Bull. Math. Soc. Sci. Math. Roumanie
Tome 59 (107) No. 4, 2016, 303–315

Multiple low-energy solutions of a Neumann problem with variable exponents

by

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Abstract

We study a class of nonhomogeneous elliptic problems with Neumann boundary condition and involving the $p(x)$-Laplace operator and power-type nonlinear terms with variable exponent. The main result of this paper establishes a sufficient condition for the existence of infinitely many weak solutions, provided that the positive parameter is sufficiently small. We also prove that these solutions are low-energy solutions, that is, they converge to zero in an appropriate function space with variable exponent. The proof combines variational arguments with a recent symmetric version of the mountain pass lemma.

Key Words: nonhomogeneous elliptic problem; variable exponent; Neumann boundary condition; mountain pass.
2010 Mathematics Subject Classification: Primary 35J60, Secondary 58E05

1 Introduction

Let $\Omega \subset \mathbb{R}$ be a bounded domain with smooth boundary. Consider the following nonhomogeneous eigenvalue problem

$$\begin{cases} -\Delta_{p(x)} u = \lambda |u|^{q(x)-2}u, & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $p, q : \overline{\Omega} \to \mathbb{R}$ are continuous functions and $\Delta_{p(x)}$ denotes the $p(x)$-*Laplace operator*, which is defined by

$$\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u).$$

The case $p(x) = q(x)$ was considered by Fan, Zhang and Zhao [5], who established the existence of a countable family of eigenvalues of problem (1.1). Their arguments rely essentially on the Ljusternik-Schnirelmann theory. A more interesting case corresponds to *different variable exponents* p and q, due to multiple *competition effects* that can appear. To the best of our knowledge, the first such research is developed by Mihăilescu and Rădulescu [8]. They assumed that

$$1 < \min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x) < \max_{x\in\overline{\Omega}} q(x). \tag{1.2}$$

Observe that this condition cannot hold in the case of constant exponents. Under hypothesis (1.2), it is established (in the subcritical setting) that there exists $\lambda^* > 0$ such that any

$\lambda \in (0, \lambda^*)$ is an eigenvalue of problem (1.1). In particular, this result two facts:
(i) the principal eigenvalue associated to problem (1.1) is zero;
(ii) problem (1.1) has a continuous spectrum in a neighborhood of the origin, which is contrast with the cases described by the Laplace or p-Laplace operators.

A central role in the arguments developed in [8] is played by the Ekeland variational principle [4]. We also refer to Rădulescu [10] and the monograph by Rădulescu and Repovš [11], where there are established related qualitative properties, provided that condition (1.2) is replaced with other growth assumptions for the variable exponents p and q.

The present paper is inspired by the aforementioned works [8], [10] and [11]. We are concerned with a related eigenvalue problem that involves several variable exponents. This time we are interested in the influence of combined effects for a class of nonhomogeneous eigenvalue problems with *Neumann boundary condition.* We study in this paper the following nonlinear problem

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = \lambda|u|^{q(x)-2}u, & x \in \Omega \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \nu} = f(u), & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where λ is a positive parameter and ν denotes the exterior outward normal to $\partial\Omega$.

Under suitable hypotheses on the variable exponents p, q and the nonlinear boundary term f, we prove that problem (1.3) has infinitely many solutions, provided that $\lambda > 0$ is small enough.

The arguments developed in this paper show that a similar result holds if the $p(x)$-Laplace operator is replaced with other nonhomogeneous differential operators with variable exponent, for instance the *generalized mean curvature operator* defined by

$$\operatorname{div}\left((1 + |\nabla u|^2)^{[p(x)-2]/2}\nabla u\right).$$

2 Function spaces with variable exponent

In this section we recall some basic definitions and properties concerning the Lebesgue and Sobolev spaces with variable exponent. We refer to the monographs of Diening, Hästo, Harjulehto and Ruzicka [3] and Rădulescu and Repovš [11] for proofs and additional results.

