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Surfaces in the nearly Sasakian 5-sphere
by

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Abstract

We investigate surfaces in the nearly Sasakian 5-sphere for which the structure vector field ξ is normal to the surface and which are anti-invariant with respect to the nearly Sasakian structure. We show that such surfaces are always minimal. We moreover obtain a correspondence between such surfaces and minimal Lagrangian surfaces in the complex projective space. We also show the same results for surfaces in the nearly cosymplectic 5-sphere.

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1 Introduction

The notion of a nearly Sasakian structure on an almost contact metric manifold has been introduced by Blair, Showers and Yano in [3]. The basic properties of such a manifold will be recalled in Section 2. They also give necessary and sufficient condition for when a hypersurface of a nearly Kaehler manifold inherits a nearly Sasakian structure. An example of such hypersurface is the 5-dimensional sphere S^5, with radius $\frac{1}{\sqrt{2}}$ umbilically embedded at an angle of $\frac{\pi}{4}$. As on this sphere all sectional curvatures are equal to 2, it immediately follows that its inherited structure is not a Sasakian structure.

The notion of nearly cosymplectic structure on an almost contact metric manifold was introduced and studied by Blair and Showers some years earlier in [1] and [2]. They show also that the totally geodesic 5-sphere in the nearly Kaehler 6-sphere, has a nearly cosymplectic structure.

Submanifolds of the nearly Kaehler sphere S^6 have been investigated by many authors leading to many classification results. The existence of such a structure for the 6-sphere was proved by Fukami and Ishihara [8] by using the properties of the Cayley algebra. The study of the surfaces in nearly Keahlerian 6-dimensional sphere were already realized by Bolton, Dillen, Opozda, Verstraelen, Vrancken and Woodward (see [4], [6]).

In contrast to previous submanifolds of the nearly Kaehler sphere S^6, we have few results about nearly Sasakian manifolds. For example, Cappelletti-Montano and Dileo focused on the 5-dimensional case and proved that there exists a one-to-one correspondence between nearly Sasakian structures and some special class of SU(2)-structures, (see [5]). Moreover, almost nothing is known about submanifolds of the nearly Sasakian S^5. This despite the fact that it is by far the easiest example of a non trivial nearly Sasakian manifold.

In this paper we will focus on surfaces of the nearly Sasakian S^5 and the nearly cosymplectic S^5 for which the structure vector field ξ is normal to the surface. In the Sasakian case, submanifolds of dimension n of a $(2n+1)$-dimensional Sasakian sphere for which ξ is normal are called, depending on the literature, C-totally real or horizontal submanifolds. In that case, it is known that such submanifolds are always anti invariant, i.e. the structure φ maps tangent vectors to normal vectors. In this paper we first show that this is no longer the case in the nearly Sasakian 5-sphere or the nearly cosymplectic 5-sphere.

In view of this, we suggest to call a surface of the 5-sphere with nearly Sasakian structure, or nearly cosymplectic structure, totally real if and only if the structure vector field ξ is normal to the surface and φ maps tangent vectors to normal vectors. Note that in the Sasakian case the second condition is redundant.

The main result we will prove about such surfaces is the following:

Theorem 1. *A totally real surface of the nearly Sasakian S^5 is always minimal.*

Note that this result is also valid for the surfaces in nearly cosymplectic 5-sphere, (see theorem 2). This result is neither true for C-totally real surfaces in Sasakian manifolds or for totally real surfaces of the nearly Kaehler 6-sphere.

As a consequence of the minimality, we can also obtain a local correspondence between totally real surfaces of the S^5 with nearly Sasakian structure, or nearly cosymplectic structure, and minimal Lagrangian surfaces of the complex projective space $\mathbb{C}P^2$ (see theorem 3).

2 Preliminaries

We begin by recalling some fundamental proprieties of almost contact manifolds, nearly Sasakian manifolds and nearly cosymplectic manifolds.

2.1 Almost contact manifolds

A $(2n+1)$-dimensional manifold $\bar{M}^{2n+1}$ of class C^∞ is said to have an almost contact structure with an associated Riemannian metric g if there exist on $\bar{M}^{2n+1}$ a tensor field φ of type (1,1), a unit vector field ξ and dual 1-form η with respect to which the fallowing are satisfied : for any vector field X, Y on $\bar{M}^{2n+1}$
$\varphi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$, and it get that $\varphi(\xi) = 0,\ \eta(\varphi X) = 0,\ rank\,\varphi = 2n$. For details we refer to [9] and [11].

2.2 Nearly Sasakian manifolds

An almost contact metric structure (φ, ξ, η, g) is said to be nearly Sasakian, if
$(\nabla_X\varphi)Y + (\nabla_Y\varphi)X = -2g(X,Y)\xi + \eta(X)Y + \eta(Y)X$, where ∇ denotes covariant differentiation with respect to the Levi Civita connection of g. On a nearly Sasakian manifold, we know that the vector field ξ is Killing. Note that a vector field ξ is called a

Killing vector field if and only if $\mathcal{L}_\xi g(X,Y) = g(\nabla_\xi X, Y) + g(X, \nabla_\xi Y) = 0$, for all vectors fields X, Y, where $\mathcal{L}$ denotes Lie differentiation. It follows from [3] and [11] that $(\nabla_\xi \varphi)\xi = 0$, $\varphi \nabla_\xi \xi = 0$, $\nabla_\xi \xi = 0$, $\nabla_\xi \eta = 0$. In [3] the authors proved the theorem:

Theorem A. *Let M^{2n+1} be a hypersurface of a nearly Kaehler manifold M^{2n+2}. Then the induced structure on M^{2n+1} is nearly Sasakian if and only if*
$h(X,Y) = g(X,Y) + (h(\xi,\xi) - 1)\eta(X)\eta(Y)$, for all X, Y vectors fields in M^{2n+1}, where h denotes the second fundamental form.

