Bull. Math. Soc. Sci. Math. Roumanie
Tome 60 (108) No. 2, 2017, 113–125

On Logical Notions
in the Fraenkel-Mostowski Cumulative Universe

by

Andrei Alexandru, Gabriel Ciobanu

Abstract

Fraenkel-Mostowski set theory represents a tool for managing infinite structures in terms of finite objects. In this paper we provide a connection between the concept of logical notions invariant under permutations introduced by Tarski and Fraenkel-Mostowski set theory. More precisely, we prove that some particular sets defined by using the axioms of Fraenkel-Mostowski set theory are logical notions in Tarski's sense. We also investigate whether a new and specific Fraenkel-Mostowski binding operator is logical in Tarski's sense.

Key Words: Fraenkel-Mostowski set theory, Tarski logicality, invariance, nominal sets, invariant sets.
2010 Mathematics Subject Classification: Primary 00A30, Secondary 03E30.

1 Introduction

It does not exist a full philosophical consensus on the distinction between logical and non-logical notions. This fact leads to certain doubts regarding our understanding of the nature of logic and its relationship to mathematics. Some logicians have suggested that what is distinctive about logical notions is their invariance under permutations of the domain of objects. In order to clarify this invariance under permutations, we mention that the set of numbers between 1 and 9 is invariant under the permutation of these numbers (it does not matter how we switch these numbers, we end up with the same set), but the set of prime numbers between 1 and 9 is not invariant under any permutation of these first 9 numbers (for instance, the related set formed by 2, 3, 5 and 7 is not invariant under the permutation that switches numbers 5 and 8 and maps all the other numbers to themselves).

In a logical sentence, signs for negation, conjunction, disjunction and the quantifiers should be invariant under any permutation of words, and so they count as logical notions (or logical constants), while words like "dog", "tall" and "blue" cannot be invariant under permutations of (a larger set of) words, and so they are not logical notions. The invariance criterion seems to fit with common intuition about logical notions. Certain technical results increase our confidence in this invariance criterion: in [19] it is proved that all of the relations definable in the language of Principia Mathematica are invariant under permutations, while in [16] every permutation-invariant operation can be defined in terms of logical operations such as identity, variable substitution, disjunction, negation and existential quantification, and each operation so definable is invariant under permutations.

Alfred Tarski gave a lecture in 1966 for a general audience at Bedford College in London entitled “What are logical notions?” Tarski’s answer to this question is presented in [20]. Essentially, logical notions are considered to be relations between individuals and classes, as well as relations over an arbitrary non-empty domain D of individuals. Tarski identified logical relations as exactly those invariant under arbitrary permutations of D. This thesis characterizes logical notions and logical operations by invariance under permutations.

As Tarski himself pointed out, the permutation invariance criterion for logical notions can be seen as a generalization of Felix Klein’s idea that different geometries can be distinguished by the groups of transformations under which their basic notions are invariant [14]. In his Erlangen Program, Klein classified the notions to be studied in various geometries (such as Euclidean, affine and projective geometry) according to the groups of (one-one and onto) transformations under which they are invariant. With logic thought of as the most general theory, logical notions should be those invariant under the largest group of transformations, namely the class of permutations. The “generality” argument for Tarski’s thesis is given by Bonnay in [9] as follows:

1. The distinctive feature of logic among other theories is that it is the most general theory one can think of.

2. The bigger the group of transformations associated with a theory, the more general the theory.

3. The biggest group of transformations is the class of all permutations.

Thus, it is concluded that logical notions are those invariant under permutation.

Tarski’s thesis and related results assimilate logical notions to mathematics. From Whitehead and Russell’s Principia Mathematica, we know that the whole of mathematics can be formalized within set theory. In [19], set theory is described as a mathematically universal language. For Tarski, this universality provides a foundational status in mathematics (and metamathematics) to set theory, and so the whole of mathematics can be expressed in the language of an appropriate set theory. In Tarski’s words, “... we need only one non-logical constant (...) for a two-termed relation which holds between an element and a set (...). Then the only concern lies in a careful selection of the axioms. They must be weak enough to escape the antinomies, but at the same time they must be strong enough to ensure, within our universe of discourse, the existence of sets which correspond to as large a class of sentential functions as possible.”

