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A rigorous derivation of certain equations arising in the lifting wing theory

by

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Abstract

Multhopp's lifting surface equation is obtained after passing to the limit in the z-component of the fluid velocity. The difficulty occurs because the involved integrals become singular. Some authors either ignore this issue or they use, without any justification in their proofs, interchanges of limit and integration or differentiation operations. In this paper, we prove rigorously the validity of the limit in the frame of the distribution theory.

Key Words: lifting wing, singular integral, theory of distribution
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1 Introduction

We consider a steady, incompressible, inviscid flow about an infinitely thin wing of finite span. Far upstream, the fluid velocity is a constant $\mathbf{V}_\infty$. Let L_0 be a characteristic length (maximum semichord of the wing sections, for instance), and p_∞, ρ be the fluid pressure and, respectively,the mass density far upstream. We work with dimensionless quantities. Let (x_1, y_1, z_1) be the dimensional Cartesian coordinates and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the unit vectors parallel to the axis Ox_1, Oy_1 and Oz_1, respectively. The direction of Ox_1 is taken parallel to the direction of the undisturbed flow. If $\mathbf{v}$ and p are the dimensionless perturbation velocity and, respectively, dimensionless perturbation pressure, then the dimensional quantities, marked with subscript "1", are related to the dimensionless quantities through the relations:

$$(x_1, y_1, z_1) = L_0(x, y, z), \quad p_1 = p_\infty + \rho V_\infty^2 p, \quad \mathbf{v}_1 = V_\infty(\mathbf{i} + \mathbf{v}). \tag{1.1}$$

We deal with the following hypothesis:

Assumption 1: Wing geometry (Figure 1)
The wing surface W is slightly deviated from a surface S lying on a cylindrical surface Σ, with the directrix parallel to the undisturbed flow and with the generatrix $\mathcal{C}$ beeing a simple curve of class C^2 situated in a plane perpendicular to Ox. The parametric equations of the generatrix $\mathcal{C}$ and of the surfaces S and W are:

$$\mathcal{C}: \quad y = y_s(\eta), \qquad z = z_s(\eta), \qquad -\theta \le \eta \le \theta, \tag{1.2}$$

$$S: \begin{cases} x = \xi, \\ y = y_s(\eta), (\xi, \eta) \in D, \\ z = z_s(\eta), \end{cases} \quad W: \begin{cases} x = \xi, \\ y = y_s(\eta) + y_\varepsilon(\xi, \eta), (\xi, \eta) \in D, \\ z = z_s(\eta) + z_\varepsilon(\xi, \eta), \end{cases} \tag{1.3}$$

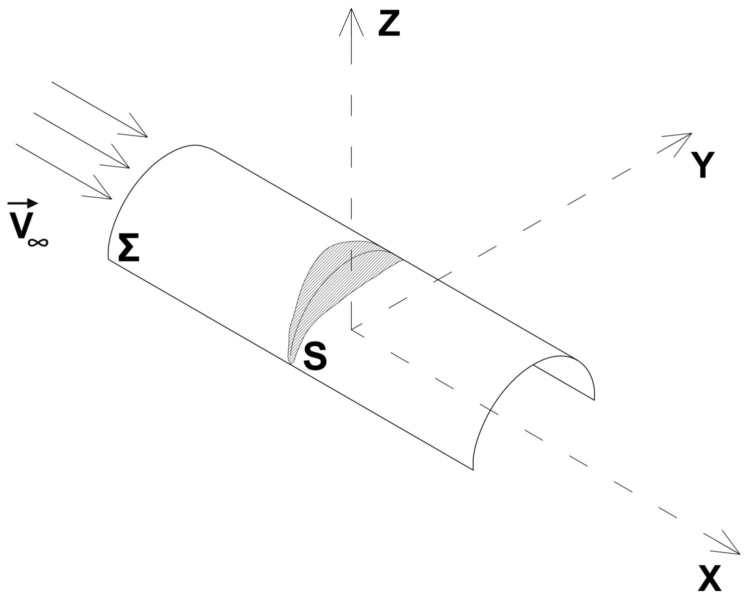

Figure 1: Wing geometry

where $\varepsilon \ll 1$ is a parameter that models the magnitude of the deviation of the wing surface W from the cylindrical surface Σ.