2.3 Nearly cosymplectic manifolds

An almost contact metric manifold whose tensors are Killing field is called nearly cosymplectic, if: $(\nabla_X \varphi)X = 0$, or $(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = 0$, where ∇ denotes covariant differentiation with respect to the Levi Civita connection of g. On a nearly cosymplectic manifold we know that the vector field ξ is Killing. In [1] the authors proved the theorem:

Theorem B. *Let M^{2n} a nearly Kaehler manifold and M^{2n-1} a C^∞ orientable hypersurface. Let η denote the induced almost contact form and suppose the second fundamental form h is proportional to $\eta \otimes \eta$. Then η is Killing and in particular, on M^{2n-1}, it is nearly cosymplectic.*

2.4 Cayley algebra on $\mathbb{R}^7$

The multiplication on the Cayley numbers may be used to define a vector cross product on the purely imaginary Cayley numbers $\mathbb{R}^7$ using the formula $u \times v = 1/2(uv - vu)$, while the standard inner product on $\mathbb{R}^7$ is given by $\langle u, v\rangle = -1/2(uv + vu)$. An ordered orthonormal basis, $e_1, e_2, ..., e_7$ is called a G_2-frame if $e_3 = e_1 \times e_2$, $e_5 = e_1 \times e_4$, $e_6 = e_2 \times e_4$, $e_7 = e_3 \times e_4$. We refer the reader to [7] for more details.

2.5 Nearly Sasakian structure on S^5

In [3] the authors show how to induce a nearly Sasakian structure on S^5. In order to do so, they look at S^5 as a hypersurface in S^6 equipped with its nearly Kaehler structure. We have $S^5 \hookrightarrow S^6 \hookrightarrow \mathbb{R}^7$, where S^6 is the unit sphere in $\mathbb{R}^7$ with its cross product $\times$ induced by the Cayley algebra. We denote by P the unit outer normal. It is well known that S^6 has a nearly Kaehler structure with respect to the induced metric, we now consider S^5 umbilically embedded in S^6 at a latitude of 45°, and with normal unit N such that the second fundamental form $\tilde{h}(X,Y) = g(X,Y)$. Then we see that the induced structure on S^5 from the nearly Kaehler structure is nearly Sasakian. In this case we have: for $P \in S^5$, V tangent vector to S^5 and N the normal vector of S^5 in S^6 : $P = \frac{1}{\sqrt{2}}(x_1, x_2, x_3, 1, x_5, x_6, x_7)$, $N = -\frac{1}{\sqrt{2}}(x_1, x_2, x_3, -1, x_5, x_6, x_7)$, $V = \frac{1}{\sqrt{2}}(v_1, v_2, v_3, 0, v_5, v_6, v_7)$, and ξ and φ from the nearly Sasakian structure are respectively given by

$$\xi = -P \times N, \quad \varphi(V) = P \times V - \eta(V)N, \quad \eta(V) = \langle \xi, V\rangle.$$

With the cross product, we obtain that $\xi = (x_5, x_6, x_7, 0, -x_1, -x_2, -x_3)$.

2.6 Nearly cosymplectic structure on S^5

In [1] the author show how to induce a nearly cosymplectic structure on S^5. In order to do so, they look at S^5 as a hypersurface in S^6 equipped with its nearly Kaehler structure. We have $S^5 \hookrightarrow S^6 \hookrightarrow \mathbb{R}^7$, where S^6 has the same structure as constructed before. We denote by P the unit outer normal. We now consider S^5 as a totally geodesic hypersurface of S^6. Then the induced structure on S^5 from the nearly Kaehler structure is nearly cosymplectic. We denote by V and N a tangent vector to S^5 and the normal vector of S^5 in S^6 at the point $P \in S^5$. $P = (x_1, x_2, x_3, 0, x_5, x_6, x_7)$, $N = (0, 0, 0, 1, 0, 0, 0)$, $V = (v_1, v_2, v_3, 0, v_5, v_6, v_7)$, and ξ and φ from the nearly cosymplectic structure are respectively given by

$$\xi = -P \times N, \quad \varphi(V) = P \times V - \eta(V)N, \quad \eta(V) = \langle \xi, V \rangle.$$

With the cross product, we obtain that $\xi = (-x_5, -x_6, -x_7, 0, x_1, x_2, x_3)$.

2.7 Totally real surfaces

Definition 1. *Let M a surface of S^5 with nearly Sasakian structure or nearly cosymplectic structure , we say that M is totally real submanifold of S^5 if for all $P \in M$ we have*

$$\xi \in N_pM \quad \text{and} \quad \varphi(T_pM) \subset N_pM, \tag{2.1}$$

where N_PM and T_PM denote respectively the normal space and the tangent space to M at the point P.

Let D be the standard Riemannian connection in $\mathbb{R}^7$. We denote the induced connections in S^6, S^5 and M by the previously mentioned immersions, respectively by $\tilde{\nabla}$, $\bar{\nabla}$ and ∇. Using the Gauss formula, we have

$$D_XY = \tilde{\nabla}_XY - \langle X, Y\rangle P, \ \tilde{\nabla}_XY = \bar{\nabla}_XY + \tilde{h}(X, Y)N, \ \bar{\nabla}_XY = \nabla_XY + h(X, Y),$$

where P denotes the position vector of the immersion of M into $\mathbb{R}^7$, and h, $\tilde{h}$ are the second fundamental forms of M and S^5 respectively, and X, Y are tangent vectors fields on M. It then follows that

$$D_XY = \underbrace{\underbrace{\underbrace{\underbrace{\nabla_XY}_{T_pM} + \underbrace{h(X,Y)}_{N_pM}}_{T_pS^5} + \underbrace{\tilde{h}(X,Y)N}_{N_pS^5}}_{T_pS^6} - \underbrace{\langle X, Y\rangle P}_{N_pS^6}}_{T_p\mathbb{R}^7}. \tag{2.2}$$

Remarks :

1. As we have previously mentioned, a n-dimensional manifold of a Sasakian manifold, for which ξ is normal, is always anti-invariant, i.e. $\varphi(T_PM) \subset N_PM$. In the following examples we show that this result is no longer true in the 5-sphere with nearly Sasakian structure or nearly cosymplectic structure.