Consider the set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For all $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \qquad \text{and} \qquad h^- = \inf_{x\in\Omega} h(x).$$

The real numbers h^+ and h^- will play a crucial role in our arguments and usually the *gap* between these quantities produces new results, which are no longer valid for constant exponents.

For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \{u;\ u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}.$$

This vector space is a Banach space if it is endowed with the *Luxemburg norm*, which is defined by

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)}\ dx \le 1\right\}.$$

Then $L^{p(x)}(\Omega)$ is reflexive if and only if $1 < p^- \le p^+ < \infty$ and continuous functions with compact support are dense in $L^{p(x)}(\Omega)$ if $p^+ < \infty$.

The inclusion between Lebesgue spaces also generalizes the classical framework, namely if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1 \le p_2$ in Ω then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

We denote by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ then the following Hölder-type inequality holds:

$$\left|\int_\Omega uv\ dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)} |v|_{q(x)}\,. \tag{2.1}$$

The *modular* of $L^{p(x)}(\Omega)$ is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\ dx.$$

If (u_n), $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations are true:

$$|u|_{p(x)} > 1 \quad\Rightarrow\quad |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+} \tag{2.2}$$

$$|u|_{p(x)} < 1 \quad\Rightarrow\quad |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-} \tag{2.3}$$

$$|u_n - u|_{p(x)} \to 0 \quad\Leftrightarrow\quad \rho_{p(x)}(u_n - u) \to 0. \tag{2.4}$$

We define the variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) :\ |\nabla u| \in L^{p(x)}(\Omega)\}.$$

On $W^{1,p(x)}(\Omega)$ we may consider one of the following equivalent norms

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$$

or

$$\|u\| = \inf\left\{\mu > 0;\ \int_\Omega \left(\left|\frac{\nabla u(x)}{\mu}\right|^{p(x)} + \left|\frac{u(x)}{\mu}\right|^{p(x)}\right)\ dx \le 1\right\}.$$

We define $W_0^{1,p(x)}(\Omega)$ as the closure of the set of compactly supported $W^{1,p(x)}$-functions with respect to the norm $\|u\|_{p(x)}$. When smooth functions are dense, we can also use the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

The space $(W^{1,p(x)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space. Moreover, if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1 \le p_2$ in Ω then there exists the continuous embedding $W^{1,p_2(x)}(\Omega) \hookrightarrow W^{1,p_1(x)}(\Omega)$.

Set

$$\varrho_{p(x)}(u)=\int_\Omega \left(|\nabla u(x)|^{p(x)}+|u(x)|^{p(x)}\right)dx. \tag{2.5}$$

If (u_n), $u\in W^{1,p(x)}(\Omega)$ then the following properties are true:

$$\|u\|>1 \ \Rightarrow\ \|u\|^{p^-}\le \varrho_{p(x)}(u)\le \|u\|^{p^+}\,, \tag{2.6}$$

$$\|u\|<1 \ \Rightarrow\ \|u\|^{p^+}\le \varrho_{p(x)}(u)\le \|u\|^{p^-}\,, \tag{2.7}$$

$$\|u_n-u\|\to 0 \quad\Leftrightarrow\quad \varrho_{p(x)}(u_n-u)\to 0\,. \tag{2.8}$$

Set

$$p^*(x)=\begin{cases} \dfrac{Np(x)}{N-p(x)} & \text{if } p(x)<N\\ +\infty & \text{if } p(x)\ge N.\end{cases}$$

We point out that if $p,q\in C_+(\overline{\Omega})$ and $q(x)<p^\star(x)$ for all $x\in\overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$ is compact.

Next, we define for all $x\in\overline{\Omega}$

$$p_*(x)=\begin{cases} \dfrac{(N-1)p(x)}{N-p(x)} & \text{if } p(x)<N\\ +\infty & \text{if } p(x)\ge N.\end{cases}$$

If $p,q\in C_+(\overline{\Omega})$ and $q(x)<p_*(x)$ for all $x\in\overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\partial\Omega)$ is compact.

