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Intersections and Unions of Critical Independent Sets in Bipartite Graphs

by

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Abstract

Let G be a simple graph with vertex set $V(G)$, and let $\mathrm{Ind}(G)$ denote the family of all independent sets of G. The number $d(X) = |X| - |N(X)|$ is the *difference* of $X \subseteq V(G)$, and a set $A \in \mathrm{Ind}(G)$ is *critical* whenever $d(A) = \max\{d(I) : I \in \mathrm{Ind}(G)\}$ [10].

In this paper we establish various relations between intersections and unions of all critical independent sets of a bipartite graph in terms of its bipartition.

Key Words: Independent set, critical set, ker, core, diadem

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1 Introduction

Throughout this paper G is a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. If $X \subseteq V(G)$, then $G[X]$ is the subgraph of G induced by X. The *neighborhood* of $v \in V(G)$ is the set $N(v) = \{w : w \in V(G) \text{ and } vw \in E(G)\}$. The *neighborhood* of $A \subseteq V(G)$ is $N(A) = \{v \in V(G) : N(v) \cap A \neq \emptyset\}$. A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent; by $\mathrm{Ind}(G)$ we mean the family of all the independent sets of G. Let $\Omega(G)$ be the family of all maximum independent sets, and $\alpha(G) = \max\{|S| : S \in \mathrm{Ind}(G)\}$. We denote $\mathrm{core}(G) = \bigcap\{S : S \in \Omega(G)\}$ [3], and $\mathrm{corona}(G) = \bigcup\{S : S \in \Omega(G)\}$. Let $\mu(G)$ be the size of a *maximum matching*. For $X \subseteq V(G)$, the number $d(X) = |X| - |N(X)|$ is the *difference* of X. The *critical difference* $d(G)$ is $\max\{d(X) : X \subseteq V(G)\}$. An independent set $A \subseteq V(G)$ with $d(A) = d(G)$ is a *critical independent set* [10].

Theorem 1. *[1] Each critical independent set is included in some $S \in \Omega(G)$.*

Recall that if $\alpha(G) + \mu(G) = |V(G)|$, then G is a *König-Egerváry graph*. As a well-known example, each bipartite graph is a König-Egerváry graph.

Theorem 2. (i) *[2, 5] G is a König-Egerváry graph if and only if each of its maximum independent sets is critical.*

(ii) *[7] $|\mathrm{corona}(G)| + |\mathrm{core}(G)| = 2\alpha(G)$ holds for any König-Egerváry graph.*

For a graph G, let $\ker(G)$ ($\mathrm{diadem}(G)$) be the intersection (the union, respectively) of all critical independent sets of G.

Theorem 3. (i) *[4] Every graph G has a unique minimal independent critical set, namely, $\ker(G)$, and $\ker(G) \subseteq \mathrm{core}(G)$.*

(ii) *[6] If G is a bipartite graph, then $\ker(G) = \mathrm{core}(G)$.*

In this paper we demonstrate some properties of $\ker(G)$ and $\mathrm{diadem}(G)$, in König-Egerváry graphs, with emphasis on bipartite graphs.

2 Results

It is known that intersections and unions of critical sets are critical as well [4]. Consequently, diadem(G) and ker(G) are critical for every graph. The sets corona(G) and core(G) are critical for each König-Egerváry graph, but not for all graphs. Moreover, we have the following.

Theorem 4. *If G is a König-Egerváry graph, then*
(i) $\text{diadem}(G) = \text{corona}(G)$*;*
(ii) $|\ker(G)| + |\text{diadem}(G)| \leq 2\alpha(G)$*.*

Proof. *(i)* Every $S \in \Omega(G)$ is a critical set, by Theorem 2*(i)*. Hence we deduce that corona$(G) \subseteq$ diadem(G). On the other hand, for every graph each critical independent set is included in a maximum independent set, in accordance with Theorem 1. Thus, we infer that diadem$(G) \subseteq$ corona(G). Consequently, the equality diadem$(G) =$ corona(G) holds.
(ii) It follows by combining Theorem 2*(ii)*, part *(i)* and Theorem 3*(i)*. □

Following Ore [8, 9], the number $\delta(X) = d(X) = |X| - |N(X)|$ is the *deficiency* of X, where $X \subseteq A$ or $X \subseteq B$ and $G = (A, B, E)$ is a bipartite graph. Let $\delta_0(A) = \max\{\delta(X) : X \subseteq A\}$ and $\delta_0(B) = \max\{\delta(Y) : Y \subseteq B\}$. A set $X \subseteq A$ with $\delta(X) = \delta_0(A)$ is *A-critical*, while $Y \subseteq B$ with $\delta(B) = \delta_0(B)$ is *B-critical*. For a bipartite graph $G = (A, B, E)$ let us denote $\ker_A(G) = \cap\{S : S \text{ is } A\text{-}critical\}$ and $\text{diadem}_A(G) = \cup\{S : S \text{ is } A\text{-}critical\}$; $\ker_B(G)$ and $\text{diadem}_B(G)$ are defined similarly. It is convenient to define $d(\emptyset) = \delta(\emptyset) = 0$.

Theorem 5. *Let $G = (A, B, E)$ be a bipartite graph.*
(i) *[8]* $\ker_A(G) \cap N(\ker_B(G)) = N(\ker_A(G)) \cap \ker_B(G) = \emptyset$*;*
(ii) *[9] If Y is a B-critical set, then* $\ker_A(G) \cap N(Y) = N(\ker_A(G)) \cap Y = \emptyset$*.*

As expected, there is a close relationship between critical independent sets and A-critical or B-critical sets.

