_1,\ \ S_{11}=4a_1,\ \ S_{20}=2a_1(3a_1+a_2),
\end{aligned}\tag{2.7}$$

where $S_{ij}=\frac{\partial^{i+j}}{\partial h^i\partial k^j}S(h,\ k)|_{(h=0,\ k=0)}$ for $i,j=1,\ 2$.

Proof. Let us consider $e_n=x_n-r$ be the error in the n^{th} iteration. The Taylor's series expansion of the function $f(x_n)$ and its first order derivative $f'(x_n)$ around $x=r$ with the assumption $f'(r)\neq 0$ leads us to:

$$f(x_n)=f'(r)\left[\sum_{i=1}^{16}c_ie_n^i+O(e_n^{17})\right],\tag{2.8}$$

and

$$f'(x_n)=f'(r)\left[\sum_{i=1}^{16}ic_ie_n^{i-1}+O(e_n^{17})\right],\tag{2.9}$$

where $c_i = \frac{f^{(i)}(r)}{i!f'(r)}$ for $i = 1,\ 2,\ \ldots,\ 16$.
By using the equations (2.8) and (2.9) in the first sub step of scheme (2.6), we have

$$w_n - r = c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (4c_2^3 - 7c_3c_2 + 3c_4)e_n^4 + \sum_{i=5}^{16} A_i e_n^i + O(e_n^{17}), \tag{2.10}$$

where $A_5 = -8c_2^4 + 20c_3c_2^2 - 10c_4c_2 - 6c_3^2 + 4c_5,\ A_6 = 16c_2^5 - 52c_3c_2^3 + 28c_4c_2^2 + (33c_3^2 - 13c_5)c_2 - 17c_3c_4 + 5c_6$, etc.

Now, we expand the function $f(w_n)$ about $x = r$, by using Taylor series expansion. Then, we obtain

$$f(w_n) = f'(r)\left[c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (5c_2^3 - 7c_3c_2 + 3c_4)e_n^4 + \sum_{i=5}^{16} B_i e_n^i + O(e_n^{17})\right], \tag{2.11}$$

where $B_5 = -2(6c_2^4 - 12c_3c_2^2 + 5c_4c_2 + 3c_3^2 - 2c_5)$, $B_6 = 28c_2^5 - 73c_3c_2^3 + 34c_4c_2^2 + \left(37c_3^2 - 13c_5\right)c_2 - 17c_3c_4 + 5c_6$, etc.

Further, by using equations (2.8) and (2.11), we have

$$\begin{aligned} h =& \frac{c_2}{a_1}e_n + \frac{2a_1c_3 - c_2^2(3a_1 + a_2)}{a_1^2}e_n^2 + \frac{c_2^3\left(8a_1^2 + 6a_1a_2 + a_2^2\right) + 3a_1^2c_4 - 2a_1c_3c_2(5a_1 + 2a_2)}{a_1^3}e_n^3 \\ &+ \sum_{k=4}^{16} C_i e_n^i + O(e_n^{17}), \end{aligned} \tag{2.12}$$

where $C_4 = \frac{1}{a_1^4}\left[a_1c_3c_2^2(37a_1^2 + 32a_1a_2 + 6a_2^2) - 2a_1^2c_4c_2(7a_1 + 3a_2) - 4a_1^2\left(c_3^2(2a_1 + a_2) - a_1c_5\right) - c_2^4(20a_1^3 + 25a_1^2a_2 + 9a_1a_2^2 + a_2^3)\right]$, etc.