For a constant function p, the variable exponent Lebesgue and Sobolev space coincide with the standard Lebesgue and Sobolev space. Cf. [11], the function spaces with variable exponent have some striking properties, such as:

(i) If $1<p^-\le p^+<\infty$ and $p:\overline{\Omega}\to[1,\infty)$ is smooth, then the formula

$$\int_\Omega |u(x)|^p dx = p\int_0^\infty t^{p-1}\left|\{x\in\Omega;\ |u(x)|>t\}\right| dt$$

has no variable exponent analogue.

(ii) Variable exponent Lebesgue spaces do *not* have the *mean continuity property*: if p is continuous and nonconstant in an open ball B, then there exists a function $u\in L^{p(x)}(B)$ such that $u(x+h)\notin L^{p(x)}(B)$ for all $h\in\mathbb{R}^N$ with arbitrary small norm.

(iii) An argument in the development of the theory of function spaces with variable exponent is the fact that these spaces are never translation invariant. The use of convolution is also limited, for instance the Young inequality

$$|f*g|_{p(x)}\le C\,|f|_{p(x)}\,\|g\|_{L^1}$$

holds if and only if p is constant.

3 The main result

We first introduce the main hypotheses on the variable exponents p, q and the nonlinear function f involved in problem (1.3). The growth assumptions on f are in strong relationship with the decay of the potentials p and q.

We suppose that

$$p, q \in C_+(\overline{\Omega}),\ q^- > p^+ \text{ and } q(x) < p^*(x) \text{ for all } x \in \overline{\Omega}. \tag{3.1}$$

Throughout this paper, we assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous, odd, bounded function that fulfills the following hypotheses:

$$\lim_{u \to 0+} \frac{f(u)}{u^{p^- -1}} = +\infty \tag{3.2}$$

and for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|F(u)| \leq \varepsilon |u|^{q^+} + C_\varepsilon \quad \text{for all } u \in \mathbb{R}, \text{ where } F(u) := \textstyle\int_0^t f(s)ds. \tag{3.3}$$

The energy functional associated to problem (1.3) is $E : W^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$E(u) = \int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx - \int_{\partial\Omega} F(u) d\sigma - \lambda \int_\Omega \frac{|u|^{q(x)}}{q(x)} dx,$$

for all $u \in W^{1,p(x)}(\Omega)$.

Assumptions (3.1) and (3.3) imply that E is well-defined and of class C^1. Moreover, $E(0) = 0$ and E is even. By straightforward computation we deduce that

$$\begin{aligned} E'(u)(v) \quad = \quad & \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uv \right) dx - \int_{\partial\Omega} f(u) v d\sigma - \\ & \lambda \int_\Omega |u|^{q(x)-2} uv dx, \end{aligned}$$

for all $v \in W^{1,p(x)}(\Omega)$.

We are looking for weak solutions of problem (1.3), that is, nontrivial critical points of E. Equivalently, we say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.3) if there exists $u \in W^{1,p(x)}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uv \right) dx - \int_{\partial\Omega} f(u) v d\sigma - \lambda \int_\Omega |u|^{q(x)-2} uv dx = 0,$$

for all $v \in W^{1,p(x)}(\Omega)$.

In such a case, the corresponding eigenfunction $u \in W^{1,p(x)}(\Omega) \setminus \{0\}$ is a solution of problem (1.3).

The main result of this paper is stated in what follows and it asserts two basic facts:
(i) any $\lambda > 0$ sufficiently small is an eigenvalue of problem (1.3);
(ii) to every such eigenvalue there correspond infinitely many eigenfunctions with lower and lower energies with respect to suitable topology. This fact is essentially due to the sublinear behavior of the nonlinear term f. In the case of a superlinear growth, we expect the existence of infinitely many solutions with higher and higher energies.