Logical operations and notions in Tarski’s sense meet the permutation invariance criterion. If they are described set-theoretically, they should have the same meaning (independent of the set-theoretical universe). By considering an appropriate set theory, we also take into account that if semantic concepts cannot be reduced to logical concepts, then we cannot proceed in “harmony with the postulates of the unity of science and of physicalism”. This is why Tarski preferred to link mathematical universality to domain universality [19].

According to [18], invariance under permutation reflects the formality of notions from logic. Thus, invariant notions are formal in the sense that they do not depend on the identity of objects. For example, in a formal language the extension of the existential quantifier $\exists$ consists of all non-empty subsets of a specific domain. Obviously, all the one-to-one applications of this domain onto itself transform any non-empty subset of the domain in another non-empty subset of the domain. Therefore, the interpretation of the existential quantifier is invariant under any permutation.

Tarski's thesis makes sense for objects in the finite relational type structure over a domain of basic objects D, where the objects at each level are relations of one or more arguments between objects of lower levels. Rather than considering the entire set of permutations of a universe built as a cumulative hierarchy over D, we can consider only those permutations of the objects in D. Thus, the logical notions (and the logical operations) are those invariant under arbitrary permutations of objects from D.

Based on the previous remark and the approach in [18], we can provide a practical method of verifying logicality in cumulative hierarchies. The general idea is to study the effect of permutations over sets. More precisely, we consider an initial domain D of basic objects, and construct a hierarchy of sets starting with the objects in D. After that we consider any permutation of objects from D, and see what effect have these permutations on the sets of various levels. The sets which are fixed under all permutations are exactly the sets that can be denoted by a logical symbol. It is easy to see that both the identity relation between basic objects and its negation are fixed under every permutation. At a higher level, the sets of sets that are fixed include the set of all non-empty sets (which is related to the existential quantifier) and the set consisting of just the empty set (which is related to the negated universal quantifier). This means that all of them can be denoted by logical symbols, and thus they represent formal notions.

Formally, Tarski's logicality criterion can be expressed as
"Given a domain D of basic objects, an operation f in the type hierarchy over D is logical if and only if it is invariant under all permutations on D."

We relate all these aspects to a recently developed Fraenkel-Mostowski set theory [11]. Our goal is to connect the concept of logical notions in Tarski sense to the sets of the Fraenkel-Mostowski universe. We also show that the newly developed Fraenkel-Mostowski axiomatic set theory has historical roots in Tarski's approach regarding logicality.

2 Fraenkel-Mostowski Cumulative Universe

First-order Fraenkel-Mostowski (FM) set theory has its origins in an approach developed initially in the 1930s [10, 15] in order to prove the independence of the axiom of choice and other axioms in Zermelo-Fraenkel with atoms (ZFA) set theory, where ZFA is Zermelo-Fraenkel set theory with the Axiom of Extensionality modified to allow the existence of atoms. In 2001, the basic Fraenkel model of ZFA without axiom of choice (model $\mathcal{N}1$ in [12]) was axiomatized and presented as an independent set theory named FM axiomatic set theory [11]. The axioms of FM set theory are precisely those of ZFA over an infinite set of atoms [11], together with the special axiom of finite support which claims that for each element x in an arbitrary set we can find a finite set supporting x. The original purpose of the FM axiomatic set theory was to provide a mathematical model for variables in a certain syntax. Atoms have the same properties as variables and names; they do not have internal structure, and are used for their ability to identify and for their distinctness. The finite support axiom is motivated by the fact that syntax can only involve finitely many distinct (free) names.

The construction of the universe of all FM sets, i.e. sets defined according to FM axioms [11], is inspired by the construction of the universe of all admissible sets over an arbitrary collection of atoms [6]. The FM sets represent a generalization of hereditary finite sets (which are particular admissible sets); actually, any FM set is an hereditary finitely supported set.

The FM set theory is also related to the alternative theory of nominal sets [17]. Nominal sets provide a formalism for describing λ-terms modulo α-conversion [11]. They also have a lot of applications in algebra, semantics, logic, topology, proof theory and in domain theory (see [5] for applications of nominal sets in calculability theory and [1] for a survey on some applications of nominal sets). Nominal sets can be defined both in the ZF framework [17] and in the FM framework [11]. In ZF, a fixed infinite set A is considered as a set of names, and a nominal set is defined as a usual ZF set endowed with a particular group action of the group of permutations over A that satisfies a certain a finite support requirement. There exists also an alternative definition for nominal sets in the FM framework. They can be defined as sets constructed according to the FM axioms with the additional property of being empty supported (invariant under all permutations). These two ways of defining nominal sets finally lead to similar properties as it is proved in [4].