We assume that y_ε and z_ε are smooth functions such that $|y_\varepsilon|$, $|z_\varepsilon|$,$\left|\dfrac{\partial y_\varepsilon}{\partial \xi}\right|$, $\left|\dfrac{\partial y_\varepsilon}{\partial \eta}\right|$, $\left|\dfrac{\partial z_\varepsilon}{\partial \xi}\right|$ and $\left|\dfrac{\partial z_\varepsilon}{\partial \eta}\right|$ are of the order of $\mathcal{O}(\varepsilon)$ on D.

If the surface Σ is contained in the xy-plane (i.e. $z_s = 0$), we are in the classical case of planar wing. Otherwise, W is a nonplanar wing.

The components of the unit normal to the wing surface W are:

$$\begin{aligned} n_x^W &= \frac{\frac{\partial y_\varepsilon}{\partial \xi} z_s'(\eta) - \frac{\partial z_\varepsilon}{\partial \xi} y_s'}{\pm\sqrt{y_s'^2(\eta) + z_s'^2(\eta)}} + \mathcal{O}(\varepsilon^2), \\ n_y^W &= \frac{-z_s'(\eta)}{\pm\sqrt{y_s'^2(\eta) + z_s'^2(\eta)}} + \mathcal{O}(\varepsilon), \\ n_z^W &= \frac{y_s'(\eta)}{\pm\sqrt{y_s'^2(\eta) + z_s'^2(\eta)}} + \mathcal{O}(\varepsilon). \end{aligned} \tag{1.4}$$

Assumption 2: Small perturbations hypothesis

We assume that, throughout the fluid, $|p| \ll 1$ and $|\mathbf{v}| \ll 1$. More precisly, we suppose that the perturbations p and $\mathbf{v}$ are of the order of $\mathcal{O}(\varepsilon)$, and that they vanish far downstream, i.e.

$$\lim \mathbf{v} = \mathbf{0}, \quad \lim p = 0, \quad \text{as } \|(x,y,z)\| \to \infty. \tag{1.5}$$

Assumption 3: Lifting wing
There is a pressure jump over S, so if we denote p_+ and p_- the perturbation pressure on the outer side and, respectively, on the inner side of the surface S, we have

$$[[p]] = p_+ - p_- = f, \text{ on } S. \tag{1.6}$$

Assumption 4: Linearized equations
The linearized equations, around the undisturbed flow, governing the perturbed motion (see [3, 7]), are

$$\begin{cases} \operatorname{div}\mathbf{v} = 0, \\ \dfrac{\partial \mathbf{v}}{\partial x} + \operatorname{grad}p = 0. \end{cases} \tag{1.7}$$

The slip condition $\mathbf{v}_1 \cdot \mathbf{n}_1 = 0$ on the wing is written in dimensionless variables

$$(1+u)n_x^W + vn_y^W + wn_z^W = 0 \text{ on } W. \tag{1.8}$$

Taking into account relations (1.4) and neglecting the products of the perturbations, we obtain the linearized slip condition:

$$y_s^{'}w - z_s^{'}v = \frac{\partial z_\varepsilon}{\partial \xi}y_s^{'} - \frac{\partial y_\varepsilon}{\partial \xi}z_s^{'} \text{ on } S. \tag{1.9}$$

The direct problem of aerodynamics consists in determining the perturbation $\{\mathbf{v}, \text{p}\}$ and the action of the fluid against the wing. In the planar case, the problem was studied by Multhopp [9]. In his theory, the wing is replaced by a sheet of doublets with their axes parallel to z-axis. Ashley and Landahl ([1], Chapter 11) extended Multhopp's theory to the nonplanar wing.
Dragoş [3] and Homentcovschi [7] utilized the fundamental solutions of the system of linearized equations (1.7) and (1.6). In all these approaches, after imposing the slip condition (1.9), a hypersingular integral equation regarding the jump pressure f is obtained.
The rest of the paper is organized as follows: in the next section we extend the *method of fundamental solutions* to the case of the nonplanar wing in order to express the perturbation in terms of the jump pressure f and in the third section, for the planar wing, we derive Multhopp's lifting surface equation, in the frame of theory of distributions.