2. In the case of nearly cosymplectic structure, S^5 is totally geodesic in S^6, then $\tilde{h}$ is vanishes, and the Gauss formula is given by :

$$D_X Y = \nabla_X Y + h(X, Y) - \langle X, Y \rangle P.$$

Example 1 : We look at S^2, which we parametrize in the usual way by $(cos\theta \cos\psi, \sin\theta\cos\psi, \sin\psi)$. We define a 1-parameter family of immersions in the nearly Sasakian $S^5 \subset S^6(1) \subset \mathbb{R}^7$ by

$$P = \frac{1}{\sqrt{2}}(\cos a\cos\theta\cos\psi,\ \cos a\sin\theta\cos\psi,\ \cos a\sin\psi, 1,\\ \sin a\cos\theta\cos\psi,\ \sin a\sin\theta\cos\psi,\ \sin a\sin\psi),$$

where a is an arbitrary constant. Straightforward computations yield that :

$$\frac{\partial P}{\partial\theta} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\cos a\sin\theta\cos\psi \\ \cos a\cos\theta\cos\psi \\ 0 \\ 0 \\ -\sin a\sin\theta\cos\psi \\ \sin a\cos\theta\cos\psi \\ 0 \end{pmatrix},\ \varphi(\frac{\partial P}{\partial\theta}) = \frac{1}{2}\begin{pmatrix} -\sqrt{2}\sin a\sin\theta\cos\psi - \frac{1}{2}\cos 2a\cos\theta\sin 2\psi \\ \cos\psi\left(\sqrt{2}\sin a\cos\theta - \cos 2a\sin\theta\sin\psi\right) \\ \cos 2a\cos^2\psi \\ 0 \\ \cos a\cos\psi\left(\sqrt{2}\sin\theta + 2\sin a\cos\theta\sin\psi\right) \\ \cos a\cos\psi\left(2\sin a\sin\theta\sin\psi - \sqrt{2}\cos\theta\right) \\ -2\cos a\cos^2\psi\sin a \end{pmatrix},$$

$$\frac{\partial P}{\partial\psi} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\cos a\cos\theta\sin\psi \\ -\cos a\sin\theta\sin\psi \\ \cos a\cos\psi \\ 0 \\ -\sin a\cos\theta\sin\psi \\ -\sin a\sin\theta\sin\psi \\ \sin a\cos\psi \end{pmatrix},\ \varphi(\frac{\partial P}{\partial\psi}) = \frac{1}{2}\begin{pmatrix} \cos 2a\sin\theta - \sqrt{2}\sin a\cos\theta\sin\psi \\ -\cos 2a\cos\theta - \sqrt{2}\sin a\sin\theta\sin\psi \\ \sqrt{2}\sin a\cos\psi \\ 0 \\ \cos a\left(\sqrt{2}\cos\theta\sin\psi - 2\sin a\sin\theta\right) \\ \cos a\left(2\sin a\cos\theta + \sqrt{2}\sin\theta\sin\psi\right) \\ -\sqrt{2}\cos a\cos\psi \end{pmatrix}.$$

We also get that $\xi = (-\sin a\cos\theta\cos\psi,\ -\sin a\sin\theta\cos\psi,\ -\sin a\sin\psi, 0,$
$\cos a\cos\theta\cos\psi,\ \cos a\sin\theta\cos\psi,\ \cos a\sin\psi),$

from which it follows that
$\langle\varphi(\frac{\partial P}{\partial\theta}), \xi\rangle = \langle\varphi(\frac{\partial P}{\partial\psi}), \xi\rangle = \langle\varphi(\frac{\partial P}{\partial\theta}), \varphi(\frac{\partial P}{\partial\psi})\rangle = \langle\varphi(\frac{\partial P}{\partial\theta}), \frac{\partial P}{\partial\theta}\rangle = \langle\varphi(\frac{\partial P}{\partial\psi}), \frac{\partial P}{\partial\psi}\rangle = 0$, and
$\langle\varphi(\frac{\partial P}{\partial\theta}), \frac{\partial P}{\partial\psi}\rangle = -\frac{1}{2\sqrt{2}}\cos 3a\cos\psi,\ \langle\varphi(\frac{\partial P}{\partial\psi}), \frac{\partial P}{\partial\theta}\rangle = \frac{1}{2\sqrt{2}}\cos a(1 - 2\cos 2a)\cos\psi.$
So we see that for all of these examples ξ is a normal vector to the immersion. However $\varphi(T_pM) \nsubseteq N_pM$, unless $a = \pm\frac{\pi}{6} + k\pi$, or $a = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$.