The theory of nominal sets over a fixed set A whose elements can be checked only for equality is extended to generalized nominal sets in [7] by using new data symmetries over arbitrary sets of data which may have a certain internal structure. Generalized nominal sets are used to study automata on data words [7], languages over infinite alphabets [7], and Turing machines operating over finite alphabets [8].

We consider that FM set theory is an appropriate framework for experimental sciences, providing a characterization of infinite structures by using a 'finitary' representation. More precisely, in the FM framework we can model infinite structures by using the notion of finite support [3, 5]. Note that FM set theory provides a balance between an informal reasoning and a rigorous representation; this is discussed in [17], where principles of structural recursion and induction are explained in the FM framework.

Let A be a fixed infinite ZF set. The following results make also sense if A is considered to be the set of atoms in the ZFA framework (characterized by the axiom "$y \in x \Rightarrow x \notin A$"), and if 'ZF' is replaced by 'ZFA' in their statement. This means that the theory of nominal sets makes sense in both ZF and ZFA.

A *transposition* is a function $(a\,b) : A \to A$ defined by $(a\,b)(a) = b$, $(a\,b)(b) = a$, and $(a\,b)(n) = n$ for $n \neq a, b$. A *finite permutation* of A is generated by composing finitely many transpositions. Let S_A be the set of all finite permutations of A (i.e. the set of all bijections on A which leave unchanged all but finitely many elements).

Definition 1. 1. *Let X be a ZF set. An S_A-*action *on X is a function $\cdot : S_A \times X \to X$ having the properties that $Id \cdot x = x$ and $\pi \cdot (\pi' \cdot x) = (\pi \circ \pi') \cdot x$ for all $\pi, \pi' \in S_A$ and $x \in X$. An S_A-*set *is a pair $(X, \cdot)$, where X is a ZF set and $\cdot : S_A \times X \to X$ is an S_A-action on X.*

2. *Let $(X, \cdot)$ be an S_A-set; we say that $S \subset A$* supports *x whenever for each $\pi \in Fix(S)$ we have $\pi \cdot x = x$, where $Fix(S) = \{\pi \,|\, \pi(a) = a$, for all $a \in S\}$.*

3. *Let $(X, \cdot)$ be an S_A-set; we say that X is a* nominal set *if for each $x \in X$, there exists a finite set $S_x \subset A$ supporting x.*

4. *Let X be an S_A-set and $x \in X$. If there exists a finite set supporting x, then there exists a least finite set supporting x which is called* the support of x *and is denoted by $supp(x)$. According to [11], the support of x is defined as the intersection of all finite sets of atoms supporting x. An empty supported element is called* equivariant.

Proposition 1. *Let $(X, \cdot)$ be an S_A-set and $\pi \in S_A$. If $x \in X$ is finitely supported, then $\pi \cdot x$ is also finitely supported, and $supp(\pi \cdot x) = \pi(supp(x))$.*

Example 1.

1. *The set A of atoms is an S_A-set with the S_A-action $\cdot : S_A \times A \to A$ defined by $\pi \cdot a := \pi(a)$ for all $\pi \in S_A$ and $a \in A$. $(A, \cdot)$ is a nominal set because for each $a \in A$ we have that $\{a\}$ supports a. Moreover, $supp(a) = \{a\}$.*

2. *The set S_A is an S_A-set with the S_A-action $\cdot : S_A \times S_A \to S_A$ defined by $\pi \cdot \sigma := \pi \circ \sigma \circ \pi^{-1}$ for all $\pi, \sigma \in S_A$. $(S_A, \cdot)$ is a nominal set because for each $\sigma \in S_A$ we have that the finite set $\{a \in A \,|\, \sigma(a) \neq a\}$ supports σ. Moreover, $supp(\sigma) = \{a \in A \,|\, \sigma(a) \neq a\}$ for each $\sigma \in S_A$.*

3. *Any ordinary ZF-set X is an S_A-set with the S_A-action $\cdot : S_A \times X \to X$ defined by $\pi \cdot x := x$ for all $\pi \in S_A$ and $x \in X$. Moreover, X is a nominal set because for each $x \in X$ we have that $\emptyset$ supports x.*