2 Perturbation velocity and pressure fields

In distributional sense, the equations (1.7) subject to (1.6), can be written (see [4], [7]):

$$\begin{cases} \operatorname{div}\mathbf{v} = 0, \\ \dfrac{\partial \mathbf{v}}{\partial x} + \operatorname{grad}p = f\delta_S\mathbf{n}, \end{cases} \tag{2.1}$$

where $\mathbf{n} = (n_x, n_y, n_z)$ is the outward unit normal to the surface S and $f\delta_S$ represents the simple layer distribution with density f. Here, n_x=0.

The system of equations (2.1) was solved by Dragoş ([4], Chapter 2), by means of the Fourier transform, for some distribution $\mathbf{f}$ in the right hand side of the equation $(2.1)_2$. For our particular case, we obtain the perturbation pressure in fluid

$$p = \frac{\partial}{\partial \mathbf{n}} (f\delta_S) * \mathcal{E} = -\frac{1}{4\pi} \iint_S f(x_1, y_1, z_1) \frac{\partial}{\partial \mathbf{n_1}} \left(\frac{1}{|\mathbf{r} - \mathbf{r_1}|} \right) \mathrm{d}\sigma_1 \tag{2.2}$$

and the perturbation velocity of the fluid

$$\mathbf{v} = H(x) * f\delta_S \mathbf{n} + \mathrm{grad}\varphi, \tag{2.3}$$

where $\mathcal{E} = -\frac{1}{4\pi r}$ is the fundamental solution of Laplace operator, H is the Heaviside step function and

$$\varphi = -\left(f\delta_S n_y * \frac{\partial}{\partial y} + f\delta_S n_z * \frac{\partial}{\partial z} \right) \int_{-\infty}^{x} \mathcal{E}\mathrm{d}x \tag{2.4}$$

is the perturbation velocity potential.
Since

$$\frac{\partial}{\partial y} \int_{-\infty}^{x} \mathcal{E}\mathrm{d}x = \frac{1}{4\pi} \frac{y}{y^2 + z^2} \left(1 + \frac{x}{r}\right) \text{ and } \frac{\partial}{\partial z} \int_{-\infty}^{x} \mathcal{E}\mathrm{d}x = \frac{1}{4\pi} \frac{z}{y^2 + z^2} \left(1 + \frac{x}{r}\right), \tag{2.5}$$

we derive

$$\varphi(x, y, z) = \frac{-1}{4\pi} \iint_S f(x_1, y_1, z_1) \frac{(\mathbf{r} - \mathbf{r_1}) \cdot \mathbf{n_1}}{(y - y_1)^2 + (z - z_1)^2} \left(1 + \frac{x - x_1}{|\mathbf{r} - \mathbf{r_1}|}\right) \mathrm{d}\sigma_1. \tag{2.6}$$

The formula (2.6) is similar to the formula (11-5) from [1].

Remark 1. *i) From the equation (2.2), we see that the pressure field is represented as a continuous superposition of doublets all over S, having their axes oriented in the direction of the local normal vector.*
ii) Since supp$(f * g) \subset$ supp$f$+supp$g$, from the equation (2.3) we deduce that the flow is everywhere potential, except of a subset $\Sigma' \subset \Sigma$ (the wake behind the wing) starting from the trailing edge and extending far downstream.
iii) The aforementioned facts were postulated by Multhopp [9] for planar wing, and by Ashley and Landahl ([1], Chapter 11), for nonplanar wing. As in the planar case (see [3]), using the method of fundamental solution, they are derived only from the equations of fluid motion.

3 Derivation of the lifting surface equation in the case of the planar wing

For the planar wing, we denote with D the orthogonal projection onto the xy-plane of the wing surface W, called wing planform. Let

$$W : z = h(x, y), \quad (x, y) \in D, \tag{3.1}$$

be the cartesian equation of the wing surface W.