Theorem 6. *[6] For a bipartite graph $G = (A, B, E)$, the following are true:*
(i) $\alpha(G) = |A| + \delta_0(B) = |B| + \delta_0(A) = \mu(G) + \delta_0(A) + \delta_0(B) = \mu(G) + d(G)$*;*
(ii) *if X is A-critical and Y is B-critical, then $X \cup Y$ is a critical set;*
(iii) *if Z is a critical independent set, then $Z \cap A$ is an A-critical set and $Z \cap B$ is a B-critical set.*

Now we are ready to describe both ker and diadem of a bipartite graph in terms of its bipartition.

Theorem 7. *Let $G = (A, B, E)$ be a bipartite graph. Then the following assertions are true:*
(i) $\ker_A(G) \cup \ker_B(G) = \ker(G)$*;*
(ii) $|\ker(G)| + |\text{diadem}(G)| = 2\alpha(G)$*;*
(iii) $|\ker_A(G)| + |\text{diadem}_B(G)| = |\ker_B(G)| + |\text{diadem}_A(G)| = \alpha(G)$*;*
(iv) $\text{diadem}_A(G) \cup \text{diadem}_B(G) = \text{diadem}(G)$*.*

Proof. *(i)* By Theorem 6*(ii)*, $\ker_A(G) \cup \ker_B(G)$ is critical in G. Moreover, the set $\ker_A(G) \cup \ker_B(G)$ is independent in accordance with Theorem 5*(i)*. Assume that $\ker_A(G) \cup \ker_B(G)$ is not minimal. Therefore, the unique minimal d-critical set of G, say Z, is a proper subset of

$\ker_A(G) \cup \ker_B(G)$, by Theorem 3*(i)*.
According to Theorem 6*(iii)*, $Z_A = Z \cap A$ is an A-critical set, which implies $\ker_A(G) \subseteq Z_A$, and similarly, $\ker_B(G) \subseteq Z_B$. Consequently, we get that $\ker_A(G) \cup \ker_B(G) \subseteq Z$, in contradiction with the fact that

$$\ker_A(G) \cup \ker_B(G) \neq Z \subset \ker_A(G) \cup \ker_B(G).$$

(ii), (iii), (iv) By Theorem 5*(ii)*, we have

$$|\ker_A(G)| - \delta_0(A) + |\mathrm{diadem}_B(G)| = |N(\ker_A(G))| + |\mathrm{diadem}_B(G)| \leq |B|.$$

Thus, Theorem 6*(i)* implies $|\ker_A(G)| + |\mathrm{diadem}_B(G)| \leq |B| + \delta_0(A) = \alpha(G)$. Changing the roles of A and B, we obtain $|\ker_B(G)| + |\mathrm{diadem}_A(G)| \leq \alpha(G)$. By Theorem 6*(iii)*, $\mathrm{diadem}(G) \cap A$ is A-critical and $\mathrm{diadem}(G) \cap B$ is B-critical. Hence $\mathrm{diadem}(G) \cap A \subseteq \mathrm{diadem}_A(G)$ and $\mathrm{diadem}(G) \cap B \subseteq \mathrm{diadem}_B(G)$. It implies both the inclusion $\mathrm{diadem}(G) \subseteq \mathrm{diadem}_A(G) \cup \mathrm{diadem}_B(G)$, and the inequality $|\mathrm{diadem}(G)| \leq |\mathrm{diadem}_A(G)| + |\mathrm{diadem}_B(G)|$. Combining Theorem 3*(ii)*, Theorem 4*(i),(ii)*, and part *(i)* with the above inequalities, we deduce

$$\begin{aligned} 2\alpha(G) &\geq |\ker_A(G)| + |\ker_B(G)| + |\mathrm{diadem}_A(G)| + |\mathrm{diadem}_B(G)| \geq \\ &\geq |\ker(G)| + |\mathrm{diadem}(G)| = |\mathrm{core}(G)| + |\mathrm{corona}(G)| = 2\alpha(G). \end{aligned}$$

Consequently, we infer that

$$\begin{gathered} |\mathrm{diadem}_A(G)| + |\mathrm{diadem}_B(G)| = |\mathrm{diadem}(G)|, |\ker(G)| + |\mathrm{diadem}(G)| = 2\alpha(G), \\ |\ker_A(G)| + |\mathrm{diadem}_B(G)| = |\ker_B(G)| + |\mathrm{diadem}_A(G)| = \alpha(G). \end{gathered}$$

Finally, based on the facts: $\mathrm{diadem}(G) \subseteq \mathrm{diadem}_A(G) \cup \mathrm{diadem}_B(G)$ and $\mathrm{diadem}_A(G) \cap \mathrm{diadem}_B(G) = \emptyset$, we get $\mathrm{diadem}_A(G) \cup \mathrm{diadem}_B(G) = \mathrm{diadem}(G)$, as claimed. □

According to Theorem 4*(i)*, the equality $\mathrm{diadem}(G) = \mathrm{corona}(G)$ holds for every König-Egerváry graph. We propose the following.

Problem 1. *Characterize graphs satisfying* $\mathrm{diadem}(G) = \mathrm{corona}(G)$.

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