4. *If $(X, \cdot)$ is an S_A-set, then $\wp(X) = \{Y \,|\, Y \subseteq X\}$ is also an S_A-set with the S_A-action $\star : S_A \times \wp(X) \to \wp(X)$ defined by $\pi \star Y := \{\pi \cdot y \,|\, y \in Y\}$ for all π of A and all subsets Y of X. For each nominal set $(X, \cdot)$ we denote by $\wp_{fs}(X)$ the set formed from those subsets of X which are finitely supported according to S_A-action $\star$. According to Proposition 1, $(\wp_{fs}(X), \star|_{\wp_{fs}(X)})$ is a nominal set, where $\star|_{\wp_{fs}(X)}$ represents the action $\star$ restricted to $\wp_{fs}(X)$.*

5. *Let $(X, \cdot)$ and $(Y, \diamond)$ be two S_A-sets. The Cartesian product $X \times Y$ is also an S_A-set with the S_A-action $\star : S_A \times (X \times Y) \to (X \times Y)$ defined by $\pi \star (x, y) = (\pi \cdot x, \pi \diamond y)$ for all $\pi \in S_A$ and all $x \in X$, $y \in Y$. If $(X, \cdot)$ and $(Y, \diamond)$ are nominal sets, then $(X \times Y, \star)$ is also a nominal set.*

6. *Let $(X, \cdot)$ and $(Y, \diamond)$ be two S_A-sets. The disjoint union of X and Y is defined by $X + Y = \{(0, x) \,|\, x \in X\} \cup \{(1, y) \,|\, y \in Y\}$. $X + Y$ is an S_A-set with the S_A-action $\star : S_A \times (X + Y) \to (X + Y)$ defined by $\pi \star z = (0, \pi \cdot x)$ if $z = (0, x)$ and $\pi \star z = (1, \pi \diamond y)$ if $z = (1, y)$. If $(X, \cdot)$ and $(Y, \diamond)$ are nominal sets, then $(X + Y, \star)$ is also a nominal set: each $z \in X + Y$ is either of the form $(0, x)$ and so supported by the finite set supporting x in X, or of the form $(1, y)$ and so supported by the finite set supporting y in Y.*

Definition 2. *Let $(X, \cdot)$ be a nominal set. A subset Z of X is called* finitely supported *if and only if $Z \in \wp_{fs}(X)$ defined in Example 1(4).*

Since functions are particular relations, we have the following results.

Definition 3. *Let X and Y be nominal sets. A function $f : X \to Y$ is* finitely supported *if $f \in \wp_{fs}(X \times Y)$.*

We use the notation $Y^X = \{f \subseteq X \times Y \mid f \text{ is a function from } X \text{ to } Y\}$.

Proposition 2. *Let $(X, \cdot)$ and $(Y, \diamond)$ be nominal sets. Then Y^X is an S_A-set with the S_A-action $\star : S_A \times Y^X \to Y^X$ defined by $(\pi \star f)(x) = \pi \diamond (f(\pi^{-1} \cdot x))$ for all $\pi \in S_A$, $f \in Y^X$ and $x \in X$. A function $f : X \to Y$ is finitely supported in the sense of Definition 3 if and only if it is finitely supported with respect the permutation action $\star$.*

Proposition 3. *Let $(X, \cdot)$ and $(Y, \diamond)$ be nominal sets. Let $f \in Y^X$ and $\sigma \in S_A$ be arbitrary elements. Let $\star$ be the S_A-action on Y^X defined in Proposition 2. Then $\sigma \star f = f$ if and only if for all $x \in X$ we have $f(\sigma \cdot x) = \sigma \diamond f(x)$.*

Let us consider the set A of atoms in the ZFA framework. As in [11], we can take a set-theoretic approach and construct a single 'large' S_A-set, i.e. a class $FM(A)$ equipped with an S_A-action and in which all the elements have the finite support property. One benefit is that if a particular construction can be expressed in this language, then the action of finite permutations can be obtained from the universe $FM(A)$ without defining it explicitly, and without being necessary to prove the associated finite support property.

Recall the usual von Neumann cumulative hierarchy of sets:

$$\nu_0 = \emptyset, \quad \nu_{\alpha+1} = \wp(\nu_\alpha), \quad \nu_\lambda = \bigcup_{\alpha<\lambda} \nu_\alpha \ (\lambda \text{ a limit ordinal}).$$

More generally, it can be analogously defined a cumulative hierarchy of sets involving atoms from a certain set of atoms U, as in [11]:

- $\nu_0(U) = \emptyset$;
- $\nu_{\alpha+1}(U) = U + \wp(\nu_\alpha(U))$, where $+$ is the disjoint union from Example 1(6);
- $\nu_\lambda(U) = \bigcup_{\alpha<\lambda} \nu_\alpha(U)$.