Therefore, in the nearly Sasakian case we define the notion of a totally real submanifold by demanding that ξ is normal and $\varphi(T_PM) \subset N_PM$. So for the immersions in our family which are totally real, i.e. when $a = \pm\frac{\pi}{6} + k\pi$, or $a = \frac{\pi}{2} + k\pi$, we will now compute the second fundamental form. Note that in this case, $\{\varphi(\frac{\partial P}{\partial\theta}), \varphi(\frac{\partial P}{\partial\psi}), \xi\}$ is a frame of normal space consisting of the mutually orthogonal vectors. Therefore the second fundamental form of the immersion is given by

$$h(\frac{\partial P}{\partial\zeta_1}, \frac{\partial P}{\partial\zeta_2}) = \frac{\langle\frac{\partial^2 P}{\partial\zeta_1\partial\zeta_2}, \varphi(\frac{\partial P}{\partial\theta})\rangle}{\left|\frac{\partial P}{\partial\theta}\right|^2}\varphi(\frac{\partial P}{\partial\theta}) + \frac{\langle\frac{\partial^2 P}{\partial\zeta_1\partial\zeta_2}, \varphi(\frac{\partial P}{\partial\psi})\rangle}{\left|\frac{\partial P}{\partial\psi}\right|^2}\varphi(\frac{\partial P}{\partial\psi}) + \langle\frac{\partial^2 P}{\partial\zeta_1\partial\zeta_2}, \xi\rangle\xi,$$

where $\zeta_1, \zeta_2 \in \{\theta, \psi\}$. Before fixing a, straightforward computations, show that $\langle \frac{\partial^2 P}{\partial \theta^2}, \varphi(\frac{\partial P}{\partial \psi})\rangle, \langle \frac{\partial^2 P}{\partial \theta^2}, \xi\rangle, \langle \frac{\partial^2 P}{\partial \psi^2}, \varphi(\frac{\partial P}{\partial \theta})\rangle, \langle \frac{\partial^2 P}{\partial \psi^2}, \varphi(\frac{\partial P}{\partial \psi})\rangle, \langle \frac{\partial^2 P}{\partial \psi^2}, \xi\rangle, \langle \frac{\partial^2 P}{\partial \psi \partial \theta}, \varphi(\frac{\partial P}{\partial \theta})\rangle, \langle \frac{\partial^2 P}{\partial \psi \partial \theta}, \xi\rangle$ are all vanished, and

$$\langle \frac{\partial^2 P}{\partial \theta^2}, \varphi(\frac{\partial P}{\partial \theta})\rangle = -\frac{1}{2\sqrt{2}} \cos a(1 - 2\cos 2a)\cos^2\psi \sin\psi,$$
$$\langle \frac{\partial^2 P}{\partial \psi \partial \theta}, \varphi(\frac{\partial P}{\partial \psi})\rangle = -\frac{1}{2\sqrt{2}} \cos a(1 - 2\cos 2a)\sin\psi.$$

Then, if M is totally real we find $h(\frac{\partial P}{\partial \theta}, \frac{\partial P}{\partial \theta}) = h(\frac{\partial P}{\partial \theta}, \frac{\partial P}{\partial \psi}) = h(\frac{\partial P}{\partial \psi}, \frac{\partial P}{\partial \psi}) = 0$.
Therefore we obtain our examples of totally real surfaces S^2 which are totally geodesic in the nearly Sasakian S^5.

Example 2 : We look at S^2, which we parametrize in the usual way by $(cos\theta \cos\psi, \sin\theta\cos\psi, \sin\psi)$. We define a 1-parameter family of immersions in the nearly cosymplectic $S^5 \subset S^6(1) \subset \mathbb{R}^7$ by

$$P = (\cos a\cos\theta\cos\psi,\ \cos a\sin\theta\cos\psi,\ \cos a\sin\psi, 0,$$
$$\sin a\cos\theta\cos\psi,\ \sin a\sin\theta\cos\psi,\ \sin a\sin\psi),$$

where a is an arbitrary constant.

In the same way as in the Example I, we prove that $\varphi(T_P M) \not\subset N_P M$ unless $a = \frac{\pi}{2} + k\pi$ or $a = \pm\frac{\pi}{3} + k\pi$, where $k \in \mathbb{Z}$.

Therefore we obtain our examples of totally real surfaces S^2 which are totally geodesic in the nearly cosymplectic S^5.

3 Main results

In this section we have always two possible cases : the surface M is a submanifold of the nearly Sasakian sphere S^5 or a submanifold of the nearly cosymplectic sphere S^5.

3.1 Surfaces in the nearly Sasakian 5-sphere

In this subsection, M will always denote a totally real surface of the 5-dimension nearly Sasakian sphere S^5 which we consider as a subset of $\mathbb{R}^7$. The structure of M is, as previously, built with the immersions :

$$M \hookrightarrow S^5 \hookrightarrow S^6 \hookrightarrow \mathbb{R}^7. \tag{3.1}$$

We now will start the proof of theorem 1. We divide it into three lemmas.

Lemma 1. *Let M be a totally real surface of the 5-dimensional nearly Sasakian sphere and let $\{u, v\}$ be a local orthonormal basis of tangent vector fields on M. Then $\{\xi, \varphi u, \varphi v\}$ is an orthonormal basis of the normal space $N_p M$. Moreover a basis of $\mathbb{R}^7$ is given by*

$$\{\underbrace{u,\quad v}_{T_pM}, \underbrace{\xi\quad,\quad \varphi u\quad,\quad \varphi v}_{N_pM}, \underbrace{-N}_{N_pS^5}, \underbrace{p}_{N_pS^6}\},$$

where we denote $\varphi V := \varphi(V)$.

Proof: We have $\xi = -P \times N$ and $\varphi u = P \times u - \eta(u)N$. As $\eta(u) = \langle u, \xi\rangle = 0$, it follows $\xi = -P \times N,\ \varphi u = P \times u,\ \varphi v = P \times v$. Using the properties of the cross product we get :

$$\begin{aligned}
\langle \xi, \varphi u\rangle &= \langle -P \times N, P \times u\rangle = -\langle P, P\rangle\langle N, u\rangle + \langle P, N\rangle\langle P, u\rangle = 0,\\
\langle \xi, \varphi v\rangle &= \langle -P \times N, P \times v\rangle = -\langle P, P\rangle\langle N, v\rangle + \langle P, N\rangle\langle P, v\rangle = 0,\\
\langle \varphi u, \varphi v\rangle &= \langle P \times u, P \times v\rangle = \langle P, P\rangle\langle u, v\rangle - \langle P, u\rangle\langle P, v\rangle = 0.
\end{aligned}$$

thus $\xi \perp \varphi u$, $\xi \perp \varphi v$ and $\varphi u \perp \varphi v$. In order to be able to use the multiplication table for the cross product in $\mathbb{R}^7$, we first remark that $\xi = -P \times N = u \times v$ and therefore $\{u = e_2,\ v = e_4,\ P \times u = e_3,\ P \times v = e_5,\ \xi = e_6,\ -N = e_7,\ P = e_1\}$ is a G_2 basis along the surface. □

