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On the monopolies of lexicographic product graphs: bounds and closed formulas

by

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Abstract

We consider a simple graph $G = (V, E)$ without isolated vertices and of minimum degree $\delta(G)$. Let k be an integer number such that $k \in \{1 - \lceil \delta(G)/2 \rceil, \dots, \lfloor \delta(G)/2 \rfloor\}$. A vertex v of G is said to be k-controlled by a set $M \subset V$, if $\delta_M(v) \geq \frac{\delta(v)}{2} + k$ where $\delta_M(v)$ represents the number of neighbors v has in M and $\delta(v)$ the degree of v. The set M is called a k-monopoly if it k-controls every vertex v of G. The minimum cardinality of any k-monopoly in G is the k-monopoly number of G. In this article we study the k-monopolies of the lexicographic product of graphs. Specifically we obtain several relationships between the k-monopoly number of this product graph and the k-monopoly numbers and/or order of its factors. Moreover, we bound (or compute the exact value) of the k-monopoly number of several families of lexicographic product graphs.

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1 Introduction and preliminaries

The k-monopolies of graphs were recently introduced in [11], as a generalization of the standard monopolies already known from several articles. From the best of our knowledge, the first description of a monopoly-type concept comes from [14], where the authors described some structures regarding sphere packing and local majorities. Concepts like (small) coalitions, alliances are also closely related and they have been appearing relatively frequently in some connected investigations. A remarkable case is [16], where several of that connections were reviewed.

The monopolies of graphs have been related to several problems regarding overcoming failures, in concordance with the fact that they very frequently have some common approaches in the field of majorities. For instance, some connections with consensus problems [3], diagnosis problems [17] or voting systems [5], among other applications and references are already described. The k-monopolies in graphs are also closely related to different parameters in graphs like global alliances and signed domination (see [11]). On the other hand, it is also known that computing the k-monopoly number of a graph is an NP-complete problem for $k \geq 0$ (see [11]). In this sense, it would be desirable to reduce the problem of computing or bounding the k-monopoly number in some complex graph classes into other ones much simpler classes. An interesting case of that are product graphs, where it is often possible to reduce the problem on the product graph to the problem on the factor graphs. Such a studies have been initiated in the works [12] and [13], for the direct and strong product of graphs, respectively. To continue pursuing this goal, in this work we obtain several relationships between the k-monopoly number of the lexicographic product graphs and the k-monopoly number of its factors.

We consider here only simple graphs $G = (V, E)$. Given a set $S \subset V$ and a vertex $v \in V$, we denote by $\delta_S(v)$ the number of neighbors v has in S. If $S = V$, then $\delta_V(v)$ is the degree of v and we just write $\delta(v)$. The *minimum degree* of G is denoted by $\delta(G)$ and the *maximum degree* by $\Delta(G)$. Given an integer $k \in \left\{1 - \left\lceil \frac{\delta(G)}{2} \right\rceil, \dots, \left\lfloor \frac{\delta(G)}{2} \right\rfloor \right\}$ and a set $M \subset V$, a vertex v of G is said to be k-*controlled* by M if $\delta_M(v) \geq \frac{\delta(v)}{2} + k$. The set M is called a k-*monopoly* if it k-controls every vertex v of G. The minimum cardinality of any k-monopoly is the k-*monopoly number* and it is denoted by $\mathcal{M}_k(G)$. A k-monopoly of cardinality $\mathcal{M}_k(G)$ is called a $\mathcal{M}_k(G)$-*set*. In particular notice that for a graph with a leaf (vertex of degree one), there exist only 0-monopolies and the neighbor of every leaf is in each $\mathcal{M}_0$-set. Monopolies in graphs were defined first in [14] and they were generalized to k-monopolies recently in [11]. Among other studies about monopolies in graphs and some of its applications we mention for instance [4, 9, 15, 16, 19].

If $\overline{M}$ represents the *complement* of the set M, then we can use the following equivalent definition for a k-monopoly in G. A set of vertices M is a k-monopoly in G if and only if for every vertex v of G, $\delta_M(v) \geq \delta_{\overline{M}}(v) + 2k$ (from now we will call this expression *the k-monopoly condition*) and we will say that M is a k-monopoly in G if and only if every v of G satisfies the k-monopoly condition for M.

