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On complete pure-injectivity in locally finitely presented categories

by

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Abstract

Let $\mathcal{C}$ be a locally finitely presented additive category, and let E be a finitely presented pure-injective object of $\mathcal{C}$. We prove that E has an indecomposable decomposition if and only if every pure epimorphic image of E is pure-injective if and only if the endomorphism ring of E is semiperfect. This extends a module-theoretic result which generalises the classical Osofsky Theorem.

Key Words: Locally finitely presented category, Krull-Schmidt category, indecomposable decomposition, (completely) pure-injective object, semiperfect ring, semisimple ring, Osofsky theorem.
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Running title Complete pure-injectivity

1 Introduction

For a finitely accessible additive category $\mathcal{C}$ with products and a family $(U_i)_{i\in I}$ of representative classes of finitely presented objects in $\mathcal{C}$ such that each object U_i is pure-injective, $\mathcal{C}$ is a Krull-Schmidt category if and only if each object U_i is completely pure-injective, in the sense that every pure epimorphic image of the objects U_i is pure-injective [5, Theorem 3.6]. Recall that the category $\mathcal{C}$ is Krull-Schmidt if every finitely presented object of $\mathcal{C}$ is a finite direct sum of indecomposable objects with local endomorphism ring [1, Section 2].

The motivation of the present paper is to establish a local version of the aforementioned result in a locally finitely presented additive category. More precisely, we show that, for a locally finitely presented additive category $\mathcal{C}$ and a finitely presented pure-injective object E of $\mathcal{C}$, E has an indecomposable decomposition if and only if E is completely pure-injective. Then each summand of the indecomposable decomposition of E must have local endomorphism ring, and we may deduce [5, Theorem 3.6] as a corollary of our theorem.

We use functorial techniques, and we show that, under the above hypotheses, E has an indecomposable decomposition if and only if its endomorphism ring S is semiperfect, whereas E is completely pure-injective if and only if S is a completely pure-injective right S-module. These reduce the problem to a module-theoretic theorem of Gómez Pardo and Guil Asensio [9, Corollary 2.3].

As a consequence, we show that if $\mathcal{C}$ is a locally finitely presented additive category and E is a finitely presented pure-injective object of $\mathcal{C}$ with von Neumann regular endomorphism ring

S, then E is completely pure-injective if and only if S is semisimple. This extends a result of Gómez Pardo, Dung and Wisbauer [8, Theorem 1], which is a general version of the Osofsky Theorem [12] for Grothendieck categories (see also [6]).

2 Locally finitely presented additive categories

We recall some terminology on finitely accessible additive categories, following [13]. An additive category $\mathcal{C}$ is called *finitely accessible* (*locally finitely presented* in the terminology of [3]) if it has direct limits, the class of finitely presented objects is skeletally small, and every object is a direct limit of finitely presented objects. The category $\mathcal{C}$ will be called *locally finitely presented* if it is finitely accessible and cocomplete (i.e., it has all colimits), or equivalently, it is finitely accessible and complete (i.e., it has all limits).

Let $\mathcal{C}$ be a finitely accessible additive category. By a *sequence*

$$0 \to X \stackrel{f}{\to} Y \stackrel{g}{\to} Z \to 0$$

in $\mathcal{C}$ we mean a pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$ such that $gf = 0$. The above sequence in $\mathcal{C}$ is called *pure exact* if it induces a short exact sequence of abelian groups

$$0 \to \mathrm{Hom}_{\mathcal{C}}(P, X) \to \mathrm{Hom}_{\mathcal{C}}(P, Y) \to \mathrm{Hom}_{\mathcal{C}}(P, Z) \to 0$$

for every finitely presented object P of $\mathcal{C}$. This implies that f and g form a kernel-cokernel pair, that f is a monomorphism and g an epimorphism. In such a pure exact sequence f is said to be a *pure monomorphism* and g a *pure epimorphism*. Pure-injectivity and pure-projectivity in $\mathcal{C}$ are defined in the usual way.