Let $\nu(U)$ be the union of all $\nu_\alpha(U)$. The class of sets built on atoms U is $\nu(U)$. We define the notions of S_A-set and finite support property in such a hierarchy by considering U to be the S_A-set A of atoms, and replacing $\wp(-)$ by $\wp_{fs}(-)$ (using the notations of Example 1). Thus, the FM cumulative hierarchy is:

- $FM_0(A) = \emptyset$;
- $FM_{\alpha+1}(A) = A + \wp_{fs}(FM_\alpha(A))$;
- $FM_\lambda(A) = \bigcup_{\alpha<\lambda} FM_\alpha(A)$, where λ is a limit ordinal.

According to Example 1, each $FM_\alpha(A)$ is a nominal set. When we consider the union of all $FM_\alpha(A)$, we get one 'large' S_A-set (i.e. an S_A-class) in which every element has finite support property. The union of all $FM_\alpha(A)$ is called the *Fraenkel-Mostowski universe*; it is denoted by $FM(A)$. Using the notations of Example 1 and names *atm* and *set* for the functions $x \mapsto (0, x)$ and $x \mapsto (1, x)$, it follows that every element x of $FM(A)$ is either of the form $atm(a)$ with $a \in A$, or of the form $set(X)$ where X is a finitely supported set formed at an earlier ordinal stage than x. The elements of the form $set(X)$ are called *FM sets*, while the elements of the form $atm(a)$ are called *atoms*.

The S_A-action $\cdot$ on the FM universe $FM(A)$ is defined recursively by:

- $\pi \cdot atm(a) = atm(\pi(a))$
- $\pi \cdot set(X) = set(\{\pi \cdot x \,|\, x \in X\})$.

We can say that $x \in \nu(A)$ is an FM set (i.e. $x \in FM(A)$) if and only if the following conditions are satisfied:

- y is a FM set or an atom for all $y \in x$,
- x has finite support property.

Thus, we can say that a ZFA set is an FM set if and only if it has finite support and all its elements have hereditary finite supports. More precisely, any FM set is a finitely supported element of the large nominal set $FM(A)$ which additionally has a recursive property of finite support for its elements. An FM set X is not itself closed under the S_A-action on $FM(A)$, unless $supp(X) = \emptyset$. Hence an FM set is not necessarily equivariant in $FM(A)$ in the sense of Definition 1. This means that the restriction on a certain FM set X of the S_A-action $\cdot$ on $FM(A)$ does not necessarily lead to a new group action of S_A on X (since the codomain of the function $\cdot|_X$ is not necessarily X). Only an FM set with empty support is itself closed under the restriction of the S_A-action $\cdot$ on it. According to these remarks, and because nominal sets need to be closed under the actions with who they are equipped (meaning that nominality requires equivariance at the following order stage in an hierarchical construction), the nominal sets in the FM cumulative hierarchy are defined as those equivariant (i.e. empty supported) elements of the Fraenkel-Mostowski universe $FM(A)$. This means that an FM set X is nominal if and only if the restriction $\cdot|_X$ of $\cdot$ on X is itself an S_A-action on X in the sense of Definition 1(1).

Definition 4. *The following axioms define the Fraenkel-Mostowski set theory:*

1. $\forall x.(\exists y.y \in x)$ implies $x \notin A$ (only non-atoms can have elements)
2. $\forall x, y.(x \notin A \, and \; y \notin A \; and \; \forall z.(z \in x$ iff $z \in y))$ implies $x = y$ (axiom of extensionality)
3. $\forall x, y.\exists z.z = \{x, y\}$ (axiom of pairing)
4. $\forall x.\exists y.y = \{z \,|\, z \subseteq x\}$ (axiom of powerset)
5. $\forall x.\exists y.y \notin A \, and \, y = \{z \,|\, \exists w.(z \in w \, and \, w \in x)\}$ (axiom of union)
6. $\forall x.\exists y.(y \notin A \, and \, y = \{f(z) \,|\, z \in x\})$, *for each functional formula* $f(z)$ (axiom of replacement)
7. $\forall x.\exists y.(y \notin A \, and \, y = \{z \,|\, z \in x \, and \, p(z)\})$, *for each formula* $p(z)$ (axiom of separation)
8. $(\forall x.(\forall y \in x.p(y))$ implies $p(x))$ *implies* $\forall x.p(x)$ (induction principle)
9. $\exists x.(\emptyset \in x \, and \, (\forall y.y \in x$ implies $y \cup \{y\} \in x))$ (axiom of infinity)

