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On the Derivatives of a Polynomial

by
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Abstract

For a polynomial $p(z)$ of degree n, having all its zeros in $|z| \le k, (k \ge 1)$, we obtain a refinement of known result [2]

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{1+k^n} \max_{|z|=1} |p(z)|,$$

(by using certain coefficients of $p(z)$), and an inequality, similar to known result involving s^{th} derivative, $(2 \le s \le n)$, instead of the first derivative of $p(z)$ (and better than the similar inequality, obtained by repeated applications of known result, in many cases).

Key Words: polynomial, derivatives, zeros in $|z| \le k, k \ge 1$, certain coefficients, generalization of Schwarz's lemma.
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1 Introduction and statement of results

For an arbitrary polynomial $f(z)$ let $M(f, r) = \max_{|z|=r} |f(z)|$. Further let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. Concerning the estimate of $|p'(z)|$ on $|z| \le 1$, we have firstly obtained

Theorem 1. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, having all its zeros in $|z| \le k,\ (k \ge 1)$. Then*

$$M(p', 1) \ge n \frac{|a_0| + |a_n| k^{n+1}}{|a_0|(1 + k^{n+1}) + |a_n|(k^{n+1} + k^{2n})} M(p, 1).$$

The result is best possible with equality for the polynomial $p(z) = z^n + k^n$.

Secondly we have obtained a result, similar to Theorem 1, involving the s^{th} derivative of $p(z)$, $(2 \le s \le n)$, instead of the first derivative of $p(z)$. More precisely we have proved

Theorem 2. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \le k, (k \ge 1)$ then for $2 \le s \le n$*

$$\begin{aligned} M(p^{(s)}, 1) \quad \ge \quad & n(n-1)(n-2)\dots(n-s+1)[k^n + \\ & \sum_{t=1}^{s-1} \tbinom{s}{t} \{(1+k^{n-t})(1+k^{n-t-1})\dots(1+k^{n-s+1})\} + \tbinom{s}{s}]^{-1} M(p, 1). \end{aligned}$$

Remark 1. *Theorem 1 is a refinement of Govil's result [2]*

$$M(p',1) \geq \frac{n}{1+k^n} M(p,1). \tag{1.1}$$

Remark 2. *Theorem 2 is better than the result*

$$M(p^{(s)},1) \geq \frac{n(n-1)(n-2)\dots(n-s+2)(n-s+1)}{(1+k^n)(1+k^{n-1})\dots(1+k^{n-s+2})(1+k^{n-s+1})} M(p,1), \tag{1.2}$$

(obtained by repeated applications of ineq.(1.1)), for

$$k > k_0,$$

where $k_0(>1)$ is the greatest positive root of the equation

$$\{(1+k^{n-s+1})\dots(1+k^{n-1})(1+k^n)\} - [k^n + \sum_{t=1}^{s-1} \tbinom{s}{t} \{(1+k^{n-t})(1+k^{n-t-1})\dots(1+k^{n-s+1})\} + 1] = 0,$$

(it being easily observed that left hand side of the equation is negative for $k = 1$).

Remark 3. *For the polynomial*

$$p(z) = (z+1)^2(z+2),$$

having all its zeros in $|z| \leq 2$,

$$M(p',1) \geq 4, \ (\textit{by ineq. } (1.1)),$$
$$M(p',1) \geq 5.684, \ (\textit{by Theorem 1}),$$

thereby implying that we get better estimate for $M(p',1)$ by Theorem 1 than by ineq.(1.1) and similarly

$$M(p'',1) \geq 1.6, \ (\textit{by ineq. } (1.2)),$$
$$M(p'',1) \geq 3.789, \ (\textit{by Theorem 2}),$$

thereby implying that we get better estimate for $M(p'',1)$ by Theorem 2 than by ineq. (1.2).

2 Lemmas

For the proofs of the theorems we require the following lemmas.

Lemma 1. *If $p(z)$ is a polynomial of degree n then*

$$\max_{|z|=R} |p(z)| \leq R^n (\max_{|z|=1} |p(z)|), R \geq 1,$$

with equality only for $p(z) = \lambda z^n$.

