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Neighborhood Union Conditions for Hamiltonicity of P_3-dominated Graphs II

by
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Abstract

A graph G is called P_3-dominated ($P3D$) if it satisfies $J(x,y) \cup J'(x,y) \neq \emptyset$ for every pair (x,y) of vertices at distance 2, where $J(x,y) = \{u|u \in N(x) \cap N(y),\ N[u] \subseteq N[x] \cup N[y]\}$ and $J'(x,y) = \{u|u \in N(x) \cap N(y)|$ if $v \in N(u) \backslash (N[x] \cup N[y])$, then $(N(u) \cup N(x) \cup N(y)) \backslash \{x,y,v\} \subseteq N(v)\}$ for $x,y \in V(G)$ at distance 2$\}$. For a noncomplete graph G, the number NC is defined as $NC = \min\{|N(x) \cup N(y)| : x,y \in V(G)$ and $xy \notin E(G)\}$, for a complete graph G, set $NC = |V(G)|-1$. In this paper, we prove that a 2-connected P_3-dominated graph G of order n is hamiltonian if $G \notin \{K_{2,3}, K_{1,1,3}\}$ and $NC(G) \geq (2n-5)/3$, moreover it is best possible.

Key Words: P_3-dominated graph, quasi claw-free graph, neighborhood union, hamiltonicity.
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1 Introduction

We shall closely follow [9] for graph-theoretical terminology and notation not defined here. Let $G = (V, E)$ be a finite graph of order n without loops and multiple edges, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set. For any $u \in V(G)$, $N(u) = \{v \mid uv \in E(G)\}$ and $N[u] = N(u) \cup \{u\}$ and $d_G(u) = |N(u)|$. For subgraphs H and K of G, let $G-H$ denote the subgraph of G which is induced by $V(G) \backslash V(H)$, and let $N_H(K)$ denote the set of vertices in H that are adjacent to some vertex in K. A set $A \subseteq V(G)$ is *independent* if any vertices $x, y \in A$ are nonadjacent in G. The *independence number* $\alpha(G)$ of G is the cardinality of a maximum independent set in G. We denote by $\sigma_k(G)$ the minimum value of the degree-sum of any k pairwise non-adjacent vertices if $k \leq \alpha(G)$; if $k > \alpha(G)$, we set $\sigma_k(G) = k(n-1)$. For a graph G, we denote by $\delta(G)$ *the minimum degree*. If G is a noncomplete graph, then NC is defined as $NC = \min\{|N(x) \cup N(y)| : x, y \in V(G),\ xy \notin E(G)\}$, for a complete graph G, set $NC = |V(G)| - 1$. A cycle containing all the vertices of the graph is said to be *a Hamilton cycle*. A graph containing a Hamilton cycle is said to be *hamiltonian*.

A graph G is said to belong to the class $\mathcal{CF}$ of *claw-free* graphs if G does not contain an induced subgraph isomorphic to a claw ($K_{1,3}$). While a large number of results have been obtained on claw-free graphs, during the last two decades several extensions of claw-free graphs have been introduced and many known results, concerning matching and hamiltonicity, on claw-free graphs have been extended to these classes. We refer to [1], [2], [4], [6]-[8], [11]-[12] and [15]-[16] for more details.

Following Ainouche [1], for each pair (x,y) of vertices at distance 2, we set $J(x,y) = \{u \mid u \in N(x) \cap N(y),\ N[u] \subseteq N[x] \cup N[y]\}$. A graph G is *quasi-claw-free* if $J(x,y) \neq \emptyset$

for each pair (x, y) of vertices at distance 2 in G. As an extention of quasi-claw-free graphs, P_3-dominated graphs are introduced by Broersma and Vumar [5]. The class $\mathcal{P3D}$ of P_3-dominated graphs is defined below.