10. *A is not finite.*

11. $\forall x.\, \exists\, S \subset A.\, S$ *is finite and* S *supports* x. (finite support axiom)

It follows that $\nu(A)$ is a model of ZFA set theory (whose axioms are precisely those in Definition 4, excepting axiom 11), and $FM(A)$ is a model of FM set theory.

As mentioned in the opening paragraph of Section 2, FM axiomatic set theory is inspired by the basic Fraenkel model of ZFA. On the other hand, FM set theory, ZFA set theory and ZF set theory are independent axiomatic set theories. All of these theories are described by axioms, and all of them have distinct models. For example, $FM(A)$ is a model of FM set theory, while detailed lists of Cohen models of ZF and Fraenkel-Mostowski permutation models of ZFA can be found in [12] (see also [13] for detailed descriptions and proofs of results on certain ZF and ZFA models).

3 Logical Notions in the FM Cumulative Universe

We start with a lemma allowing us to say that in the FM universe the permutations of the sets of atoms are necessarily finite permutations.

Lemma 1. *Let $f : A \to A$ be a finitely supported permutation of A. Then $\{a \in A \,|\, f(a) \neq a\}$ is finite, and $supp(f) = \{a \in A \,|\, f(a) \neq a\}$.*

Proof. First we prove that for each $a \in A$, if $a \notin supp(f)$ then $f(a) = a$. Let $a \notin supp(f)$. Assume that $f(a) \neq a$. Let us consider two atoms $b, c \notin supp(f)$ such that a, b, c, are all different (such atoms exist because $supp(f)$ is finite, while A is infinite). Since $supp(f)$ supports f and $(a\, b) \in Fix(supp(f))$, we have $(a\, b) \star f = f$, where $\star$ is the S_A-action on A^A presented in Proposition 2. Analogously, $(a\, c) \star f = f$. According to Proposition 3, we have $f(b) = f((a\, b)(a)) = (a\, b)(f(a))$. However, $f(a) \neq a$. Since f is an injection, it follows that $f(a) = b$ (otherwise, we would have $f(b) = f(a)$ with $b \neq a$). However, from $f((a\, c)(a)) = (a\, c)(f(a))$, it follows that $f(c) = (a\, c)(b) = b = f(a)$, which contradicts the injectivity of f. Thus, $f(a) = a$. This means $S = \{a \in A \,|\, f(a) \neq a\} \subseteq supp(f)$. Since $supp(f)$ is finite, it follows that S is finite.

Now we prove that the finite set S supports f. Indeed, let us consider $\pi \in Fix(S)$, i.e. $\pi(a) = a$ whenever $f(a) \neq a$. We claim that $f(\pi(x)) = \pi(f(x))$ for all $x \in A$. Indeed, let us fix an arbitrary element $x \in A$. If $f(x) \neq x$, then $\pi(x) = x$ and $f(\pi(x)) = f(x)$. However, since f is injective, we also have $f(f(x)) \neq f(x)$ and so $\pi(f(x)) = f(x)$. Thus, $f(\pi(x)) = \pi(f(x))$. On the other hand, if $f(x) = x$, then $\pi(f(x)) = \pi(x)$. Suppose that $f(\pi(x)) \neq \pi(x)$. This means $\pi(\pi(x)) = \pi(x)$, and so $\pi(x) = x$. Then $f(\pi(x)) = f(x) = x = \pi(x)$, which contradicts the assumption that $f(\pi(x)) \neq \pi(x)$. It follows that $f(\pi(x)) = \pi(x)$, and so $f(\pi(x)) = \pi(f(x))$. According to Proposition 3, we have that S supports f. Since $supp(f)$ is minimal between the finite sets supporting f, it follows that $S = supp(f)$. ∎

According to Lemma 1 a permutation of (the nominal set) A is finitely supported if and only if it is a finite permutation. Since in the FM universe $FM(A)$ any element has to be finitely supported, we conclude that any permutation of A which is an element of $FM(A)$

has to be finitely supported. Therefore, by Lemma 1, any such permutation of A is an element of S_A. Thus, we get the following corollary to Lemma 1.