Since, it is clear from the equations (2.12), $h = O(e_n)$. Therefore, we can expand the weight function $P(h)$ in the neighborhood of zero by Taylor series expansion up to second terms as follows:

$$P(h) = P(0) + P'(0)h + \frac{1}{2!}P''(0). \tag{2.13}$$

Using the values $P(0) = 1,\ \ P'(0) = 2a_1,\ \ P''(0) = 4a_1^2 + 2a_1a_2$ (which are defined in (2.7)) and equations (2.8), (2.9), (2.11) –(2.13) in the second sub step of scheme (2.6), we obtain bi-quadratic convergence

$$z_n - r = \frac{c_2^3(3a_1 + a_2) - a_1c_2c_3}{a_1}e_n^4 + \sum_{i=5}^{16} D_i e_n^i + O(e_n^{17}), \tag{2.14}$$

where $D_5 = -\frac{2}{a_1}\left[c_2^4(8a_1 + 3a_2) - c_3c_2^2(10a_1 + 3a_2) + a_1c_4c_2 + a_1c_3^2\right]$, etc.

Now, we obtain the following expansion of $f(z_n)$ about $x = r$, by using Taylor series expansion

$$f(z_n) = f'(r)\left[\frac{c_2^3(3a_1 + a_2) - a_1c_2c_3}{a_1}e_n^4 + \sum_{i=4}^{16} \bar{D}_i e_n^i + O(e_n^{17})\right]. \tag{2.15}$$

In addition, we define the new variable $t = \frac{f(z_n)}{f(w_n)} = \frac{3a_1c_2^2-a_1c_3+a_2c_2^2}{a_1}e_n^2 + O(e_n^3)$ and expand the weight function $S(h,\ t)$ in the neighborhood of $(0,\ 0)$, up to second-order term with the help Taylor series expansion. Then ,we have

$$S(h,\ t) = S_{00} + S_{10}h + S_{01}t + \frac{1}{2!}\big(S_{20}h^2 + 2S_{11}ht + S_{02}t^2\big). \tag{2.16}$$

Then, from the third step $k_n = z_n - \frac{f(z_n)}{f'(x_n)}S(h,\ t)$, we further yields

$$k_n - r = \sum_{i=8}^{16} E_i e_n^i + O(e_n^{17}), \tag{2.17}$$

where $E_8 = -\frac{1}{2a_1^3}c_2((3a_1+a_2)c_2^2-a_1c_3)\big[c_2^4\left\{9a_1^2(S_{02}-6) + 2a_1a_2(3S_{02}-13) + a_2^2(S_{02}-2)\right\}+ a_1^2c_3^2(S_{02}-2) - 2a_1^2c_4c_2 - 2a_1c_3c_2^2\{a_1(3S_{02}-17) + a_2(S_{02}-5)\}\big]$, etc.

Again, we obtain the following expansion of $f(k_n)$ about $x = r$, by using the Taylor series expansion,

$$f(k_n) = f'(r)\left[E_8e_n^9 + \sum_{i=9}^{16}\bar{E}_i e_n^i + O(e_n^{17})\right]. \tag{2.18}$$

In order to obtain optimal sixteenth-order of convergence, we will use equations (2.7) – (2.18), in the fourth-step of scheme (2.6) and after some simplification, we have