Let $\mathcal{C}$ be a locally finitely presented additive category, and let E be an object of $\mathcal{C}$ with endomorphism ring $S = \mathrm{End}_{\mathcal{C}}(E)$. We denote by $\mathrm{Mod}(S)$ the category of right S-modules. Since $\mathcal{C}$ is complete, Freyd's Adjoint Functor Theorem [7, p. 84] implies the existence of a left adjoint $T : \mathrm{Mod}(S) \to \mathcal{C}$ for the functor $H = \mathrm{Hom}_{\mathcal{C}}(E, -) : \mathcal{C} \to \mathrm{Mod}(S)$ such that $TH(E) \cong E$ (see also [2]). As a left adjoint, T preserves direct limits. If E is finitely presented, then H preserves direct limits as well. In this case, both T and H preserve pure exact sequences by the following result.

Lemma 1. *[5, Lemma 2.1] Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between locally finitely presented additive categories such that F is left or right exact and preserves direct limits. Then F preserves pure short exact sequences.*

We are also interested in preservation of pure-injectivity and pure-projectivity by some adjoint functors, as shown in the following lemma.

Lemma 2. *Let $L : \mathcal{A} \to \mathcal{B}$ be a functor between locally finitely presented additive categories having a right adjoint $R : \mathcal{B} \to \mathcal{A}$. Then:*

(1) R preserves pure-injective objects.

(2) If R preserves direct limits, then L preserves pure-projective objects.

Proof. (1) This is [5, Lemma 2.2].

(2) This is basically dual to (1), but we sketch a proof for completeness. Let A be a pure-projective object in $\mathcal{A}$ and let $0 \to X \to Y \to Z \to 0$ be a pure short exact sequence in $\mathcal{B}$. The functor R is left exact as a right adjoint, and preserves direct limits, hence Lemma 1 yields the pure exact sequence

$$0 \to R(X) \to R(Y) \to R(Z) \to 0$$

in $\mathcal{A}$. This induces the short exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(A, R(X)) \to \mathrm{Hom}_{\mathcal{A}}(A, R(Y)) \to \mathrm{Hom}_{\mathcal{A}}(A, R(Z)) \to 0.$$

Using the adjointness we have the short exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{B}}(L(A), X) \to \mathrm{Hom}_{\mathcal{B}}(L(A), Y) \to \mathrm{Hom}_{\mathcal{B}}(L(A), Z) \to 0,$$

which shows that $L(A)$ is pure-projective in $\mathcal{B}$. □

Let $\mathcal{C}$ be a locally finitely presented additive category, and let E be an object of $\mathcal{C}$. Following [4, Section 3], we denote by $\mathrm{PAdd}(E)$ the class of pure epimorphic images of direct sums of copies of E. As usual, $\mathrm{Add}(E)$ denotes the class of objects isomorphic to direct summands of direct sums of copies of E. The properties of the class $\mathrm{PAdd}(E)$ presented in the rest of the section will be essential for proving our main theorem.

Lemma 3. *Let $\mathcal{C}$ be a locally finitely presented additive category, and let E be a finitely presented object of $\mathcal{C}$. Then* $\mathrm{PAdd}(E)$ *is closed under pure epimorphic images and direct limits.*

Proof. This follows similarly as [4, Lemma 3.1]. □

Theorem 1. *Let $\mathcal{C}$ be a locally finitely presented additive category, and let E be a finitely presented object of $\mathcal{C}$ with endomorphism ring $S = \mathrm{End}_{\mathcal{C}}(E)$. Consider the functor $H = \mathrm{Hom}_{\mathcal{C}}(E, -) : \mathcal{C} \to \mathrm{Mod}(S)$ and its left adjoint $T : \mathrm{Mod}(S) \to \mathcal{C}$. Then the adjoint pair (T, H) induces an equivalence between* $\mathrm{PAdd}(E)$ *and the flat right S-modules.*