In the relations (2.2),(2.6) we set

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \quad \mathbf{r}_1 = \xi\mathbf{i} + \eta\mathbf{j}, \tag{3.2}$$

and we denote

$$x_0 = x - \xi, \qquad y_0 = y - \eta, \qquad R_1 = |r - r_1| = \sqrt{x_0^2 + y_0^2 + z^2}. \tag{3.3}$$

Particularly, from (2.3) and (2.6) we obtain w, the z-component of the perturbation velocity of the fluid,

$$\begin{aligned} w(x,y,z) = &-\frac{1}{4\pi}\iint_D f(\xi,\eta)\frac{y_0^2 - z^2}{(y_0^2+z^2)^2}\left(1 + \frac{x_0}{R_1}\right)\mathrm{d}\xi\mathrm{d}\eta + \\ &+\frac{z^2}{4\pi}\iint_D \frac{f(\xi,\eta)x_0}{(y_0^2+z^2)R_1^3}\mathrm{d}\xi\mathrm{d}\eta. \end{aligned} \tag{3.4}$$

The formula (3.4) is similar to the formula (5.1.11) from [3].
The slip condition (1.9) becomes

$$w(x,y,0) = \frac{\partial h}{\partial x}(x,y), (\forall)(x,y) \in D. \tag{3.5}$$

If we set $z = 0$ in (3.4), the integrals become singular. Having in mind the definition of the *finite part* in the Hadamard sense of the integral (see Fox [5]), we introduce the following singular distribution:

Definition 1. *For any test function* $\varphi \in \mathcal{D}\left(\mathbb{R}^2\right)$*, supp* $\varphi \subset [-A, A]^2$*, we define*

$$\begin{aligned} &\left\langle \mathcal{FP}\frac{1}{y^2}\left(1 + \frac{x}{\sqrt{x^2+y^2}}\right), \varphi\right\rangle = \iint_D^* \frac{\varphi(x,y)}{y^2}\left(1 + \frac{x}{\sqrt{x^2+y^2}}\right)\mathrm{d}x\mathrm{d}y = \\ &= \lim_{\varepsilon\to 0_+}\left[\iint_{D_\varepsilon}\frac{\varphi(x,y)}{y^2}\left(1 + \frac{x}{\sqrt{x^2+y^2}}\right)\mathrm{d}x\mathrm{d}y - \frac{2}{\varepsilon}\int_{-A}^{A}\varphi(x,0)(1 + \mathrm{sgn}x)\mathrm{d}x\right] \end{aligned} \tag{3.6}$$

where $D_\varepsilon = [-A, A] \times \{[-A, A] \setminus [-\varepsilon, \varepsilon]\}$.

Lemma 1. *For any test function* $\varphi \in \mathcal{D}\left(\mathbb{R}^2\right)$*, supp* $\varphi \subset [-A, A]^2$*, the following formula holds true:*

$$\begin{aligned} &\left\langle \mathcal{FP}\frac{1}{y^2}\left(1 + \frac{x}{\sqrt{x^2+y^2}}\right), \varphi\right\rangle = \\ &= \int_{-A}^{A}\int_{-A}^{A} \frac{\varphi(x,y) - \varphi(x,0) - y\dfrac{\partial\varphi}{\partial y}(x,0)}{y^2}\left(1 + \frac{x}{\sqrt{x^2+y^2}}\right)\mathrm{d}x\mathrm{d}y - \\ &- \frac{2}{A}\int_{-A}^{A}\frac{\varphi(x,0) - \varphi(0,0)}{x}\left(x + \sqrt{A^2+x^2}\right)\mathrm{d}x - 4\varphi(0,0). \end{aligned} \tag{3.7}$$

Proof: We extract the singular term from equation (3.6). For any test function $\varphi \in \mathcal{D}\left(\mathbb{R}^2\right)$, supp $\varphi \subset [-A, A]^2$, we have

$$\begin{aligned}
&\iint_{D_\varepsilon} \frac{\varphi(x,y)}{y^2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)\mathrm{d}x\mathrm{d}y - \frac{2}{\varepsilon}\int_{-A}^{A}\varphi(x,0)(1+\mathrm{sgn}x)\mathrm{d}x = \\
&= \iint_{D_\varepsilon} \frac{\varphi(x,y)-\varphi(x,0)-y\frac{\partial\varphi}{\partial y}(x,0)}{y^2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)\mathrm{d}x\mathrm{d}y+ \\
&+\iint_{D_\varepsilon} \frac{y\frac{\partial\varphi}{\partial y}(x,0)}{y^2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)\mathrm{d}x\mathrm{d}y+ \\
&+\iint_{D_\varepsilon} \frac{\varphi(x,0)}{y^2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)\mathrm{d}x\mathrm{d}y - \frac{2}{\varepsilon}\int_{-A}^{A}\varphi(x,0)(1+\mathrm{sgn}x)\mathrm{d}x.
\end{aligned} \tag{3.8}$$