Lemma 2. *On the surface M, for any tangent vector fields X, Y, we have $h(X, Y) \perp \xi$. Moreover, we get :*

$$\langle h(u, v), \varphi v\rangle = \langle h(v, v), \varphi u\rangle, \qquad \langle h(u, v), \varphi u\rangle = \langle h(u, u), \varphi v\rangle.$$

Proof: We know that ξ is in the normal space of M in S^5, then $\langle \xi, u\rangle = \langle \xi, v\rangle = 0$.
From $D_v\langle u, \xi\rangle = 0$ we have $\underbrace{\langle h(u, v), P \times N\rangle}_{Symmetric} + \underbrace{\langle u \times v, N\rangle}_{antisymmetric} = 0$,
so $\langle h(u, v), P \times N\rangle = 0$ and $D_v\langle u, \xi\rangle = 0 \Rightarrow h(u, v) \perp \xi$.
In the same way, from $D_u\langle u, \xi\rangle = 0$ and $D_v\langle v, \xi\rangle = 0$ respectively, we get $h(u, u) \perp \xi$ and $h(v, v) \perp \xi$. Therefore $h(X, Y) \perp \xi$, for any tangent vector fields X and Y on M. Next using that the immersion is anti-invariant, i.e. $\langle u, \varphi v\rangle = 0$ and $\langle v, \varphi u\rangle = 0$, we deduce that:

$$\begin{aligned}
D_v\langle u, \varphi v\rangle = 0 &\Rightarrow \langle h(u, v), P \times v\rangle = \langle h(v, v), P \times u\rangle,\\
D_u\langle v, \varphi u\rangle = 0 &\Rightarrow \langle h(u, v), P \times u\rangle = \langle h(u, u), P \times v\rangle.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 3. *We have $\nabla_u u = \alpha\, v,\ \nabla_v u = \beta\, v,\ \nabla_v v = -\beta\, u,\ \nabla_u v = -\alpha\, u$, where α and β are local functions on M.*

Proof: We have $\langle u, u\rangle = \langle v, v\rangle = 1$ and $\langle u, v\rangle = 0$. From $D_u\langle u, u\rangle = 2\langle D_u u, u\rangle = 0$ we have $\langle \nabla_u u, u\rangle = 0$. Then $D_u\langle u, u\rangle = 0 \Rightarrow \nabla_u u = \alpha\, v$, where $\alpha \in \mathcal{F}(M)$. In the same way, the fact that $D_v\langle u, u\rangle = D_v\langle v, v\rangle = D_u\langle v, v\rangle = D_v\langle u, v\rangle = D_u\langle u, v\rangle = 0$, we get that there exist local functions α', β and β' on M such that: $\nabla_v u = \beta\, v,\ \nabla_v v = \alpha'\, u,\ \nabla_u v = \beta'\, u$ and $\alpha' = -\beta,\ \beta' = -\alpha$. □

Now we give the proof of the theorem 1.

Proof: From lemmas 2 and 3, we have that $h \perp \xi$, therefore we get :

$$h(u,u) = a_1\varphi u + a_2\varphi v, \quad h(u,v) = b_1\varphi u + b_2\varphi v, \quad h(v,v) = c_1\varphi u + c_2\varphi v, \tag{3.2}$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ are local functions on M.
As $\xi = -P \times N = u \times v$, we can compute the covariant derivative of ξ in two different ways. Indeed, we have that $D_u\xi = D_u(-P \times N) = D_u(u \times v)$. So, we see that

$$-D_u(P \times N) = -P \times v - P \times u$$
and
$$D_u(u \times v) = (-a_1 - b_2)N + (-a_2 + b_1)P - P \times u - P \times v.$$

From these equalities, we get $(-a_1 - b_2)N + (-a_2 + b_1)P = 0$, so $\begin{cases} b_1 = a_2, \\ b_2 = -a_1. \end{cases}$

In a similar way, we obtain : $D_v\xi = D_v(-P \times N) = D_v(u \times v) \Rightarrow \begin{cases} c_1 = b_2, \\ c_2 = -b_1. \end{cases}$

Finally we get $\begin{cases} c_1 = b_2 = -a_1, \\ c_2 = -b_1 = -a_2. \end{cases}$

Therefore we have $h(u,u) = -h(v,v)$ i.e M is a minimal surface. ∎

3.2 Surface in the nearly cosymplectic 5-sphere

We use the same kind of arguments as in subsection 3.1 and M will denote a totally real surface of the nearly cosymplectic sphere S^5 which we consider as a subset of $\mathbb{R}^7$. The structure of M is built as previously, with the immersions : $M \hookrightarrow S^5 \hookrightarrow S^6 \hookrightarrow \mathbb{R}^7$. We obtain the following theorem :

Theorem 2. *A totally real surface of the nearly cosymplectic S^5 is always minimal.*

Remark

1. Changing the expression "nearly Sasakian" with "nearly cosymplectic" in the subsection 3.1, the lemmas 1, 2 and 3 remain valid and their proofs are similar. But, in the nearly cosymplectic case, $\tilde{h}$ vanishes.

2. The proof of theorem 2 is the consequence of the previous lemmas as in the subsection 3.1.

3.3 Minimal surfaces in the 5-sphere with nearly Sasakian structure or nearly cosymplectic structure

As indicated in the following proposition, we can further improve our choice of basis.