Proof of Lemma 1. It is a simple consequence of maximum modulus principle, (see [4]).

Lemma 2. *Let $f(z)$ be analytic in $|z| < 1$, with $f(0) = a$ and $|f(z)| \le M$, $|z| < 1$. Then*

$$|f(z)| \le M \frac{M|z| + |a|}{|a||z| + M},\ |z| < 1.$$

Lemma 2 is a well-known generalization of Schwarz's lemma, (see [5, p.212]).

Lemma 3. *Let $f(z)$ be analytic in $|z| \le 1$, with $f(0) = a$ and $|f(z)| \le M$, $|z| \le 1$. Then $|f(z)| \le M \frac{M|z|+|a|}{|a||z|+M}$, $|z| \le 1$.*

Proof of Lemma 3. It easily follows from Lemma 2.

Lemma 4. *Let $p(z)$ be a polynomial of degree n and*

$$q(z) = z^n \overline{p(1/\overline{z})}.$$

Then for $1 \le s \le n$

$$\begin{aligned} q^{(s)}(z) \quad = \quad & (n(n-1)\dots(n-\overline{s-1}))z^{n-s}\ \overline{p(1/\overline{z})} - \\ & \tbinom{s}{1}((n-1)(n-2)\dots(n-\overline{s-1}))z^{n-s-1}\ \overline{p'(1/\overline{z})} + \\ & \tbinom{s}{2}((n-2)(n-3)\dots(n-\overline{s-1}))z^{n-s-2}\ \overline{p''(1/\overline{z})} - \\ & \tbinom{s}{3}((n-3)(n-4)\dots(n-\overline{s-1}))z^{n-s-3}\ \overline{p'''(1/\overline{z})} + \dots + \\ & (-1)^t\tbinom{s}{t}((n-t)\dots(n-\overline{s-1}))z^{n-s-t}\ \overline{p^{(t)}(1/\overline{z})} + \dots + \\ & (-1)^{s-1}\tbinom{s}{s-1}(n-\overline{s-1})z^{n-s-\overline{s-1}}\ \overline{p^{(s-1)}(1/\overline{z})} + \\ & (-1)^s\tbinom{s}{s}z^{n-2s}\ \overline{p^{(s)}(1/\overline{z})}. \end{aligned}$$

Proof of Lemma 4. It follows easily by mathematical induction.

Lemma 5. *If $p(z)$ is polynomial of degree n, having all its zeros in $|z| \le k$, $(k \ge 1)$ then for $2 \le s \le n$ and $1 \le t \le s-1$*

$$\{(n-t)(n-t-1)\dots(n-\overline{s-1})\}M(p^{(t)}, 1) \le \{(k^{n-t}+1)(k^{n-t-1}+1)\dots(k^{n-\overline{s-1}}+1)\}M(p^{(s)}, 1).$$

Proof of Lemma 5. It follows by repeated applications of ineq.(1.1).

Lemma 6. *Let $T(z)$ be a polynomial of degree n, having all its zeros in $|z| \le 1$ and let $R(z)$ be a polynomial with its degree $\le n$. If*

$$|R(z)| \le |T(z)|,\ |z| = 1 \tag{2.1}$$

then for $0 \le s \le n$

$$|R^{(s)}(z)| \le |T^{(s)}(z)|, |z| \ge 1, (R^{(0)}(z) = R(z), T^{(0)}(z) = T(z)).$$

Proof of Lemma 6. Using (2.1) we can say that the zero z'_j, (with $|z'_j| = 1$ and multiplicity t_j), of $T(z)$ will also be a zero, (with muliplicity $(\ge t_j)$), of $R(z)$, thereby helping us to write

$$\begin{aligned} T(z) &= \phi_0(z)T_1(z), && (2.2) \\ R(z) &= \phi_0(z)R_1(z), && (2.3) \\ \phi_0(z) &= \begin{cases} \Pi_{j=1}^m (z-z'_j)^{t_j}; |z'_j| = 1\ \forall j \text{ with } \sum_{j=1}^m t_j = t, T(z) \text{ has certain zeros on } |z| = 1, \\ 1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad , T(z) \ne 0 \text{ on } |z| = 1, \end{cases} && (2.4) \\ T_1(z) &\ne 0,\ |z| = 1, && (2.5) \end{aligned}$$

with

$$|R_1(z)| \leq |T_1(z)|,\ |z| = 1, (\text{by } (2.1), (2.2), (2.3) \text{ and } (2.4)). \tag{2.6}$$