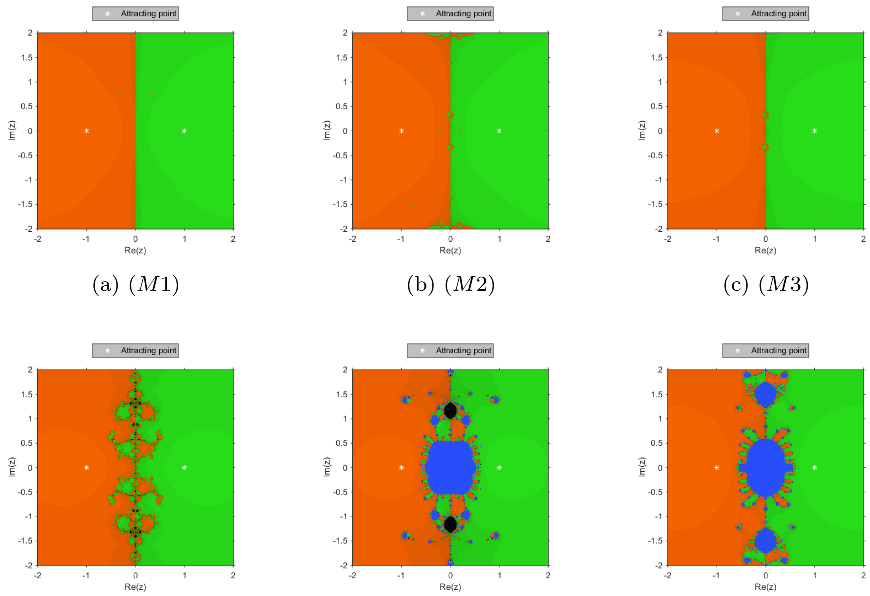
$$\begin{aligned} e_{n+1} = &-\frac{c_2^3\left(c_2^2(3a_1+a_2) - a_1c_3\right)^2}{2a_1^4}\Big[c_2^4\{S_{02}(3a_1+a_2)^2 - 2(27a_1^2 + 13a_1a_2 + a_2^2)\} + a_1^2c_3^2(S_{02}-2) \\ &- 2a_1^2c_4c_2 - 2a_1c_3c_2^2\{a_1(3S_{02}-17) + a_2(S_{02}-5)\}\Big](c_2^4 - 3c_3c_2^2 + 2c_4c_2 + c_3^2 - c_5)e_n^{16} \\ &+ O(e_n^{17}). \end{aligned} \tag{2.19}$$

Finally, we obtain the above error equation which reveals that the our proposed scheme (2.6) reaches at optimal sixteenth-order convergence by using only five functional evaluations per iteration. This completes the proof. □

3 Basins of attraction

In this section, we will compare the stability and reliability of the proposed methods with other existing ones, in terms of dependence on initial estimations. In order to get this aim, the performance of all proposed and known methods will be checked on a simple nonlinear function $x^2 - 1$, and also on a much more complex one, $\frac{x^3}{x^4+1} + \sqrt{x^4+8}\sin\left(\frac{\pi}{x^2+2}\right) - \sqrt{6} + \frac{8}{17}$ (this one will be also used in the numerical section). For a wide set of initial estimation in the complex plane, all the methods will be tested by drawing their corresponding dynamical phase spaces.

We shall compare the proposed schemes with the optimal sixteenth-order method (5), for (a=1), of Neta [14], denoted by (N_{16}). we will also compare them with the optimal families



(a) $(M1)$ (b) $(M2)$ (c) $(M3)$

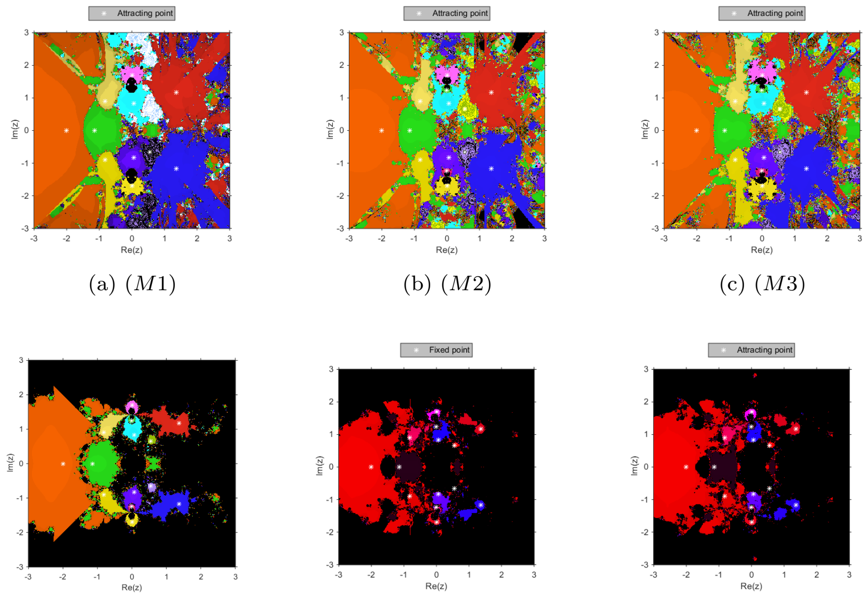
(d) (N_{16}) (e) $(G1_{16})$ (f) $(G2_{16})$