Corollary 1. *In the Fraenkel-Mostowski universe $FM(A)$, the set S_A of all finite permutations of A is exactly the set of all permutations of A which belong to $FM(A)$. Thus, in $FM(A)$, S_A coincides with the set of all permutations of A in Tarski's view (i.e. with the set of all one-to-one transformations of A onto itself).*

Note that the basic Fraenkel model $\mathcal{N}$ is determined by a countably infinite set A of atoms, the group G of all permutations of A, and the finite support (normal) filter Γ. If one considers S_A (i.e. the group of all permutations of A which move only finitely many atoms) instead of G, then the resulting permutation model $\mathcal{N}'$ determined by A, S_A and Γ is equal to $\mathcal{N}$. Corollary 1 is the reason for considering S_A-actions rather than G-actions in the construction of $FM(A)$.

The Fraenkel-Mostowski approach corresponds to Tarski's view. In order to define the cumulative Fraenkel-Mostowski universe $FM(A)$, we started with a collection of basic objects (set A of atoms) and constructed a cumulative hierarchy of sets above them. According to the recursive definition of the S_A action $\cdot$ on $FM(A)$, we can say that an element having the form $\pi \cdot x$ (where $\pi \in S_A$ and $x \in FM(A)$) is a new element $y \in FM(A)$ obtained by replacing each atom a from the structure of x by $\pi(a)$. Thus, an element of the form $\pi \cdot x$ can be associated with 'the effect of the transformation π on the element x' in Tarski's view. We conclude that the empty-supported elements in $FM(A)$ are invariant under all permutations of A, and so the related elements are *logical notions*. We can present this in a more formal way.

Theorem 1. *Nominal sets defined in the FM cumulative hierarchy are logical (in Tarski's sense). In particular, the nominal set S_A of all finite permutations of A is logical (in Tarski's sense).*

The FM sets, i.e. the arbitrary elements from the FM cumulative universe, are not necessary logical in Tarski's sense. They satisfy only a "weak" form of logicality, meaning that they are fixed only by those permutations satisfying an additional requirement. More precisely, an FM set x is invariant under all permutations fixing its support pointwise. Furthermore, this is the "strongest" possible form of invariance because the support of an element is the least set supporting it. However, given a nominal set X from the FM cumulative universe, the set of all finitely supported subsets of X (generally denoted by $\wp_{fs}(X)$) is logical, i.e. invariant under all permutations of atoms. This follows by a direct refinement of Proposition 1 (adapted for the FM cumulative universe) which states than for any finitely supported subset Y of X we have that $\pi \star Y$ is supported by $\pi \star supp(Y)$ for any $\pi \in S_A$. Thus, the set of all finitely supported subsets of X is closed under the effect of any permutation from S_A. We describe this formally in the following result.

Proposition 4. *Let X be a nominal set from the FM cumulative hierarchy. Then the set $\wp_{fs}(X)$ of all finitely supported subsets of X is logical (in Tarski's sense).*

In the framework of FM set theory, a new quantifier И is introduced in [11]. Formally, if P is a predicate over A, we say that $\text{И}a.P(a)$ is true if $P(a)$ is true for all but finitely many elements of A. In a formal language, the extension of the quantifier И consists of all cofinite subsets of the domain A. Obviously, all the one-to-one mappings of A onto itself transform any cofinite subset of the domain in another cofinite subset of the domain. Therefore, the interpretation of the quantifier И is invariant under every permutation. We describe this formally in the following result.

Theorem 2. *The quantifier* И *of the FM set theory is logical (in Tarski's sense).*

4 Logical Notions in Semantics of Process Calculi

The particular class of FM sets formed by those FM sets which are furthermore logical notions (in [1] the related equivariant FM sets are called IFM sets) can be used in defining new semantics for various process calculi. More precisely, in [1], by using the IFM sets, we were able to present more compact semantics (formed by transition rules presented without assuming additional freshness/side conditions) for the π-calculus, πI-calculus and fusion calculus. The central idea was to use the IFM set of atoms in order to represent variable symbols, and the nominal abstraction defined in [11] to represent the binding operators in these process calculi. The terms (processes) in each of these process calculi form an IFM set, and the set of terms modulo α-conversion in each of these process calculi can also be represented as an IFM set. A mixture of $\forall$ and И quantifiers was used to replace the side conditions in the transition rules of the previously mentioned process calculi.