Theorem 1. *Suppose that hypotheses* (3.1), (3.2) *and* (3.3) *are satisfied. Then the following properties ar true.*

(i) There exists $\lambda^* > 0$ *such that for all* $\lambda \in (0, \lambda^*)$ *there are infinitely many solutions* (u_n) *of problem* (1.3).

(ii) The sequence (u_n) *converges to 0 in* $W^{1,p(x)}(\Omega)$.

We first observe that $E(0) = 0$ and E is even. We now prove that E is bounded from below.

3.1 Verification of the Palais-Smale condition

We first recall that the functional $E : W^{1,p(x)}(\Omega) \to \mathbb{R}$ satisfies the Palais-Smale condition if any sequence $(u_n) \subset W^{1,p(x)}(\Omega)$ such that

$$E(u_n) = O(1) \text{ and } \|E'(u_n)\|_{(W^{1,p(x)}(\Omega))^*} = o(1) \text{ as } n \to \infty, \tag{3.4}$$

is relatively compact.

Let $(u_n) \subset W^{1,p(x)}(\Omega)$ be a sequence such that (3.4) is satisfied. We claim that

$$(u_n) \text{ is bounded in } W^{1,p(x)}(\Omega). \tag{3.5}$$

Arguing by contradiction and passing eventually to a subsequence, we can assume that

$$\|u_n\| \to \infty \quad \text{as } n \to \infty.$$

We normalize this sequence and we set

$$v_n := \frac{u_n}{\|u_n\|} \quad \text{for all } n \geq 1.$$

Thus, up to a subsequence

$$\begin{array}{ll} v_n \to v & \text{weakly in } W^{1,p(x)}(\Omega) \\ v_n \to v & \text{in } L^{p(x)}(\Omega) \text{ and } L^{q(x)}(\Omega) \\ v_n \to v & \text{a.e. in } \Omega. \end{array}$$

Returning now to (3.4) we deduce that

$$\int_\Omega \frac{|\nabla u_n|^{p(x)} + |u_n|^{p(x)}}{p(x)} dx - \int_{\partial\Omega} F(u_n) d\sigma - \lambda \int_\Omega \frac{|u_n|^{q(x)}}{q(x)} dx = O(1) \quad \text{as } n \to \infty.$$

It follows that there exists $C_1 > 0$ such that for all $n \geq 1$

$$\frac{1}{p^+} \int_\Omega \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right) dx - \int_{\partial\Omega} F(u_n) d\sigma - \frac{\lambda}{q^-} \int_\Omega |u_n|^{q(x)} dx \leq C_1.$$

Let c be the best constant of the compact embedding of $W^{1,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$. We deduce that

$$\left(\frac{1}{p^+} - \frac{\lambda c}{q^-} \right) \int_\Omega \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right) dx - \int_{\partial\Omega} F(u_n(x)) d\sigma \leq C_1. \tag{3.6}$$

In particular, we have

$$\frac{1}{p^+} - \frac{\lambda c}{q^-} > 0 \quad \text{for all } \lambda < \lambda^*,$$

where

$$\lambda^* := \frac{q^-}{cp^+}.$$

Since (u_n) is unbounded, we can assume, up to a subsequence, that $\|u_n\| > 1$ for all $n \geq 1$. By relations (2.6) and (3.6) w deduce that for all $n \geq 1$ we have

$$\left(\frac{1}{p^+} - \frac{\lambda c}{q^-}\right) \|u_n\|^{p^-} - \int_{\partial\Omega} F(u_n(x)) d\sigma \leq C_1. \tag{3.7}$$

Dividing with $\|u_n\|^{p^-}$ in (3.7) we obtain

$$\left(\frac{1}{p^+} - \frac{\lambda c}{q^-}\right) \int_\Omega \left(|\nabla v_n|^{p^-} + |v_n|^{p^-}\right) dx - \int_{\partial\Omega} \frac{F(u_n(x))}{\|u_n\|^{p^-}} d\sigma \leq \frac{C_1}{\|u_n\|^{p^-}} \to 0 \tag{3.8}$$

as $n \to \infty$.