Figure 1: Dynamical planes of 16th-order methods on $x^2 - 1$

of sixteenth-order methods proposed by Geum and Kim in [7, 6], out of these families we shall choose the expression (Y1) (defined in Table 1 of Geum and Kim [7]) and expression (K2) (for details of this method please see Table 1 of Geum and Kim [6]), respectively called by $(G1_{16})$ and $(G2_{16})$. These methods will be also used for numerical performances in the following section.

These dynamical planes have been generated by using the routines appearing in [3]. They are generated by using each point of the complex plane as initial estimation (we have used a mesh of 400×400 points). We paint in different colors the points whose orbit converges to different attracting fixed points (all fixed points appear marked as a white star in the figures) and in black if it reaches the maximum number of 40 iterations without converging to any of the fixed points. The basin of attraction of each fixed points is defined by the set of initial estimations that converge to it. So, wider basins of attraction mean more stable behavior and lower dependence on initial estimations of the methods.

In Figure 1, the basins of attraction of the different 16th-order schemes are represented. In this case, there are only two real fixed points, $x = \pm 1$, and the basins of attraction of these points are plotted in green and orange. Methods $(M1)$, $(M2)$ and $(M3)$ show a very stable behavior (see Figures 1a to 1c), specially $(M1)$ whose basins of attraction are apparently the same as one of Newton's scheme, but with 16-th order of convergence. Case of (N_{16}) (Figure 1d) is similar, but with small black regions of no convergence to the roots. More different are the basins of attraction of schemes $(G1_{16})$ and $(G2_{16})$ (Figures 1e and 1f), as in addition of green and orange basins, a blue basin of convergence to infinity



(a) $(M1)$ (b) $(M2)$ (c) $(M3)$

(d) (N_{16}) (e) $(G1_{16})$ (f) $(G2_{16})$

Figure 2: Dynamical planes of 16th-order methods on $\frac{x^3}{x^4+1} + \sqrt{x^4+8}\sin\left(\frac{\pi}{x^2+2}\right) - \sqrt{6} + \frac{8}{17}$

(divergence) is shown and also black areas of convergence to other attracting points different from the roots appear.

When a much more complicated function is analyzed, the basins of attraction are also much more devious. In this function, two real and twelve complex roots are plotted as white stars. Their respective basins of attraction are plotted in different colors. It can be observed as these basins are much more wider in case of proposed methods (Figures 2a to 2c) than in case of the rest of schemes that, although converge to the real roots in all cases, have tiny basins of attraction of some of the complex roots and wide black areas of no convergence (see Figures 2d to 2f).

4 Numerical experiments

This section is fully devoted to check the effectiveness and validity of our theoretical results which have been proposed in section 2. Therefore, we consider the following special cases of our proposed scheme to see the comparison of them with the other existing optimal methods of order sixteen

$$\begin{cases} P(h) = 1 + 2a_1h + a_1(2a_1 + a_2)h^2, \\ S(h,\ k) = 1 + 2a_1h + k + a_1(3a_1 + a_2)h^2 + 4a_1hk. \end{cases} \tag{4.1}$$

Using the above weight function in the scheme (2.6) with $a_1 = 1$ and $a_2 = -2$, we obtain a new sixteenth-order modification of Ostrowski method, denoted by $(M1)$. In addition, we also insert the above weight function in the proposed scheme (2.6) with ($a_1 = 1$ & $a_2 = -3$) and ($a_1 = 1$ and $a_2 = -\frac{17}{5}$), respectively, called by $(M2)$ and $(M3)$. As in case of the comparison by means of basins of attraction, we shall compare them with the optimal sixteenth-order method (N_{16}) and those from Geum and Kim $(G1_{16})$ and $(G2_{16})$.

