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Conformally flat CR-integrable almost Kenmotsu manifolds

by

Yaning Wang

Abstract

In this paper, we prove that a CR-integrable almost Kenmotsu manifold of dimension greater than three with scalar curvature invariant along the characteristic vector field is conformally flat if and only if it is of constant sectional curvature -1.

Key Words: Almost Kenmotsu manifold, CR-integrable structures, conformally flat, hyperbolic space.
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1 Introduction

In 1969, Tanno in [19] studied almost contact metric manifolds whose automorphism groups attain the maximum dimensions and proved that such manifolds can be classified into three classes. In 1972, K. Kenmotsu in [14] introduced a new class of almost contact metric manifolds which characterized the third case of Tanno's classification theorem, and such manifolds were firstly called Kenmotsu manifolds by Janssens and Vanhecke [13]. With regard to the classification results of Kenmotsu manifolds, it is well-known that K. Kenmotsu in [14] proved the following result.

Theorem 1 ([14, Proposition 11]). *Let M^{2n+1} be a Kenmotsu manifold of dimension > 3. If M^{2n+1} is conformally flat, then it is a space of constant negative curvature -1.*

In 1981, Janssens and Vanhecke in [13] first introduced the notion of almost Kenmotsu manifolds which is a generalization of Kenmotsu manifolds. Almost Kenmotsu manifolds were recently studied by many authors, see, for example, Kim and Pak [15], Dileo and Pastore [7, 8, 9], Pastore and Saltarelli [17, 18] and Wang and Liu [20, 21, 22]. Following [18], we observe that on an almost Kenmotsu manifold M^{2n+1} the $(1,1)$-type tensor field h (defined in Section 2) vanishes if and only if the Reeb foliation is conformal. In addition, the vanishing of h on M^{2n+1} implies that equations (2.5)-(2.9) in Binh-Tamássy-De-Tarafdar [2] keep correct. Therefore, we obtain immediately from [2, Theorem 3] that the following result is true.

Theorem 2. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold of dimension > 3 with conformal Reeb foliation. Then M^{2n+1} is conformally flat if and only if it is a Kenmotsu manifold of constant sectional curvature -1.*

Remark 1. *In fact, making use of Theorem 2 of Dileo and Pastore [7] and [18], we know that any almost Kenmotsu manifold with conformal Reeb foliation is locally isometric to a warped product $\mathbb{R}\times_{ce^t} N^{2n}$, where N^{2n} denotes an almost Kähler manifold. On the one hand, according to the Case (i) of Theorem 1 of Brozos-Vázquez, García-Río and Vázquez-Lorenzo [4], we obtain that the warped product $\mathbb{R}\times_f N^{2n}$ is conformally flat if and only if the fiber (N^{2n}, g_N) is a space of constant sectional curvature. On the other hand, following Oguro and Sekigawa [16, Corollary], we know that any almost Kähler manifold of constant sectional curvature of dimension >2 is locally flat. Based on the above statements, we conclude that any conformally flat almost Kenmotsu manifold of dimension >3 with conformal Reeb foliation is of constant sectional curvature -1.*

P. Dacko and Z. Olszak in [6] proved that any conformally flat almost cosymplectic manifold of dimension >3 with Kählerian leaves is locally flat and cosymplectic. Moreover, Ghosh, Koufogiorgos and Sharma in [11] proved that any conformally flat contact metric manifold of dimension >3 with Kählerian leaves is a Sasakian manifold of constant sectional curvature 1. Motivated by these results, in this paper we aim to generalize the above Theorems 1 and 2 on a special class of almost Kenmotsu manifolds, namely almost Kenmotsu manifolds with CR-integrable structures (or equivalently, almost Kenmotsu manifolds with Kählerian leaves).

Very recently, the present author and X. Liu in [20] proved that any CR-integrable almost Kenmotsu manifold of dimension >3 is locally symmetric if and only it is locally isometric to either the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$ (which is a strictly almost Kenmotsu manifold). A Schur-type theorem for CR-integrable almost Kenmotsu manifolds was also obtained by the present author and X. Liu in [22]. Making use of some key results provided in [20], our main result in the present paper is stated as follows.