The first integral from the right hand side of the equation (3.8) is regular. Since the domain D_ε is symmetric with respect to the x-axis and the integrand is an odd function with respect to y, we deduce that

$$\iint_{D_\varepsilon} \frac{y\frac{\partial\varphi}{\partial y}(x,0)}{y^2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)\mathrm{d}x\mathrm{d}y = 0.$$

The last two integrals from (3.8) can be written as

$$\begin{aligned}
&\iint_{D_\varepsilon} \frac{\varphi(x,0)}{y^2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)\mathrm{d}x\mathrm{d}y - \frac{2}{\varepsilon}\int_{-A}^{A}\varphi(x,0)(1+\mathrm{sgn}x)\mathrm{d}x = \\
&= \int_{-A}^{A}\varphi(x,0)\left[\frac{-2\varepsilon(x+\sqrt{A^2+x^2})+2A(x+\sqrt{\varepsilon^2+x^2})}{A\varepsilon x} - \frac{2(1+\mathrm{sgn}x)}{\varepsilon}\right]\mathrm{d}x = \\
&= \int_{-A}^{A}\frac{\varphi(x,0)-\varphi(0,0)}{x}\left[\frac{-2\varepsilon(x+\sqrt{A^2+x^2})+2A(x+\sqrt{\varepsilon^2+x^2})}{A\varepsilon} - \frac{2x(1+\mathrm{sgn}x)}{\varepsilon}\right]\mathrm{d}x \\
&\qquad - 4\varphi(0,0).
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{-2\varepsilon(x+\sqrt{A^2+x^2})+2A(x+\sqrt{\varepsilon^2+x^2})}{A\varepsilon} - \frac{2x(1+\mathrm{sgn}x)}{\varepsilon} = -\frac{2}{A}\left(x+\sqrt{A^2+x^2}\right)+ \\
&+\frac{2\varepsilon}{|x|+\sqrt{x^2+\varepsilon^2}} \quad \text{and} \quad \left|\frac{2\varepsilon}{|x|+\sqrt{x^2+\varepsilon^2}}\right| < 2, \text{ for } \varepsilon>0, A>0, x\in[-A,A],
\end{aligned}$$

we complete the proof after applying Lebesgue's dominated convergence theorem. ∎

Lemma 2. *In the space of distributions $\mathcal{D}'(\mathbb{R}^2)$, the following limits are true:*

$$\begin{aligned}
&a)\ \lim_{z\to 0}\frac{y^2-z^2}{(y^2+z^2)^2}\left(1+\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = \mathcal{FP}\frac{1}{y^2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right); \\
&b)\ \lim_{z\to 0}\frac{xz^2}{(y^2+z^2)(x^2+y^2+z^2)^{3/2}} = 0.
\end{aligned} \tag{3.9}$$

Proof: **a)** Let $\varphi \in \mathcal{D}\left(\mathbb{R}^2\right)$, supp $\varphi \subset [-A, A]^2$, be a test function.
We denote $r = \sqrt{x^2 + y^2 + z^2}$. Then,

$$\begin{aligned}
&\left\langle \frac{y^2 - z^2}{(y^2 + z^2)^2}\left(1 + \frac{x}{r}\right), \varphi \right\rangle = \int\limits_{-A}^{A}\int\limits_{-A}^{A} \varphi(x,y)\frac{y^2 - z^2}{(y^2 + z^2)^2}\left(1 + \frac{x}{\mathrm{r}}\right)\mathrm{d}x\mathrm{d}y = \\
&= \int\limits_{-A}^{A}\int\limits_{-A}^{A} \frac{\varphi(x,y) - \varphi(x,0) - y\dfrac{\partial\varphi}{\partial y}(x,0)}{y^2}\ \frac{y^2(y^2 - z^2)}{(y^2 + z^2)^2}\left(1 + \frac{x}{r}\right)\mathrm{d}x\mathrm{d}y + \\
&+ \int\limits_{-A}^{A}\int\limits_{-A}^{A} \varphi(x,0)\ \frac{(y^2 - z^2)}{(y^2 + z^2)^2}\left(1 + \frac{x}{r}\right)\mathrm{d}x\mathrm{d}y + \\
&+ \int\limits_{-A}^{A}\int\limits_{-A}^{A} \frac{\partial\varphi}{\partial y}(x,0)\ \frac{y(y^2 - z^2)}{(y^2 + z^2)^2}\left(1 + \frac{x}{r}\right)\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{3.10}$$