Further, we consider the approximated zero of test functions when the exact zero is not available, which is corrected up to 200 significant digits to calculate $|x_n - r|$. For the computer programming, all computations have been performed using the programming package $Mathematica$ 9 with multiple precision arithmetic. The test functions to be used and the searched roots in the numerical performances are:

1. $f_1(x) = \frac{x^3}{x^4+1} + \sqrt{x^4+8}\sin\left(\frac{\pi}{x^2+2}\right) - \sqrt{6} + \frac{8}{17}$, $r = 2$.

2. $f_2(x) = e^{-x^2}\frac{\sin(x)}{x^2-1} + x^2\log(x - \pi + 1)$, $r = \pi$.

3. $f_3(x) = -\log\left(4x^2 - \pi + 1\right) + \sin\left(2x^2\right) - 1$, $r = \sqrt{\frac{\pi}{4}}$.

4. $f_4(x) = e^{2x} + \sin^{-1}(x^2 - 1) - 7$, $r \approx 0.97629186887861075372580403259$0.

5. $f_5 = 10xe^{-x^2} - 1$, $r \approx 1.67963061042844994067492033884$.

Methods	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_n - r\rvert$	$\frac{e_n}{e_{n-1}^{16}}$	η	$\frac{\log\lvert e_n/\eta\rvert}{\log\lvert e_{n-1}\rvert}$
N_{16}	0	-1.9	$3.2e(-2)$	$1.0e(-1)$			
	1	-2.00000000000	$2.0e(-16)$	$6.1e(-16)$	6.087155557	0.2361281910	14.589
	2	-2.00000000000	$2.8e(-245)$	$8.4e(-245)$	0.2361281910		16.000
$G1_{16}$	0	-1.9	$3.2e(-2)$	$1.0e(-1)$			
	1	-2.00000000000	$6.5e(-16)$	$2.0e(-15)$	19.69885532	0.08418846348	13.631
	2	-2.00000000000	$1.4e(-237)$	$4.3e(-237)$	0.08418846348		16.000
$G2_{16}$	0	-1.9	$3.2e(-2)$	$1.0e(-1)$			
	1	-2.00000000000	$1.4e(-14)$	$4.1e(-14)$	414.1970377	3.625713950	13.942
	2	-2.00000000000	$9.0e(-215)$	$2.7e(-214)$	3.625713950		16.000
$M1$	0	-1.9	$3.2e(-2)$	$1.0e(-1)$			
	1	-2.00000000000	$1.8e(-17)$	$5.3e(-17)$	0.5313647842	0.04662092834	14.943
	2	-2.00000000000	$6.3e(-263)$	$1.9e(-262)$	0.04662092834		16.000
$M2$	0	-1.9	$3.2e(-2)$	$1.0e(-1)$			
	1	-2.00000000000	$1.6e(-18)$	$4.7e(-18)$	0.04723703076	0.01317792297	15.446
	2	-2.00000000000	$2.7e(-280)$	$8.1e(-280)$	0.01317792297		
$M3$	0	-1.9	$3.2e(-2)$	$1.0e(-1)$			
	1	-2.00000000000	$2.1e(-19)$	$6.3e(-19)$	0.006263947505	0.005798398834	15.966
	2	-2.00000000000	$1.1e(-294)$	$3.3e(-294)$	0.005798398834		16.000

Table 1: Convergence for $f_1(x)$ with $r = -2$

For better comparisons of our proposed methods with the other existing ones, we have displayed the number of iteration indexes (n), approximated zeros (x_n), absolute residual error of the corresponding function $(|f(x_n)|)$, errors $|e_n|$ (where $e_n = x_n - r$), $\left|\frac{e_{n+1}}{e_n^{16}}\right|$ and the asymptotic error constant $\eta = \lim_{n\to\infty} \left|\frac{e_{n+1}}{e_n^{16}}\right|$. In order to calculate the asymptotic computational order of convergence (ρ) , we use the formula $p \approx \rho = \frac{\log|e_n/\eta|}{\log|e_{n-1}|}$. We calculate the computational order of convergence and asymptotic error constant and other constants up to several number of significant digits (minimum 1000 significant digits) to minimize the round off error.