We use for a graph G standard notations $N_G(g)$ for the *open neighborhood* of a vertex g: $N_G(g) = \{g' : gg' \in E(G)\}$ and $N_G[g]$ for the *closed neighborhood* of g: $N_G[g] = N_G(g) \cup \{g\}$. Let $S \subset V(G)$. Neighborhoods over S form a *subpartition* of a graph G, if $N_G(u) \cap N_G(v) = \emptyset$ for every different $u, v \in S$. A set S forms a *maximum subpartition* if S forms a subpartition where $V(G) - \bigcup_{v \in S} N_G(v)$ has the minimum cardinality among all possible sets $S' \subset V(G)$. In the extreme case where $\bigcup_{v \in S} N_G(v) = V(G)$ we call G an *efficient open domination graph* and the set S an *efficient open dominating set* of G. Efficient open domination graphs were first studied in [6]. The work was continued in [7], where all efficient open domination trees have been inductively described. Also, there was shown that deciding whether a graph is an efficient open domination graph is an NP-complete problem. Recently, in [18], the efficient open dominating sets of Cayley graphs were considered and, a discussion with respect to product graphs can be found in [10]. Clearly not all graphs are efficient open domination graphs.

The *lexicographic product* $G \circ H$ (also sometimes denoted by $G[H]$ and called composition) of graphs G and H is a graph with $V(G \circ H) = V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in $G \circ H$ whenever $gg' \in E(G)$ or ($g = g'$ and $hh' \in E(H)$). For a fix $h \in V(H)$ we call $G^h = \{(g, h) \in V(G \circ H) : g \in V(G)\}$ a G-*layer* in $G \circ H$. An H-*layer* ${}^gH$ for a fix $g \in V(G)$ is defined symmetrically. Notice that the subgraph induced by G^h or ${}^gH$ is isomorphic to G or H, respectively. The lexicographic product is clearly not commutative, while it is associative [8].

2 Upper bounds

We begin with the following general result in which the k-monopoly number of the second factor of the lexicographic product graph contains at least half its order.

Theorem 1. *Let G and H be graphs and let $k \in \left\{1 - \left\lceil \frac{\delta(H)}{2} \right\rceil, \ldots, \left\lfloor \frac{\delta(H)}{2} \right\rfloor \right\}$. If $\mathcal{M}_k(H) \geq \frac{|V(H)|}{2}$, then we have*

$$\mathcal{M}_k(G \circ H) \leq \mathcal{M}_k(H)|V(G)|.$$

Proof. Let M_H be a $\mathcal{M}_k(H)$-set where $|M_H| \geq \frac{|V(H)|}{2}$. Notice that H has no isolated vertex, since $\mathcal{M}_k(H)$ exists. We claim that $M = V(G) \times M_H$ is a k-monopoly set of $G \circ H$. For $(g, h) \in V(G \circ H)$ we have

$$\begin{aligned} \delta_M(g,h) &= \delta_G(g)|M_H| + \delta_{M_H}(h) \\ &\geq \delta_G(g)\frac{|V(H)|}{2} + \frac{\delta_H(h)}{2} + k \\ &= \frac{\delta_{G \circ H}(g,h)}{2} + k. \end{aligned}$$

Therefore the result follows. □

The expectation, that the condition $\mathcal{M}_k(H) \geq \frac{|V(H)|}{2}$ from the above theorem is fulfilled, is greater when k is "big". Hence, this theorem is more useful for positive k. For instance, if H is a graph with even number of vertices with an universal vertex, then $\mathcal{M}_k(H) \geq \frac{|V(H)|}{2}$ for $k \geq 0$.

If $\mathcal{M}_k(H) > \frac{|V(H)|}{2}$, then we can add (at least one) isolated vertex to H to obtain graph H^+ and the upper bound from Theorem 1 still holds for $G \circ H^+$.

Next we continue with a general result for non negative values of k.

Theorem 2. *Let G and H be graphs and let $\ell = \min\{\delta(G), \delta(H)\}$. For every $k \in \{0, \ldots, \left\lfloor \frac{\ell}{2} \right\rfloor\}$,*

$$\mathcal{M}_{2k^2}(G \circ H) \leq \mathcal{M}_k(G)|V(H)|.$$

Proof. Let M_G be a $\mathcal{M}_k(G)$-set. We will show that $M = (M_G \times V(H))$ is a $2k^2$-monopoly set of $G \circ H$. Let $(g, h) \in V(G \circ H)$. We consider the following cases.

Case 1: $g \in M_G$. Hence,

$$\begin{aligned} \delta_M(g,h) &= \delta_{M_G}(g)|V(H)| + \delta_H(h) \\ &\geq \delta_{\overline{M_G}}(g)|V(H)| + 2k|V(H)| + \delta_H(h) \\ &= \delta_{\overline{M}}(g,h) + 2k|V(H)| + \delta_H(h) \\ &\geq \delta_{\overline{M}}(g,h) + 2k\delta_H(h) + \delta_H(h) \\ &\geq \delta_{\overline{M}}(g,h) + 4k^2 + 2k. \end{aligned}$$