Let (x, y) be a pair of vertices at distance 2 in G. We consider a common neighbor u of x and y with the following property.

$$\text{If } v \in N(u) \setminus \{x, y\} \text{ is neither adjacent to } x \text{ nor to } y, \text{ then it is adjacent to all vertices of } N(x) \cup N(y) \cup N(u) \setminus \{x, y, v\}. \tag{1.1}$$

For a pair (x, y) of vertices at distance 2 in G, set $J'(x, y) = \{u \in N(x) \cap N(y) \mid u \text{ satisfies (1.1)}\}$. We say that G is in the class $\mathcal{P3D}$ of *P_3-dominated* graphs if $J(x, y) \cup J'(x, y) \neq \emptyset$ for every pair (x, y) of vertices at distance 2 in G.

In [5], [10], [13]-[14] and [17]-[19], some known results on claw-free graphs have extended to P_3-dominated graphs. Particularly, the 3-connected case concerning hamiltonicity of P_3-dominated graphs is shown [17]. However, in this paper we mainly discuss 2-connected case, which is different from the above work in [17]. Meanwhile, their neighborhood union conditions for hamiltonicity of P_3-dominated graphs are also different.

The objective of this paper is also to extend the following result on claw-free graphs, which was obtained by Bauer et al. [3], to P_3-dominated graphs. The main results of this paper are the following Theorem 2 and Corollary 1, and the proofs are given in Section 2.

Theorem 1 (Bauer et al. [3]). *Let G be 2-connected claw-free graph of order n. If $NC(G) \geq (2n-5)/3$, then G is Hamiltonian.* □

Theorem 2. *If $G \notin \{K_{1,1,3}, K_{2,3}\}$ is a 2-connected P_3-dominated graph of order n such that $NC(G) \geq (2n-5)/3$, then G is Hamiltonian.* □

Since the class of P_3-dominated graphs contain all quasi claw-free graphs, we have:

Corollary 1. *If G is a 2-connected quasi claw-free graph of order n such that $NC(G) \geq (2n-5)/3$, then G is Hamiltonian.* □

Some ideas and proof techniques demonstrated by Broersma [7] are adopted in the proof of Theorem 2. Also some results obtained by Bauer [3] are used in the proof of Theorem 2. They are stated as lemmas in the following section.

2 Proof of Theorem 2

Before starting the proof of Theorem 2, we present some necessary notations and preliminary lemmas.

Let C be a cycle in G with an inherent clockwise orientation and H be a component of $G - C$. For $x, y \in V(C)$, let x^+ and x^- be the successor and predecessor of x along the orientation of C, respectively. Set $x^{++} = (x^+)^+$, $x^{--} = (x^-)^-$. If $x, y \in V(C)$, then $C[x, y]$ denotes the consecutive vertices on C from x to y in the chosen direction of C, and $C(x, y)$=$C[x, y] - \{x, y\}$. Then same vertices in the reverse order are respectively denoted by $\overleftarrow{C}[y, x]$ and $\overleftarrow{C}(y, x)$. Both $C[x, y]$ and $\overleftarrow{C}[y, x]$ are considered as paths as well as vertex sets. In this section we will use such symbols for a given cycle without giving the definition.

Lemma 1. *Let $G \notin \{K_{1,1,3}, K_{2,3}\}$ be a 2-connected P_3-dominated graph and let C be a longest cycle with a cyclic order in G, and let H be a component of $G - C$. Then*
(a) $x^- x^+ \in E(G)$ for each $x \in N_C(H)$;
(b) $N(x^-) \cap \{y, y^-, y^{--}\} = \emptyset$, $N(x^{--}) \cap \{y, y^-, y^{--}\} = \emptyset$ for each $x, y \in N_C(H)$ with $x \neq y$.

Proof : For the proof of (a) see [5], and the proof of (b) is straightforward, hence we omit it. □

Lemma 2 (Bauer et al. [3]). $\sigma_3(G) \geq 3NC(G) - n + 3$ *for any graph G of order $n \geq 3$.* □

Lemma 3. *Let $G \notin \{K_{2,3}, K_{1,1,3}\}$ be a 2-connected P_3-dominated graph of order n. If $\sigma_3(G) \geq n - 2$, then G is Hamiltonian.* □

Combining Lemmas 2 and 3, we obtain the main result Theorem 2.