Methods	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_n - r\rvert$	$\frac{e_n}{e_{n-1}^{16}}$	η	$\frac{\log\lvert e_n/\eta\rvert}{\log\lvert e_{n-1}\rvert}$
	0	4	9.9	$8.6e(-1)$			
N_{16}	1	3.14159265392	$3.3e(-9)$	$3.4e(-10)$	$3.876697e(-9)$	$2.895997e(-8)$	29.171
	2	3.14159265358	$7.9e(-159)$	$8.0e(-160)$	$2.895997e(-8)$		16.000
	0	4	9.9	$8.6e(-1)$			
$G1_{16}$	1	3.14159265422	$6.6e(-9)$	$6.3e(-10)$	$7.299548e(-9)$	$1.462815e(-10)$	−9.6099
	2	3.14159265358	$9.9e(-157)$	$1.0e(-157)$	$1.462815e(-10)$		16.000
	0	4	9.9	$8.6e(-1)$			
$G2_{16}$	1	3.14159265116	$2.4e(-8)$	$2.4e(-9)$	$2.790258e(-8)$	$1.458588e(-10)$	−18.412
	2	3.14159265358	$2.1e(-147)$	$2.1e(-148)$	$1.458588e(-10)$		16.000
	0	4	9.9	$8.6e(-1)$			
$M1$	1	3.14159265358	$1.3e(-13)$	$1.3e(-14)$	$1.467022e(-13)$	$2.668433e(-8)$	95.326
	2	3.14159265358	$1.3e(-229)$	$1.3e(-230)$	$2.668436e(-8)$		16.000
	0	4	9.9	$8.6e(-1)$			
$M2$	1	3.14159265373	$1.4e(-9)$	$1.4e(-10)$	$1.623296e(-9)$	$1.241222e(-8)$	29.324
	2	3.14159265358	$3.0e(-165)$	$3.1e(-166)$	$1.241222e(-8)$		16.000
	0	4	9.9	$8.6e(-1)$			
$M3$	1	3.14159265392	$3.3e(-9)$	$3.3e(-10)$	$3.850528e(-9)$	$1.832566e(-9)$	11.137
	2	3.14159265358	$4.5e(-160)$	$4.5e(-161)$	$1.832566e(-9)$		16.000

Table 2: Convergence for $f_2(x)$ with $r = \pi$

As we mentioned in the above paragraph that we calculate the values of all the constants and functional residuals up to several number of significant digits but due to the limited paper space, we display the value of x_n up to 15. In addition, ρ and $\left(\left|\frac{e_{n+1}}{e_n^{16}}\right| \text{ and } \eta\right)$ are presented up to 5 and 10 significant digits, respectively. Moreover, all the other constants namely, $|e_n|$, and absolute residual error in the function $|f(x_n)|$, are display up to 2 significant digits with exponent power which are mentioned in the Tables 1 − 5. Furthermore, the approximated zeros up to 35 significant digits are also displayed in the caption of the Tables 1 − 5, although minimum 1000 significant digits are available with us.

In 1, results from the test made on function $f_1(x)$ are shown. In it, is clear that the theoretical 16th-order of convergence is reached and the numerical results obtained after three iterations are similar. However, proposed methods $M1$, $M2$ and $M3$ show better precision and lower asymptotic error constant. Similar results can be observed in Table 2.