Proof. Let $C \in \mathrm{PAdd}(E)$. Then there is a pure epimorphism $E^{(I)} \to C$ in $\mathcal{C}$. Since H preserves direct limits, the induced epimorphism

$$S^{(I)} = H(E)^{(I)} \cong H(E^{(I)}) \to H(C)$$

is pure in $\mathrm{Mod}(S)$ by Lemma 1. Then $H(C)$ is flat in $\mathrm{Mod}(S)$ [14, 36.6]. Also, we have $C = \varinjlim C_j$, where each C_j is a finite direct sum of copies of E by [14, 34.2], whose proof works for any locally finitely presented category. Since $TH(E) \cong E$ and both T and H preserve direct sums, the restriction of TH to $\mathrm{Add}(E)$ is the identity, hence $TH(C_j) \cong C_j$ for each C_j. It follows that

$$C \cong \varinjlim TH(C_j) = TH(\varinjlim C_j) = TH(C),$$

because H and T preserve direct limits.

Let Z be a flat right S-module. Then $Z = \varinjlim P_k$, where each P_k is a finitely generated projective right S-module. Hence $T(Z) \cong \varinjlim T(P_k)$. There are split epimorphisms $S^{n_k} \to P_k$ in $\mathrm{Mod}(S)$, hence split epimorphisms

$$E^{n_k} \cong T(S)^{n_k} \cong T(S^{n_k}) \to T(P_k).$$

It follows that each $T(P_k) \in \mathrm{PAdd}(E)$. Then $T(Z) \cong \varinjlim T(P_k) \in \mathrm{PAdd}(E)$ by Lemma 3. Also, since $HT(S) \cong S$ and both T and H preserve direct sums, the restriction of HT to projective right S-modules is the identity. It follows that

$$Z = \varinjlim P_k \cong \varinjlim HT(P_k) \cong HT(\varinjlim P_k) = HT(Z),$$

because H and T preserve direct limits. □

Corollary 1. *Let $\mathcal{C}$ be a locally finitely presented additive category, and let E be a finitely presented object of $\mathcal{C}$. Then* $\mathrm{PAdd}(E)$ *is closed under pure subobjects.*

Proof. Consider the functor $H = \mathrm{Hom}_{\mathcal{C}}(E, -) : \mathcal{C} \to \mathrm{Mod}(S)$ and its left adjoint $T : \mathrm{Mod}(S) \to \mathcal{C}$. Let $0 \to A \to B \to C \to 0$ be a pure short exact sequence in $\mathcal{C}$ with $B \in \mathrm{PAdd}(E)$. Then $C \in \mathrm{PAdd}(E)$ by Lemma 3. We have an induced pure short exact sequence

$$0 \to H(A) \to H(B) \to H(C) \to 0$$

in $\mathrm{Mod}(S)$ by Lemma 1. Since $B \in \mathrm{PAdd}(E)$, $H(B)$ is flat in $\mathrm{Mod}(S)$ by Theorem 1, hence $H(A)$ must be flat in $\mathrm{Mod}(S)$ [13, Proposition 5.9]. Thus we have $H(A) \cong H(A')$ for some object $A' \in \mathrm{PAdd}(E)$ by Theorem 1. This implies that $A \cong A' \in \mathrm{PAdd}(E)$. □

3 Complete pure-injectivity of finitely presented objects

Following [9], an object E of a locally finitely presented additive category is called *completely pure-injective* if every pure epimorphic image of E is pure-injective. Also, a ring R is called *semilocal* if R is semisimple modulo its Jacobson radical [11, p. 7].

Now we are in a position to establish the main result of the paper, which generalises [9, Corollary 2.5].

Theorem 2. *Let $\mathcal{C}$ be a locally finitely presented additive category, and let E be a finitely presented pure-injective object of $\mathcal{C}$ with endomorphism ring $S = \mathrm{End}_{\mathcal{C}}(E)$. Then the following statements are equivalent:*

(1) E has an indecomposable decomposition.

(2) E is completely pure-injective.

(3) Every pure subobject of E is pure-projective.

(4) S is a semilocal ring.

(5) S is a semiperfect ring.

(6) S is a completely pure-injective right S-module.

(7) Every pure right ideal of S is pure-projective.