Proposition 1. *Let M be a totally real surface of the sphere S^5 with its nearly Sasakian structure, respectively nearly cosymplectic structure. Then if necessary by restricting to an open dense subset there exists a local orthonormal frame $\{u, v\}$ of M at each point P of M such that*

$$\nabla_u u = \alpha\, v, \quad \nabla_v u = \beta\, v, \quad \nabla_v v = -\beta\, u, \quad \nabla_u v = -\alpha\, u,$$
$$h(u,u) = aP \times u, \quad h(u,v) = -aP \times v,$$

where α, β are the functions defined before and a is a function on this open dense subset of M satisfying :

$$\begin{cases} v(\alpha) - u(\beta) + 2a^2 - \alpha^2 - \beta^2 - \epsilon = 0, \\ v(\beta) + u(\alpha) = 0, \\ v(a) = 3\alpha a, \\ u(a) = -3\beta a, \end{cases} \tag{3.3}$$

where $\epsilon = 2$ in the nearly Sasakian case and 1 in the nearly cosymplectic case.

First, note that if the immersion is totally geodesic on an open part, then we can take $a_1 = a_2 = 0$ (where a_1, a_2 are defined in (3.2)) and it follows from the Gauss equation that the first equations in (3.3) are satisfied.
Else, by restricting to an open dense subset, we may assume that $a_1^2 + a_2^2 \neq 0$.

Lemma 4. *Let M be a totally real surface. In a neighborhood of a non totally geodesic point of M, there exist a local function a such that the second fundamental form h can be written as*

$$h(u,u) = a\varphi u, \quad h(u,v) = -a\varphi v, \quad h(v,v) = -a\varphi u,$$

where $\{u, v\}$ is the frame of M at each point P of the neighborhood.

Proof: Using a rotation of the orthonormal frame, we write

$$\begin{cases} U = \cos\theta\, u - \sin\theta\, v, \\ V = \sin\theta\, u + \cos\theta\, v, \end{cases}$$

where θ is a differentiable function. Then we have $h(U,U) = A_1 P \times U + A_2 P \times V$.
We now want to find a function θ such that A_2 vanishes. As
$h(u,u) = a_1\varphi u + a_2\varphi v, \quad h(u,v) = a_2\varphi u - a_1\varphi v, \quad h(v,v) = -a_1\varphi u - a_2\varphi v,$
we get :

$$h(U,U) = (a_1(\cos^2\theta - \sin^2\theta) - 2a_2\cos\theta\sin\theta)P \times u \\ + (a_2(\cos^2\theta - \sin^2\theta), +2a_1\cos\theta\sin\theta)P \times v.$$

We have $P \times V = \sin\theta\, P \times u + \cos\theta\, P \times v$ and $A_2 = \langle h(U,U), P \times V\rangle = 0$ then

$$A_2 = a_1 \sin 3\theta + a_2 \cos 3\theta = 0.$$

As by assumption a_1 and a_2 do not both vanish, we see that it is sufficient to take θ such that $\cos 3\theta = -\frac{a_1}{a_1^2+a_2^2}$ and $\sin 3\theta = \frac{a_2}{a_1^2+a_2^2}$ and take $a = A_1$. □

Remark : This lemma is true regardless of whether the sphere S^5 has the structure nearly Sasakian or nearly cosymplectic.
Using the previous lemma, we complete the proof of proposition 1.

Proof: Using the frame vectors along our surface, we can compute the curvature tensor of $\mathbb{R}^7$. So, we take X and Y tangent vector fields to the surface and for Z we take any vector field belonging to our frame of $\mathbb{R}^7$. As the connection on $\mathbb{R}^7$ is flat, we have that : $R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z = 0$.
Using lemma 3 and 4 and Gauss formula, straightforward computations in the nearly Sasakian case show that

$$\begin{aligned}
R(u,v)P &= R(u,v)(P \times N) = 0,\\
R(u,v)u &= (u(\beta) - v(\alpha) - 2a^2 + \alpha^2 + \beta^2 + 2)v\\
&\quad + (3\alpha a - v(a))P \times u + (-3\beta a - u(a))P \times v,\\
R(u,v)v &= (-u(\beta) + v(\alpha) + 2a^2 - \alpha^2 - \beta^2 - 2)u\\
&\quad + (-3\beta a - u(a))P \times u + (-3\alpha a + v(a))P \times v,\\
R(u,v)(P \times u) &= (-3\alpha a + v(a))u + (3\beta a + u(a))v\\
&\quad + (u(\beta) - v(\alpha) - 2a^2 + \alpha^2 + \beta^2 + 2)P \times v,\\
R(u,v)(P \times v) &= (3\beta a + u(a))u + (3\alpha a - v(a))v\\
&\quad + (-u(\beta) + v(a) + 2a^2 - \alpha^2 - \beta^2 - 2)P \times u.
\end{aligned}$$

We deduce that :

$$\begin{cases}
v(\alpha) - u(\beta) + 2a^2 - \alpha^2 - \beta^2 - 2 = 0,\\
v(a) = 3\alpha a,\\
u(a) = -3\beta a.
\end{cases}$$

Note that, as we are working with the Levi Civita connection, we have that

$$[v,u](f) = v(u(f)) - u(v(f)) = (\nabla_v u - \nabla_u v)(f),$$

where f is an arbitrary function. With these two ways of computing the Lie bracket for the function a, we deduce that

$$-4\beta v(a) - 4\alpha u(a) - 3v(\beta)a - 3u(\alpha)a = 0.$$

From this equation, we obtain $v(\beta) + u(\alpha) = 0$.

In the same way, we get the system (3.3), with $\epsilon = 1$, in the nearly cosymplectic case. This completes the proof of the proposition. □

In order to relate our surfaces to minimal Lagrangian surfaces in the complex projective space, we will use this proposition in order to introduce suitable coordinates on the surface.