Next, using the fact that f is bounded, the mean value theorem yields

$$|F(u_n)| \leq \|f\|_{L^\infty} \cdot |u_n|.$$

It follows that

$$|\int_{\partial\Omega} \frac{F(u_n(x))}{\|u_n\|^{p^-}} d\sigma| \leq C \int_{\partial\Omega} \frac{|u_n(x)|}{\|u_n\|^{p^-}} d\sigma \to 0 \quad \text{as } n \to \infty, \tag{3.9}$$

since $p^- > 1$.

Fix $\lambda \in (0, \lambda^*)$. Returning now to relation (3.8) and using (3.9) in combination with $q^- > 1$, we deduce that (v_n) converges to zero, contradiction. This concludes the proof of our claim (3.5). Thus, up to a subsequence Thus, up to a subsequence

$$\begin{array}{ll} u_n \to u & \text{weakly in } W^{1,p(x)}(\Omega) \\ u_n \to u & \text{in } L^{p(x)}(\Omega),\ L^{q(x)}(\Omega) \text{ and } L^1(\partial\Omega) \\ u_n \to u & \text{a.e. in } \Omega. \end{array}$$

Using relation (3.4), we have

$$\|E'(u_n)\|_{(W^{1,p(x)}(\Omega))^*} = o(1) \text{ as } n \to \infty.$$

Therefore

$$\int_\Omega \left(|\nabla u_n|^{p(x)-2} \nabla u_n \nabla v + |u_n|^{p(x)-2} u_n v\right) dx - \int_{\partial\Omega} f(u_n) v d\sigma - \\ \lambda \int_\Omega |u_n|^{q(x)-2} u_n v dx \to 0 \tag{3.10}$$

as $n \to \infty$ for all $v \in W^{1,p(x)}(\Omega)$.

Taking $v = u_n - u$ in relation (3.10) we obtain

$$\begin{aligned}&\int_\Omega \Big(|\nabla u_n|^{p(x)-2}\nabla u_n \nabla(u_n-u) + |u_n|^{p(x)-2}u_n(u_n-u)\Big)\,dx - \\ &\int_{\partial\Omega} f(u_n)(u_n-u)d\sigma - \lambda\int_\Omega |u_n|^{q(x)-2}u_n(u_n-u)dx \to 0\end{aligned} \tag{3.11}$$

as $n \to \infty$.

Since $u_n \to u$ in $L^{q(x)}(\Omega)$ it follows that

$$\int_\Omega |u_n|^{q(x)-2}u_n(u_n-u)dx \to 0 \quad \text{as } n \to \infty.$$

Next, since f is bounded and $u_n \to u$ in $L^1(\partial\Omega)$, we obtain

$$\int_{\partial\Omega} f(u_n)(u_n-u)d\sigma \to 0 \quad \text{as } n \to \infty.$$

Returning now to relation (3.11) we deduce that

$$\limsup_{n\to\infty}\int_\Omega \Big(|\nabla u_n|^{p(x)-2}\nabla u_n \nabla(u_n-u) + |u_n|^{p(x)-2}u_n(u_n-u)\Big)\,dx \le 0. \tag{3.12}$$

Since $u_n \to u$ weakly in $W^{1,p(x)}(\Omega)$, relation (3.12) yields

$$\limsup_{n\to\infty}\|u_n\| \le \|u\|.$$

Using now Proposition III.30 of Brezis [2] in combination with the fact that $W^{1,p(x)}(\Omega)$ is a reflexive Banach space, we deduce that

$$u_n \to u \quad \text{in } W^{1,p(x)}(\Omega).$$

This shows that the energy functional E satisfies the Palais-Smale condition.

3.2 An auxiliary symmetric mountain pass lemma

We first recall the notion of *genus* of a set, see Struwe [12, pp. 108-124] for more details.