Proof. Consider the functor $H = \mathrm{Hom}_{\mathcal{C}}(E, -) : \mathcal{C} \to \mathrm{Mod}(S)$ and its left adjoint $T : \mathrm{Mod}(S) \to \mathcal{C}$. By Lemma 2 it follows that $S = H(E)$ is a pure-injective right S-module.

(5)$\Leftrightarrow$(6)$\Leftrightarrow$(7) These follow by [9, Corollary 2.3].

(1)$\Leftrightarrow$(5) Since both functors T and H preserve direct sums, E has an indecomposable decomposition in $\mathcal{C}$ if and only if so does $S = H(E)$ in $\mathrm{Mod}(S)$. Since S is a pure-injective right S-module, it follows by [15, Theorem 9] and [14, 42.6] that E has an indecomposable decomposition in $\mathcal{C}$ if and only if S is semiperfect.

(2)$\Leftrightarrow$(6) Assume first that E is completely pure-injective. Let $S \to Z$ be a pure epimorphism in $\mathrm{Mod}(S)$. Lemma 1 yields a pure epimorphism $E \cong T(S) \to T(Z)$ in $\mathcal{C}$, whence $T(Z)$ is pure-injective in $\mathcal{C}$ by hypothesis. Then $HT(Z)$ is pure-injective in $\mathrm{Mod}(S)$ by Lemma 2. But Z is flat in $\mathrm{Mod}(S)$ by [14, 36.6], hence we have $Z \cong HT(Z)$ by Theorem 1. It follows that S is a completely pure-injective right S-module.

Conversely, assume that S is a completely pure-injective right S-module. Let $E \to C$ be a pure epimorphism in $\mathcal{C}$. Lemma 1 yields a pure epimorphism $S = H(E) \to H(C)$ in $\mathrm{Mod}(S)$, whence $H(C)$ is pure-injective in $\mathrm{Mod}(S)$ by hypothesis. Then $H(C)$ is flat cotorsion in $\mathrm{Mod}(S)$, which implies that C is pure-injective in $\mathcal{C}$ [10, Lemma 3]. Hence E is completely pure-injective.

(3)$\Leftrightarrow$(7) Assume first that every pure subobject of E is pure-projective. Let I be a pure right ideal of S, and consider the inclusion monomorphism $I \to S$. Lemma 1 yields a pure monomorphism $T(I) \to T(S) \cong E$ in $\mathcal{C}$. Then $T(I)$ is pure-projective in $\mathcal{C}$ by hypothesis, which implies that $HT(I)$ is projective in $\mathrm{Mod}(S)$ [3, (3.1) Lemma]. Since I is a flat right S-module, $HT(I) \cong I$ by Theorem 1. Hence I is projective.

Conversely, assume that every pure right ideal of S is pure-projective. Let A be a pure subobject of E. Lemma 1 yields a pure monomorphism $H(A) \to H(E) = S$ in $\mathrm{Mod}(S)$, where $H(A)$ is pure-projective by hypothesis. Since H preserves direct limits, $TH(A)$ is pure-projective in $\mathcal{C}$ by Lemma 2. But $A \in \mathrm{PAdd}(E)$ by Corollary 1, hence we have $TH(A) \cong A$ by Theorem 1. Thus A is pure-projective.

(4)$\Leftrightarrow$(5) Note that an arbitrary ring S is semiperfect if and only if S is semilocal and idempotents lift modulo the Jacobson radical of S (e.g. see [11, p. 363]). Since S is right pure-injective, idempotents lift modulo any ideal of S [15, Theorem 9], hence the required equivalence follows. $\square$

Remark 1. Each summand of the indecomposable decomposition of E must have local endomorphism ring. Indeed, if C is such an indecomposable pure-injective object of $\mathcal{C}$, then $H(C)$ is an indecomposable pure-injective right S-module by Lemma 2. But every indecomposable pure-injective module has a local endomorphism ring [15, Theorem 9], hence $\mathrm{End}_{\mathcal{C}}(C) \cong \mathrm{End}_S(H(C))$ is local.