Theorem 3. *Let M be totally real surface of the 5 dimension sphere S^5 with nearly Sasakian structure or nearly cosymplectic structure, then around each non totally geodesic point, M locally corresponds to a minimal Lagrangian in the complex projective space $\mathbb{C}P^2$ with the group $SU(3)$.*

Proof: In the nearly Sasakian case, we have the differential system (3.3) :

$$\begin{cases} v(\alpha) - u(\beta) + 2a^2 - \alpha^2 - \beta^2 - 2 = 0, \\ v(\beta) + u(\alpha) = 0, \\ v(a) = 3\alpha a, \\ u(a) = -3\beta a. \end{cases}$$

As we are working on a neighborhood of a non totally geodesic point, we have that $a \neq 0$. We define a function ρ on M by $\rho = a^{-\frac{1}{3}}$. It then follows that

$$\begin{cases} u(\ln \rho) = \beta, \\ v(\ln \rho) = -\alpha, \end{cases} \qquad \text{or equivalently} \qquad \begin{cases} u(\rho) = \rho\beta, \\ v(\rho) = -\rho\alpha. \end{cases}$$

Computing now $\nabla_{\rho v}\rho u$ and $\nabla_{\rho u}\rho v$, we get :

$$\begin{aligned} \nabla_{\rho v}\rho u &= \rho\, v(\rho) u + \rho^2 \beta v = -\rho^2 \alpha u + \rho^2 \beta v, \\ \nabla_{\rho u}\rho v &= \rho\, u(\rho) v - \rho^2 \alpha u = \rho^2 \beta v - \rho^2 \alpha u. \end{aligned}$$

Hence $[\rho u, \rho v] = 0$. This implies that there exist local coordinates x and y on M such that $\frac{\partial}{\partial x} = \rho u$ and $\frac{\partial}{\partial y} = \rho v$. It now follows that $\alpha = -\frac{1}{\rho^2}\frac{\partial}{\partial y}(\rho)$ and $\beta = \frac{1}{\rho^2}\frac{\partial}{\partial x}(\rho)$. We compute $v(\alpha)$ and $u(\beta)$:

$$\begin{aligned} v(\alpha) &= \frac{1}{\rho}\frac{\partial}{\partial y}(\alpha) = \frac{2}{\rho^4}\left(\frac{\partial}{\partial y}(\rho)\right)^2 - \frac{1}{\rho^3}\frac{\partial^2}{\partial y^2}(\rho), \\ u(\beta) &= \frac{1}{\rho}\frac{\partial}{\partial x}(\beta) = \frac{-2}{\rho^4}\left(\frac{\partial}{\partial x}(\rho)\right)^2 + \frac{1}{\rho^3}\frac{\partial^2}{\partial x^2}(\rho). \end{aligned}$$

Replacing it all in the equation $v(\alpha) - u(\beta) - \alpha^2 - \beta^2 + 2a^2 - 2 = 0$, it reduces to the following differential equation :

$$-\rho\Delta\rho + \left(\frac{\partial}{\partial x}(\rho)\right)^2 + \left(\frac{\partial}{\partial y}(\rho)\right)^2 + 2\rho^{-2} - 2\rho^4 = 0.$$

If we write now $\rho = e^{\psi}$, we get :

$$\begin{cases} \frac{\partial}{\partial x}(\rho) = \frac{\partial}{\partial x}(\psi)e^{\psi}, \\ \frac{\partial}{\partial y}(\rho) = \frac{\partial}{\partial y}(\psi)e^{\psi}, \end{cases} \qquad \text{and} \qquad \begin{cases} \frac{\partial^2}{\partial x^2}(\rho) = \left(\frac{\partial}{\partial x}(\psi)\right)^2 e^{\psi} + \frac{\partial^2}{\partial x^2}(\psi)e^{\psi}, \\ \frac{\partial^2}{\partial y^2}(\rho) = \left(\frac{\partial}{\partial y}(\psi)\right)^2 e^{\psi} + \frac{\partial^2}{\partial y^2}(\psi)e^{\psi}, \end{cases}$$

and the differential equation reduces to : $2\Delta\psi - 4e^{-4\psi} + 4e^{2\psi} = 0$.
In this last equation, applying the change of coordinates $\psi = \gamma + d$, where d is constant gives : $2\Delta\gamma - 4e^{-4d}e^{-4\gamma} + 4e^{2d}e^{2\gamma} = 0$.
For $d = -\ln 2$, we get the final equation :

$$2\Delta\gamma - 64\, e^{-4\gamma} + e^{2\gamma} = 0, \tag{3.4}$$

with $\gamma = -\frac{1}{3}\ln a + \ln 2$.

In the nearly cosymplectic case, from the differential system of the Proposition 1, we get in the same way than before :

$$2\Delta\gamma - 16\, e^{-4\gamma} + e^{2\gamma} = 0, \tag{3.5}$$

with $\gamma = -\frac{1}{3}\ln a + \frac{1}{2}\ln 2$.