Let A be a subset of a Banach space X. We say that A is symmetric if $u \in A$ implies $-u \in A$. Assuming that A is symmetric, closed and $0 \notin A$, we define the *genus* $\gamma(A)$ of A as the smallest integer k such that there exists an odd continuous mapping from A into $\mathbb{R}^k \setminus \{0\}$. If there is no such an integer k, we set $\gamma(A) = +\infty$. We also set $\gamma(\emptyset) = 0$.

We denote by Γ_k the family of all symmetric closed subsets of A such that $0 \notin A$ and $\gamma(A) \ge k$.

Our arguments in what follows use the following symmetric mountain pass lemma established by Kajikiya [6], which provides a sequence of critical values converging to zero.

Theorem 2. *Let X be an infinite dimensional Banach space. Assume that E is real-valued functional of class C^1 defined on X such that the following hypotheses are fulfilled:*

(A1) $E(0) = 0$, E is even, and bounded from below;

(A2) E satisfies the Palais-Smale condition;

(A3) for all positive integer k, there exists $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} E(u) < 0$.

Then either (i) or (ii) below holds.

(i) There exists a sequence (u_k) such that $E'(u_k) = 0$, $E(u_k) < 0$ and (u_k) converges to zero.

(ii) There exist two sequences (u_k) and (v_k) such that $E'(u_k) = 0$, $E(u_k) = 0$, $u_k \neq 0$, $\lim_{k\to\infty} u_k = 0$, $E'(v_k) = 0$, $E(v_k) < 0$, $\lim_{k\to\infty} E(v_k) = 0$, and (v_k) converges to a nonzero limit.

If the assumptions of Theorem 2 are satisfied then the energy functional E has a sequence (u_n) of critical points (hence, solutions of problem (1.3)) with nonnegative critical values $E(u_n) \leq 0$. However, E is not bounded from below in $W^{1,p(x)}(\Omega)$. Since we are looking for nonnegative critical values, the idea is to construct a related "energy functional" $\mathcal{E}$ with good properties in the sense of Theorem 2 and such that $E(u) = \mathcal{E}(u)$ for all $u \in W^{1,p(x)}(\Omega)$ such that $E(u) < M_0$, where M_0 is a positive constant.

We first establish a lower bound of $E(u)$ for $\|u\|$ large enough. We have

$$\begin{aligned} E(u) &\geq \frac{1}{p^+}\|u\|_{p(x)}^{p^+} - \frac{\lambda}{q^-}|u|_{q(x)}^{q^+} - \int_{\partial\Omega}(\varepsilon|u|^{q^+} + C_\varepsilon)d\sigma \\ &\geq \frac{1}{p^+}\|u\|_{p(x)}^{p^+} - \left(\frac{\lambda}{q^-} + \varepsilon\right)|u|_{q(x)}^{q^+} - C_\varepsilon|\partial\Omega \\ &\geq \frac{1}{p^+}\|u\|_{p(x)}^{p^+} - c\left(\frac{\lambda}{q^-} + \varepsilon\right)\|u\|_{p(x)}^{q^+} - C_\varepsilon|\partial\Omega \\ &= C_1\|u\|_{p(x)}^{p^+} - C_2\|u\|_{p(x)}^{q^+} - C_3, \end{aligned}$$

where C_1, C_2, C_3 are positive constants.

Since $p^+ < q^+$, the function

$$\psi(t) = C_1 t^{p^+} - C_2 t^{q^+} - C_3, \quad \text{for } t \geq 0$$

achieves its (positive) maximum at some point $t_M > 0$. Set $M = \psi(t_M) > 0$. Fix $0 < M_0 < M$, hence there exists $t_0 \in (0, t_M)$ such that $\psi(t_0) = M_0$.