Firstly let $T(z)$ have certain zeros on $|z| = 1$. Then by (2.2), (2.4) and (2.5) we can say that $T_1(z)$ is a polynomial of degree $(n - t)$, having all its zeros in $|z| < 1$. Now by (2.6) we have for λ with $|\lambda| > 1$

$$|R_1(z)| < |\lambda T_1(z)|,\ |z| = 1.$$

Therefore by Rouché's theorem, the polynomial $\lambda T_1(z) - R_1(z)$ will have $(n-t)$ zeros in $|z| < 1$ and accordingly the polynomial $\lambda T(z) - R(z)$ will have all its zeros in $|z| \leq 1$. Further by using Gauss-Lucas' theorem we can say that for $0 \leq s \leq n$, the polynomial $\lambda T^{(s)}(z) - R^{(s)}(z)$ will have all its zeros in $|z| \leq 1$ and therefore $|R^{(s)}(z)| \leq |T^{(s)}(z)|,\ |z| > 1$. On using continuity Lemma 6 follows for the possibility under consideration. Finally let $T(z) \neq 0$ on $|z| = 1$. Then $T_1(z) = T(z)$, (a polynomial of degree n), (by (2.2) and (2.4)), $R_1(z) = R(z)$, (by (2.3) and (2.4)). Now Lemma 6 for the present possibility will follow similar to previous possibility. This completes the proof of Lemma 6.

Remark 4. *Lemma 6 is a generalization of Bernstein's result([1], [3, Theorem C]).*

Lemma 7. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq k$, $(k \geq 1)$ and*

$$q(z) = z^n \overline{p(1/\overline{z})} \tag{2.7}$$

then for $0 \leq s \leq n$

$$M(q^{(s)}, 1) \leq k^n M(p^{(s)}, 1).$$

Proof of Lemma 7. We observe that

$$P(z) = p(kz) \tag{2.8}$$

is a polynomial of degree n, having all its zeros in $|z| \leq 1$ and

$$\begin{aligned} Q(z) &= z^n \overline{P(1/\overline{z})}, && (2.9)\\ &= k^n (\frac{z}{k})^n \overline{p(k/\overline{z})}, (\text{by } (2.8)), &&\\ &= k^n q(z/k), (\text{by } (2.7)) && (2.10) \end{aligned}$$

is a polynomial of degree $\leq n$, with the characteristic $|Q(z)| = |P(z)|, |z| = 1$. Therefore by Lemma 6 we can say that for $0 \leq s \leq n$

$$|Q^{(s)}(z)| \leq |P^{(s)}(z)|, |z| \geq 1, \tag{2.11}$$

i.e.

$$k^{n-2s} |q^{(s)}(\frac{z}{k})| \leq |p^{(s)}(kz)|, |z| \geq 1, (\text{by } (2.8) \text{ and } (2.10)). \tag{2.12}$$

On taking $z = ke^{i\theta},\ 0 \leq \theta \leq 2\pi$, in (2.12) we get for $0 \leq s \leq n$

$$k^{n-2s} |q^{(s)}(e^{i\theta})| \leq |p^{(s)}(k^2 e^{i\theta})|,\ 0 \leq \theta \leq 2\pi,$$

which implies

$$\begin{aligned} k^{n-2s} M(q^{(s)}, 1) &\leq M(p^{(s)}, k^2),\\ &\leq (k^2)^{n-s} M(p^{(s)}, 1),\ (\text{by Lemma 1}) \end{aligned}$$

and Lemma 7 follows.