When $z \to 0$, the limit of the first integral from (3.10) is

$$\int\limits_{-A}^{A}\int\limits_{-A}^{A} \frac{\varphi(x,y) - \varphi(x,0) - y\dfrac{\partial\varphi}{\partial y}(x,0)}{y^2}\left(1 + \frac{x}{\sqrt{x^2 + y^2}}\right)\mathrm{d}x\mathrm{d}y.$$

Due to symmetry reasons, the last integral from (3.10) is zero. If we denote

$$\begin{aligned}
\Psi(x,z) &= \int_{-A}^{A} \frac{y^2 - z^2}{(y^2 + z^2)^2}\left(1 + \frac{x}{r}\right)\mathrm{d}y = \\
&= -\frac{2A(x + \sqrt{A^2 + x^2 + z^2})}{x(A^2 + z^2)} + \frac{2z}{x^2}\mathrm{arctg}\left[\frac{Ax}{z\sqrt{A^2 + x^2 + z^2}}\right],
\end{aligned}$$

the second integral from (3.10) can be written as

$$\begin{aligned}
&\int\limits_{-A}^{A} \varphi(x,0)\left(\int\limits_{-A}^{A} \frac{y^2 - z^2)}{(y^2 + z^2)^2}\left(1 + \frac{x}{r}\right)\mathrm{d}y\right)\mathrm{d}x = \int_{-A}^{A} \varphi(x,0)\Psi(x,z)\mathrm{d}x = \\
&= \int_{-A}^{A} \frac{\varphi(x,0) - \varphi(0,0)}{x} x\Psi(x,z)\mathrm{d}x + \varphi(0,0)\int_{-A}^{A} \Psi(x,z)\mathrm{d}x.
\end{aligned}$$

Since

$$\varphi(0,0)\int_{-A}^{A} \Psi(x,z)\mathrm{d}x = -\frac{4A^2\varphi(0,0)}{A^2 + z^2} \xrightarrow{z \to 0} -4\phi(0,0),$$

$$\left|\frac{z}{x}\mathrm{arctg}\frac{Ax}{z\sqrt{A^2 + x^2 + z^2}}\right| \leq \frac{A}{\sqrt{A^2 + x^2 + z^2}},\ \text{for } \varepsilon > 0, A > 0, x \in [-A, A]$$

and

$$\lim_{z\to 0}\frac{z}{x}\operatorname{arctg}\frac{Ax}{z\sqrt{A^2+x^2+z^2}}=0 \text{ a.e.},$$

it follows that

$$\lim_{z\to 0}\int_{-A}^{A}\int_{-A}^{A}\varphi(x,0)\ \frac{y^2-z^2)}{(y^2+z^2)^2}\left(1+\frac{x}{r}\right)\mathrm{d}x\mathrm{d}y=$$

$$=-\frac{2}{A}\int_{-A}^{A}\frac{\varphi(x,0)-\varphi(0,0)}{x}(x+\sqrt{A^2+x^2})\mathrm{d}x-4\varphi(0,0).$$

Based on Lemma 1, the first part of Lemma 2 is proven.

b) Let φ be a test function, $\operatorname{supp}\varphi\subset[-A,A]^2$, and let $a>0$ be a small enough number such that the disk of radius a centered at the origin, denoted D_a, is contained in the square $[-A,A]^2$. If we set

$$r=\sqrt{x^2+y^2+z^2},\quad \rho=\sqrt{x^2+y^2}\quad\text{and}\quad \psi(x,y)=\frac{\varphi(x,y)-\varphi(0,0)}{\rho},$$

then

$$\left\langle\frac{xz^2}{(y^2+z^2)r^3},\varphi\right\rangle=\int_{-A}^{A}\int_{-A}^{A}\varphi(x,y)\frac{z^2}{y^2+z^2}\frac{x}{r^3}\mathrm{d}x\mathrm{d}y=$$

$$=\left(\iint_{[-A,A]^2\setminus D_a}+\iint_{D_a}\right)\varphi(x,y)\frac{z^2}{y^2+z^2}\frac{x}{r^3}\mathrm{d}x\mathrm{d}y.$$