Proof of Lemma 3

Assume, to the contrary, that G is not hamiltonian. Let C be a longest cycle of G and H be a component of $G - C$. Fix an orientation on C. By assumption C is not a Hamilton cycle of G, there exists a vertex $u \in V(H)$. Since G is 2-connected, $G \notin \{K_{2,3}, K_{1,1,3}\}$, there exist at least 2 distinct vertices $w_1, w_2, \ldots, w_k$ of C such that $uw_i \in E(G)$ $(i = 1, 2, \ldots, k)$. Let $\{w_i | \ i = 1, 2, \ldots, k\}$ be chosen such that k is maximum $(k \geq 2)$. By the maximality of k, u has no neighbors in $V(C) - \{w_1, w_2, \ldots, w_k\}$. Let the order of occurrence on C of the vertices w_i, $i = 1, 2, \ldots, k$, be according to their indices. From the choice of C it follows that, for $1 \leq i \leq k$, $w_i w_{i+1} \notin E(C)$ (indices mod k), $uw_i^+ \notin E(G)$ and $uw_i^- \notin E(G)$. By Lemma 1 (a), we have $w_i^+ w_i^- \in E(G)$ $(i = 1, 2, \ldots, k)$. From the choice of C it also follows that w_i^+ and w_{i+1}^- cannot coincide and $w_i^+ w_{i+1}^- \notin E(C)$ (indices mod k) for $1 \leq i \leq k$. If $w_i^+ w_{i+1}^- \in E(C)$, then the cycle $uw_i w_i^+ \overleftarrow{C}[w_i^-, w_{i+1}^+] w_{i+1}^- w_{i+1} u$ contradicts the choice of C. By Lemma 1 (b), we have $w_i^- w_j^-, w_i^{--} w_j^-, w_i^{--} w_j^{--}, w_i^{--} w_j$ and $w_i^- w_j \notin E(G)$ where $i, j = 1, 2, \ldots, k$ and $i \neq j$.

Let s_i be a vertex of $C[w_i, w_{i+1}]$ such that
(i) s_i^- is adjacent to w_i^{--}, w_i^- or w_i;
(ii) s_i is adjacent to none of w_i^{--}, w_i^- and w_i;
(iii) $|C[s_i, w_{i+1}]|$ is minimum (indices mod k).

Since w_i^+ is adjacent to w_i, and w_{i+1}^- is adjacent none of w_i^{--}, w_i^- and w_i, there exists at least one vertex of $C(w_i^{++}, w_{i+1}^-)$ that satisfies both (i) and (ii). Thus s_i is well-defined.

Now we continue our proof for Lemma 3 with the following claims.

Claim 1. s_i is not adjacent to w_j or w_j^+.

If $s_i w_j \in E(G)$, we consider the following cases.

Case	Cycle C'
$s_i^- w_i \in E(G)$	$uw_i \overleftarrow{C}[s_i^-, w_i^+] \overleftarrow{C}[w_i^-, w_j^+] \overleftarrow{C}[w_j^-, s_i] w_j u$
$s_i^- w_i^- \in E(G)$	$uC[w_i, s_i^-] \overleftarrow{C}[w_i^-, w_j^+] \overleftarrow{C}[w_j^-, s_i] w_j u$
$s_i^- w_i^{--} \in E(G)$	$uw_i w_i^- C[w_i^+, s_i^-] \overleftarrow{C}[w_i^{--}, w_j^+] \overleftarrow{C}[w_j^-, s_i] w_j u$

If $s_i w_j^+ \in E(G)$, we consider the following cases.

Case	Cycle C''
$s_i^- w_i \in E(G)$	$uw_i \overleftarrow{C}[s_i^-, w_i^+] \overleftarrow{C}[w_i^-, w_j^+] C[s_i, w_j] u$
$s_i^- w_i^- \in E(G)$	$uC[w_i, s_i^-] \overleftarrow{C}[w_i^-, w_j^+] C[s_i, w_j] u$
$s_i^- w_i^{--} \in E(G)$	$uw_i w_i^- C[w_i^+, s_i^-] \overleftarrow{C}[w_i^{--}, w_j^+] C[s_i, w_j] u$

In each of these cases, the cycle C' and C'' are longer than C, a contradiction.