Theorem 3. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a CR-integrable almost Kenmotsu manifold of dimension >3 with scalar curvature invariant along the characteristic vector field. Then M^{2n+1} is conformally flat if and only if it is a Kenmotsu manifold of constant sectional curvature -1.*

The present paper is arranged as follows. In Section 2, we first recall some well-known basic formulas and properties of almost Kenmotsu manifolds. Next, in Section 3, we shall introduce the notion of CR-integrable almost Kenmotsu manifolds together with some examples. Later, we may present some key lemmas on CR-integrable almost Kenmotsu manifolds under additional conditions. Finally, we give our main results with the detailed proofs. In addition, some corollaries of our main results are presented.

2 Almost Kenmotsu manifolds

Following Blair [3], an *almost contact structure* on a $(2n+1)$-dimensional smooth manifold M^{2n+1} is denoted by a triplet (ϕ,ξ,η), where ϕ is a $(1,1)$-type tensor field, ξ a global vector field (which is called the characteristic or the Reeb vector field) and η a 1-form, such that

$$\phi^2=-\mathrm{id}+\eta\otimes\xi,\ \eta(\xi)=1, \tag{2.1}$$

where id denotes the identity endomorphism of the tangent bundle. Using the relation (2.1), it follows that $\phi(\xi) = 0$, $\eta \circ \phi = 0$ and $\mathrm{rank}(\phi) = 2n$. If a Riemannian metric g on M^{2n+1} satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all differentiable vector fields on M^{2n+1}, then g is said to be *compatible* with the almost contact structure (ϕ, ξ, η). An almost contact structure endowed with a compatible Riemannian metric is said to be an *almost contact metric structure*. A smooth manifold endowed with an almost contact metric structure is called an *almost contact metric manifold* and it is denoted by $(M^{2n+1}, \phi, \xi, \eta, g)$. The *fundamental 2-form* Φ of an almost contact metric manifold M^{2n+1} is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y. We may define, on the product manifold $M^{2n+1} \times \mathbb{R}$, an almost complex structure J by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where X denotes the vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^{2n+1} \times \mathbb{R}$. An almost contact structure is said to be *normal* if the above almost complex structure is integrable. Following Blair [3], the normality of an almost contact structure is expressed by $[\phi, \phi] = -2d\eta \otimes \xi$, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ defined by

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

for any vector fields $X, Y \in \mathfrak{X}(M)$.

Following Blair [3], an almost contact metric manifold with $d\eta = \Phi$ is called a *contact metric manifold* and a normal contact metric manifold is called a *Sasakian manifold*. On an almost contact metric manifold M^{2n+1} if both η and Φ are closed, then M is said to be an *almost cosymplectic manifold* and a normal almost cosymplectic manifold is said to be a *cosymplectic manifold*. According to Janssens and Vanhecke [13], an almost contact metric manifold satisfying $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ is called an *almost Kenmotsu manifold* and a normal almost Kenmotsu manifold is said to be a *Kenmotsu manifold*.

We consider now two tensor fields $l = R(\cdot, \xi)\xi$ and $h = \frac{1}{2}\mathcal{L}_\xi \phi$ on an almost Kenmotsu manifold M^{2n+1}, where R denotes the curvature tensor of M^{2n+1} and $\mathcal{L}$ is the Lie differentiation. Then, following [15, 7, 8], we see that the two $(1,1)$-type tensor fields l and h are symmetric and satisfy the following equations.

$$h\xi = l\xi = 0,\ \mathrm{tr}h = \mathrm{tr}(h\phi) = 0,\ h\phi + \phi h = 0, \tag{2.3}$$

$$\nabla_X \xi = X - \eta(X)\xi - \phi h X, \tag{2.4}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.5}$$

$$\nabla_\xi h = -\phi - 2h - \phi h^2 - \phi l, \tag{2.6}$$

$$\mathrm{tr}(l) = S(\xi, \xi) = g(Q\xi, \xi) = -2n - \mathrm{tr}h^2, \tag{2.7}$$

$$R(X, Y)\xi = \eta(X)(Y - \phi h Y) - \eta(Y)(X - \phi h X) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \tag{2.8}$$

for any vector fields X and $Y \in \mathfrak{X}(M)$, where S, Q, ∇ and tr denote the Ricci curvature tensor, the Ricci operator with respect to g, the Levi-Civita connection of g and the trace operator, respectively.