When $z\to 0$, the limit of the first integral is zero. Furthermore,

$$\iint_{D_a}\varphi(x,y)\frac{z^2}{y^2+z^2}\frac{x}{r^3}\mathrm{d}x\mathrm{d}y=\iint_{D_a}\psi(x,y)\frac{z^2}{y^2+z^2}\frac{x}{r}\frac{\rho^2}{r^2}\frac{1}{\rho}\mathrm{d}x\mathrm{d}y+$$

$$+\varphi(0,0)\iint_{D_a}\frac{z^2}{y^2+z^2}\frac{x}{r}\frac{1}{r^2}\mathrm{d}x\mathrm{d}y.$$

Due to symmetry reasons, the second integral is zero. Since ψ is bounded, the proof ends after we apply again the Lebesgue's theorem in the first integral. □

The main result of this paper is stated in the following theorem:

Theorem 1. *If f is a locally integrable function with compact support, then*

$$\lim_{z\to 0}w(x,y,z)=-\frac{1}{4\pi}\overset{*}{\iint_D}\frac{f(\xi,\eta)}{(y-\eta)^2}\left[1+\frac{x-\xi}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\right]\mathrm{d}\xi\mathrm{d}\eta,\tag{3.11}$$

in distributional sense.

Proof: Since the regular distribution f has compact support and

$$w(x,y,z) = f * \frac{y^2 - z^2}{(y^2+z^2)^2}\left(1 + \frac{x}{\sqrt{x^2+y^2+z^2}}\right) + f * \frac{xz^2}{(y^2+z^2)(x^2+y^2+z^2)^{3/2}},$$

taking into account Lemma 2, the proof is completed.

□

Hence, from the slip condition (3.5) and Theorem 1, we obtain the famous Multhopp's lifting surface equation [9]:

$$-\frac{1}{4\pi}\iint_D^* \frac{f(\xi,\eta)}{(y-\eta)^2}\left(1+\frac{x_0}{R}\right)\mathrm{d}\xi\mathrm{d}\eta = \frac{\partial h}{\partial x}(x,y), (x,y) \in \mathring{D}. \tag{3.12}$$

This is an integral equation with hypersingular kernel and the singular term has to be evaluated as a Hadamard *finite part* integral.

In the following, we shall prove another form of the Multhopp's equation, often utilized, whose justification is formal in many papers, being based only on the identity

$$\frac{\partial}{\partial y}\left[\frac{1}{y}\left(1+\frac{\sqrt{x^2+y^2}}{x}\right)\right] = -\frac{1}{y^2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right). \tag{3.13}$$

That form of the equation contains weaker singularities, of Cauchy type.

Definition 2. *For any test function* $\varphi \in \mathcal{D}\left(\mathbb{R}^2\right)$, *supp* $\varphi \subset [-A,A]^2$, *we define*

$$\left\langle \mathcal{PV}\left[\frac{1}{y}\left(1+\frac{\sqrt{x^2+y^2}}{x}\right)\right], \varphi\right\rangle = \lim_{\varepsilon\to 0_+}\iint_{D_{1\varepsilon}} \frac{\varphi(x,y)}{y}\left(1+\frac{\sqrt{x^2+y^2}}{x}\right)\mathrm{d}x\mathrm{d}y, \tag{3.14}$$

where $D_{1\varepsilon} = ([-A,-\varepsilon]\cup[\varepsilon,A]) \times ([-A,-\varepsilon]\cup[\varepsilon,A])$.

Lemma 3.

$$\frac{\partial}{\partial y}\mathcal{PV}\left[\frac{1}{y}\left(1+\frac{\sqrt{x^2+y^2}}{x}\right)\right] = -\mathcal{FP}\left[\frac{1}{y^2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)\right], \text{ in } \mathcal{D}'(\mathbb{R}^2) \tag{3.15}$$

Proof: Let φ be a test function, supp$\varphi \subset [-A,A]^2$. Then, one has:

$$-\left\langle \frac{\partial}{\partial y}\mathcal{PV}\left[\frac{1}{y}\left(1+\frac{\sqrt{x^2+y^2}}{x}\right)\right], \varphi\right\rangle = \left\langle \mathcal{PV}\left[\frac{1}{y}\left(1+\frac{\sqrt{x^2+y^2}}{x}\right)\right], \frac{\partial\varphi}{\partial y}\right\rangle =$$

$$= \lim_{\varepsilon\to 0_+}\iint_{D_{1\varepsilon}} \frac{\partial\varphi}{\partial y}\left[\frac{1}{y}\left(1+\frac{\sqrt{x^2+y^2}}{x}\right)\right]\mathrm{d}x\mathrm{d}y.$$