Claim 2. $us_i \notin E(G)$ and $N(u) \cap N(s_i) = \emptyset$ for $i = 1, 2, \ldots, k$.

Claim 3. $s_i s_j \notin E(G)$.

Assume $s_i s_j \in E(G)$. If $s_i^- w_i^{--} \in E(G)$, then we discuss the following cases.

Case	Cycle C'
$s_j^- w_j \in E(G)$	$uw_i w_i^- C[w_i^+, s_i^-]\ \overleftarrow{C}[w_i^{--}, s_j] C[s_i, w_j^-] C[w_j^+, s_j^-] w_j u$
$s_j^- w_j^- \in E(G)$	$uw_i w_i^- C[w_i^+, s_i^-]\ \overleftarrow{C}[w_i^{--}, s_j] C[s_i, w_j^-] \overleftarrow{C}[s_j^-, w_j] u$
$s_j^- w_j^{--} \in E(G)$	$uw_i w_i^- C[w_i^+, s_i^-]\ \overleftarrow{C}[w_i^{--}, s_j] C[s_i, w_j^{--}] \overleftarrow{C}[s_j^-, w_j^+] w_j^- w_j u$

Obviously, the cycle C' contradicts the choice of C in each cases. The other cases $s_i^- w_i^- \in E(G)$ or $s_i^- w_i \in E(G)$ are similar.

Claim 4. For similar reasons, $N_{G-C}(s_i) \cap N_{G-C}(s_j) = \emptyset$ for $i, j = 1, 2, \ldots, k$ and $i \neq j$.

Claim 5. If $v, v^+ \in C[s_i^+, w_j^-]$, then at most one of edges $s_j v$ and $s_i v^+$ is present in G.

Suppose $s_j v \in E(G)$ and $s_i v^+ \in E(G)$. If $s_i^- w_i^{--} \in E(G)$, then we have the following cases.

Case	Cycle C'
$s_j^- w_j \in E(G)$	$uw_i w_i^- C[w_i^+, s_i^-]\ \overleftarrow{C}[w_i^{--}, s_j] \overleftarrow{C}[v, s_i] C[v^+, w_j^-] C[w_j^+, s_j^-] w_j u$
$s_j^- w_j^- \in E(G)$	$uw_i w_i^- C[w_i^+, s_i^-]\ \overleftarrow{C}[w_i^{--}, s_j] \overleftarrow{C}[v, s_i] C[v^+, w_j^-] \overleftarrow{C}[s_j^-, w_j] u$
$s_j^- w_j^{--} \in E(G)$	$uw_i w_i^- C[w_i^+, s_i^-]\ \overleftarrow{C}[w_i^{--}, s_j] \overleftarrow{C}[v, s_i] C[v^+, w_j^{--}] \overleftarrow{C}[s_j^-, w_j^+] w_j^- w_j u$

The cycle C' contradicts the choice of C in each cases. The other cases $s_i^- w_i^- \in E(G)$ or $s_i^- w_i \in E(G)$ are similar.

Claim 6. If v, $v^+ \in C[w_i^+, s_i^-]$ and $w_i s_i^- \in E(G)$, then at most one of the edges $s_i v$ and $s_j v^+$ is present in G.

If $s_i v \in E(G)$ and $s_j v^+ \in E(G)$, e.g., $s_j^- w_j^{--} \in E(G)$, then the cycle $uw_i \overleftarrow{C}[s_i^-, v^+] C[s_j, w_i^-]$ $C[w_i^+, v] C[s_i, w_j^{--}] \overleftarrow{C}[s_j^-, w_j^+] w_j^- w_j u$ is longer than C, a contradiction. The other cases are similar.

Claim 7. If v, $v^+ \in C[w_i^+, s_i^-]$ and $w_i s_i^- \notin E(G)$, then at most one of the edges $s_j v$ and $s_i v^+$ is present in G.

The proof of this Claim is similar to that of Claim 5. So we omit it.