3 Conformally flat CR-integrable almost Kenmotsu manifolds

Throughout this paper, we shall denote by $\mathcal{D}$ the distribution which is defined by $\mathcal{D} = \ker(\eta)$ on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$. Then there exists on M^{2n+1} an almost CR-structure $(\mathcal{D}, \phi_{\mathcal{D}})$, where $\phi_{\mathcal{D}}$ denotes the restriction of ϕ on distribution $\mathcal{D}$. From Dileo and Pastore [8], the above almost CR-structure is integrable if and only if $[\phi X, \phi Y] - [X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] = 0$ for any $X, Y \in \mathcal{D}$, and this is also equivalent to that the integral manifolds of $\mathcal{D}$ are Kählerian (or equivalently, the $(1, 1)$-type tensor field ϕ is η-parallel, i.e., $g((\nabla_X \phi)Y, Z) = 0$ for any $X, Y, Z \in \mathcal{D}$).

If the integral manifolds of $\mathcal{D}$ are Kählerian, in this paper we say that M^{2n+1} is a *CR-integrable almost Kenmotsu manifold* (see [8, Section 3]). Some examples of CR-integrable (almost) Kenmotsu manifolds are presented as follows.

Any normal almost contact metric manifold is a CR-integrable manifold (see Blair [3]). This implies that a Kenmotsu manifold is always CR-integrable.

Following Dileo and Pastore [8] and Pastore and Saltarelli [17], on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, if the characteristic vector field ξ satisfies

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{3.1}$$

for any $X, Y \in \mathfrak{X}(M)$ and smooth functions k and μ, then M^{2n+1} is called a *generalized (k, μ)-almost Kenmotsu manifold*. Similarly, if on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ the characteristic vector field ξ satisfies

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y) \tag{3.2}$$

for any $X, Y \in \mathfrak{X}(M)$, smooth functions k and μ and $h' = h \circ \phi$, then M^{2n+1} is called a *generalized $(k, \mu)'$-almost Kenmotsu manifold.* If both k and μ in relations (3.1) or (3.2) are constants, then M^{2n+1} is said to be a *(k, μ) or $(k, \mu)'$-almost Kenmotsu manifold.* According to [17, Remark 3.1], when $h \neq 0$ (which is equivalent to $h' \neq 0$), then both a generalized (k, μ) and a generalized $(k, \mu)'$-almost Kenmotsu manifold are CR-integrable. Using this property, we remark that some examples of CR-integrable almost Kenmotsu manifolds in any odd dimensions can also be seen in [8] and [17, Section 6].

In addition, by [9, Proposition 4], we know that an almost Kenmotsu manifold for which h' is η-parallel (i.e., $g((\nabla_X h')Y, Z) = 0$ for any $X, Y, Z \in \mathcal{D}$) and 0 is a simple eigenvalue of h' is a CR-integrable manifold.

In what follows, we present some useful properties of CR-integrable almost Kenmotsu manifolds.

Lemma 1 ([7, 15])**.** *An almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is a Kenmotsu manifold if and only if it is CR-integrable and h vanishes.*

The above lemma implies that the above Theorem 2 is a generalization of [14, Theorem 1]. The following lemma can be regarded as a special case of Proposition 2.2 of [10] proved by Falcitelli and Pastore, which characterized the CR-integrability of almost Kenmotsu manifolds.

Lemma 2 ([8, Proposition 2.3])**.** *In an almost Kenmotsu manifold M^{2n+1} the distribution $\mathcal{D}$ has Kählerian leaves if and only if*

$$(\nabla_X \phi)Y = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX) \tag{3.3}$$

for any $X, Y \in \mathfrak{X}(M)$.

By applying the above Lemma 2, the present author and X. Liu in [20] obtained the following result to classify locally symmetric almost Kenmotsu manifolds with CR-integrable structures.

Lemma 3 ([20, Lemma 3.3])**.** *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost Kenmotsu manifold with CR-integrable structure, then we have*

$$Q\phi - \phi Q = l\phi - \phi l + 4(n-1)h + (\eta \circ Q\phi) \otimes \xi - \eta \otimes \phi Q\xi, \tag{3.4}$$

where $n \geq 1$.