If we denote $A_\varepsilon = [-A, -\varepsilon] \cup [\varepsilon, A]$ and we take into account the identity (3.13), after an integration by parts, we can write

$$\int_{A_\varepsilon} \frac{\partial \varphi}{\partial y}\left[\frac{1}{y}\left(1+\frac{\sqrt{x^2+y^2}}{x}\right)\right]\mathrm{d}y = \int_{A_\varepsilon} \frac{\varphi(x,y)}{y^2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)\mathrm{d}y -$$
$$-\frac{\varphi(x,\varepsilon)+\varphi(x,-\varepsilon)}{\varepsilon}\left(1+\frac{\sqrt{x^2+\varepsilon^2}}{x}\right) =$$
$$= -\frac{\varphi(x,\varepsilon)+\varphi(x,-\varepsilon)-2\varphi(x,0)}{\varepsilon}\left(1+\frac{\sqrt{x^2+\varepsilon^2}}{x}\right) + \tag{3.16}$$
$$+\int_{A_\varepsilon} \frac{\varphi(x,y)}{y^2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)\mathrm{d}y - \frac{2\varphi(x,0)}{\varepsilon}(1+\mathrm{sign}x) -$$
$$-2\frac{\varphi(x,0)-\varphi(0,0)}{x}\frac{\sqrt{x^2+\varepsilon^2}-|x|}{\varepsilon} + 2\frac{\varphi(0,0)}{x}\frac{\sqrt{x^2+\varepsilon^2}-|x|}{\varepsilon}.$$

Since

$$\left|\frac{\sqrt{x^2+\varepsilon^2}-|x|}{\varepsilon}\right| = \frac{\varepsilon}{|x|+\sqrt{x^2+\varepsilon^2}} \le 1,$$

then

$$\lim_{\varepsilon\to 0}\int_{A_\varepsilon} \frac{\varphi(x,0)-\varphi(0,0)}{x}\frac{\sqrt{x^2+\varepsilon^2}-|x|}{\varepsilon}\mathrm{d}x = 0. \tag{3.17}$$

If $x \in A_\varepsilon$, then $\left|\frac{\sqrt{x^2+\varepsilon^2}}{x}\right| \le \sqrt{2}$. Hence, via Lebesgue's theorem,

$$\lim_{\varepsilon\to 0}\int_{A_\varepsilon} \frac{\varphi(x,\varepsilon)+\varphi(x,-\varepsilon)-2\varphi(x,0)}{\varepsilon}\left(1+\frac{\sqrt{x^2+\varepsilon^2}}{x}\right)\mathrm{d}x = 0. \tag{3.18}$$

Integrating equation (3.16) with respect to x, taking into account the limits (3.17),(3.18) and the fact that

$$\int_{A_\varepsilon} 2\frac{\varphi(0,0)}{x}\frac{\sqrt{x^2+\varepsilon^2}-|x|}{\varepsilon}\mathrm{d}x = 0,$$

$$\lim_{\varepsilon\to 0_+}\int_{-\varepsilon}^{\varepsilon}\left[\int_{A_\varepsilon}\frac{\varphi(x,y)}{y^2}\left(1+\frac{x}{\sqrt{x^2+y^2}}\right)\mathrm{d}y - \frac{2}{\varepsilon}\varphi(x,0)(1+\mathrm{sgn}x)\right]\mathrm{d}x = 0,$$

by means of Definition 1, the proof is completed. ∎

Hence, we can derive immediately the second main result of this paper.

Theorem 2. *Another form of the lifting surface equation is*

$$\frac{1}{4\pi}\frac{\partial}{\partial y}\iint_D^{*} \frac{\mathrm{f}(\xi,\eta)}{y-\eta}\left(1+\frac{R}{x-\xi}\right)\mathrm{d}\xi\mathrm{d}\eta = \frac{\partial h}{\partial x}(x,y), \quad (x,y) \in \mathring{D}. \tag{3.19}$$

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