3 Proofs of the theorems

Proof of Theorem 1. By symbols used in Proof of Lemma 7 and inequality (2.11) we can say that

$$|Q'(z)| \leq |P'(z)|,\ |z| = 1. \tag{3.1}$$

Using (3.1) we can say that a zero z_j, (with $|z_j| = 1$ and multiplicity m_j), of $P'(z)$ will also be a zero, with multiplicity ($\geq m_j$), of $Q'(z)$, thereby helping us to write

$$P'(z) = \phi(z)P_1(z), \tag{3.2}$$
$$Q'(z) = \phi(z)Q_1(z), \tag{3.3}$$

where

$$\phi(z) = \begin{cases} 1 & , P'(z) \neq 0 \text{ on } |z| = 1, \\ \Pi_{j=1}^{p}(z - z_j)^{m_j}; |z_j| = 1 \forall j & , P'(z) \text{ has certain zeros on } |z| = 1, \end{cases} \tag{3.4}$$

$$P_1(z) \neq 0,\ |z| = 1 \tag{3.5}$$

and

$$|Q_1(z)| \leq |P_1(z)|,\ |z| = 1, (\text{by } (3.1), (3.2) \text{ and } (3.3)). \tag{3.6}$$

Now as $P(z)$ has all its zeros in $|z| \leq 1$, we can say by Gauss-Lucas' theorem that $P'(z)$ will also have all its zeros in $|z| \leq 1$. Therefore by (3.2), (3.4) and (3.5), we can say that

$$\psi(z) = \frac{Q_1(z)}{P_1(z)} \tag{3.7}$$

is analytic in $|z| > r'$, (for certain r', with $(0 < r' < 1)$), including ∞ and accordingly

$$f(z) = \psi(1/z), \tag{3.8}$$

with

$$\begin{aligned} f(0) = \psi(\infty) &= lt_{z\to\infty}\psi(z), \\ &= lt_{z\to\infty}\frac{Q'(z)}{P'(z)},\ (\text{by } (3.7), (3.2) \text{ and } (3.3)), \\ &= \frac{\overline{a_0}}{a_n k^n},\ (\text{by } (2.8) \text{ and } (2.9)) \end{aligned} \tag{3.9}$$

is analytic in $|z| < \frac{1}{r'}, (\frac{1}{r'} > 1)$. Further $|\psi(z)| \leq 1, |z| = 1$, (by (3.6)) and therefore

$$|f(z)| \leq 1,\ |z| = 1,\ (\text{by } (3.8)), \tag{3.10}$$

which, by (3.9) and Lemma 3, helps us to write

$$|f(z)| \leq \frac{|z| + |\frac{a_0}{a_n k^n}|}{|\frac{a_0}{a_n k^n}||z| + 1},\ |z| \leq 1,$$

i.e.

$$|f(re^{i\theta})| \leq \frac{|a_n|k^n r + |a_0|}{|a_0|r + |a_n|k^n},\ r \leq 1 \text{ and } 0 \leq \theta \leq 2\pi, \tag{3.11}$$

i.e.

$$|\psi(\frac{1}{r}e^{-i\theta})| \leq \frac{|a_n|k^n r + |a_0|}{|a_0|r + |a_n|k^n},\ 0 < r \leq 1 \text{ and } 0 \leq \theta \leq 2\pi,\ (\text{by } (3.8)),$$

i.e.

$$|\psi(Re^{-i\theta})| \leq \frac{|a_n|k^n + |a_0|R}{|a_0| + |a_n|k^n R},\ R \geq 1 \text{ and } 0 \leq \theta \leq 2\pi,$$

i.e.

$$|Q_1(Re^{-i\theta})| \leq \frac{|a_n|k^n + |a_0|R}{|a_0| + |a_n|k^n R}|P_1(Re^{-i\theta})|,\ R \geq 1,\ (\text{by } (3.7)),$$

i.e.

$$|Q'(Re^{-i\theta})| \leq \frac{|a_n|k^n + |a_0|R}{|a_0| + |a_n|k^n R}|P'(Re^{-i\theta})|,\ R \geq 1, (\text{by } (3.2) \text{ and } (3.3)),$$

i.e.

$$|Q'(z)| \leq \frac{|a_n|k^n + |a_0||z|}{|a_0| + |a_n|k^n|z|}|P'(z)|, |z| \geq 1,$$

i.e.

$$k^{n-2}|q'(z/k)| \leq \frac{|a_n|k^n + |a_0||z|}{|a_0| + |a_n|k^n|z|}|p'(kz)|, |z| \geq 1, (\text{by } (2.8) \text{ and } (2.10)). \tag{3.12}$$