Generally, a Riemanian manifold M of dimension $m > 3$ is said to be *conformally flat* if and only if the Weyl conformal curvature tensor $\mathcal{C}$, defined by

$$\begin{aligned} &\mathcal{C}(X,Y)Z \\ =&R(X,Y)Z + \frac{r}{(m-1)(m-2)}\{g(Y,Z)X - g(X,Z)Y\} \\ &- \frac{1}{m-2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\}, \end{aligned} \tag{3.5}$$

vanishes for any vector fields X, Y, Z on M, where S, Q and r denote the Ricci curvature tensor, the Ricci operator with respect to g and the scalar curvature, respectively. The Weyl conformal curvature tensor $\mathcal{C}$ is said to be *harmonic* if it is divergence free, i.e., $\mathrm{div}\mathcal{C} = 0$. For any Riemanian manifold M of dimension $m > 1$, by a straightforward calculation it is easy to see that the Weyl curvature tensor $\mathcal{C}$ is harmonic if and only if

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{2(m-1)}\{X(r)g(Y,Z) - Y(r)g(X,Z)\} \tag{3.6}$$

for any vector fields X, Y, Z on M.

Lemma 4. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold of dimension > 3 with CR-integrable structure. If M^{2n+1} is conformally flat, then the characteristic vector field ξ is an eigenvector field of the Ricci operator.*

Proof. Suppose that M^{2n+1} is a CR-integrable almost Kenmotsu manifold. Taking into account (3.3), it follows from (2.8) that

$$\begin{aligned} &(\nabla_X h)Y - (\nabla_Y h)X \\ =&\phi R(X,Y)\xi - \eta(X)(\phi Y + hY) + \eta(Y)(\phi X + hX) + 2g(X, h^2\phi Y)\xi \end{aligned} \tag{3.7}$$

for any $X, Y \in \mathfrak{X}(M)$. On the other hand, taking the covariant derivative of (3.3) we obtain

$$\begin{aligned} \nabla_Y(\nabla_X \phi) =&(\nabla_Y g(\phi X + hX, \cdot))\xi + g(\phi X + hX, \cdot)\nabla_Y \xi \\ &- (\nabla_Y g(\xi, \cdot))(\phi X + hX) - g(\xi, \cdot)\nabla_Y(\phi X + hX) \end{aligned} \tag{3.8}$$

for any $X, Y \in \mathfrak{X}(M)$. Applying relations (3.7) and (3.8) into the well-known Ricci identity

$$\begin{aligned}&(R(X,Y)\phi)Z\\=&(\nabla_X(\nabla_Y\phi))Z-(\nabla_Y(\nabla_X\phi))Z-(\nabla_{\nabla_XY}\phi)Z+(\nabla_{\nabla_YX}\phi)Z\end{aligned}\tag{3.9}$$

for any $X, Y, Z \in \mathfrak{X}(M)$, we obtain that

$$\begin{aligned}&R(X,Y)\phi Z-\phi R(X,Y)Z\\=&[g(\phi R(X,Y)\xi,Z)-\eta(X)g(\phi Y+hY,Z)+\eta(Y)g(\phi X+hX,Z)]\xi\\&+g(\phi Y+hY,Z)(X-\phi hX)-g(\phi X+hX,Z)(Y-\phi hY)\\&+g(Y-\phi hY,Z)(\phi X+hX)-g(X-\phi hX,Z)(\phi Y+hY)\\&-\eta(Z)[\phi R(X,Y)\xi-\eta(X)(\phi Y+hY)+\eta(Y)(\phi X+hX)]\end{aligned}\tag{3.10}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. Next, replacing X by ξ in (3.10) we obtain

$$R(\xi,Y)\phi Z-\phi R(\xi,Y)Z=-g(\phi lY,Z)\xi+\eta(Z)\phi lY\tag{3.11}$$

for any $Y, Z \in \mathfrak{X}(M)$. We now consider a local orthogonal ϕ-basis $\{E_i : 1 \le i \le 2n+1\}$ on M^{2n+1}. Taking into account $\phi\xi = h\xi = 0$ and putting $Y = Z = E_i$ into equation (3.11), and summing over $i = 1, 2, \dots, 2n+1$, we obtain

$$\sum_{i=1}^{2n+1}R(\xi,E_i)\phi E_i=\phi Q\xi-\mathrm{tr}(\phi l)\xi\tag{3.12}$$

On the other hand, because of the conformal flatness of M^{2n+1} we have $\mathcal{C}(X,Y)Z = 0$ for any $X, Y, Z \in \mathfrak{X}(M)$. Then, putting $X = \xi$, $Y = E_i$ and $Z = \phi E_i$ into equation (3.5) and summing over $i = 1, 2, \dots, 2n+1$ we obtain

$$\sum_{i=1}^{2n+1}R(\xi,E_i)\phi E_i=\frac{1}{2n-1}\left(\sum_{i=1}^{2n+1}S(E_i,\phi E_i)\xi+\phi Q\xi\right).\tag{3.13}$$