Regarding Table 3, the showed results give us similar information about the same aspects of the test: estimated errors, approximated order of convergence; however, the asymptotic error constant of methods $G1_{16}$, $G2_{16}$ and $M3$ are extremely high, meanwhile that of N_{16}, $M1$ and $M2$ remain in reasonable values.

In case of function $f_4(x)$, the results obtained by $M1$, $M2$ and $M3$ are clearly better than those got by known schemes, including the error estimations and the asymptotic constant

Methods	n	x_n	$\vert f(x_n)\vert$	$\vert x_n - r\vert$	$\frac{e_n}{e_{n-1}^{16}}$	η	$\frac{\log\vert e_n/\eta\vert}{\log\vert e_{n-1}\vert}$
	0	0.9	$9.5e(-2)$	$1.4e(-2)$			
N_{16}	1	0.886226925452	$1.0e(-24)$	$1.5e(-25)$	87098.28165	9012.374837	15.471
	2	0.886226925452	$2.7e(-393)$	$3.9e(-394)$	9012.374837		16.000
	0	0.9	$9.5e(-2)$	$1.4e(-2)$			
$G1_{16}$	1	0.886226925452	$2.9e(-23)$	$4.1e(-24)$	$2.431552414e(+6)$	$1.558675557e(+7)$	16.434
	2	0.886226925452	$6.4e(-367)$	$9.1e(-368)$	$1.558675557e(+7)$		16.000
	0	0.9	$9.5e(-2)$	$1.4e(-2)$			
$G2_{16}$	1	0.886226925452	$7.0e(-22)$	$9.8e(-23)$	$5.857298039e(+7)$	$2.712508318e(+8)$	16.358
	2	0.886226925452	$1.4e(-343)$	$2.0e(-344)$	$2.712508318e(+8)$		16.000
	0	0.9	$9.5e(-2)$	$1.4e(-2)$			
$M1$	1	0.886226925452	$1.0e(-24)$	$1.4e(-25)$	85238.09626	140880.4065	16.117
	2	0.886226925452	$3.0e(-392)$	$4.3e(-393)$	140880.4065		16.000
	0	0.9	$9.5e(-2)$	$1.4e(-2)$			
$M2$	1	0.886226925452	$3.3e(-23)$	$4.7e(-24)$	0.08655188030	0.06653652987	16.218
	2	0.886226925452	$2.7e(-366)$	$3.8e(-367)$	0.06653652987		16.000
	0	0.9	$9.5e(-2)$	$1.4e(-2)$			
$M3$	1	0.886226925452	$5.8e(-23)$	$8.2e(-24)$	$4.914422055e(+6)$	$1.351536840e(+7)$	16.000
	2	0.886226925452	$4.3e(-362)$	$6.1e(-363)$	$1.351536840e(+7)$		16.000