By taking $z = ke^{i\theta}$ in (3.12) we get

$$k^{n-3}|q'(e^{i\theta})| \leq \frac{|a_n|k^{n-1} + |a_0|}{|a_0| + |a_n|k^{n+1}}|p'(k^2e^{i\theta})|,\ 0 \leq \theta \leq 2\pi,$$

which implies

$$k^{n-3}M(q', 1) \leq \frac{|a_n|k^{n-1} + |a_0|}{|a_0| + |a_n|k^{n+1}}M(p', k^2)$$

and therefore

$$M(q', 1) \leq \frac{|a_n|k^{n-1} + |a_0|}{|a_0| + |a_n|k^{n+1}}k^{n+1}M(p', 1),\ (\text{by Lemma 1}). \tag{3.13}$$

Now by (2.7), we get

$$|q'(e^{i\theta})| + |p'(e^{i\theta})| \geq n|p(e^{i\theta})|,\ 0 \leq \theta \leq 2\pi,$$

which implies

$$M(q', 1) + M(p', 1) \geq nM(p, 1)$$

and on using (3.13) we get

$$M(p', 1) \geq n\frac{|a_0| + |a_n|k^{n+1}}{|a_0|(1 + k^{n+1}) + |a_n|(k^{n+1} + k^{2n})}M(p, 1). \tag{3.14}$$

This completes the proof of Theorem 1.

Remark 5. *If instead of, the polynomial $p(z)=\sum_{j=0}^{n} a_j z^j$, (of degree n), having all its zeros in*
$|z|\leq k,\ (k\geq 1)$ we consider the polynomial $p(z)=\sum_{j=m}^{n} a_j z^j,\ (0\leq m<n)$, (of degree n), having all its zeros in $|z|\leq k,\ (k\geq 1)$, then, by thinking of, the function

$$\chi(z) = \frac{Q'(z)}{P'(z)}, \left(\begin{array}{l} (P(z) \text{ and } Q(z), \text{ as in (2.8) and (2.9)}), \\ \text{analytic in } 1<|z|<\infty, \text{ as well as in } r'<|z|<1, \\ (\text{for certain } r', \text{ with } (0<r'<1)) \end{array}\right), \tag{3.15}$$

$$= \frac{(n-m)\overline{a_m}k^m z^{n-m-1}+\ldots+\overline{a_{n-1}}k^{n-1}}{na_n k^n z^{n-1}+\ldots+a_m k^m m z^{m-1}}, \left(\begin{array}{l} \text{analytic in } 1<|z|<\infty, \\ \text{as well as in } r'<|z|<1 \end{array}\right), \tag{3.16}$$

along with

$$f(z)=\psi(1/z),\ (\textit{as in } (3.8)), \tag{3.17}$$

the relation

$$\begin{aligned} f(z) &= \chi(1/z),\ (0<|z|<1, 1<|z|<1/r'), \\ &= z^m \frac{(n-m)\overline{a_m}k^m+\ldots+\overline{a_{n-1}}k^{n-1}z^{n-m-1}}{na_n k^n+\ldots+a_m k^m m z^{n-m}}, (0<|z|<1, 1<|z|<1/r'), \\ &\qquad \text{(by (3.16))}, \qquad (3.18) \\ &= z^m T(z),\ \text{(say)}\ , (0<|z|<1,\ 1<|z|<1/r'), \qquad (3.19) \end{aligned}$$

with

$$\begin{aligned} T(0) &= \frac{(n-m)\overline{a_m}k^m}{na_n k^n},\ \text{(by (3.19))}, \qquad (3.20) \\ &= d,\ \text{(say)}, \\ T(z) &= f(z)/z^m, (0<|z|<1, 1<|z|<1/r'), \qquad (3.21) \\ T(z) &= lt_{\zeta\to z} T(\zeta),\ |z|=1, (\text{by using (3.21) and (3.17)}), \qquad (3.22) \\ |T(z)| &\leq 1,\ |z|\leq 1, \left(\begin{array}{l} \text{as } T(z) \text{ is analytic in } |z|<1, \text{ by (3.15), (3.19), (3.18)} \\ \text{and (3.20), and } T(z) \text{ is continuous in } |z|\leq 1, \text{ by (3.21),} \\ \text{(3.22) and the fact that } T(z) \text{ is analytic in } |z|<1 \end{array}\right) \end{aligned}$$