Subtracting (3.12) from (3.13) yields

$$2(n-1)\phi Q\xi=\left(\sum_{i=1}^{2n+1}S(E_i,\phi E_i)+(2n-1)\mathrm{tr}(\phi l)\right)\xi.$$

In view of the hypothesis $n > 1$, using $\phi\xi = 0$ and equation (2.7), then the action of ϕ on the above equation yields

$$Q\xi=\mathrm{tr}(l)\xi.\tag{3.14}$$

This means that ξ is an eigenvector field of the Ricci operator. This completes the proof. □

Lemma 5. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold for which the characteristic vector field ξ is an eigenvector field of the Ricci operator. If the Weyl conformal curvature tensor is harmonic, then we have*

$$d\left(\mathrm{tr}(l)-\frac{r}{4n}\right)\wedge\eta=0,\tag{3.15}$$

where r denotes the scalar curvature of M^{2n+1}.

Proof. Let ξ be an eigenvector field of the Ricci operator, then it follows from relation (2.7) that (3.14) holds. Taking the covariant differentiation of (3.14) along an arbitrary vector field $X \in \mathfrak{X}(M)$ and using (2.4) we have

$$(\nabla_X Q)\xi = X(\mathrm{tr}(l))\xi + Q(\phi^2 X + \phi h X) - \mathrm{tr}(l)(\phi^2 X + \phi h X). \tag{3.16}$$

Since the Weyl curvature tensor $\mathcal{C}$ is harmonic, substituting $Z = \xi$ into (3.6) and taking into account $g((\nabla_X Q)Y, Z) = (\nabla_X S)(Y, Z)$ we obtain

$$g((\nabla_X Q)Y - (\nabla_Y Q)X, \xi) = \frac{1}{4n}\{X(r)\eta(Y) - Y(r)\eta(X)\} \tag{3.17}$$

for any $X, Y \in \mathfrak{X}(M)$. From the symmetry $g((\nabla_X Q)Y, Z) = g((\nabla_X Q)Z, Y)$ and equation (3.17) we obtain

$$g((\nabla_X Q)\xi, Y) - g((\nabla_Y Q)\xi, X) = \frac{1}{4n}\{X(r)\eta(Y) - Y(r)\eta(X)\} \tag{3.18}$$

for any $X, Y \in \mathfrak{X}(M)$. Putting (3.16) into (3.18) we get

$$\begin{aligned} &X(\mathrm{tr}(l))\eta(Y) - Y(\mathrm{tr}(l))\eta(X) + g(Y, Q\phi h X) - g(X, Q\phi h Y) \\ =&\frac{1}{4n}\{X(r)\eta(Y) - Y(r)\eta(X)\} \end{aligned} \tag{3.19}$$

for any $X, Y \in \mathfrak{X}(M)$. Replacing X and Y by ϕX and ϕY in (3.19), respectively, we obtain

$$g(\phi Y, Q\phi h\phi X) = g(\phi X, Q\phi h\phi Y)$$

for any $X, Y \in \mathfrak{X}(M)$. Thus, it follows from the above equation that $h'Q = Qh'$ and hence by substituting $Y = \xi$ into (3.19) we obtain

$$X(\mathrm{tr}(l)) - \frac{1}{4n}X(r) = \left(\xi(\mathrm{tr}(l)) - \frac{1}{4n}\xi(r)\right)\eta(X) \tag{3.20}$$

for any $X \in \mathfrak{X}(M)$. Thus we complete the proof. □

Finally, applying the above preliminaries we may present the detailed proof of our main result and its corollaries as follows.

Theorem 4. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a CR-integrable almost Kenmotsu manifold of dimension > 3 with scalar curvature invariant along the characteristic vector field. If M^{2n+1} is conformally flat, then it is a Kenmotsu manifold of constant sectional curvature -1.*