We present an example of how a transition rule in the (monadic) fusion calculus is rephrased in the FM framework by using Tarski logical symbols.

The rule U-PASS in the original semantics uc of the fusion calculus

$$\frac{P \xrightarrow[uc]{[x/y]} P'}{(z)P \xrightarrow[uc]{[x/y]} (z)P'},\ z \neq x, y\,;\ x \neq y$$

becomes the following rule in the new "logical" FM semantics nuc of the fusion calculus.

$$\forall x.\text{И}y.\text{И}z.\forall P, P'.\frac{P \xrightarrow[nuc]{[x/y]} P'}{[z]P \xrightarrow[nuc]{[x/y]} [z]P'}\ .$$

The update action in the monadic fusion calculus, generally denoted by $[z/t]$ indicates the replacement of all t by z in both uc and nuc. In uc, the scope operator $(x)Q$ limits the scope of x to Q; scopes can be used to delimit the extent of updates (that is, the update effects with respect to x are limited to Q). In nuc, the bindings represented by the scope operator are associated to nominal abstractions. Technical details can be found in [1] or [2]. We were able to prove that the new semantics are equivalent with (i.e. they have the same expressive power as) the original semantics of the related process calculi. However, the newly defined semantics of these process calculi were presented by involving only logical notions and symbols, whereas the original (old) semantics of these process calculi were presented by assuming additional freshness conditions for each transition rule, and so they were not logical.

5 Conclusion and Related Work

The plurality of geometrical systems raises the question which of these systems is the “true” geometry. Felix Klein developed a mathematical framework in which, by studying the properties remaining invariant under different transformations, it is possible to classify systematically various geometries. Similarly to what happens in geometry, the plurality of set theories raises the question which of these is the “true” one. From the point of view of pure mathematics, every mathematical theory is acceptable; this does not mean that each such theory is equally significant or mathematically relevant. Somehow similarly to Klein’s approach, Tarski defined the logical notions as those invariant under all possible one-one transformations of the universe of discourse onto itself. The question is what are the set theoretical notions appropriate to the Tarski logical notions. The answer is important because the language of set theory is able to express the whole mathematics. In some sense, the question is whether certain foundational mathematical notions are logical.

The aim of this paper was to establish a connection between the theory of nominal sets and the concept of logical notion presented by Tarski. We know that nominal sets can be defined both in ZF and FM set theories, and they have similar properties in these two frameworks. More exactly, the ZF nominal sets can similarly be represented in the FM framework, and the ZF properties of nominal sets can be naturally translated in FM (see [1]). Here we proved that those nominal sets defined in the FM set theory have a special property. More precisely, we proved that any nominal set defined in the FM cumulative hierarchy (i.e. any equivariant FM set) is a logical notion according to Tarski’s view. Moreover, the new defined quantifier И (associated to the FM set theory) is logical. This means that the criterion of Tarski’s logical notions corresponds properly to the new operation of finding fresh names in syntax. Informally, we can think of the elements of a nominal set as having a finite set of ‘free names’ (which are related to the notions of ‘support’). The action of a permutation on such an element actually represents the renaming of the ‘bound names’. Thus, the notion of renaming (often used in computer science) is related to the concept of logical notion (in Tarski’s sense) when it is managed by involving the FM set theory.

Related to the results presented in this paper, we developed a new set theory which is consistent with Tarski’s idea regarding logicality and which deals with a more relaxed notion of finiteness [1]. We called it the ‘Finitely Supported Mathematics’ (FSM for short). Informally, in FSM we can model infinite structures by using a finite number of observations. Moreover, we (re)describe some parts of algebra by replacing ‘(infinite) sets’ with the so called ‘invariant sets’, where the invariant sets are related to the concept of logical notions in Tarski’s sense. This approach allows us to work with infinite structures by only using their finite supports. More exactly, in FSM we admit the existence of infinite structures, but for any infinite structure only of a finite number of its elements (i.e. its support) is “really important” in order to characterize the whole structure, while the other elements are somehow “similar”.

Acknowledgements. The authors are grateful to the anonymous reviewer for several comments and suggestions which improved the paper.

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Received: 14.07.2017
Revised: 11.03.2017
Accepted: 04.04.2017

Institute of Computer Science
Romanian Academy, Iaşi
E-mail:
andrei.alexandru@iit.academiaromana-is.ro
gabriel@info.uaic.ro