and on applying Lemma 2 to $T(z)$ we get

$$|T(z)|\leq \frac{|z|+|d|}{1+|z||d|},\ |z|<1,$$

which, by (3.21), implies that

$$|f(z)|\leq |z|^m \frac{|z|+|d|}{1+|z||d|},\ 0<|z|<1$$

and therefore

$$|f(z)|\leq |z|^m \frac{|z|+|d|}{1+|z||d|},\ |z|<1,$$

as well as

$$|f(z)| \leq |z|^m \frac{|z|+|d|}{1+|z||d|},\ |z| \leq 1,\ (\text{by } (3.10)),$$

i.e.

$$|f(re^{i\theta})| \leq \frac{n|a_n|k^n r^{m+1} + (n-m)|a_m|k^m r^m}{n|a_n|k^n + (n-m)|a_m|k^m r},\ r \leq 1,$$

thereby giving, (on repeating steps from (3.11) to (3.14), (of, Proof of Theorem 1))

$$M(p',1) \geq n \frac{n|a_n|k^{n+1} + (n-m)|a_m|k^m}{n|a_n|(k^{2n-m}+k^{n+1}) + (n-m)|a_m|(k^{n+1}+k^m)} M(p,1), \tag{3.23}$$

a generalization of Theorem 1. (Please note that (3.23) is trivially true for $m = n$ also, thereby suggesting that (3.23) is true for the polynomial $p(z) = \sum_{j=m}^{n} a_j z^j,\ (0 \leq m \leq n)$, (of degree n), having all its zeros in $|z| \leq k,\ (k \geq 1)$).

Proof of Theorem 2. Using Lemma 4 we get for $2 \leq s \leq n$

$$\begin{aligned} |q^{(s)}(z)| \quad &+ \quad \sum_{t=1}^{s-1} \tbinom{s}{t} ((n-t)(n-t-1)\dots(n-s+1))|p^{(t)}(z)| + \tbinom{s}{s}|p^{(s)}(z)| \\ &\geq \quad (n(n-1)\dots(n-s+1))|p(z)|, \qquad |z| = 1, \end{aligned}$$

which implies

$$\begin{aligned} M(q^{(s)},1) \quad &+ \quad \sum_{t=1}^{s-1} \tbinom{s}{t} ((n-t)(n-t-1)\dots(n-s+1))M(p^{(t)},1) + \tbinom{s}{s}M(p^{(s)},1) \\ &\geq \quad (n(n-1)\dots(n-s+1))M(p,1). \end{aligned} \tag{3.24}$$

Now by combining (3.24) with Lemma 5 and Lemma 7 we get for $2 \leq s \leq n$

$$\begin{aligned} [k^n \quad &+ \quad \sum_{t=1}^{s-1} \tbinom{s}{t} \{(1+k^{n-t})(1+k^{n-t-1})\dots(1+k^{n-s+1})\} + \tbinom{s}{s}] M(p^{(s)},1) \\ &\geq \quad (n(n-1)\dots(n-s+1))M(p,1) \end{aligned}$$

and Theorem 2 follows.

References

[1] S. N. Bernstein, Sur la limitation des derivees des polynômes, *C. R. Acad. Sc. Paris* **190** (1930), 338–340.

[2] N. K. Govil, On the derivative of a polynomial, *Proc. Amer. Math. Soc.* **41** (1973), 543–546.

[3] M. A. Malik, M. C. Vong, Inequalities concerning the derivative of polynomials, *Rendi. Cir. Mat. Palermo Serie II* **34** (1985), 422–426.

[4] G. Polya, G. Szegö, *Problems and Theorems in analysis*, Vol. 1, Springer-Verlag, Berlin-Heidelberg, 1972.

[5] E. C. Titchmarsh, *The theory of functions*, The English Language Book Society and Oxford University Press, London, 1962.

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