Proof. Under the hypotheses of the theorem, we may apply Lemma 4 to obtain that ξ is an eigenvector field of the Ricci operator. Since M^{2n+1} is conformally flat, i.e., $\mathcal{C}(X, Y)Z = 0$ for any vector fields $X, Y, Z \in \mathfrak{X}(M)$, putting $Y = Z = \xi$ into equation (3.5) we obtain

$$lX = \frac{1}{2n-1}(\mathrm{tr}(l)X - 2\mathrm{tr}(l)\eta(X)\xi + QX) - \frac{r}{2n(2n-1)}(X - \eta(X)\xi) \tag{3.21}$$

for any $X \in \mathfrak{X}(M)$. Replacing X by ϕX in (3.21) gives an equation; and separately, the action of ϕ on (3.21) gives another; subtracting these two equations we obtain

$$l\phi - \phi l = \frac{1}{2n-1}(Q\phi - \phi Q). \tag{3.22}$$

We remark that Lemma 3 is applicable in this context. Hence, using equation (3.22) in (3.4) and taking into account the hypothesis $n > 1$ we get

$$Q\phi - \phi Q = 2(2n-1)h. \tag{3.23}$$

On the other hand, by a straightforward calculation, it follows from relations (2.5) and (2.6) that

$$\phi l - l\phi = -4h - 2\nabla_\xi h. \tag{3.24}$$

Using (3.23) in (3.22) we get $l\phi - \phi l = 2h$ and comparing this equation with (3.24) we have

$$\nabla_\xi h = -h. \tag{3.25}$$

In terms of (3.25) we obtain from equation (2.6) that $l = \phi^2 - h' - h^2$. Making use of this equation in (3.21) we obtain

$$Q = (\mathrm{tr}(l) - c)\mathrm{id} + c\eta \otimes \xi - (2n-1)(h' + h^2), \tag{3.26}$$

where $c := 2n - 1 - \frac{r}{2n} + 2\mathrm{tr}(l)$. The conformal flatness of M^{2n+1} means that the Weyl curvature tensor is harmonic, in view of Lemma 4 we know that Lemma 5 is applicable in this context. Then, from (3.15) we see that c is a constant along the distribution $\mathcal{D}$.

As M^{2n+1} is a CR-integrable almost Kenmotsu manifold, by replacing X by ξ in (3.3) we obtain $\nabla_\xi \phi = 0$. Hence, by (3.25) we have $\nabla_\xi h' = -h'$ and $\nabla_\xi h^2 = -2h^2$. Consequently, taking the covariant derivative of (3.26) in direction of ξ and using (2.4) we get

$$\nabla_\xi Q = \xi(\mathrm{tr}(l) - c)\mathrm{id} + \xi(c)\eta \otimes \xi + (2n-1)(h' + 2h^2). \tag{3.27}$$

Since the Weyl curvature tensor is harmonic, taking into account equation (3.6) and the symmetry of the Ricci operator we obtain

$$(\nabla_\xi Q)X = (\nabla_X Q)\xi + \frac{1}{4n}(\xi(r)X - X(r)\xi) \tag{3.28}$$

for any $X \in \mathfrak{X}(M)$. Using relation (3.26) in (3.16) and then, replacing the new equation in (3.28), we get

$$\begin{aligned}(\nabla_\xi Q)X =& X\left(\mathrm{tr}(l) - \frac{r}{4n}\right)\xi + \left(c + \frac{1}{4n}\xi(r)\right)X - c\eta(X)\xi \\ &+ ch'X + (2n-1)(h'X + 2h^2X + h^3\phi X)\end{aligned} \tag{3.29}$$

for any $X \in \mathfrak{X}(M)$. In view of $c = 2n - 1 - \frac{r}{2n} + 2\mathrm{tr}(l)$, comparing (3.27) with (3.29) we may obtain

$$\frac{1}{2}X(c)\xi + \left(c + \frac{1}{2}\xi(c)\right)X - (c + \xi(c))\eta(X)\xi + ch'X + (2n-1)h^3\phi X = 0 \tag{3.30}$$

for any $X \in \mathfrak{X}(M)$. Applying (3.15) we know that $\mathrm{tr}(l) - \frac{r}{4n}$ is a constant along the distribution $\mathcal{D}$. Then, replacing X by ϕX in (3.30) yields

$$\left(c + \frac{1}{2}\xi(c)\right)\phi X - chX - (2n-1)h^3X = 0 \tag{3.31}$$

for any $X \in \mathfrak{X}(M)$. On the other hand, the action of ϕ on equation (3.30) gives that

$$\left(c + \frac{1}{2}\xi(c)\right)\phi X + chX + (2n-1)h^3X = 0 \tag{3.32}$$

for any $X \in \mathfrak{X}(M)$. Consequently, subtracting (3.31) from (3.32) we have $ch+(2n-1)h^3 = 0$, which is also equivalent to $ch' + (2n-1)h'^3 = 0$ because of $h^2 = h'^2$. Obviously, adding (3.31) to (3.32) yields $\xi(c) = -2c$. Now, we shall separate our arguments into two cases as follows.