Following [1, Section 2], a locally finitely presented additive category $\mathcal{C}$ is called *Krull-Schmidt* if every finitely presented object of $\mathcal{C}$ is a finite direct sum of indecomposable objects with local endomorphism ring. Now we obtain the following corollary, which generalises the module-theoretic result [9, Corollary 2.6].

Corollary 2. *[5, Theorem 3.6] Let $\mathcal{C}$ be a locally finitely presented additive category with a family $(U_i)_{i\in I}$ of representative classes of finitely presented objects in $\mathcal{C}$ such that each object U_i is pure-injective. Then each object U_i is completely pure-injective if and only if $\mathcal{C}$ is a Krull-Schmidt category.*

Proof. The direct implication follows by Theorem 2 and Remark 1. For the converse, see the proof of [5, Theorem 3.6]. □

Remark 2. The hypothesis that each finitely presented object is completely pure-injective from Corollary 2 is satisfied, for instance, if the locally finitely presented additive category is pure-semisimple.

The next consequence is related to [8, Theorem 1], whose context of a locally finitely generated Grothendieck category is replaced here by that of a locally finitely presented additive category. It follows by Theorem 2, because a von Neumann regular semiperfect ring is semisimple. As in [8], an object C of $\mathcal{C}$ is called *completely injective* if every epimorphic image of C is injective.

Corollary 3. *Let $\mathcal{C}$ be a locally finitely presented additive category, and let E be a finitely presented pure-injective object of $\mathcal{C}$ with von Neumann regular endomorphism ring S. Then E is completely pure-injective if and only if S is semisimple.*

Corollary 4. *Let $\mathcal{C}$ be a locally finitely presented additive category, and let E be a finitely generated projective injective object of $\mathcal{C}$ with endomorphism ring S. Then E is completely injective if and only if S is semisimple.*

Proof. This is the same as the proof of [8, Corollary 2], using Corollary 3. □

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References

[1] M. Auslander, Representation theory of artin algebras II, *Comm. Algebra*, **1** (1974), 269–310.

[2] M.K. Berktaş, P.A. Guil Asensio, The functor category of a finitely accessible additive category, *JP J. Algebra Number Theory Appl.*, **25** (2012), 113-132.

[3] W. Crawley-Boevey, Locally finitely presented additive categories, *Comm. Algebra*, **22** (1994), 1641–1674.

[4] S. Crivei, Σ-extending modules, Σ-lifting modules, and proper classes, *Comm. Algebra*, **36** (2008), 529–545.

[5] S. Crivei, On Krull-Schmidt finitely accessible categories, *Bull. Aust. Math. Soc.*, **84** (2011), 90–97.

[6] S. Crivei, C. Năstăsescu, B. Torrecillas, On the Osofsky-Smith theorem, *Glasgow Math. J.*, **52A** (2010), 61–67.

[7] P. Freyd, *Abelian categories*, Harper and Row, New York, 1964.

[8] J.L. Gómez Pardo, N.V. Dung, R. Wisbauer, Complete pure injectivity and endomorphism rings, *Proc. Amer. Math. Soc.*, **118** (1993), 1029–1034.

[9] J.L. Gómez Pardo, P.A. Guil Asensio, Indecomposable decompositions of finitely presented pure-injective modules, *J. Algebra*, **192** (1997), 200–208.

[10] I. Herzog, Pure-injective envelopes, *J. Algebra Appl.*, **4** (2003), 397–402.

[11] T.Y. Lam, *Lectures on modules and rings*, Springer, New York, 1999.

[12] B.L. Osofsky, Rings all of whose finitely generated modules are injective, *Pacific J. Math.*, **14** (1964), 645–650.

[13] M. Prest, Definable additive categories: purity and model theory, *Mem. Amer. Math. Soc.* **210**, No. 987 (2011).

[14] R. Wisbauer, *Foundations of module and ring theory*, Gordon and Breach, Reading, 1991.

[15] B. Zimmermann-Huisgen, W. Zimmermann, Algebraically compact rings and modules, *Math. Z.*, **161** (1978), 81–93.

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