Claim 8. If $w_i s_i^- \notin E(G)$, then at most one of the edges $s_i^- s_j$ and $w_i^+ s_i$ is present in G.
The proof of this Claim is similar to that of Claim 6. So we omit it.

If S is a subset of $V(G)$, then $d_S(s_i) = |N(s_i) \cap S|$. We consider the sets $I_1 = C[w_1^+, s_1^-]$ and $I_2 = C[s_1^+, w_2^-]$ and let $A_1 = \{v \in I_1 | \ vs_1 \in E(G)\}$, $B_1 = \{v \in I_2 | \ vs_1 \in E(G)\}$ and $B_2 = \{v \in I_2 | \ v^- s_2 \in E(G)\}$. If $w_1 s_1^- \in E(G)$, then let $A_2 = \{v \in I_1 | \ v^+ s_2 \in E(G)\}$; if $w_1 s_1^- \notin E(G)$ let $A_2 = \{v \in I_1 | \ v^- s_2 \in E(G)\}$. $B_1 \cap B_2 = \emptyset$ by Claim 5, hence we have $d_{I_2}(s_1) + d_{I_2}(s_2) = |B_1| + |B_2| = |B_1 \cup B_2| \leq |I_2|$.

By the similar arguments, if $w_1 s_1^- \in E(G)$, $A_1 \cap A_2 = \emptyset$ by Claims 6, then $d_{I_1}(s_1) + d_{I_1}(s_2) \leq |I_1|$. For $w_1 s_1^- \notin E(G)$, by Claim 7, we get $A_1 \cap A_2 = \emptyset$. Then we consider two possibilities:
(a) $s_1^- s_2 \notin E(G)$. Then $d_{I_1}(s_1) + d_{I_1}(s_2) = |A_1| + |A_2| = |A_1 \cup A_2| \leq |I_1|$;
(b) $s_1^- s_2 \in E(G)$. Then $d_{I_1}(s_1) + d_{I_1}(s_2) = |A_1| + |A_2| + 1 = |A_1 \cup A_2| + 1$. In addition, $w_1^+ \notin A_1 \cup A_2$ by Claim 8. Hence $d_{I_1}(s_1) + d_{I_1}(s_2) \leq |I_1|$.

Similarly, we have $d_{I_3}(s_1) + d_{I_3}(s_2) \leq |I_3|$ for $I_3 = C[w_2^+, s_2^-]$. Finally, we consider $I_4 = C[s_2^+, w_1^-]$, and let $D_1 = \{v \in I_4 | \ vs_2 \in E(G)\}$ and $D_2 = \{v \in I_4 | \ v^- s_1 \in E(G)\}$. $D_1 \cap D_2 = \emptyset$ By Claim 5 and if $k \geq 3$, then $w_i^+ \notin D_1 \cup D_2$ by Claim 1, for $3 \leq i \leq k$. So $d_{I_4}(s_1) + d_{I_4}(s_2) = |D_1| + |D_2| = |D_1 \cup D_2| \leq |I_4| - (k-2)$. In addition, by Claims 1 and 3, $d_{V(C)}(s_1) + d_{V(C)}(s_2) \leq |I_1| + |I_2| + |I_3| + |I_4| - (k-2) = |V(C)| - k - 2$. Hence, by Claims 2 and 4, $d(u) + d(s_1) + d(s_2) \leq (n-1-|V(C)|) + k + |V(C)| - k - 2 = n-3$ which contraries to $\sigma_3(G) \geq n-2$. Thus the Lemma is proved. □

Note that Lemma 3 and Theorem 2 are best possible. This can be seen from the P_3-dominated graph G obtained as follows: take three copies of the complete graph K_t, say, K_t^1, K_t^2 and K_t^3 $(t \geq 3)$, pick 2 distinct vertices x_i, y_i from $K_t^i (i = 1, 2, 3)$ and then form 2 triangles $x_1x_2x_3$ and $y_1y_2y_3$. This graph G is 2-connected P_3-dominated graph, and we have $\sigma_3 = n-3$, $NC = (2n-6)/3$, but G is not hamiltonian.

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