Case 1: c is a constant. In this case, by $c + \frac{1}{2}\xi(c) = 0$ we know $c = 0$ and hence from $ch + (2n-1)h^3 = 0$ we obtain $h^3 = 0$. As h is self-adjoint then we conclude that $h = 0$. Making use of $h = 0$ in equation (3.3) we obtain

$$(\nabla_X\phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X$$

for any $X, Y \in \mathfrak{X}(M)$. This implies that M^{2n+1} is a Kenmotsu manifold (see Janssens and Vanhecke [13]). Therefore, the proof follows from Theorem 1.

Case 2: c is not a constant. As Lemma 5 implies that c is a constant along the distribution $\mathcal{D}$, i.e., $X(c) = 0$ for any vector field X orthogonal to ξ. We remark that c is not a constant on M^{2n+1} if and only if $\xi(c) \neq 0$. As locally we have $\xi(c) = -2c \neq 0$, we may write $\xi = \frac{\partial}{\partial_t}$ and hence we obtain $c = \alpha e^{-2t}$, where α is a negative constant. Moreover, we also have that the tensor field h is non-vanishing everywhere, otherwise the proof follows from case 1. If we denote by $\lambda > 0$ an eigenvalue of h', then it follows from $ch'+(2n-1)h'^3 = 0$ that $\lambda^2 = -\frac{c}{2n-1}$. Thus, by using relation (2.7) we may obtain $\mathrm{tr}(l) = -2n - \mathrm{tr}(h'^2) = -2n + \frac{2(n-n_0)c}{2n-1}$, where we have assumed that the spectrum of h' is $\{0, \lambda, -\lambda\}$ and the multiplicity of eigenvalue 0 of h' is $2n_0 + 1$ satisfying $0 \leq n_0 \leq n-1$. By applying this in $c = 2n - 1 - \frac{r}{2n} + 2\mathrm{tr}(l)$ we may get $r = \frac{2n(2n-4n_0+1)}{2n-1}c - 2n(2n+1)$ and hence we have

$$\xi(r) = -\frac{4n(2n-4n_0+1)}{2n-1}c.$$

Since the scalar curvature r is invariant along the characteristic vector field ξ, it follows from the above equation that either $c = 0$ or $4n_0 = 2n+1$. In fact, the first case implies that $h = 0$ and by the second case we get a contradiction. This completes the proof. □

Theorem 3 follows from Theorem 4 since the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ is obviously conformally flat.

Clearly, if $h = 0$ then we have from (2.7) that $R(X,Y)\xi = \eta(X)Y - \eta(Y)X$ for any vector fields X, Y. This means that ξ belongs to the k-nullity distribution with $k = -1$. We have to point out that ξ belonging to the k-nullity distribution gives that $h = 0$ (see [18, Theorem 4.1]). Thus, the following two corollaries can be regarded as some generalizations of Theorem 2.

Corollary 1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a generalized (k, μ)-almost Kenmotsu manifold of dimension > 3. Then M^{2n+1} is conformally flat if and only if it is a Kenmotsu manifold of constant sectional curvature -1.*

Proof. Suppose that h vanishes, the proof follows from Theorem 2. Next, we assume that $h \neq 0$ and hence by Remark 3.1 of [17] we know that M^{2n+1} is a CR-integrable almost Kenmotsu manifold. Also, from Proposition 3.1 of [17] we obtain $\nabla_\xi h = -2h + \mu h'$. On the other hand, from the proof of Theorem 4 we have equation (3.25), i.e., $\nabla_\xi h = -h$. It follows from the above two equations that $h = 0$, a contradiction. This completes the proof. ∎

Corollary 2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold of dimension > 3 with ξ belonging to the generalized $(k, \mu)'$-nullity distribution. If scalar curvature is invariant along the characteristic vector field, then M^{2n+1} is conformally flat if and only if it is a Kenmotsu manifold of constant sectional curvature -1.*

Proof. If $h = 0$, from Theorem 2 we know that M^{2n+1} is a Kenmotsu manifold of constant sectional curvature -1. If $h \neq 0$, Remark 3.1 of [17] implies that M^{2n+1} is CR-integrable. Thus, from the proof of Theorem 4 we have $\nabla_\xi h' = -h'$. Using this in Proposition 3.1 of [17] gives that $\mu = -1$.