Let $g : [0,\infty) \to \mathbb{R}$ be a C^1-function such that $g \equiv 1$ in $[0, t_0]$ and

$$g(t) = \frac{C_1 t^{p^+} - C_3 - M}{C_2 t^{q^+}} \quad \text{if } t \geq t_M.$$

Define the functional

$$\mathcal{E}(u) = \int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)}dx - g(\|u\|)\int_{\partial\Omega} F(u)d\sigma - g(\|u\|)\lambda\int_\Omega \frac{|u|^{q(x)}}{q(x)}dx,$$

for all $u \in W^{1,p(x)}(\Omega)$. Then $\mathcal{E}$ is of class C^1, is even and satisfies the Palais-Smale. The definition of $\mathcal{E}$ shows that this functional is bounded from below. Moreover, if $\mathcal{E}(u) < M_0$ then $\|u\| < t_0$, hence $g(\|u\|) = 1$ and $E(u) = \mathcal{E}$.

We check in what follows hypothesis (A3) in the statement of Theorem 2. Fix a positive integer k. We show that there exists a positive number ρ_k such that $\gamma(A_k) \geq k$, where A_k is the symmetric closed set

$$A_k := \left\{u \in W^{1,p(x)}(\Omega);\ E(u) \leq -\rho_k\right\} \setminus \{0\}.$$

Let V denote an arbitrary k-dimensional subspace of $W^{1,p(x)}(\Omega)$. For our purpose, we claim that there exists $t > 0$ small enough such that

$$\{u \in V;\ \|u\| = t\} \subset A_k.$$

This claim follows from the fact that

$$\gamma\left(\{u \in V;\ \|u\| = t\}\right) = k$$

combined with the property $\gamma(S_1) \leq \gamma(S_2)$, provided that there exists an odd continuous mapping $\varphi : S_1 \to S_2$.

Fix $u \in W^{1,p(x)}(\Omega) \setminus \{0\}$. Using hypothesis (3.2) we have

$$\lim_{t \to 0+} A(t) = +\infty,$$

where

$$A(t) := \frac{F(tu)}{(t|u|)^{p^-}}\,.$$

Using (2.7) we obtain for $t > 0$ small enough

$$\begin{aligned} E(tu) &= \int_\Omega t^{p(x)} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx - \int_{\partial\Omega} F(tu) d\sigma - \lambda \int_\Omega \frac{t^{q(x)}|u|^{q(x)}}{q(x)} dx \\ &\leq \frac{t^{p^-}}{p^-} \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx - \frac{\lambda t^{q^+}}{q^+} \int_\Omega |u|^{q(x)} dx - t^{p^-} A(t) \int_{\partial\Omega} |u|^{p^-} d\sigma. \end{aligned}$$

Set

$$m_{k1} := \inf\left\{\int_\Omega |u|^{q(x)} dx;\ u \in V,\ \|u\| = 1\right\}$$

and

$$m_{k2} := \inf\left\{\int_\Omega |u|^{u^{p^-}} dx;\ u \in V,\ \|u\| = 1\right\}.$$

Since V is a finite dimensional vector space, it follows that m_{k1} and m_{k2} are positive numbers.

Assuming without loss of generality that $\|u\| = 1$, it follows that

$$\begin{aligned} E(tu) &\leq \frac{t^{p^-}}{p^-} - m_{k1} \frac{t^{q^+}}{q^+} - m_{k2} t^{p^-} A(t) \\ &= t^{p^-} \left(\frac{1}{p^-} - A(t) m_{k2}\right) - \frac{t^{q^+}}{q^+} m_{k1}. \end{aligned}$$

Since $p^- < q^+$, $\lim_{t\to 0+} A(t) = +\infty$, and m_{k1}, m_{k2} are positive numbers, we deduce that there exists $t > 0$ small enough such that

$$t^{p^-}\left(\frac{1}{p^-} - A(t)m_{k2}\right) - \frac{t^{q^+}}{q^+}m_{k1} =: -\rho_k < 0.$$

The same arguments show that $\mathcal{E}$ also satisfies hypothesis (A3) in the statement of Theorem 2.