Case 2: $g \notin M_G$. Hence,

$$\begin{aligned}
\delta_M(g,h) &= \delta_{M_G}(g)|V(H)| \\
&\geq \delta_{\overline{M_G}}(g)|V(H)| + 2k|V(H)| \\
&= \delta_{\overline{M_G}}(g)|V(H)| + \delta_H(h) + 2k|V(H)| - \delta_H(h) \\
&= \delta_{\overline{M}}(g,h) + 2k|V(H)| - \delta_H(h) \\
&\geq \delta_{\overline{M}}(g,h) + 2k(\delta_H(h)+1) - \delta_H(h) \\
&\geq \delta_{\overline{M}}(g,h) + \delta_H(h)(2k-1) + 2k \\
&\geq \delta_{\overline{M}}(g,h) + 2k(2k-1) + 2k \\
&= \delta_{\overline{M}}(g,h) + 4k^2.
\end{aligned}$$

Thus, we have that M is a $2k^2$-monopoly in $G \circ H$, which completes the proof. □

3 Lower bounds

The following observation and its corollary presented in [12] are useful to deduce a lower bound for $\mathcal{M}_k(G \circ H)$.

Remark 1. [12] *Let G be a graph. If $S \subset V(G)$ forms a subpartition of G, then*

$$\mathcal{M}_k(G) \geq k|S| + \sum_{v \in S} \left\lceil \frac{\delta(v)}{2} \right\rceil .$$

Corollary 1. [12] *Let G be a graph. If $S \subset V(G)$ forms a subpartition of G, then*

$$\mathcal{M}_k(G) \geq |S|(k + \lceil \delta(G)/2 \rceil).$$

In [10] all efficient open domination graphs among lexicographic products have been characterized. It was shown that $G \circ H$ is an efficient open domination graph if and only if one of the following conditions is fulfilled:

(i) G is a graph without edges and H an efficient open domination graph, or

(ii) G is an efficient open domination graph and H contains an isolated vertex.

Notice that (ii) can be generalized to every graph for maximum subpartitions. The following proposition describes this case.

Proposition 1. *Let G and H be graphs, where H contains an isolated vertex h and G is without them. If S_G forms a subpartition of G, then $S = S_G \times \{h\}$ forms a subpartition of $G \circ H$.*

Proof. Since h has no neighbor in H and $N_G(g) \cap N_G(g') = \emptyset$ for every different vertices $g, g' \in S_G$, it follows that $N_{G\circ H}(g,h) \cap N_{G\circ H}(g',h) = \emptyset$. So $S_G \times \{h\}$ form a subpartition of $G \circ H$. □

The following theorem follows directly from Observation 1, Proposition 1 and the fact that $\delta_{G\circ H}(g,h) = \delta_G(g)|V(H)| + \delta_H(h)$. (Notice that h is an isolated vertex of H and hence $\delta_H(h) = 0$.)

Theorem 3. *Let G and H be graphs, H with an isolated vertex h and G without them and let $\ell = \delta(G)|V(H)| + \delta(H)$. If S_G forms a subpartition of G, then for every $k \in \{1 - \lceil \frac{\ell}{2} \rceil, \dots, \lfloor \frac{\ell}{2} \rfloor\}$,*

$$\mathcal{M}_k(G \circ H) \geq k|S_G| + \sum_{(g,h) \in S_G \times \{h\}} \left\lceil \frac{\delta(g)|V(H)|}{2} \right\rceil .$$

Finally we consider the case where H has no isolated vertices. To do so we need to introduce some notation. Given a graph G and a set $S \subset V(G)$, we say that the closed neighborhoods of the vertices of S form a *closed subpartition* for G if it is satisfied $N_G[u] \cap N_G[v] = \emptyset$ for every two different vertices $u, v \in S$. A closed subpartition for G, formed by a set S, is a *maximum closed subpartition*, if $V(G) - \bigcup_{v \in S} N_G[v]$ has the minimum possible cardinality among all closed subpartitions for G formed by sets $S' \subset V(G)$. It is already known that, if S forms a maximum closed subpartition which is also a partition of $V(G)$, then the set S is called a *perfect code* [2] or an *efficient dominating set* [1] in G. The result, regarding the maximum closed subpartitions of graphs, yields an interesting consequence for our purposes.

Proposition 2. *Let G and H be graphs, both without isolated vertices. If S_G forms a closed subpartition of G and h is a vertex of maximum degree in H, then $S = S_G \times \{h\}$ forms a subpartition of $G \circ H$.*

Proof. The result follows similarly as in Proposition 1. Since G^h is isomorphic to G and S_G forms a maximum closed subpartition of G, we have that $N_{G \circ H}(g, h) \cap N_{G \circ H}(g', h) = \emptyset$ for every different vertices g and g' from S_G. Hence, we obtain that $S_G \times \{h\}$ forms a subpartition of $G \circ H$. □

Again, the following theorem is obtained directly from the Observation 1, Proposition 2 and the fact that $\delta_{G \circ H}(g, h) = \delta_G(g)|V(H)| + \delta_H(h)$. (Notice that we can choose the vertex of maximum degree $\Delta(H)$ in H.)