Since r is invariant along the characteristic vector field ξ, it follows from Case *c*) of Proposition 5.2 of [17] that $\xi(k) = 0$. Also, from Proposition 3.2 of [17] we have $X(k) = 0$ for any X orthogonal to ξ. This implies that k is a constant.

However, from Proposition 4.2 of [8] we know that ξ belonging to the $(k, \mu)'$-nullity distribution on an almost Kenmotsu manifold implies $\mu = -2$, a contradiction. This completes the proof. ∎

Before closing this paper, we present some examples of CR-integrable almost Kenmotsu manifolds as follows.

Example 1 ([8, Example 1])**.** *Let us consider $\mathbb{R}^{2n+1}$ $(n \geq 2)$ whose standard basis is denoted by $\{\xi, X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$. We denote by $\mathfrak{g}$ the Lie algebra defined by*

$$[\xi, X_i] = -[X_i, \xi] = -(1+\lambda)X_i,\ [\xi, Y_i] = -[Y_i, \xi] = -(1-\lambda)Y_i,$$
$$[X_i, X_j] = [X_i, Y_j] = [Y_i, X_j] = [Y_i, Y_j] = 0$$

for any $1 \leq i, j \leq n$ and a positive constant $\lambda \in \mathbb{R}$. We consider the endomorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ and the 1-form $\eta : \mathfrak{g} \to \mathbb{R}$ defined by

$$\phi(\xi) = 0,\ \phi(X_i) = Y_i,\ \phi(Y_i) = -X_i,\ \eta(\xi) = 1,\ \eta(X_i) = 0,\ \eta(Y_i) = 0$$

for any $i = 1, \cdots, n$.

We consider an inner product g on $\mathfrak{g}$ such that the basis $\{\xi, X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$ is orthonormal. Next, we denote by G a connected Lie group whose Lie algebra is $\mathfrak{g}$. For simplicity, we shall denote by ξ, X_1, $\cdots$, X_n, Y_1, $\cdots$, Y_n and ϕ, η, g the left-invariant vector fields and the left-invariant tensor fields on G determined by the corresponding tensors, respectively.

One can check that (ϕ, ξ, η, g) defines an almost contact metric structure on G satisfying $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. Then G is an almost Kenmotsu manifold. We denote by ∇ the Levi-Civita connection of g. Following [8], we have

$$\nabla_\xi \xi = 0,\ \nabla_{X_i}\xi = (1+\lambda)X_i,\ \nabla_{Y_i}\xi = (1-\lambda)Y_i,$$
$$\nabla_{X_i}X_j = -(1+\lambda)\delta_{ij}\xi,\ \nabla_{Y_i}Y_j = -(1-\lambda)\delta_{ij}\xi,\ \nabla_{X_i}Y_j = \nabla_{Y_i}X_j = 0$$

for any $1 \leq i, j \leq n$. Using the above relation, we remark that (3.3) *holds and this implies that G is a CR-integrable almost Kenmotsu manifold. In fact, Dileo and Pastore in [8] proved that G is a $(-1-\lambda^2, -2)'$-almost Kenmotsu manifold. Then the CR-integrability of G can be confirmed by using Proposition 4.1 of [8]. Moreover, from [8, Proposition 4.3] we know that the scalar curvature r of G is a negative constant $r = -2n(\lambda^2 + 2n + 1)$.*

If G is conformally flat, using (3.6) we remark that the Ricci tensor of G is of Codazzi type. This is equivalent to that the curvature tensor of G is harmonic. Therefore, from [21, Corollary 3.3] we obtain that G is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. However, applying **1.167** in Besse [1] we know that the above product is not conformally flat for $n \geq 2$, a contradiction. Thus, the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$, $n \geq 2$, is a CR-integrable almost Kenmotsu manifold (non-Kenmotsu) with scalar curvature invariant along the Reeb vector field which is not conformally flat. We also refer the reader to [7] for the almost Kenmotsu structure constructed on such product.

Remark 2. *On the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ there do exists a Kenmotsu structure which is obviously CR-integrable and conformally flat. For more details on this Kenmotsu structure, we refer the reader to Chinea and Gonzalez [5, Example].*

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Yaning Wang
Henan Engineering Laboratory for Big Data
Statistical Analysis and Optimal Control
School of Mathematics and Information Sciences
Henan Normal University
Xinxiang 453007, Henan, P. R. China
E-mail: `wyn051@163.com`