3.3 Proof of Theorem 1 completed

We have established that $\mathcal{E}$ satisfies the hypotheses of Theorem 2. It follows that $\mathcal{E}$ has infinitely many critical points (u_k) with negative critical values, hence $\mathcal{E}(u_k) = c_k < 0$, where

$$c_k := \inf_{A\in\Gamma_k} \sup_{u\in A} \mathcal{E}(u).$$

The above construction shows that $\mathcal{E}(u_k) = E(u_k)$. This shows that u_k are critical points of E thus weak solutions of problem (1.3). Moreover, again by Theorem 2, the sequence (u_k) of nontrivial solutions converges to zero in $W^{1,p(x)}(\Omega)$.

Conclusions and open problems

The study developed in this paper is inspired by some of the results contained in the monograph [11]. However, the present paper is concerned with a class of *Neumann problems* with variable exponents, while the aforementioned book is mainly devoted to *Dirichlet problems* with variable exponent.

The presence of the term $|u|^{p(x)-2}u$ in the left-hand side of problem (1.3) is due to the fact that the analysis is developed in the function space $W^{1,p(x)}(\Omega)$ and not in $W_0^{1,p(x)}(\Omega)$. The reason for this choice is the same as in Example 4 of Brezis [2, Chapter VIII.4]. The analysis of the proof of Theorem 1 shows that the result remains true if the left-hand side of problem (1.3) is replaced with

$$-\Delta_{p(x)}u + \alpha|u|^{p(x)-2}u,$$

where α is a real number such that the operator $-\Delta_{p(x)}u + \alpha|u|^{p(x)-2}u$ is *coercive* in $W^{1,p(x)}(\Omega)$, that is, there exists $C > 0$ such that

$$\int_\Omega \left(|\nabla u|^{p(x)} + \alpha|u|^{p(x)}\right) dx \geq C\varrho_{p(x)}(u),$$

where $\varrho_{p(x)}(u)$ is defined in relation (2.5).

The analysis developed in this paper can be extended in a *nonsmooth multi-valued setting*, namely under weaker assumptions on the nonlinear boundary data f. This corresponds to *variational-hemivariational inequalities.* We refer to Motreanu and Rădulescu [9] for a related inequality problem with boundary contribution.

We now raise some open problems in relationship with the study developed in this paper.

Open problem 1. Problem (1.3) has been studied in the *subcritical case*, namely under the basic assumption $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, which is crucial for the verification of the Palais-Smale compactness condition. We consider that a very interesting research direction is to study the same problem in the *almost critical* setting, hence under the following assumption: there exists $x_0 \in \Omega$ such that

$$q(x) < p^*(x) \text{ for all } x \in \overline{\Omega} \setminus \{x_0\} \text{ and } q(x_0) = p^*(x_0). \tag{3.13}$$

Of course, this hypothesis is not possible if the functions p and q are *constant*. We conjecture that the result stated in Theorem 1 remains true under assumption (3.13).

Open problem 2. Inspired by the study developed in [11, Chapter 3.3], we propose to study problem (1.3) if the $p(x)$-Laplace operator is replaced with an operator with *several variable exponents*, for instance

$$\operatorname{div}\left((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u\right).$$

Open problem 3. The main result of this paper establishes sufficient conditions for the existence of a continuous spectrum in a neighbourhood of the origin. An interesting open problem is to reconsider the Neumann problem (1.3) in order to find some circumstances such that there is some $\Lambda > 0$ for which all $\lambda > \Lambda$ is an eigenvalue of problem (1.3). This situation corresponds to a concentration of the spectrum at infinity and we conjecture that the nonlinear term f should have a superlinear growth.

Open problem 4. We conjecture that the multiplicity result stated in Theorem 1 or related properties are true if the $p(x)$-Laplace operator is replaced with a general class of Leray-Lions type operators, as defined in [11, pp. 27-28]. We also refer to the pioneering paper by Leray and Lions [7] for basic properties of these operators and some relevant applications.

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Received: 18.12.2015

Accepted: 16.01.2016

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