Theorem 4. *Let G and H be graphs, both without isolated vertices and let $\ell = \delta(G)|V(H)| + \delta(H)$. If S_G forms a closed subpartition of G and h is a vertex of maximum degree in H, then for $k \in \{1 - \lceil \frac{\ell}{2} \rceil, \dots, \lfloor \frac{\ell}{2} \rfloor\}$ we have*

$$\mathcal{M}_k(G \circ H) \geq k|S_G| + \sum_{(g,h) \in S_G \times \{h\}} \left\lceil \frac{\delta_G(g)|V(H)| + \Delta(H)}{2} \right\rceil .$$

Notice that Theorems 3 and 4 behave better in the case the (closed) subpartitions considered in G will be maximum.

The next result, presented in [12], is also useful to deduce a bound for the k-monopoly number of lexicographic product graphs.

Proposition 3. [12] *If G is a graph of order n, then for any $k \in \left\{1 - \left\lceil \frac{\delta(G)}{2} \right\rceil, \dots, \left\lfloor \frac{\delta(G)}{2} \right\rfloor \right\}$,*

$$\mathcal{M}_k(G) \geq \left\lceil \frac{n}{\Delta(G)} \left(\left\lceil \frac{\delta(G)}{2} \right\rceil + k \right) \right\rceil .$$

Corollary 2. *Let G and H be two graphs without isolated vertices of order n and m, respectively. For any $k \in \left\{1 - \left\lceil \frac{m\delta(G)+\delta(H)}{2} \right\rceil, \dots, \left\lfloor \frac{m\delta(G)+\delta(H)}{2} \right\rfloor \right\}$ we have*

$$\mathcal{M}_k(G \circ H) \geq \left\lceil \frac{nm}{m\Delta(G) + \Delta(H)} \left(\left\lceil \frac{m\delta(G) + \delta(H)}{2} \right\rceil + k \right) \right\rceil .$$

4 Bounds and exact values for some specific families of lexicographic product graphs

The following result from [11] will be useful in this section.

Lemma 1. [11] *For every integer $r \geq 3$,*

$$\mathcal{M}_0(C_r) = \mathcal{M}_0(P_r) = \begin{cases} \frac{r}{2}, & \text{if } r \equiv 0 \pmod 4, \\ \frac{r+2}{2}, & \text{if } r \equiv 2 \pmod 4, \\ \frac{r+1}{2}, & \text{if } r \equiv x \pmod 4, x \in \{1,3\}. \end{cases}$$

It is clear from the previous section that lower bounds for $\mathcal{M}_k(G \circ H)$ depends on (non) existence of an isolated vertex in H. This is the reason that we start with an empty graph E_n as second factor. By E_n we mean a graph on n vertices without any edges.

Proposition 4. *Let $r, t \geq 3$ be integers.*

(i) If $r \equiv 0 \pmod 4$, then $\mathcal{M}_0(C_r \circ E_t) = \frac{rt}{2}$.

(ii) If $r \equiv x \pmod 4$, $x \in \{1,3\}$, then $\frac{rt}{2} - \frac{t}{2} \leq \mathcal{M}_0(C_r \circ E_t) \leq \frac{rt}{2} + \frac{t}{2}$.

(iii) If $r \equiv 2 \pmod 4$, then $\frac{rt}{2} - t \leq \mathcal{M}_0(C_r \circ E_t) \leq \frac{rt}{2} + t$.

Proof. The upper bounds follow directly from Theorem 2 and Lemma 1. For the lower bounds we use Theorem 3. For this we need the cardinality of a maximum subpartition S of C_r. It is easy to see that

$$|S| = \begin{cases} \frac{r}{2}, & \text{if } r \equiv 0 \pmod 4, \\ \frac{r-2}{2}, & \text{if } r \equiv 2 \pmod 4, \\ \frac{r-1}{2}, & \text{if } r \equiv x \pmod 4, x \in \{1,3\} \end{cases}$$

and the result follows. Notice that the lower and upper bound obtained as above mentioned equal the same value for the case of (i). □

Proposition 5. *Let $r, t \geq 3$ be integers.*

(i) If $r \equiv 0 \pmod 4$, then $\mathcal{M}_0(P_r \circ E_t) = \frac{rt}{2}$.

(ii) If $r \equiv 1 \pmod 4$*, then* $\frac{rt}{2} - \frac{3t}{2} + 2\left\lceil\frac{t}{2}\right\rceil \leq \mathcal{M}_0(P_r \circ E_t) \leq \frac{rt}{2} + \frac{t}{2}$.

(iii) If $r \equiv 2 \pmod 4$*, then* $\frac{rt}{2} - t + 2\left\lceil\frac{t}{2}\right\rceil \leq \mathcal{M}_0(P_r \circ E_t) \leq \frac{rt}{2} + t$.