Table 3: Convergence for $f_3(x)$ with $r \approx \sqrt{\frac{\pi}{4}}$

Methods	n	x_n	$\vert f(x_n)\vert$	$\vert x_n - r\vert$	$\frac{e_n}{e_{n-1}^{16}}$	η	$\frac{\log\vert e_n/\eta\vert}{\log\vert e_{n-1}\vert}$
	0	1.2	4.5	$2.2e(-1)$			
N_{16}	1	0.976291868946	$1.1e(-9)$	$6.8e(-11)$	1.731187559	7.703357883	16.997
	2	0.976291868878	$2.7e(-161)$	$1.7e(-162)$	7.703357883		16.000
	0	1.2	4.5	$2.2e(-1)$			
$G1_{16}$	1	0.976291868713	$2.7e(-9)$	$1.7e(-10)$	4.208028085	15.27838262	16.861
	2	0.976291868878	$7.8e(-155)$	$4.9e(-156)$	15.27838262		16.000
	0	1.2	4.5	$2.2e(-1)$			
$G2_{16}$	1	0.976291868102	$1.2e(-8)$	$7.8e(-10)$	19.73714501	2414.275399	19.210
	2	0.976291868878	$6.8e(-142)$	$4.2e(-143)$	2414.275399		16.000
	0	1.2	4.5	$2.2e(-1)$			
$M1$	1	0.976291868878	$1.6e(-12)$	$1.0e(-13)$	0.002590514342	0.001630150285	15.691
	2	0.976291868878	$3.6e(-210)$	$2.2e(-211)$	0.001630150285		16.000
	0	1.2	4.5	$2.2e(-1)$			
$M2$	1	0.976291868882	$5.5e(-11)$	$3.4e(-12)$	0.08655188030	0.06653652987	15.824
	2	0.976291868878	$3.5e(-184)$	$2.2e(-185)$	0.06653652987		16.000
	0	1.2	4.5	$2.2e(-1)$			
$M3$	1	0.976291868897	$1.9e(-10)$	$1.2e(-11)$	0.3071658708	0.976291868878611	15.738
	2	0.976291868878	$6.9e(-175)$	$4.3e(-176)$	0.2073478134		16.000

Table 4: Convergence for $f_4(x)$ with $r \approx 0.976291868878610753725804032590$

error. Nevertheless, all of them hold the theoretical order of convergence.

The differences among the methods in terms of stability are clearly stated in Table 5. In it, best results of $M1$, $M2$ and $M3$ in all the checked elements of the numerical process can be observed. Also better, but in a smaller amount, are the results got by $M1$, $M2$ and $M3$ respect to the ones of the comparison methods N_{16}, $G1_{16}$ and $G2_{16}$ (see Table 5).

Methods	n	x_n	$\lvert f(x_n)\rvert$	$\lvert x_n - r\rvert$	$\frac{e_n}{e_{n-1}^{16}}$	η	$\frac{\log\lvert e_n/\eta\rvert}{\log\lvert e_{n-1}\rvert}$
	0	1.5	$5.8e(-1)$	$1.8e(-1)$			
N_{16}	1	1.67963061042	$1.8e(-11)$	$6.5e(-12)$	5.537170610	153.6631200	17.936
	2	1.67963061042	$4.4e(-177)$	$1.6e(-177)$	153.6631200		16.000
	0	1.5	$5.8e(-1)$	$1.8e(-1)$			
$G1_{16}$	1	1.67963061043	$1.0e(-11)$	$3.6e(-12)$	3.083765922	100.3740333	18.029
	2	1.67963061042	$2.5e(-181)$	$8.9e(-181)$	100.3740333		16.000
	0	1.5	$5.8e(-1)$	$1.8e(-1)$			
$G2_{16}$	1	1.67963061036	$1.7e(-10)$	$6.0e(-11)$	50.84310947	10753.24541	19.119
	2	1.67963061042	$7.8e(-160)$	$2.8e(-160)$	10753.24541		16.000
	0	1.5	$5.8e(-1)$	$1.8e(-1)$			
$M1$	1	1.67963061042	$2.1e(-14)$	$7.5e(-15)$	0.006390604103	0.06863566058	17.383
	2	1.67963061042	$1.9e(-227)$	$7.0e(-228)$	0.06863566058		16.000
	0	1.5	$5.8e(-1)$	$1.8e(-1)$			
$M2$	1	1.67963061042	$1.6e(-15)$	$5.8e(-16)$	0.0004898530879	0.0004802161904	15.988
	2	1.67963061042	$1.9e(-247)$	$7.0e(-248)$	0.0004802161904		16.000
	0	1.5	$5.8e(-1)$	$1.8e(-1)$			
$M3$	1	1.67963061042	$1.4e(-15)$	$5.2e(-16)$	0.0004453340558	0.03036170171	18.459
	2	1.67963061042	$2.7e(-246)$	$9.6e(-247)$	0.03036170171		16.000

Table 5: Convergence for $f_5(x)$ with $r \approx 1.679630610428449940674920338\,84$

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