The differential equations (3.4) and (3.5) are elliptic equations of Ţiţeica type of the form :

$$2\Delta\psi + \epsilon Q^2 e^{-4\psi} + \lambda e^{2\psi} = 0.$$

Using the classification in [10] for both cases, we obtain that our surface M is **minimal Lagrangian in** $\mathbb{C}P^2$ with $SU(3)$ group.
This completes the proof of the theorem. $\square$

We give now two examples of totally real surfaces which are not totally geodesic in S^5, equipped with the nearly Sasakian structure or nearly cosymplectic structure.
Example 3 : We consider the sphere S^5 equipped with the nearly Sasakian structure constructed before, and the surface M defined by the position vector :

$$P = \begin{pmatrix} \frac{\cos\left(\sqrt{2}(x-y)\right)+\cos\left(\sqrt{2+\sqrt{3}}x+\sqrt{2-\sqrt{3}}y\right)+\cos\left(\sqrt{2-\sqrt{3}}x+\sqrt{2+\sqrt{3}}y\right)}{3\sqrt{2}} \\ \frac{2\sin\left(\sqrt{2}(x-y)\right)+\left(1+\sqrt{3}\right)\sin\left(\sqrt{2+\sqrt{3}}x+\sqrt{2-\sqrt{3}}y\right)+\left(\sqrt{3}-1\right)\sin\left(\sqrt{2-\sqrt{3}}x+\sqrt{2+\sqrt{3}}y\right)}{6\sqrt{2}} \\ -\frac{2\cos\left(\sqrt{2}(x-y)\right)+\left(\sqrt{3}-1\right)\cos\left(\sqrt{2+\sqrt{3}}x+\sqrt{2-\sqrt{3}}y\right)-\left(1+\sqrt{3}\right)\cos\left(\sqrt{2-\sqrt{3}}x+\sqrt{2+\sqrt{3}}y\right)}{6\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{\sin\left(\sqrt{2}(x-y)\right)-\sin\left(\sqrt{2+\sqrt{3}}x+\sqrt{2-\sqrt{3}}y\right)+\sin\left(\sqrt{2-\sqrt{3}}x+\sqrt{2+\sqrt{3}}y\right)}{3\sqrt{2}} \\ -\frac{2\cos\left(\sqrt{2}(x-y)\right)-\left(1+\sqrt{3}\right)\cos\left(\sqrt{2+\sqrt{3}}x+\sqrt{2-\sqrt{3}}y\right)+\left(\sqrt{3}-1\right)\cos\left(\sqrt{2-\sqrt{3}}x+\sqrt{2+\sqrt{3}}y\right)}{6\sqrt{2}} \\ \frac{-2\sin\left(\sqrt{2}(x-y)\right)+\left(\sqrt{3}-1\right)\sin\left(\sqrt{2+\sqrt{3}}x+\sqrt{2-\sqrt{3}}y\right)+\left(1+\sqrt{3}\right)\sin\left(\sqrt{2-\sqrt{3}}x+\sqrt{2+\sqrt{3}}y\right)}{6\sqrt{2}} \end{pmatrix}.$$

In the same way as in the example 1, we prove that this surface is totally real in S^5 but not totally geodesic. In fact, we find that the second fundamental form h is given by : $h(u,u) = \varphi u, \quad h(u,v) = -\varphi v, \quad h(v,v) = -\varphi u.$

Example 4 : We consider the sphere S^5 equipped with the nearly cosymplectic structure constructed before, and the surface M defined by the position vector

$$P = \begin{pmatrix} \frac{1}{3}\left(2\cos\left(\sqrt{\frac{3}{2}}x\right)\cos\left(\frac{y}{\sqrt{2}}\right)+\cos\left(\sqrt{2}y\right)\right) \\ \sqrt{\frac{2}{3}}\sin\left(\sqrt{\frac{3}{2}}x\right)\cos\left(\frac{y}{\sqrt{2}}\right) \\ \frac{1}{3}\sqrt{2}\left(\cos\left(\sqrt{2}y\right)-\cos\left(\sqrt{\frac{3}{2}}x\right)\cos\left(\frac{y}{\sqrt{2}}\right)\right) \\ 0 \\ \frac{1}{3}\left(\sin\left(\sqrt{2}y\right)-2\cos\left(\sqrt{\frac{3}{2}}x\right)\sin\left(\frac{y}{\sqrt{2}}\right)\right) \\ -\sqrt{\frac{2}{3}}\sin\left(\sqrt{\frac{3}{2}}x\right)\sin\left(\frac{y}{\sqrt{2}}\right) \\ \frac{1}{3}\sqrt{2}\sin\left(\frac{y}{\sqrt{2}}\right)\left(\cos\left(\sqrt{\frac{3}{2}}x\right)+2\cos\left(\frac{y}{\sqrt{2}}\right)\right) \end{pmatrix}.$$

This surface is totally real in S^5 too and its second fundamental form h has the same form as in the previous example.

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References

[1] D.E. Blair, Almost contact manifolds with Killing structure tensors, *Pacific J. of Math.*, **39** , (1971), 285–292.

[2] D.E. Blair, D.K. Showers, Almost contact manifolds with Killing structures tensors. II, *J. Diff. Geometry*, **9** , (1974), 577–582.

[3] D.E. Blair, D.K. Showers, K. Yano Nearly Sasakian Structures, *Kodai Math. Sem.*, **27**, (1976), 175–180.

[4] J. Bolton, L. Vrancken, L.M. Woodward, Totally Real minimal surfaces with noncircular ellipse of curvature in the nearly Kähler S^6, *J. London Math. Society*, **Vol. 56, no. 3** (1997), 625–644.

[5] B. Cappelletti-Montano, G. Dileo, Nearly Sasakian geometry and SU(2)-structures, *arXiv* 1410.0942, (2015).

[6] F. Dillen, B. Opozda, L. Verstraelen, L. Vrancken, Totally Real minimal surfaces of the nearly Keahler 6-sphere, *Geom. Dedicata*, **27**, (1988), 325–334.

[7] F. Dillen, L. Vrancken, Totally Real Submanifolds in $S^6(1)$ Satisfying Chen's Equality, *Transactions of the American Math. Society* **348-4** (1996), 1633–1646.

[8] T. Fukami, S. Ishihara, Almost hermitian structure on S^6, *Tohoku Math. Journal*, **7**, (1955), 151–156.

[9] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Interscience Publishers (1969).

[10] J. Loftin, I. Mcintosh, Cubic Differentials in the Differential Geometry of Surfaces, *arXiv*1310.5120 (2013).

[11] K. Yano, M. Kon, *Structures on Manifolds*, World Scientific (1984).

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