(iv) If $r \equiv 3 \pmod 4$*, then* $\frac{rt}{2} - \frac{t}{2} + \left\lceil\frac{t}{2}\right\rceil \leq \mathcal{M}_0(P_r \circ E_t) \leq \frac{rt}{2} + \frac{t}{2}$.

Proof. Again the upper bounds follow directly from Theorem 2 and Lemma 1. For the lower bounds we need the cardinality and the structure of a maximum subpartition S of C_r. Notice that there exists S which contains

- no vertices of degree 1 and $\frac{r}{2}$ vertices of degree 2 if $r \equiv 0 \pmod 4$;
- two vertices of degree 1 and $\frac{r-3}{2}$ vertices of degree 2 if $r \equiv 1 \pmod 4$;
- two vertices of degree 1 and $\frac{r-2}{2}$ vertices of degree 2 if $r \equiv 2 \pmod 4$;
- one vertex of degree 1 and $\frac{r-1}{2}$ vertices of degree 2 if $r \equiv 3 \pmod 4$.

From this the lower bounds follow directly from Theorem 3. □

Next results are given for the lexicographic products of cycles and/or paths.

Proposition 6. *Let* $r, t \geq 3$ *be integers.*

(i) If $t \equiv 0 \pmod 4$*, then* $\mathcal{M}_0(C_r \circ C_t) = \frac{rt}{2}$.

(ii) If $t \equiv 1 \pmod 4$*, then* $\left\lceil\frac{rt}{2}\right\rceil \leq \mathcal{M}_0(C_r \circ C_t) \leq \frac{rt}{2} - \frac{r}{2} + \mathcal{M}_0(C_r)$*. Moreover,*

(a) if $r \equiv 0 \pmod 4$*, then* $\mathcal{M}_0(C_r \circ C_t) = \frac{rt}{2}$,
(b) if r *is odd, then* $\mathcal{M}_0(C_r \circ C_t) = \frac{rt+1}{2}$,
(c) if $r \equiv 2 \pmod 4$*, then* $\frac{rt}{2} \leq \mathcal{M}_0(C_r \circ C_t) \leq \frac{rt}{2} + 1$.

(iii) If $t \equiv 2 \pmod 4$*, then* $\left\lceil\frac{rt}{2}\right\rceil \leq \mathcal{M}_0(C_r \circ C_t) \leq \frac{rt}{2} + \left\lfloor\frac{r}{2}\right\rfloor$.

(iv) If $t \equiv 3 \pmod 4$*, then* $\left\lceil\frac{rt}{2}\right\rceil \leq \mathcal{M}_0(C_r \circ C_t) \leq \frac{r(t+1)}{2}$.

Proof. First notice that, since $C_r \circ C_t$ is a $(2t+2)$-regular graph, from Corollary 2 it follows that $\mathcal{M}_0(C_r \circ C_t) \geq \left\lceil\frac{rt}{2}\right\rceil$ and all the lower bounds follow. Now, let $V_1 = \{u_0, u_1, ..., u_{r-1}\}$ and $V_2 = \{v_0, v_1, ..., v_{t-1}\}$ be the vertex sets of C_r and C_t, respectively (operations with the subscripts of u_i and v_j will be done modulo r and t, respectively). We consider the following cases.

Case 1: $t \equiv 0 \pmod 4$. From Lemma 1 we have that $\mathcal{M}_0(C_t) = \frac{t}{2}$. Hence, by Theorem 1, we obtain that $\mathcal{M}_0(C_r \circ C_t) \leq r\mathcal{M}_0(C_t) = \frac{rt}{2}$ and (i) follows.

Case 2: $t \equiv 1 \pmod 4$. By Lemma 1 we have that $\mathcal{M}_0(C_t) = \frac{t+1}{2}$. Thus, there exists a $\mathcal{M}_0(C_t)$-set A such that $v_0, v_1, v_2 \in A$ and let X be a $\mathcal{M}_0(C_r)$-set. We claim that the set $M = (V_1 \times A) - (\overline{X} \times \{v_0\})$ (recall that $\overline{X}$ is the complement of X) is a 0-monopoly in $C_r \circ C_t$. To see this, notice that if $u_i \in X$, then the corresponding ${}^{u_i}C_t$-layer contains $\frac{t+1}{2}$ vertices of M and, if $u_i \notin X$, then the ${}^{u_i}C_t$-layer contains $\frac{t+1}{2} - 1$ vertices of M. Also, notice that every vertex of $(u_i, v_j) \in M$ has at least one neighbor in $M \cap {}^{u_i}C_t$ and at least one neighbor in $M \cap C_r{}^{v_j}$. Hence, for every vertex (u_i, v_j) we have that

$$\begin{aligned}\delta_M(u_i, v_j) &= \delta_{M\cap {}^{u_{i-1}}C_t}(u_i, v_j) + \delta_{M\cap {}^{u_i}C_t}(u_i, v_j) + \delta_{M\cap {}^{u_{i+1}}C_t}(u_i, v_j)\\ &\geq \frac{t+1}{2} + 1 + \frac{t+1}{2} - 1\\ &= t+1\\ &= \frac{\delta(u_i, v_j)}{2}.\end{aligned}$$

As a consequence, we obtain that every vertex of $C_r \circ C_t$ satisfies the 0-monopoly condition. Thus, the upper bound of (ii) is obtained from the fact that $|M| = \frac{r(t+1)}{2} - r + \mathcal{M}_0(C_r) = \frac{rt}{2} - \frac{r}{2} + \mathcal{M}_0(C_r)$.

Now, by making some simple calculations, from the bounds of (ii) and Lemma 1, the items (a), (b) and (c) are obtained.

Case 3: $t \equiv 2 \pmod 4$. We proceed similarly as in the Case 2. By Lemma 1 we have that $\mathcal{M}_0(C_t) = \frac{t+2}{2}$. Hence, we consider a $\mathcal{M}_0(C_t)$-set B such that $v_0, v_1, v_2 \in B$. Let $Y \subset V_1$ such that $Y = \{u_0, u_2, u_4, ..., u_{2\lceil r/2\rceil-2}\}$. We show that $M' = (V_1 \times B) - (Y \times \{v_0\})$ is a 0-monopoly in $C_r \circ C_t$. Notice that, if $u_i \in Y$, then the corresponding ${}^{u_i}C_t$-layer contains $\frac{t+2}{2} - 1$ vertices of M' and, if $u_i \notin Y$, then the ${}^{u_i}C_t$-layer contains $\frac{t+2}{2}$ vertices of M'. Also, notice that every vertex of $(u_i, v_j) \in M'$ has at least one neighbor in $M' \cap {}^{u_i}C_t$. Hence, for every vertex (u_i, v_j) we have that

$$\begin{aligned}\delta_{M'}(u_i, v_j) &= \delta_{M'\cap {}^{u_{i-1}}C_t}(u_i, v_j) + \delta_{M'\cap {}^{u_i}C_t}(u_i, v_j) + \delta_{M'\cap {}^{u_{i+1}}C_t}(u_i, v_j)\\ &\geq \frac{t+2}{2} - 1 + 1 + \frac{t+2}{2} - 1\\ &= t+1\\ &= \frac{\delta(u_i, v_j)}{2}.\end{aligned}$$

Thus, every vertex of $C_r \circ C_t$ satisfies the 0-monopoly condition. Since $|M'| = \frac{r(t+2)}{2} - \left\lceil \frac{r}{2} \right\rceil = \frac{rt}{2} + \left\lfloor \frac{r}{2} \right\rfloor$, the upper bound of (iii) is proved.

Case 4: $t \equiv 3 \pmod 4$. The upper bound of (iv) follows directly from Theorem 1 and Lemma 1. $\square$

Proposition 7. *Let $r, t \geq 3$ be integers. If H is either a path P_t or a cycle C_t, then*

$$\mathcal{M}_0(P_r \circ H) \geq \begin{cases} \frac{r}{3}(t+1), & \text{if } r \equiv 0 \pmod 3,\\ \left(\left\lceil \frac{r}{3} \right\rceil - 2\right)(t+1) + t + 2, & \text{if } r \equiv 1 \pmod 3,\\ \left(\left\lceil \frac{r}{3} \right\rceil - 1\right)(t+1) + \frac{t+2}{2}, & \text{if } r \equiv 2 \pmod 3,\end{cases}$$

and

$$\mathcal{M}_0(P_r \circ H) \leq \begin{cases} \frac{rt}{2}, & \text{if } r \equiv 0 \pmod 4, \\ \frac{r(t+2)}{2}, & \text{if } r \equiv 2 \pmod 4, \\ \frac{r(t+1)}{2}, & \text{if } r \equiv x \pmod 4, x \in \{1,3\}. \end{cases}$$

Proof. Notice that the graph P_r has always a maximum closed subpartition formed by a set S of cardinality $\lceil \frac{r}{3} \rceil$ which has the following shape.

- If $r \equiv 0 \pmod 3$, then every vertex of S has degree two.
- If $r \equiv 1 \pmod 3$, then there are two vertices in S having degree one and the other vertices of S have degree two.
- If $r \equiv 2 \pmod 3$, then there is only one vertex in S having degree one and the other vertices of S have degree two.

Hence, by using Theorem 4 we have the lower bounds. Finally, the upper bounds follow from Theorem 1 and Lemma 1. □

The above result can be compared with Proposition 5 and we can observe that the gap between upper and lower bound is bigger. The reason of this is related to Theorems 3 and 4 for the lower bounds.

Proposition 8. *Let $r, t \geq 3$ be integers.*

(i) If $t \equiv 0 \pmod 4$, then $\mathcal{M}_0(C_r \circ P_t) = \frac{rt}{2}$.

(ii) If $t \equiv x \pmod 4, x \in \{1,3\}$, then $\lceil \frac{rt}{2} \rceil \leq \mathcal{M}_0(C_r \circ P_t) \leq \frac{r(t+1)}{2}$.

(iii) If $t \equiv 2 \pmod 4$, then $\frac{rt}{2} \leq \mathcal{M}_0(C_r \circ P_t) \leq \frac{r(t+2)}{2}$.

Proof. Notice that the minimum degree of P_t is one. Since $2t+1$ is always an odd number, Corollary 2 leads to the following

$$\mathcal{M}_0(C_r \circ P_t) \geq \left\lceil \frac{rt}{2t+2} \left\lceil \frac{2t+1}{2} \right\rceil \right\rceil = \left\lceil \frac{rt}{2t+2} \frac{2t+2}{2} \right\rceil = \left\lceil \frac{rt}{2} \right\rceil.$$

The bounds follow from Theorem 1 and Lemma 1. □

The following results were obtained in [13] for the strong product of graphs. We recall that the *strong product* $G \boxtimes H$ of graphs G and H is a graph with vertex set $V(G \boxtimes H) = V(G) \times V(H)$. Two vertices (g,h) and (g',h') are adjacent in $G \boxtimes H$ whenever ($gg' \in E(G)$ and $h = h'$) or ($g = g'$ and $hh' \in E(H)$) or ($gg' \in E(G)$ and $hh' \in E(H)$). Since $C_m \boxtimes K_n \cong C_m \circ K_n$ and $P_m \boxtimes K_n \cong P_m \circ K_n$, the next two results are direct consequences of Proposition 3.5 and Proposition 3.6, respectively, from [13].

Proposition 9. *For integers $r, t \geq 3$ we have*

$$\frac{rt}{2} + \frac{r}{6} \leq \mathcal{M}_0(C_r \circ K_t) \leq r \left\lceil \frac{t+1}{2} \right\rceil - \left\lfloor \frac{r}{3} \right\rfloor .$$

Moreover, if $r \equiv 0 \pmod 3$ and t is odd, then $\mathcal{M}_0(C_r \circ K_t) = \frac{rt}{2} + \frac{r}{6}$.

Proposition 10. *For integers $r, t \geq 3$ we have*

$$\frac{tr}{2} + \frac{r}{6} - \frac{t}{3} \leq \mathcal{M}_0(P_r \circ K_t) \leq \begin{cases} \left\lfloor \frac{r}{3} \right\rfloor \cdot \left\lceil \frac{3t+1}{2} \right\rceil, & \text{if } r \equiv 0 \pmod 3, \\ \left\lfloor \frac{r}{3} \right\rfloor \cdot \left\lceil \frac{3t+1}{2} \right\rceil + 1, & \text{if } r \equiv 1 \pmod 3, \\ \left\lfloor \frac{r}{3} \right\rfloor \cdot \left\lceil \frac{3t+1}{2} \right\rceil + t, & \text{if } r \equiv 2 \pmod 3. \end{cases}$$

Nevertheless, since the lexicographic product is not commutative, we can exchange the factors to obtain $K_r \circ C_t$ and $K_r \circ P_t$, which we do at next.

Proposition 11. *Let $r, t \geq 3$ be integers.*

(i) If $t \equiv 0 \pmod 4$, then $\mathcal{M}_0(K_r \circ C_t) = \frac{rt}{2}$.

(ii) If $t \equiv x \pmod 4, x \in \{1, 3\}$, then $\left\lceil \frac{rt}{2} \right\rceil \leq \mathcal{M}_0(K_r \circ C_t) \leq \frac{r(t+1)}{2}$.

(iii) If $t \equiv 2 \pmod 4$, then $\frac{rt}{2} \leq \mathcal{M}_0(K_r \circ C_t) \leq \frac{r(t+2)}{2}$.

Proof. We can use Theorem 1 and the values of $\mathcal{M}_0(C_t)$ given in Lemma 1 to obtain upper bounds.

On the other hand, by using the Corollary 2 it follows that

$$\mathcal{M}_0(K_r \circ C_t) \geq \left\lceil \frac{rt}{t(r-1)+2} \left\lceil \frac{t(r-1)+2}{2} \right\rceil \right\rceil \geq \left\lceil \frac{rt}{t(r-1)+2} \frac{t(r-1)+2}{2} \right\rceil = \left\lceil \frac{rt}{2} \right\rceil .$$

Therefore, the lower bounds are obtained and the proof is complete. □

A similar approach can be used for the case of the graph $K_r \circ P_t$. Notice that the expression $t(r-1)+1$ is an even number if and only if t is odd and r is even. So, for t even or (t odd and r odd), since the minimum degree of P_t is one, Corollary 2 leads to the following:

$$\mathcal{M}_0(K_r \circ P_t) \geq \left\lceil \frac{rt}{t(r-1)+2} \left\lceil \frac{t(r-1)+1}{2} \right\rceil \right\rceil = \left\lceil \frac{rt}{t(r-1)+2} \frac{t(r-1)+2}{2} \right\rceil = \left\lceil \frac{rt}{2} \right\rceil .$$

According to that, we have the next result.

Proposition 12. *Let $r, t \geq 3$ be integers.*

(i) If $t \equiv 0 \pmod 4$, then $\mathcal{M}_0(K_r \circ P_t) = \frac{rt}{2}$.

(ii) If $t \equiv x \pmod 4, x \in \{1, 3\}$ and r is odd, then $\left\lceil \frac{rt}{2} \right\rceil \leq \mathcal{M}_0(K_r \circ P_t) \leq \frac{r(t+1)}{2}$.

(iii) If $t \equiv 2 \pmod 4$, then $\frac{rt}{2} \leq \mathcal{M}_0(K_r \circ P_t) \leq \frac{r(t+2)}{2}$.

(iv) If $t \equiv x \pmod 4, x \in \{1,3\}$ *and* r *is even, then* $\left\lceil \frac{rt}{t(r-1)+2} \left\lceil \frac{t(r-1)+1}{2} \right\rceil \right\rceil \leq \mathcal{M}_0(K_r \circ P_t) \leq \frac{r(t+1)}{2}$.

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References

[1] D. W. Bange, A. E. Barkauskas, and P. J. Slater, Efficient dominating sets in graphs, in: *R.D. Ringeisen and F.S. Roberts, eds., Applications of Discrete Mathematics*, SIAM, Philadelphia, (1988) 189–199.

[2] N. Biggs, Perfect codes in graphs, *Journal of Combinatorial Theory-Series B* **15** (1973) 289–296.

[3] C. Dwork, D. Peleg, N. Pippenger, and E. Upfal, Fault tolerance in networks of bounded degree, *SIAM Journal on Computing* **17** (1988) 975–988.

[4] P. Flocchini, R. Kralovic, A. Roncato, P. Ruzicka, and N. Santoro, On time versus size for monotone dynamic monopolies in regular topologies, *Journal of Discrete Algorithms* **1** (2003) 129–150.

[5] H. García-Molina and D. Barbara, How to assign votes in a distributed system, *Journal of the ACM* **32** (1985) 841–860.

[6] H. Gavlas and K. Schultz, Efficient open domination, *Electronics Notes on Discrete Mathematics* **11** (2002) 681–691.

[7] H. Gavlas, K. Schultz, and P. Slater, Efficient open domination in graphs, *Scientia Series A Mathematical Sciences* (N.S.) **6** (1994/00) (2003) 77–84.

[8] R. Hammack, W. Imrich, and S. Klavžar, *Handbook of Product Graphs, Second Edition*, CRC Press, Boca Raton, FL, 2011.

[9] K. Khoshkhah, M. Nemati, H. Soltani, and M. Zaker, A Study of Monopolies in Graphs, *Graphs and Combinatorics* **29** (2013) 1417–1427.

[10] D. Kuziak, I. Peterin, and I. G. Yero, Efficient open domination in graph products, *Discrete Mathematics and Theoretical Computer Science* **16** (2014) 105–120.

[11] D. Kuziak, I. Peterin, and I. G. Yero, k-monopolies in graphs: complexity and related concepts (available ???)

[12] D. Kuziak, I. Peterin, and I. G. Yero, Computing the (k-)monopoly number of direct product of graphs, *FILOMAT* **29** (2015)1163–1171.

[13] D. Kuziak, I. Peterin, and I. G. Yero, Bounding the (k-)monopoly number of strong product graphs, Manuscript 2014. (available ???)

[14] N. Linial, D. Peleg, Y. Rabinovich, and M. Saks, Sphere packing and local majorities in graphs, in 2^{nd} *Israel Symposium on Theory and Computing Systems* (1993) 141–149.

[15] A. Mishra and S. B. Rao, Minimum monopoly in regular and tree graphs, *Discrete Mathematics* **306** (2006) 1586–1594.

[16] D. Peleg, Local majorities, coalitions and monopolies in graphs: a review, *Theoretical Computer Science* **282** (2002) 231–257.

[17] G. F. Sullivan, *The complexity of system-level fault diagnosis and diagnosability*, Ph. D. Thesis, Yale University New Haven. CT, USA 1986.

[18] T. Tamizh Chelvam, Efficient open domination in Cayley graphs, *Applied Mathematics Letters* **25** (2012) 1560–1564.

[19] M. Zaker, On dynamic monopolies of graphs with general thresholds, *Discrete Mathematics* **312** (2012) 1136–1143.

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